Games and Economic Behavior **34,** 227–237 (2001)
doi:10.1006/game.2000.0804, available online at http://www.idealibrary.com on IDEAL®

Stability and Largeness of the Core

Amit K. Biswas[1]

Indian Statistical Institute, 110, Nelson Manickam Road, Aminjikarai, Chennai-600029, India

T. Parthasarathy[2]

Indian Statistical Institute, 7, S.J.S. Sansanwal Marg, New Delhi-110016, India

and

G. Ravindran[3]

Indian Statistical Institute, 8th Mile, Mysore Road, Bangalore 560 059, India

Received December 30, 1998

In general, there are examples of TU games where the core is stable but is not large. In this paper, we show that the extendability condition introduced by Kikuta and Shapley (1986, "Core Stability in n-Person Games," Mimeo) is sufficient for the core to be stable as well as large, for TU games with five or fewer players. We provide a counter example when the number of players is six. We then introduce a stronger extendability condition and show that it is necessary and sufficient for the core to be large. Our proof makes use of a well-known result from the theory of convex sets. *Journal of Economic Literature* Classification Number: C71.

Key Words: large core; stable core; extendability.

1. INTRODUCTION

The core of a TU game is perhaps the most intuitive and easiest solution concept in Cooperative Game Theory (Peleg, 1992). Another approach to solution concepts is the stable sets introduced by von Neumann and Morgenstern (1944). The core of a TU game is said to be stable if it is a stable set in the sense of von Neumann and Morgenstern. Sharkey (1982) introduced the notion of largeness of the core and showed that it implies

[1] E-mail: amit@isid.ac.in.
[2] E-mail: tps@isid.ac.in.
[3] To whom correspondence should be addressed. E-mail: ravi@isibang.ac.in.

0899-8256/01 $35.00

that the core is a stable set. van Gellekom *et al.* (1998) have given an example of a nonsymmetric six-person TU game where the core is stable but is not large. However, in two different papers, Biswas *et al*. (2000) and Biswas and Bhattacharya (1998) have shown for symmetric TU and NTU games, the two concepts coincide. In this paper, we are concerned with sufficient conditions for a stable core to be large for TU games in general. There have been several results which deal with sufficient conditions for the core to be stable (Gillies, 1957). The survey papers of Aumann (1967), Lucas (1990, 1992), and the references therein give an excellent review of these topics. For more recent results on core and monotonic solutions see Housman and Clark (1998) and Young (1985, 1994).

Kikuta and Shapley (1986) have introduced the notion of extendability of the game in the sense that every subgame core element (whenever it exists) can be extended to a core element of the original game. They have shown that extendability implies the stability of the core. Recently van Gellekom *et al.* (1998) have given an example to show that the stability of the core need not imply the extendability property of the game. They have also given a seven-person TU game, where the game has the extendability property but the core is not large. These results immediately raise two questions: (i) Can we find a TU game with fewer than seven players satisfying extendability without having a large core? (ii) Can we strengthen the Kikuta–Shapley notion of extendability so that it will imply the largeness of the core? We answer these two questions in this article.

We introduce a different notion of extendability property, namely, that every lower boundary point of every $(n-1)$ person subgame can be extended to a core element of the original n-person game. We will show this notion to be equivalent to the largeness of the core. Therefore, we will call this concept, a bit prematurely, strong extendability. Proof of this result depends on a well-known result from the theory of convex sets. Then we show that when the number of players is at most five, extendability of the game and largeness of the core coincide. However, when the number of players is six, we give an example to show that there exists a game that has the extendability property but does not have a large core.

Our plan for the paper is as follows: In Section 2, we give some preliminaries. In Section 3, we state and prove our main results. Section 4 contains examples and further remarks.

2. PRELIMINARIES

We start with some definitions that are needed for our discussions.

DEFINITION 1. A TU-game is a pair (N, v), where N is a finite set of players with $n = |N|$, the cardinality of N. A subset $S \subseteq N$ is called a *coali-*

tion and v assigns a real number to every coalition S with the convention that $v(\phi) = 0$. This function $v : 2^N \to \mathbf{R}$ is called *the characteristic function* and $v(S)$ is the value or worth of the coalition S. The *subgame* (S, v_S) is defined by $v_S(T) = v(T)$ for all $T \subseteq S$.

DEFINITION 2. Let (N, v) be a TU-game. We write for short $x(S) = \sum_{i \in S} x_i$, where $x \in \mathbf{R}^n$ and $S \subseteq N$. The *imputation set*, $I(v)$, is defined by $I(v) = \{x \in \mathbf{R}^n : x(N) = v(N), x_i \geq v(i) \text{ for all } i \in N\}$ and the set of all *acceptable vectors*, $A(v)$, by $A(v) = \{x \in \mathbf{R}^n : x(S) \geq v(S) \text{ for all } S \subseteq N\}$.

The core $C(v)$ of (N, v) is the intersection of these two sets, that is, $C(v) = \{x \in \mathbf{R}^n : x(N) = v(N), x(S) \geq v(S) \text{ for all } S \subset N\}$.

DEFINITION 3. For any collection $\mathcal{B} = \{S_1, S_2, \ldots, S_m\}$ of nonempty subsets of $N = \{1, 2, \ldots, n\}$, we say that $\mathcal{B}$ is *N-balanced* if there exist positive numbers $y_1, y_2, \ldots, y_m$ such that for each i, $\sum_j^{i \in S_j} y_j = 1$. The vector y is called *a balancing vector* and y_js are called *balancing coefficients*.

DEFINITION 4. A TU-game (N, v) is said to be *totally balanced* if every subgame (S, v_S) of (N, v) has a nonempty core.

DEFINITION 5. Let (N, v) be a TU-game. Let (S, v_S) be a subgame. For every subgame (S, v_S) of (N, v), we define $\bar{v}(S) = \max \sum_j x_j v(S_j)$ where the maximum is taken over all balanced collection of subsets $\{S_1, \ldots, S_j, \ldots, S_m\}$ of S and x_js are the corresponding balancing coefficients. The game $(N, \bar{v})$ is called the *totally balanced cover of* (N, v).

DEFINITION 6. Let (N, v) be a TU-game with a nonempty core. The core $C(v)$ is called *stable* if for every imputation $y \in I(v) \backslash C(v)$ there exists a vector $x \in C(v)$ and a coalition $S \subset N$ such that $x(S) = v(S)$ and $x_i > y_i$ for all $i \in S$. We then say x dominates y (via S) and write $x \succ_S y$.

It is clear that if the core is stable then it is the unique von Neumann–Morgenstern solution.

A point $x \in A(v)$ is a *lower boundary point* if there is no point $y \in A(v)$ with $y \leq x$ and $y \neq x$. Alternatively, We say that x is a lower boundary point of $A(v)$, if $A(v) \cap Q_x = \{x\}$ where $Q_x = \{y \in \mathbf{R}^N : y_i \leq x_i \text{ for all } i \in N\}$. Let us write $L(v)$ for the set of lower boundary points of $A(v)$. It is a well-known fact that $L(v)$ is a nonempty set since $A(v)$ is a nonempty convex set bounded from below (see Ferguson, 1967, p. 69, for a proof).

DEFINITION 7. Let (N, v) be a TU-game with a nonempty core. The core is called *large* if for every $y \in A(v)$, there is an element $x \in C(v)$ with $x \leq y\, (x_i \leq y_i \text{ for all } i)$.

It is not difficult to check that the core $C(v)$ is large if and only if $C(v) = L(v)$. For a proof of this assertion see Sharkey (1982). We now define the notion of extendability due to Kikuta and Shapley (1986).

DEFINITION 8. Let (N, v) be a TU-game with a nonempty core. We say that (N, v) is *extendable* if for every nonempty coalition $S \subset N$ for which $C(v_S) \neq \phi$ and for every core element $y \in C(v_S)$ there is a core element $x \in C(v)$ with $x_i = y_i$ for all $i \in S$. Hence if (N, v) is *extendable* then every core element of any subgame can be extended to a core element of (N, v).

Kikuta and Shapley (1986) have shown that if the core is large for a TU-game (N, v), then (N, v) is extendable. They have also shown that if $C(v) \neq \phi$ and if (N, v) is extendable then $C(v)$ is a stable set. In a recent paper van Gellekom *et al.* (1998) have given examples to show that in general, stability of $C(v)$ need not imply extendability with a six-person game and extendability need not imply largeness of the core with a seven-person game. Our inspiration for this article comes from these two papers. In order to state our main results we need a different notion of extendability.

DEFINITION 9. Let (N, v) be a TU-game with a nonempty core. We say (N, v) is *extendable in the stronger sense* if for every $S \subset N$ with $|S| = n - 1$ and every $y \in L(v_S)$ there is a core element $x \in C(v)$ such that $x_i = y_i$ for all $i \in S$. In other words if (N, v) is extendable in the stronger sense, then any lower boundary point of any $(n - 1)$ player subgame can be extended to a core element of (N, v).

We show in this paper the following: (i) a TU-game (N, v) is extendable in the stronger sense if and only if the core $C(v)$ of (N, v) is large and (ii) a TU-game (N, v), with $n \leq 5$, is extendable if and only if the core $C(v)$ of (N, v) is large. Finally we give an example of a six-person game which is extendable but does not have a large core. These results complement the results obtained by van Gellekom *et al.* (1998).

3. EXTENDABILITY AND LARGENESS OF THE CORE

In this section we state and prove some results connecting extendability and largeness of the core. We need the following result from the theory of convex sets (Berge, 1963).

Berge's theorem. If $D_1, \ldots, D_m \subset \mathbf{R}^m$ are closed convex sets with (a) $\bigcup_i D_i$ is convex and (b) $\bigcap_{j \neq i} D_j \neq \phi$ for every index i then $\bigcap_i D_i \neq \phi$.

THEOREM 3.1. *Let (N, v) be a TU-game with a nonempty core. Then the core $C(v)$ is large if and only if the game (N, v) is extendable in the stronger sense.*

Proof. If the core is large, it is not difficult to see that the game is extendable in the stronger sense. We need to prove only the converse.

Suppose there is an element $y \in L(v)$ with $y(N) > v(N)$ and define $D_i := \{x \in C(v) : x_i \leq y_i\}$. We will prove that $\bigcap_i D_i \neq \phi$. Every core element must belong to at least one D_i, since $y(N) > v(N) = x(N)$ for any core element x. Hence we have $\bigcup_i D_i = C(v)$, a convex set. In order to apply Berge's Theorem, we must prove that $\bigcap_{j \neq i} D_j \neq \phi$ for every player i.

So take $i \in N$. We will produce a core element belonging to $\bigcap_{j \neq i} D_j \neq \phi$. Reduce $y_{N \setminus i}$ to an element $y^*_{N \setminus i} \leq y_{N \setminus i}$ with $y^*_{N \setminus i} \in L(v_{N \setminus i})$. Then $y^*_{N \setminus i}$ can be extended into a core element $x^{(i)}$. This is an element of $\bigcap_{j \neq i} D_j$. Since i is arbitrary, $\bigcap_{j \neq i} D_j \neq \phi$ for every player i.

Hence $\bigcap_i D_i \neq \phi$. Let x be such an element in $C(v)$, that is, $x_i \leq y_i$. Since y is a lower boundary point with $y(N) > v(N)$, we arrive at a contradiction. This concludes the proof of the Theorem.

THEOREM 3.2. *Let (N, v) be a TU-game with nonempty core. Suppose $|N| = n \leq 5$. Then the core $C(v)$ is large if and only if (N, v) is extendable.*

Proof. Suppose (N, v) satisfies extendability and let $y \in L(v)$ with $y(N) > v(N)$. We define $\mathcal{T}(y)$ as the collection of coalitions T with $y(T) = v(T)$.

Then $\mathcal{T}(y)$ covers N and y_T can be extended into a core element $(y_T, x_{N \setminus T}) \in C(v)$.

The following situations are not possible:

(a) There are coalitions T_1 and T_2 in $\mathcal{T}(y)$ with $T_1 \cup T_2 = N$.

(b) There is a coalition $T^* \in \mathcal{T}(y)$ and, for each player $j \notin T^*$ a coalition $T_j \in \mathcal{T}(y)$ with $j \in T_j \subseteq T^* \cup \{j\}$.

If case (a) occurs, we write $A := N \setminus T_2$, $B := T_1 \cap T_2$ and $C := N \setminus T_1$. Then $A \cup B = T_1$ and there is a core allocation $x = (y_A, y_B, x_C)$. Then $B \cup C = T_2$ and $y_B(B) + x_C(C) \geq v(B \cup C) = y_B(B) + y_C(C)$. This means $y(N) \leq x(N) = v(N)$, a contradiction.

If case (b) occurs, there exists a core allocation $x = (y_{T^*}, \{x_j\}_{j \notin T^*})$. Then $x(T_j) = x_j + y(T^* \cap T_j) \geq v(T_j) = y(T^* \cap T_j) + y_j$. Then $x(N \setminus T^*) \geq y(N \setminus T^*)$ and $x(N) \geq y(N) > v(N)$, a contradiction.

It is also not possible to have:

(c) There is a coalition $T^* \in \mathcal{T}(y)$ with $|T^*| = n - 1$.

In case (c) we have $T_1 := N \setminus i \in \mathcal{T}(y)$ for some player $i \in N$. There is also a coalition T_2 containing i. Then T_1 and T_2 are not possible by (a). We will denote by k-coalition a coalition with k elements.

Case $n = 4$. There is at least one coalition containing 1, at least one coalition containing 2, up to a coalition containing 4 in $\mathcal{T}(y)$. By case (b) these coalitions cannot be all 1-coalitions and there are no $3 = n - 1$ coalitions. So, there is a 2-coalition, say, w.l.o.g. $T_1 = (1, 2) \in \mathcal{T}(y)$. Then by the impossibility of (a), there is no coalition $T_2 \in \mathcal{T}(y)$ containing 3 and 4 and by the impossibility of (b) there are no coalitions T_2' and T_2'' in $\mathcal{T}(y)$, one containing 3 and not 4, and one containing 4 and not 3. So, there is no possibility left for $\mathcal{T}(y)$.

Case $n = 5$. There is no 4-coalition in $\mathcal{T}(y)$. Suppose $T_1 \in \mathcal{T}(y)$ with $|T_1| = 3$. Then, w.l.o.g. $T_1 = (1, 2, 3)$. There must be a coalition $T_2 \in \mathcal{T}(y)$ containing (4, 5), which is not possible by (a), or two coalitions T_2' and T_2'' in $\mathcal{T}(y)$, one containing 4 and not 5 and one containing 5 and not 4. The latter is also not possible by (b). So, every coalition in $\mathcal{T}(y)$ has size 1 or 2 and there is at least one 2-coalition, by (b).

It is not possible to have

(d) $n = 5$ and T_1 and T_2 are two disjoint 2-coalitions in $\mathcal{T}(y)$.

In case (d) we have w.l.o.g. $T_1 = (1, 2)$ and $T_2 = (3, 4)$. There is also a coalition T_3 in $\mathcal{T}(y)$ containing 5. This can be $T_3 = (5)$ or $T_3 = (p, 5)$ with $p \in \{1, 2, 3, 4\}$. Without loss of generality, we may assume that $p = 1$. Let $x = (y_1, y_2, x_3, x_4, x_5)$ be an extension of (y_1, y_2) into a core allocation of (N, v). In the first case we have $x_3 + x_4 \geq v(34) = y_3 + y_4$ and $x_5 \geq v(5) = y_5$. Then $x(N) \geq y(N) > v(N)$ and this is a contradiction. In the second case, we also have $x_3 + x_4 \geq y_3 + y_4$ and $y_1 + x_5 \geq y_1 + y_5$. Then again $x(N) \geq y(N) > v(N)$.

Now we try to find a possibility for $\mathcal{T}(y)$.

$T_1 = (12)$. By the impossibility of (b) there must be a second 2-coalition T_2 in $\mathcal{T}(y)$ and by (d) it is not disjoint from (12). Without loss of generality $T_2 = (13)$. Again by (b) there must be a third 2-coalition $T_3 \in \mathcal{T}(y)$ containing 4 or 5. Without loss of generality it contains 4, and T_3 must intersect T_1 and T_2. So $T_3 = (14)$ is the only possibility. Finally there must be a coalition T_4 containing 5. This cannot be (5) by the impossibility of (b). It is therefore a 2-coalition intersecting T_1, T_2, and T_3. Then (15) is the only possibility for T_4, but this is after all also impossible by (b) (take $T^* = (12)$). This concludes the proof of the Theorem.

Remark 1. For $n = 6$ the collection $\mathcal{T}(y)$ consisting of (12), (13), (23), (124), (135), and (236) is not in contradiction with the impossibility of (a), (b), and (c). This fact will be used in the last section for giving an example of a six-person game satisfying extendability without having a large core.

Biswas and Parthasarathy (1998) earlier considered the extension of all lower boundary points of every subgame to the core elements and showed that the core is large. Though this appears to be rather trivial, it is worthwhile to note that the strong extendability, weaker than the above, as it may look, eventually implies the original assumption as in the above paper.

Theorem 3.1 is useful in examining whether the core is large. If for some $(n - 1)$ player subgame we find a lower boundary point which cannot be extended as a core element of the original game, then the core cannot be large. We illlustrate this with an example in the next section. Following van Gellekom *et al.* (1998), we denote by $U(v)$, the set of upper vectors, that is, $U(v) = \{x : x \in \mathbf{R}^n, x(S) \geq v(S)$ for all proper coalitions $S \subset N.\}$. It is easy to see $U(v) \supseteq A(v)$. It is shown in van Gellekom *et al.* (1998), that if the core of a TU-game is large, then $z(N) \leq v(N)$ for every extreme point z of $U(v)$.

4. EXAMPLES AND REMARKS

We first give an example to show that Theorem 3.2 may not be true when the number of players is six.

EXAMPLE 4.1. Let $N = \{1, 2, 3, 4, 5, 6\}$ and v is given by $v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2\}) = 1$, $v(\{1, 2, 4\}) = v(\{1, 3, 5\}) = v(\{2, 3, 6\}) = 2$ and $v(N)$ =4 and $v(S)$ is defined suitably for other S, so that v is monotonic. For example $v(\{1, 2, 3\}) = 1.5$, $v(\{1, 2, 3, 5, 6\}) = 2$, $v(\{i\}) = 0$ and so on.

It is not hard to check that this game is extendable but the core is not large. However, we present below an indication of the proof of extendability. One can verify that $y = (1/2, 1/2, 1/2, 1, 1, 1)$ is a lower boundary point of the game and $y(N) = 4.5 > v(N) = 4$. This example shows the sharpness of Theorem 3.2.

Proof of Extendability for Example 4.1. It is enough to prove extendability of one two-player coalition, say, $\{1, 2\}$ and one three-player coalition, say, $\{1, 2, 4\}$. Let $(x_1, x_2) = (1, 0)$. Then take $(x_3, x_4, x_5, x_6) = (1, 1, 0, 1)$. It is easy to check that $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in C(v)$. If $(x_1, x_2) = (0, 1)$, then we can take $(x_3, x_4, x_5, x_6) = (1, 1, 1, 0)$, so that $x \in C(v)$. Any other subgame core element of $\{1, 2\}$ is now extendable. Similar proofs are possible for $\{1, 3\}$ and $\{2, 3\}$.

Consider the coalition $\{1, 2, 4\}$ and let $(x_1, x_2, x_4) = (1, 1, 0)$ be a core element of the subgame. Then taking $(x_3, x_5, x_6) = (0, 1, 1)$, we find $x \in C(v)$. Similarly if $(x_1, x_2, x_4) = (1, 0, 1)$ or $(0, 1, 1)$ take $(x_3, x_5, x_6) = (1, 0, 1)$ or $(1, 1, 0)$, respectively, so that $x \in C(v)$. Thus it is easy to see that the core of the subgame $\{1, 2, 4\}$ is extendable. Similar facts are true about $\{1, 3, 5\}$ and $\{2, 3, 6\}$. The rest of the of extendability property follows from the above. The following example given in van Gellekom *et al.* (1998) with seven players also shows that the game is extendable but the core is not large.

EXAMPLE 4.2. Let $N = \{1, 2, 3, 4, 5, 6, 7\}$ and v is given by $v(17) = v(47) = 2$ and $v(127) = v(137) = v(457) = v(467) = 3$. $v(N) = 7$ and $v(S) = 0$ otherwise. In this example, $(N, \bar{v})$ is extendable but the core of

$(N, \bar{v})$ is not large. Here $(N, \bar{v})$ stands for the totally balanced cover of the TU-game (N, v).

Let (N, v) be a TU-game. The restriction of v to $2^N \backslash N$ is denoted by v^o. Then (N, v^o) is called an incomplete TU-game where the value of the grand coalition, namely N, is not specified. If the game $(N, v^o, v(N) = \beta)$ has a large core, then the game $(N, v^o, v(N) \geq \beta)$ also has a large core. That is if a TU-game at $v(N) = \beta$ has a large core then the TU-game at $v(N) > \beta$ also has a large core. In other words, largeness of the core is a prosperity property (see van Gellekom *et al.*, 1998, for the precise definition of the prosperity property). Now the following question arises naturally. Given an incomplete TU-game (N, v^o) what is the least value of β so that the TU-game $(N, v^o, v(N) = \beta)$ has a large core. In van Gellekom *et al.*, it is shown that $\beta \geq \max\{z(N) : z$ is an extreme point of $U(v)\}$. It is not easy to find the least β in practical situations. One can find the least β (easily) in symmetric TU-games using the specified vectors defined in Biswas *et al.* (2000). However, the following elementary proposition gives a rough bound for β .

PROPOSITION 4.1. *Let* (N, v) *be a TU-game with* $v(\{i\}) = 0$ *for all* i *and* $v(S) \geq 0$ *for every* S. *Let* $\alpha = \max\{v(S) : S \subset N\}$. *If* $v(N) \geq (n-1) \cdot \alpha$, *then the core of the TU-game* (N, v) *is large*.

Before giving the proof of this proposition we would like to make the following remarks.

Remark 2. It is well known that every (essential) TU-game is strategically equivalent to a TU-game where the worth of any coalition S is nonnegative and the worth of every singleton coalition is zero. For a discussion on these, refer to Owen (1995).

Remark 3. If two TU-games are strategically equivalent and if one of them has a large core, then the other game also has a large core. Combining the above two remarks, it is clear that the nonnegativity assumption of v in Proposition 4.1 is not restrictive.

Remark 4. In the class of all TU-games where the number of players is finite, it is not possible to improve α given in Proposition 4.1 as the following simple example shows.

EXAMPLE 4.3. Let $N = \{1, 2, 3\}$ and v is given by $v(12) = v(13) = 1$, $v(N) = 2$ and $v(S) = 0$ otherwise. In this case $\alpha = \max\{v(12), v(13)\} = 1$ and $v(N) = (n-1) \cdot \alpha = 2$. In this example core is stable and large when $v(N) = 2$. In other words, the core is not stable when $v(N) < 2$ and the core is stable and large whenever $v(N) \geq 2$.

We now give the proof of Proposition 4.1.

Proof. Our assumption is $v(N) \geq (n-1) \cdot \alpha$ where $\alpha = \max\{v(S) : S \subset N\}$. Clearly $\alpha \geq 0$ since $v(S) \geq 0$. We will simply show that every lower boundary point of the game is a core element. This will imply the core of the game is nonempty and large. Suppose $y \in L(v)$ with $y(N) > v(N)$. Write $y = (y_1, y_2, \ldots, y_n)$. Since y is a lower boundary point, there exists a nonempty coalition S, such that $y(S) = v(S)$. If $y_{i_o} > \alpha$ for some i_o, since $y(N) > v(N)$, we can reduce y_{i_o} by an $\varepsilon(> 0)$, so that the vector $y^* = (y_{1,}y_2, \ldots, y_{i_o} - \varepsilon, y_{i_o+1} \ldots, y_n)$ is an acceptable vector, and this contradicts our assumption that $y \in L(v)$ as $y* \leq y$. Thus $y_i \leq \alpha$ for every $i \in N$. Without loss of generality, let us suppose $S = \{1, 2, \ldots, s\}$ where $y(S) = v(S)$. Suppose $|S|$ =1. Then $y = (0, y_2, \ldots, y_n)$ since $y(1) = v(1) = 0$ and consequently $y(N) \leq (n-1) \cdot \alpha \leq v(N)$, contradicting our assumption $y(N) > v(N)$. So we shall and do assume $|S| \geq 2$, where $y(S) = v(S)$. That is, $y(N) = v(S) + y(N \backslash S) \leq v(S) + (n-s) \cdot \alpha \leq (n-s+1) \cdot \alpha$ for $v(S) \leq \alpha$. Hence $v(N) < y(N) \leq (n-s+1) \cdot \alpha \leq (n-1) \cdot \alpha$, contradicting our hypothesis $v(N) > (n-1) \cdot \alpha$. Thus every $y \in L(v)$ is a core element and consequently the core of the game is large. This completes the proof of our proposition.

Proposition 4.1 is of limited use. For instance, consider Example 4.1. In that example $\alpha = 2$. The proposition tells us that the core of the game will be large if $v(N) \geq 10$, but we know from other considerations that this six-person game has a large core if $v(N) \geq 4.5$. In fact we can define $\alpha = \max\{y(N) : y \in L(v)\}$. Then any TU-game with $v(N) \geq \alpha$ will have a large core. We end this section with the following problem: Is it possible to find this efficiently through linear programming (LP)? We can formulate it as several LP problems and get the value of α, but this method requires solving an exponential number of LP problems.

In view of the above proposition and examples one may ask whether there are games in which stability, Kikuta–Shapley extendability, and largeness of the core occur at different values of $v(N)$. Indeed this can occur as the following example shows:

EXAMPLE 4.4. We consider two games. Let $N_1 = \{1, 2, 3, 4, 5, 6\}$ and v is given by $v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2\}) = 1, v(\{1, 2, 4\}) = v(\{1, 3, 5\}) = v(\{2, 3, 6\}) = 2$, $v(N) = 4$ and $v(S)$ is defined suitably for other subsets S of N_1. For example $v(\{1, 2, 3\}) = 1.5$, $v(\{1, 2, 3, 5, 6\}) = 2$, $v(\{i\}) = 0$ and so on. Let $N_2 = \{7, 8, 9, 10, 11, 12\}$ with $v(\{7, 9\}) = v(\{7, 8\}) = v(\{10, 11\}) = v(\{10, 12\}) = 1$, $v(N_2) = 3$ and $v(S)$ is suitably defined for other subsets S of N_2. Let N $= N_1 \cup N_2$ and if $S \subseteq N$ and $S = S_1 \cup S_2 \neq N$, where $S_1 \subset N_1$ and $S_2 \subset N_2$, define $v(S) = v(S_1) + v(S_2)$.

We can easily check that this game (N, v) has a stable core when $v(N) = 7$, but the game does not have extendability property and the core is not large. The game is extendable but does not have a large core when $v(N) = 8$ and has a large core (and hence a stable core and extendability property) when $v(N) = 8.5$.

ACKNOWLEDGMENTS

We are grateful to Professor Jos Potters for extensive comments on an earlier version of the paper. We also thank Professor Kikuta and an anonymous referee for their comments which helped us in improving our presentation. The second author thanks the University of Nijmegen for their hospitality during June–July, 1998, when a part of this work was done.

REFERENCES

Aumann, R. J. (1967). "A Survey of Cooperative Games without Side Payments," in *Essays in Mathematical Economics: In honour of Oskar Morgenstern*. Princeton: Princeton Univ. Press.

Berge, C. (1963). *Topological Spaces*. Oliver and Boyd.

Biswas, A. K., and Parthasarathy, T. (1998). "Necessary and Sufficient Conditions for Large Core of TU Games," Technical Report, ISI Madras.

Biswas, A. K., Ravindran, G., and Parthasarathy, T. (2000). "Stability and Largeness of Core for Symmetric Games," *Int. J. Game Theory* **29**, 11–22.

Biswas, A. K., Parthasarathy, T., Potters, J. A. M., and Voorneveld, M. (1999). "Large Cores and Exactness," *Games Econom. Behav.* **28**, 1–12.

Biswas, A. K., and Bhattacharya, A. (1998). "Stability of the Core in a Class of NTU Games: A Characterization," Technical Report, ISI Madras.

Ferguson, T. (1967). *Mathematical Statistics*. Academic Press.

Gillies, D. B. (1959). "Solutions of General Non-zero Sum Games," in *Contributions to the Theory of Games, Vol. III*, Annals of Math. Studies, 40. pp. 47–85. Princeton: Princeton Univ. Press.

Housman, D., and Clark, L. (1998). "Core and Monotonic Allocation Methods," *Int. J. Game Theory* **27**, 611–616.

Kikuta, K., and Shapley, L. S. (1986). "Core Stability in *n*-Person Games," Mimeo.

Lucas, W. F. (1990). "Developments in Stable Set Theory" in *Game Theory and Applications* (T. Ichiishi, A. Neyman, and Y. Taumann, Eds.), pp. 300–316. New York: Academic Press.

Lucas, W. F. (1992). "von-Neumann Morgenstern Stable Sets," in *Handbook of Game Theory, Vol. 1* (R. J. Aumann and S. Hart, Eds.), pp. 543–590. Elsevier Science Publishers.

Owen, G. (1995). *Game Theory*. Third Edition, Academic Press.

Peleg, B. (1992). "Axiomatizations of Core," in *Handbook of Game Theory, Vol. 1* (R. J. Aumann and S. Hart, Eds.), pp. 398–412. Elsevier Science Publishers.

Sharkey, W. W. (1982). "Cooperative Games with Large Cores," *Int. J. Game Theory* **11**(3) 175–182.

van Gellekom, J. R. G., Potters, J. A. M., and Reijneierse, J. H. (1998). "Prosperity Properties of TU Games," *Int. J. Game Theory* **28**, 211–227.

von Neumann, J., and Morgenstern, O. (1944). *Theory of Games and Economic Behaviour*. Princeton: Princeton Univ. Press.

Young, H. P. (1985). “Monotonic Solutions of Cooperative Games,” *Int. J. Game Theory* **14**, 65–72.

Young, H. P. (1994). “Cost allocation,” in *Handbook of Game Theory, Vol. II* (R. J. Aumann and S. Hart, Eds.), pp. 1193–1235. Elsevier Science Publishers.