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WEAK MIXING AND UNITARY REPRESENTATION PROBLEM

BY

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ABSTRACT. – We give an affirmative answer to the unitary representation problem on IN-groups and extension of compact group by a nilpotent group. Thus, weak mixing problem also has a positive solution on these groups, that is any ergodic and strictly aperiodic probability measure on these groups is weakly mixing. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction and preliminaries

Let G be a locally compact, σ-compact metric group with a right invariant Haar measure m. Let $\mathcal{P}(G)$ be the space of all regular Borel probability measures on G. For μ and λ in $\mathcal{P}(G)$, the convolution $\lambda * \mu$ of λ and μ is defined by $\lambda * \mu(f) = \int\int f(st)\,\mathrm{d}\lambda(s)\,\mathrm{d}\mu(t)$ for all bounded continuous functions f on G. For $n \geqslant 1$, let μ^n denote the n-fold convolution product of μ with itself. Let $L_0^1(G)$ be the m-integrable functions on G such that $\int f(t)\,\mathrm{d}m(t) = 0$. Let $\|\cdot\|_1$ be the L^1-norm on $L^1(G)$. For $\mu \in \mathcal{P}(G)$ and $f \in L^1(G)$, define $\mu * f(s) = \int f(st)\,\mathrm{d}\mu(t)$ for all $s \in G$. Then $\mu * f \in L^1(G)$ and $\|\mu * f\|_1 \leqslant \|f\|_1$.

Asymptotic behavior of random walks on G are studied by several authors ([2] and references cited there). Ergodic and weak mixing are two important properties in the study of asymptotic behavior of random walks on G and it is also interesting to find the connections between them. We say that a $\mu \in \mathcal{P}(G)$ is *ergodic* (*by convolution*) or the random walk induced by μ on G is ergodic if

$$\left\| \frac{1}{n} \sum_{k=1}^{n} \mu^k * f \right\|_1 \to 0$$

for all $f \in L_0^1(G)$ and that μ is *weakly mixing* (*by convolution*) if

$$\frac{1}{n} \sum_{k=1}^{n} \int \left| \mu^k * f(s) g(s) \right| \mathrm{d}m(s) \to 0$$

for all $f \in L_0^1(G)$ and all $g \in L^\infty(G)$.

Let X be a Banach space and $T: G \to B(X)$ be a bounded operator representation of G. The representation is called continuous (respectively, weakly continuous) if the map $t \mapsto T(t)x$ is continuous (respectively, weakly continuous) for every $x \in X$. For a $\mu \in \mathcal{P}(G)$, the μ-average $U_\mu(x) = \int T(t)x \,\mathrm{d}\mu(t)$ is defined in the strong operator topology for strongly continuous bounded representations. If X is reflexive and the representation is weakly continuous, the $U_\mu x$ is defined in the weak operator topology.

A $\mu \in \mathcal{P}(G)$ is called *adapted* if the closed subgroup generated by the support of μ is G and μ is called *strictly aperiodic* if there are no proper closed normal subgroups a coset of which contains the support of μ.

For any $\mu \in \mathcal{P}(G)$, weak mixing of μ implies ergodicity and strict aperiodicity of μ and its converse is known as *weak mixing problem* (see [2]). Lin and Wittmann proved that any ergodic and strictly aperiodic measure is weakly mixing whenever G is a SIN group or G is a Nilpotent group (see [2]). In fact Lin and Wittmann proved that the unitary representation problem also has a positive solution for probabilities on these groups. (*Unitary representation problem*: for any $\mu \in \mathcal{P}(G)$, whether adapted and strictly aperiodicity implies the strong convergence U_μ^n for all continuous unitary representations.)

In this short article we extend Theorem 3.3 of [2] to IN-groups and extension of compact groups by nilpotent groups and obtain the following result. We now recall that a locally compact groups G is called a *IN-group* if G has a compact invariant neighborhood of identity in G.

THEOREM 1.1. – *Let G be a locally compact σ-compact metrizable group. Suppose G is a IN-group or G is a extension of a compact group by a nilpotent group. Then the following are equivalent for $\mu \in \mathcal{P}(G)$:*

(1) *μ is ergodic and strictly aperiodic*;

(2) *μ is ergodic, and for every unitary representation, U_μ^n is strongly convergent*;

(3) *for every bounded continuous representation in a Banach space X,*

$$\lim_{n\to\infty} \sup_{\|x^*\|\leqslant 1} \frac{1}{n}\sum_{k=1}^{n} \left|\langle x^*, U_\mu^k x\rangle\right| = 0$$

for all $x \in N$ where N is the closed linear manifold generated by $\bigcup_{t\in G}(I - T(t))X$.

(4) *μ is weakly mixing.*

As mentioned in Remark (2) below Theorem 3.3 of [2], to prove the Theorem, it is enough to prove that (1) implies (2), that is if μ is a ergodic and strictly aperiodic probability measure, then U_μ^n converges strongly for any unitary representation. In fact in the next section, we prove a stronger result, namely if μ is adapted and strictly aperiodic, then U_μ^n converges strongly.

2. Unitary representation problem

In this section we provide an affirmative answer to unitary representation problem for measures on certain groups which includes all IN-groups. We first prove the following lemma.

LEMMA 2.1. – *Let G be a compact metrizable group and T be any unitary representation of G in a Hilbert space $\mathcal{H}$. Suppose there exists a dense subgroup H and a sequence (x_n) in $\mathcal{H}$ such that $\|T(t)x_n - x_n\| \to 0$ for all $t \in H$ and $\|x_n\| \leqslant 1$ for all $n \geqslant 1$. Then there exists a sequence (y_n) in $\mathcal{H}$ such that $\|x_n - y_n\| \to 0$ and $T(t)y_n = y_n$ for all $t \in G$ and $x_n - y_n$ is orthogonal to all common fixed points for all $n \geqslant 1$.*

Proof. – Let P be the projection on the common fixed points. Then $(I - P)\mathcal{H}$ is a closed T-invariant subspace of $\mathcal{H}$. Let $R : G \to B((I - P)\mathcal{H})$ be the representation of G defined by $R(t)x = T(t)x$ for any $t \in G$ and any $x \in (I - P)\mathcal{H}$. Then R is a bounded continuous representation of G. Also, $\|R(t)(I-P)x_n-(I-P)x_n\|\to 0$ for all $t\in H$. Since G is metrizable, there exists an adapted and strictly aperiodic probability measure μ on G such that $\mu(B)=1$ for some (countable) Borel set $B\subset H$. This implies for all $k\geqslant 1$, that $\|(I-V_\mu^k)(I-P)x_n\|\to 0$. Since G is compact, by Corollary 2.11 of [2], $\|V_\mu^n\|\to 0$ and hence $I-V_\mu^k$ is invertible for some $k\geqslant 1$. This implies that $\|(I-P)x_n\|\to 0$. This proves the result. □

We first consider extension of compact group by a nilpotent group.

PROPOSITION 2.1. – *Let G be a locally compact σ-compact metric group and T be an irreducible unitary representation of G in a Hilbert space $\mathcal{H}$. Suppose G has a compact normal subgroup K such that G/K is a nilpotent group. Suppose there exists a dense subgroup H of G and a sequence (x_n) in $\mathcal{H}$ such that $\|T(t)x_n-x_n\|\to 0$ for all $t\in H$ and $\|x_n\|\leqslant 1$ for all $n\geqslant 1$. Then either T is trivial or $\|x_n\|\to 0$.*

Proof. – Let $L=\overline{H\cap K}$. Then L is a compact group. We denote the restriction of T to L by R. Let F be the space of all common fixed points for R. By Lemma 2.1, there exists sequence (y_n) in F such that $\|x_n-y_n\|\to 0$. Since K is a normal subgroup of G, L is also a normal subgroup of G and hence F is G-invariant. Since T is irreducible, $F=(0)$ or $F=\mathcal{H}$. If $F=(0)$, then $\|x_n\|\to 0$. If $F=\mathcal{H}$. Then T is trivial on L. Now replacing G by $G/\overline{H\cap K}$, we may assume that $H\cap K=(e)$. Let $S=G/K$ and $S^{k+1}=[S^{k-1},S]$ where $S^0=S$. Since S is nilpotent, there exists a $k\geqslant 1$ such that $S^{k+1}=[S^{k-1},S]=(e)$. Then for any $t\in H^{k+1}$, we have $t\in K$. Since $H^{k+1}\subset H$ and $H\cap K=(e)$, we have $t=e$. Thus, H is a nilpotent group and hence G is a nilpotent group.

Now, since G is metrizable, there exists an adapted and strictly aperiodic probability measure μ on G such that $\mu(B)=1$ for some (countable) Borel set $B\subset H$. This implies for all $k\geqslant 1$, that $\|(I-U_\mu^k)x_n\|\to 0$. Since G is nilpotent, if T is non-trivial, Corollary 2.6 of [2] implies that $\|U_\mu^n\|\to 0$ and hence $I-U_\mu^k$ is invertible for some $k\geqslant 1$. This implies that $\|x_n\|\to 0$. This proves the result. □

THEOREM 2.1. – *Let G be a locally compact σ-compact metrizable group and μ be an adapted and strictly aperiodic probability measure on G. Suppose G has a compact normal subgroup K such that G/K*

is nilpotent. Then $\|U_\mu^n\| \to 0$ *for any non-trivial irreducible unitary representation.*

Proof. – Let T be an irreducible unitary representation of G. Suppose $\|U_\mu^n\| \not\to 0$. Then by Theorem 2.3 of [2], there exists a dense subgroup H of G and sequence (x_n) such that $\|T(t)x_n - x_n\| \to 0$ for all $t \in H$ and $\|x_n\| = 1$ for all $n \geqslant 1$. By Proposition 2.1 we get that, T is trivial. Thus, $\|U_\mu^n\| \to 0$ for all non-trivial irreducible unitary representation of G. □

We now consider unitary representation problem for IN-groups.

PROPOSITION 2.2. – *Let G be a σ-compact locally compact metrizable group and T be a irreducible unitary representation of G. Suppose G is a IN-group, that is G has a compact invariant neighborhood of identity. Suppose there exists a dense subgroup H and sequence (x_n) such that* $\|T(t)x_n - x_n\| \to 0$ *for all* $t \in H$ *and* $\|x_n\| = 1$ *for all* $n \geqslant 1$. *Then* $G/\mathrm{Ker}(T)$ *is a SIN-group, that is group with a basis of invariant neighborhoods at identity.*

Proof. – By Theorem 2.5 of [1], there exists a compact normal subgroup K such that G/K is a SIN-group and in fact K is the intersection of all compact invariant neighborhoods of identity. Let N be the subgroup generated by a compact symmetric invariant neighborhood of e. Then N is a compactly generated open normal subgroup of G such that the closure of each conjugacy class is compact and hence by Theorem 3.20 of [1], the closure of the derived subgroup N' of N is compact.

Let L be the closure of N'. Since N is open, $N \cap H$ is dense in N. This implies that the derived subgroup of $N \cap H$, say D is dense in L. Since N is a normal subgroup of G, L is also a normal subgroup of G. Let F be the space of fixed points for L. Since L is normal, F is G-invariant. Since T is irreducible, $F = (0)$ or $F = \mathcal{H}$.

Suppose $F = (0)$, then by Lemma 2.1, $\|x_n\| \to 0$. This is a contradiction. Thus, $F = \mathcal{H}$ and hence $N' \subset \mathrm{Ker}(T)$. Let $R : G/L \to B(\mathcal{H})$ be defined $R(tL) = T(t)$ for all $t \in G$. Then $\|R(tL)x_n - x_n\| \to 0$ for all $t \in H$. Since N is metrizable, there exists an adapted and strictly aperiodic probability measure λ on N/L such that $\lambda(B) = 1$ for some (countable) Borel set $B \subset (N \cap H)L/L$. Then $\|(I - V_\lambda^k)x_n\| \to 0$ for all $k \geqslant 1$. Since $\|x_n\| = 1$, $\|V_\lambda^k\| = 1$ for any $k \geqslant 1$. Since N/L is abelian, by Theorem 2.11 of [2], there exists a sequence (y_n) in $\mathcal{H}$ such that

$\|R(tL)y_n - y_n\| \to 0$ for all $t \in N$ and $\|y_n\| = 1$ for all $n \geqslant 1$. Since N/L is a compactly generated abelian group, it has a maximal compact subgroup. Let M be a closed subgroup of N containing L such that M/L is the maximal compact subgroup of N/L. Since N/L is a normal subgroup G/L, M/L is also a normal subgroup of G/L. Let E the space of fixed points for M/L. Then E is (G/L)-invariant. Since T is irreducible, R is irreducible. This implies that $E = (0)$ or $E = \mathcal{H}$. Suppose $E = (0)$, then by Lemma 2.1, $\|y_n\| \to 0$. This is a contradiction. Thus, $E = \mathcal{H}$. This implies that $M/L \subset \mathrm{Ker}(R)$. Since $L \subset \mathrm{Ker}(T)$, $M \subset \mathrm{Ker}(T)$. Also, M is a compact normal subgroup such that N/M is an abelian group having no compact subgroups, that is M is the largest compact subgroup of N. Since K is the intersection of all compact invariant neighborhoods of identity, $K \subset N$ and hence $K \subset M$. This implies that $K \subset \mathrm{Ker}(T)$. This proves the result. □

THEOREM 2.2. – *Let G be a σ-compact locally compact metrizable group and μ be an adapted and strictly aperiodic probability measure on G. Suppose G is a IN-group, that is G has a compact invariant neighborhood of identity. Then U_μ^n converges strongly for any irreducible unitary representation.*

Proof. – Suppose $\|U_\mu^n x\| \not\to 0$ for some x, then $\|U_\mu^n\| \not\to 0$. By Theorem 2.3 of [2], there exist a dense subgroup H and a sequence (x_n) such that $\|T(t)x_n - x_n\| \to 0$ with $\|x_n\| = 1$ for all $t \in H$. By Proposition 2.2, $G/\mathrm{Ker}(T)$ is a SIN-group. Thus, T actually defines a representation R of $G/\mathrm{Ker}(T)$ which is a SIN-group, by Theorem 3.3 of [2], $U_\mu^n x = V_\lambda^n x$ converges strongly where λ is the image of μ in $G/\mathrm{Ker}(T)$ and V is associated to R. Since $U_\mu^n x \not\to 0$, T has a non-trivial common fixed point. Since T is irreducible, T is trivial. Thus, U_μ^n converges strongly. □

3. Proof of Theorem 1.1

The implications (2) ⇒ (3) ⇒ (4) ⇒ (1) hold for any metrizable group. We now prove that (1) implies (2). Suppose μ is a ergodic and strictly aperiodic probability measure. Then μ is adapted. Now, Theorem 2.1 and Theorem 2.2 imply that U_μ^n converges strongly for any irreducible unitary representation of G. An application of Theorem 2.2 of [2] yields the result.

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