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Local ergodic theorems for K-spherical averages on the Heisenberg group

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Abstract. Given a Gelfand pair (H^n, K) where H^n is the Heisenberg group and K is a compact subgroup of the unitary group $U(n)$ we consider the sphere and ball averages of certain $K-$invariant measures on H^n. We prove local ergodic theorems for these measures when $n \geq 3$. We also consider averages over annuli in the case of reduced Heisenberg group and show that when the functions have zero mean value the maximal function associated to the annulus averages behave better than the spherical maximal function. We use square function arguments which require several properties of the $K-$spherical functions.

1 Introduction and the main results

The aim of this paper is to prove local ergodic theorems for certain one parameter families of probability measures on the Heisenberg group associated to Gelfand pairs. Let H^n denote the Heisenberg group which is simply $\mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z,t)(w,s) = (z+w, t+s+\frac{1}{2}Im(z.\bar{w}))$$

where $z, w \in \mathbb{C}^n, t, s \in \mathbb{R}$. Let (H^n, K) be a Gelfand pair where K is a subgroup of the unitary group $U(n)$. Given a point $w \in \mathbb{C}^n$ there is a measure $\mu_{K.w}$ which is supported on $K.w$, the $K-$orbit through w. This measure is defined by the equation

$$f * \mu_{K.w}(z,t) = \int_K f((z,t)(k.w,0)^{-1})dk$$

where dk is the normalised Haar measure on K. This measure in general is very singular being supported on lower dimensional subsets and depends on several parameters known as the fundamental invariants associated to the Gelfand pair (see the work of Benson, Jenkins and Ratcliff [2]). Some examples are given towards the end of this section.

By averaging the measures $\mu_{K.w}$ over spherical subsets of $\mathbb{C}^n$ we can construct one parameter families of probability measures which are still singular. It is an interesting problem to study pointwise ergodic properties of such families. When we consider the average over the sphere $S_r = \{(z, 0) : |z| = r\}$ with respect to the surface measure μ_r^{2n-1} normalised so that $\mu_r^{2n-1}(S_r) = 1$ then it turns out that

$$\int_{S_r} f * \mu_{K.w}\, d\mu_r^{2n-1} = f * \mu_r$$

where $f * \mu_r$ is the spherical means of f. The ergodic properties of this family have been studied in [17]. In this paper we are interested in averages over still lower dimensional sets. More precisely we will consider averages over balls and spheres in $\mathbb{R}^n$.

Before we state our main theorems we recall a couple of definitions. Let G be a locally compact second countable group acting on a standard Borel measure space (X, B, m) where m is $\sigma-$finite. The action is denoted by $(g, x) \to g.x$ and let the action preserve m. Without loss of generality we can assume that X is a locally compact metric space and the action is jointly continuous. There is a natural isometric representation of G on $L^p(X), 1 \leq p \leq \infty$ given by $\pi(g)f(x) = f(g^{-1}.x)$. We say that the action is ergodic if there are no $G-$invariant functions in $L^2(X)$ other than constants.

Given a complex bounded Borel measure σ on G we can define an operator $\pi(\sigma)$ on $L^p(X)$ by

$$\pi(\sigma)f(x) = \int_G \pi(g)f(x)\, d\sigma.$$

When G is acting on $L^p(G)$ by left translations we use the notation $f * \sigma$ rather than $\pi(\sigma)$. If the group is unimodular then $\sigma \to \pi(\sigma)$ turns out to be a norm continuous star representation of the involutive Banach algebra $M(G)$ of complex Borel measures on G as an algebra of operators on $L^2(X)$. Consider a one parameter family of probability measures $\sigma_r, r > 0$ on G. We say that $\{\sigma_r\}$ is a local ergodic family in L^p if for every ergodic action of G on (X, B, m) and for every $f \in L^p(X)$ the limit $\lim_{r \to 0} \pi(\sigma_r)f(x) = f(x)$ exists for $m-$almost every x and also in the L^p norm.

In this paper we are concerned with measures $\mu_{K.x}$ supported on orbits through real points $x \in \mathbb{R}^n$. We study ergodic properties of the spherical averages

$$\sigma_r = \int_{|x|=r} \mu_{K.x} d\mu_r^{n-1}$$

where μ_r^{n-1} is the normalised surface measure on the sphere of radius r in $\mathbb{R}^n$ and the ball averages

$$\nu_r = \int_{|x|\leq r} \mu_{K.x} dx.$$

Note that these measures are singular and they depend on the group K. In fact, they are supported on a union of $K-$ orbits through subsets of $\mathbb{R}^n$. We now state our main results.

Theorem 1.1 *The ball averages ν_r is a local ergodic family in L^p for all $1 < p < \infty$ in any dimension.*

As we are assuming that the action of H^n on the measure space is jointly continuous, it follows that $\pi(\nu_r)f(x)$ converges to $f(x)$ pointwise for all continuous functions. Since such functions form a dense class in L^p the above theorem will follow once we prove the following maximal theorem. Let

$$M_\nu f(x) = \sup_{r>0} |\pi(\nu_r)f(x)|$$

be the associated maximal function.

Theorem 1.2 *Let $n \geq 1$ and $1 < p < \infty$. Then the maximal function $M_\nu f$ is measurable and satisfies the estimate $||M_\nu f||_p \leq C||f||_p$ for all $f \in L^p(X)$.*

As we will see in the proof the maximal theorem for ν_r is an easy consequence of Birkhoff's theorem for the action of $\mathbb{R}$. What is not so easy is the following maximal theorem for the sphere averages. Let M_σ be the maximal function associated to the family σ_r.

Theorem 1.3 *Let $n \geq 3$ and $p > \frac{n}{n-1}$. Then the maximal function $M_\sigma f$ is measurable and satisfies $||M_\sigma f||_p \leq C||f||_p$ for all $f \in L^p(X)$.*

As above the density of compactly supported continuous functions in $L^p(X)$ and the maximal theorem yields the following result.

Theorem 1.4 *Let $n \geq 3$ and $p > \frac{n}{n-1}$. Then the sphere averages σ_r is a local ergodic family in L^p.*

For the action of the Heisenberg group on itself the last theorem is an instance of sphere differentiation on the Heisenberg group. The case of $\mathbb{R}^n$ is the celebrated theorem of Stein [19]. In an earlier paper [17] we considered the sphere differentiation corresponding to the case $K = U(n)$ which gives us the spherical means as noted above. For this case it was shown that the spherical means converge almost everywhere for $f \in L^p(H^n)$ for all $p > \frac{2n-1}{2n-2}$. The sphere averages treated in this paper are more singular than the spherical means.

Pointwise ergodic theorems for various groups have been studied by several authors. The case of $\mathbb{R}^n$ is treated in [11], the case of simple Lie groups in [14] and [15]. For the case of semi simple groups see [16], and also the references given there. It would be interesting to see if we can obtain pointwise ergodic theorems which considers the limits as r tending to infinity in our set up. What is lacking is a dense class of functions in $L^p(X)$ for which the ergodic averages will converge as r goes to infinity.

In proving the maximal theorem we closely follow Stein-Wainger [19] in their proof of the spherical maximal theorem. In place of the Fourier transform we will use expansion in terms of spherical functions associated to the Gelfand pair under consideration. As the measures we consider are $K-$ invariant we can expand them in terms of spherical functions. As in [19] we use square functions and analytic interpolation.

In Sect. 4 of this paper we study the maximal function associated to shell averages or averages over annuli of fixed thickness. That is we consider the maximal function

$$Af(z,t) = sup_{r>0}\left|\int_r^{r+1} f * \mu_s(z,t)ds\right| \tag{1.1}$$

where $f * \mu_s$ are the spherical means on the Heisenberg group. In the case of $\mathbb{R}^n$ the maximal function associated to averages over annuli of thickness one are bounded on $L^p(\mathbb{R}^n)$ if and only if $p > \frac{n}{n-1}$. This can be seen by a scaling argument: any estimate for the annulus maximal function will imply the same estimate for the spherical maximal function. On the other hand the situation is different in the case of semi-simple Lie groups where the balls have exponential volume growth. It was shown recently by Nevo and Stein in [16] that in the case of semisimple Lie groups the maximal functions associated to annuli of fixed thickness and balls have the same L^p mapping properties.

Naturally one is curious to know what happens in the case of the Heisenberg group. Again a dilation argument shows that any L^p estimate for the maximal function Af leads to the same estimate for the spherical maximal function. On the other hand the situation is quite different in the case of the reduced Heisenberg group. Recall that the reduced Heisenberg group H^n_{red}

is simply the group $\mathbb{C}^n \times T$ with the group law

$$(z, e^{it})(w, e^{is}) = (z + w, e^{i(t+s+\frac{1}{2}Im(z.\bar{w}))}).$$

For functions on the reduced Heisenberg group which have mean value zero the maximal function Af has a better behaviour than the spherical maximal function. More precisely, we have the following result. Presumably, Af is not bounded on all L^p spaces though we do not have a counter example.

Theorem 1.5 *Let $n \geq 2$ and consider functions in $L^p(H^n_{red})$ which satisfy the mean zero condition $\int_0^{2\pi} f(z,t)dt = 0$. Then the maximal function Af associated to the annulus averages is bounded on $L^p(H^n_{red})$ for all $p > \frac{2n+1}{2n}$.*

In the paper [17] mentioned earlier the authors have restricted to the case $n \geq 2$. When $n = 1$ the spherical maximal function is not expected to be bounded on $L^2(H^1)$ and so we cannot make use of the square function argument. In that case it is conjectured that the spherical maximal function is bounded on L^p for all $p > 2$. The situation is very much like the Euclidean case. For spherical averages on $\mathbb{R}^n$ Stein proved his theorem only for $n \geq 3$. The case $n = 2$ was settled much later by Bourgain [1] using a different argument. However, for certain annulus averages of the spherical means on H^1 we can prove a maximal theorem.

Consider the spherical means $f * \mu_r$ on H^1 given by

$$f * \mu_r(z,t) = \frac{1}{2\pi} \int_0^{2\pi} f((z,t)(re^{i\theta}, 0)^{-1}) d\theta.$$

If $\sup_{r>0} |f * \mu_r|$ were bounded on L^p then so would be the maximal function for the annulus averages:

$$Mf(z,t) = \sup_{r \geq 1} |\int_{r-1}^{r+1} f * \mu_{\sqrt{s}}(z,t) ds|. \tag{1.2}$$

More generally, we can consider the maximal function

$$M_\varphi f(z,t) = \sup_{r>0} |\int_0^\infty \varphi(r-s) f * \mu_{\sqrt{s}}(z,t) ds|$$

where φ is an integrable function on $\mathbb{R}$.

Theorem 1.6 *Let the function φ satisfy the condition*

$$\int_{-\infty}^{\infty} |\hat{\varphi}(t) t^{-1}| dt < \infty.$$

Then the maximal function $M_\varphi f$ is bounded on $L^p(H^1)$ for all $p \geq 2$.

The function $\varphi(t) = t\chi_{(-1,1)}(t)$ where $\chi_{(-1,1)}$ is the characteristic function of the interval $(-1,1)$ satisfies the condition of the theorem. Consequently the maximal function

$$M_\varphi f(z,t) = \sup_{r>1} \left| \int_{r-1}^{r+1} (t-s) f * \mu_{\sqrt{s}}(z,t)\, ds \right|$$

is bounded on $L^p(H^1)$ for $p \geq 2$. This maximal function is not the same as (1.2). However, for the reduced Heisenberg group we do have the following result.

Corollary 1.7 *The maximal function (1.2) is bounded on $L^p(H^1_{red})$ for $p \geq 2$ provided $\int_0^{2\pi} f(z,t)dt = 0$. Otherwise, it is bounded only for $p > 2$.*

As we have already remarked we need a different argument to deal with the annulus averages when $n = 1$. We use the following simple idea. Suppose we are interested in the maximal function $\sup_{r>0} |T_r f(x)|$. Extend the definition of T_r to all $r \in \mathbb{R}$ by setting it zero for $r < 0$. If we can take the Fourier transform of $T_r f(x)$ in the r variable then we have

$$\sup_{r>0} |T_r f(x)| \leq \int_{-\infty}^{\infty} |\hat{T}_s f(x)| ds.$$

So, it is enough to show that $\hat{T}_s f$ is bounded on L^p with norm, say $C(s)$ satisfying $\int_{-\infty}^{\infty} C(s) ds < \infty$. Since the spherical means involve Laguerre functions of type zero whose Fourier transforms are explicitly known we can make use of this method. For more about this kind of philosophy to deal with maximal functions we refer to Cowling [7].

We end this section with a couple of examples. The following examples show that the measures we consider are supported on very thin sets.

As we have already remarked the Gelfand pair $(H^n, U(n))$ leads to the spherical means $f * \mu_r$ studied in [17]. Let $n = 2$ and consider the pair $(H^2, T(2))$ where $T(2)$ is the 2-torus acting on $\mathbb{C}^2$. Writing (z_1, z_2, t) for the elements of H^2 we see that the measure $\mu_{K.x}$ supported on the $K-$ orbit through $x = (x_1, x_2)$ is given by

$$\begin{aligned} &f * \mu_{K.x}(z,t) \\ &= (2\pi)^{-2} \int_0^{2\pi} \int_0^{2\pi} f((z_1, z_2, t)(e^{i\theta_1} x_1, e^{i\theta_2} x_2, 0)^{-1}) d\theta_1 d\theta_2. \end{aligned}$$

Writing $x = (r\cos\varphi, r\sin\varphi)$ and integrating with respect to φ we obtain

$$\begin{aligned} &f * \sigma_r(z,t) \\ &= (2\pi)^{-3} \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} f((z_1, z_2, t)(e^{i\theta_1} r\cos\varphi, e^{i\theta_2} r\sin\varphi, 0)^{-1}) d\theta_1 d\theta_2 d\varphi. \end{aligned}$$

Another example is provided by the group $K = SO(n, \mathbb{R}) \times T$ where $SO(n, \mathbb{R})$ is the special orthogonal group and T is the torus. Here, the action of $SO(n, \mathbb{R})$ is given by $\sigma.z = \sigma.x + i\sigma.y$ if $z = x + iy$ and T acts on $\mathbb{C}^n$ by scalar multiplication. This Gelfand pair and the associated spherical functions have been studied in [3]. In this case there are two kinds of orbits. When $w = u + iv$ with u and v linearly dependent then the orbit $K.w$ is isomorphic to $S^{n-1} \times T$. When u and v are linearly independent the orbit is isomorphic to $V_{n,2} \times T$ where $V_{n,2}$ is the compact Stiefel manifold of orthonormal two frames in $\mathbb{R}^n$. When $w = x \in \mathbb{R}^n$ the measure $\mu_{K.x}$ is given by

$$f * \mu_{K.x}(z, t) = (2\pi)^{-1} \int_{SO(n)} \int_0^{2\pi} f((z, t)(e^{i\theta} k.x, 0)^{-1}) dk d\theta.$$

From this it is clear that the measure $\mu_{K.x}$ depends only on $|x|$. If we let $x = rx'$ with $|x'| = 1$ then the sphere averages σ_r associated to $\mu_{K.x}$ is given by

$$f * \sigma_r(z, t) = (2\pi)^{-1} \int_0^{2\pi} \int_{SO(n)} f((z, t)(re^{i\theta} x', 0)^{-1}) d\theta dx'.$$

We end this section with the following remarks. We started this investigation with the aim of proving pointwise ergodic theorems for $K-$spherical means associated to Gelfand pairs (H^n, K). The particular case when $K = U(n)$ was treated in Nevo- Thangavelu [17]. When we tried to use the same circle of ideas we encountered the following problems. First one has to establish a maximal theorem and then one has to prove convergence on a dense class of functions. In this paper we have restricted ourselves to the problem of studying the L^p boundedness of the maximal functions associated to the $K-$spherical averages. The second problem should be tractable once we have fairly good estimates on the associated $K-$spherical functions. We hope to return to this problem in the future.

In order to prove the maximal theorem we use square function arguments which depend heavily on good estimates for the $K-$spherical functions. In the general situation the $K-$spherical functions have been studied by Benson, Jenkins and Ratcliff in a series of papers [2], [3] and [5]. Though their works provide us with important information on the $K-$ spherical functions, we do not have any good estimates on these functions. The only cases where we have explicit formulas and hence good estimates for the the $K-$ spherical functions are when $K = T(n)$ and $K = U(n)$. Even in these cases the known estimates are not good enough to prove the optimal results as can be seen from the work [17].

Therefore, we are forced to consider measures $\mu_{K.x}$ supported on orbits through real points. No doubt, we are excluding several interesting cases by imposing this restriction but with our present knowledge of spherical functions we cannot do better. Under the above restriction we are able to get usable formulae for the associated spherical functions which lead us to good estimates. Even then we are not sure if the results we get are optimal or not. However, this is just the beginning of our investigation and we hope to return to several problems left open in this article.

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We will be freely using the notations of [17] and [22]. For various facts about the Heisenberg group and Gelfand pairs we refer the reader to the monographs [8], [9], [21] and the paper [12].

2 Gelfand pairs and K-spherical functions

Let $H^n = \mathbb{C}^n \times \mathbb{R}$ be the $(2n+1)$ dimensional Heisenberg group and let $Aut(H^n)$ be the group of automorphisms of H^n. For each $\sigma \in U(n)$, the group of unitary matrices we have an automorphism in $Aut(H^n)$ given by $\sigma(z,t) = (\sigma z, t)$. This $U(n)$ is a maximal compact connected subgroup of $Aut(H^n)$ and it can be shown that any subgroup K of $Aut(H^n)$ is conjugate to a subgroup of $U(n)$. So without loss of generality we will only consider subgroups of the unitary group.

The Banach space $L^1(H^n)$ forms a (non-commutative) Banach algebra under convolution. Let $L^1_K(H^n)$ stand for the subspace consisting of all integrable, K invariant functions. We say that (H^n, K) is a Gelfand pair if $L^1_K(H^n)$ turns out to be a commutative Banach algebra.There are several subgroups $K \subset U(n)$ for which (H^n, K) is a Gelfand pair. For example, the full unitary group $K = U(n)$ and the torus group $K = T(n)$ give rise to Gelfand pairs.

There is a representation theoretic criterion due to Carcano [6] for (H^n, K) to be a Gelfand pair. In our case this criterion implies that (H^n, K) is a Gelfand pair if and only if the action of K on the holomorphic polynomials $P(\mathbb{C}^n)$ is multiplicity free. Let $K_C \subset GL(n, \mathbb{C})$ be the complexification of K. Then the irreducible components of $P(\mathbb{C}^n)$ with respect to K and K_C are identical. The connected groups K_C which act irreducibly and without multiplicity have been classified by V.Kac [12]. The classification of groups which act in a multiplicity free way was completed by Benson and Ratcliff in [4] and also independently by A. Leahy in a Rutgers university thesis.

Let (H^n, K) be a Gelfand pair. We say that a function φ on H^n is a $K-$spherical function if it is $K-$invariant, $\varphi(0) = 1$ and it satisfies

$$\int_K \varphi(g.kh)\,dk = \varphi(g)\varphi(h), \ \ g,h \in H^n \tag{2.1}$$

where dk is the normalised Haar measure on K. The general theory in [2] describes the bounded $K-$spherical functions for a Gelfand pair in terms of the representation theory of the Heisenberg group. There are two distinct classes of $K-$spherical functions. We record here some of their properties without any proof.

Let $P(\mathbb{C}^n) = \sum_\alpha P_\alpha$ denote the decomposition of $P(\mathbb{C}^n)$ into K-irreducibles. The type I spherical functions are parametrised by the pairs (λ, P_α) where λ is a non-zero real number. They arise from the infinite dimensional representations of the Heisenberg group and we denote them by e_α^λ. They satisfy the relation $e_\alpha^\lambda(z,t) = e_\alpha^1(\sqrt{\lambda}z, \lambda t)$ for $\lambda > 0$ and $e_\alpha^\lambda(z,t) = e_\alpha^{|\lambda|}(z,-t)$ for $\lambda < 0$. The type II spherical functions arise from the one dimensional representations and are parametrised by $\mathbb{C}^n/K$ the set of $K-$orbits in $\mathbb{C}^n$. For $w \in \mathbb{C}^n$ we denote by η_w for the associated $K-$spherical function. It is known that η_w is independent of t and is given by the Fourier transform of the unit mass on the orbit $K.w$.

We concentrate on the type I spherical functions. We require some useful formulas for them that were proved in [2]. Consider the Fock space realisation of the infinite dimensional representations of the Heisenberg group. For $\lambda > 0$ let $\mathcal{F}_\lambda$ be the space of holomorphic functions on $\mathbb{C}^n$ that are square integrable with respect to the measure $dw_\lambda = \left(\frac{\lambda}{2\pi}\right)^n e^{-\frac{1}{2}\lambda|z|^2}dw$. The space $P(\mathbb{C}^n)$ of holomorphic polynomials is dense in $\mathcal{F}_\lambda$ and contains an orthonormal basis given by

$$u_{\alpha,\lambda}(w) = \left(\frac{\lambda^{|\alpha|}}{2^{|\alpha|}\alpha!}\right)^{\frac{1}{2}} w^\alpha$$

where $\alpha \in N^n$. The representation π_λ of H^n on $\mathcal{F}_\lambda$ is given by

$$\pi_\lambda(z,t)u(w) = e^{i\lambda t - \frac{1}{2}\lambda(w,z) - \frac{1}{4}\lambda|z|^2}u(z+w).$$

For $\lambda < 0, \mathcal{F}_\lambda$ consists of anti-holomorphic functions which are square integrable with respect to $dw_{|\lambda|}$ and the representation is given by

$$\pi_\lambda(z,t)u(\bar{w}) = e^{i\lambda t + \frac{1}{2}\lambda(w,z) + \frac{1}{4}\lambda|z|^2}u(\bar{z}+\bar{w}).$$

Let $P(\mathbb{C}^n) = \sum_\alpha P_\alpha$ be the decomposition of $P(\mathbb{C}^n)$ into K-irreducible pieces. Then we have the following formula for the type I $K-$spherical functions [2].

Proposition 2.1 *Suppose $v_1, v_2, ..., v_l$ is an orthonormal basis for P_α. Then*

$$e_\alpha^\lambda(z,t) = \frac{1}{l}\sum_{j=1}^{l}(\pi_\lambda(z,t)v_j, v_j).$$

As a corollary we obtain the following result. Let K' be a compact subset of K so that (H^n, K') is another Gelfand pair. Let $P_\alpha = \sum_{i=1}^{n_\alpha} P_{\alpha,i}$ be the decomposition of P_α into $K'-$irreducible subspaces and let $e^\lambda_{\alpha,i}$ be the associated $K'-$ spherical functions. Then

$$dim(P_\alpha)e^\lambda_\alpha = \sum_{i=1}^{n_\alpha} dim(P_{\alpha,i})e^\lambda_{\alpha,i}.$$

If we let $\varphi^\lambda_\alpha = dim(P_\alpha)e^\lambda_\alpha$ then we can write the above equation as

$$\varphi^\lambda_\alpha = \sum_{i=1}^{n_\alpha} \varphi^\lambda_{\alpha,i}. \tag{2.2}$$

The $K-$spherical functions are explicitly known in two cases. When $K = U(n)$ the decomposition of $P(\mathbb{C}^n)$ is given by $P(\mathbb{C}^n) = \sum_{k=0}^{\infty} P_k$ where P_k is the set of all polynomials that are homogeneous of degree k which is spanned by $\{u_{\alpha,1} : |\alpha| = k\}$. The corresponding spherical functions are given by (for $\lambda = 1$)

$$E^1_k(z,t) = \frac{k!(n-1)!}{(k+n-1)!} e^{it}\varphi_k(z)$$

where

$$\varphi_k(z) = L^{n-1}_k\left(\frac{1}{2}|z|^2\right) e^{-\frac{1}{4}|z|^2}$$

are the Laguerre functions of type $(n-1)$. When $K = T(n)$, the subgroup of diagonal matrices in $U(n)$, the P_α in the decomposition is just the span of $u_{\alpha,1}$ where α runs through all multiindices. The corresponding spherical functions are given by

$$E^1_{\alpha,\alpha}(z,t) = e^{it}\Phi_{\alpha,\alpha}(z)$$

where

$$\Phi_{\alpha,\alpha}(z) = \Pi^n_{j=1} L_{\alpha_j}\left(\frac{1}{2}|z_j|^2\right) e^{-\frac{1}{4}|z_j|^2}$$

with $L_k(t)$ being Laguerre polynomials of type 0. For these facts we refer to [2] and [22].

The $K-$spherical functions are eigenfunctions of all $K-$invariant, left invariant differential operators on H^n. In particular, they are all eigenfunctions of the sublaplacian $\mathcal{L}$ and the operator $\frac{\partial}{\partial t}$. The joint eigenfunction expansions of these two operators have been extensively studied by Strichartz

[20],[21] and is given in terms of the $U(n)$ spherical functions. More precisely, we have the expansion (which holds for functions in $L^1 \cap L^2$)

$$f(z,t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} \left(\sum_{k=0}^{\infty} f * e_k^\lambda(z,t) \right) |\lambda|^n \, d\lambda$$

where we have written e_k^λ for $\frac{(k+n-1)!}{k!(n-1)!} E_k^\lambda$.

A similar expansion in terms of $K-$ spherical functions is also valid. In fact as noted in (2.2) we can write

$$e_k^\lambda(z,t) = \sum_{i=1}^{n_k} \varphi_{k,i}^\lambda(z,t)$$

and hence the above expansion can be rewritten as

$$f(z,t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \sum_{i=1}^{n_k} f * \varphi_{k,i}^\lambda(z,t) \right) |\lambda|^n d\lambda$$

which in short can be put in the form

$$f(z,t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} \left(\sum_{\alpha} f * \varphi_\alpha^\lambda(z,t) \right) |\lambda|^n \, d\lambda.$$

Since $\varphi_\alpha^\lambda(z,t) = e^{i\lambda t} \varphi_\alpha^\lambda(z)$ the above decomposition can be written in the form

$$f(z,t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{i\lambda t} \left(\sum_{\alpha} f^\lambda *_\lambda \varphi_\alpha^\lambda(z) \right) |\lambda|^n \, d\lambda.$$

In the above

$$f^\lambda(z) = \int_{-\infty}^{\infty} f(z,t) e^{i\lambda t} dt$$

is the partial Fourier transform in the t variable and $*_\lambda$ is the λ twisted convolution for two functions on $\mathbb{C}^n$ defined by

$$F *_\lambda G(z) = \int_{\mathbb{C}^n} F(z-w) G(w) e^{\frac{i\lambda}{2} Im(z.\bar{w})} dw$$

where dw is the Lebesgue measure on $\mathbb{C}^n$.

As $\varphi_\alpha^\lambda(z,t)$ comes from different pieces of an orthogonal decomposition, the above is an orthogonal expansion and we have the Plancherel theorem in the form

$$||f||_2^2 = (2\pi)^{-2n-1} \sum_{\alpha} \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} |f^\lambda *_\lambda \varphi_\alpha^\lambda(z)|^2 \, \lambda^{2n} dz d\lambda.$$

For more about this expansion in the case $K = U(n)$ and its applications we refer to [21] and [22].

If we want to prove pointwise ergodic theorems for $K-$spherical averages using harmonic analysis techniques, then good estimates on the associated spherical functions and their derivatives are indispensable. Unfortunately, except for the cases $K = U(n)$ and $K = T(n)$ such estimates are not known and the formulas we have for the spherical functions are not good enough to yield required estimates. However, for certain averages of the $K-$spherical functions we can get good estimates, thanks to the following formula. Let P_α and v_j be as in Proposition 2.1. Let μ_r^{n-1} be the normalised surface measure on the sphere $\{x \in \mathbb{R}^n : |x| = r\}$.

Proposition 2.2

$$\int_{|x|=r} e_\alpha^\lambda(x,0)d\mu_r^{n-1} = \frac{1}{l}\sum_{j=1}^{l}\int_{\mathbb{R}^n} c_n \frac{J_{\frac{n}{2}-1}(\sqrt{|\lambda|}r|\xi|)}{(\sqrt{|\lambda|}r|\xi|)^{\frac{n}{2}-1}}|u_j(\xi)|^2\, d\xi$$

where $J_{\frac{n}{2}-1}$ is the Bessel function of order $\left(\frac{n}{2}-1\right)$, u_j is a family of orthonormal functions in $L^2(\mathbb{R}^n)$ and $c_n = 2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)$.

Proof. It is enough to prove the proposition when $\lambda = 1$ and to do that we use the expression

$$e_\alpha^1(z,t) = \frac{1}{l}\sum_{j=1}^{l}(\pi_1(z,t)v_j, v_j).$$

We will rewrite the above expression in terms of the Schrodinger representation ρ_1. This representation which is realised on the Hilbert space $L^2(\mathbb{R}^n)$ is given by

$$\rho_1(z,t)\varphi(\xi) = e^{it}e^{i(x.\xi+\frac{1}{2}x.y)}\varphi(\xi+y)$$

where $\varphi \in L^2(\mathbb{R}^n)$ and $z = x+iy$. According to the fundamental theorem of Stone-von Neumann the representations π_1 and ρ_1 are unitarily equivalent and the intertwining operator is provided by the Bargmann transform B which takes $\mathcal{F}_1$ onto $L^2(\mathbb{R}^n)$.

Thus we have $\pi_1(z,t) = B^*\rho_1(z,t)B$ and therefore,

$$e_\alpha^1(z,0) = \frac{1}{l}\sum_{j=1}^{l}(\rho_1(z,0)u_j, u_j)$$

where $u_j = Bv_j$ is a unit vector in $L^2(\mathbb{R}^n)$. Now,

$$\int_{|x|=r} e_\alpha^1(x,0)d\mu_r^{n-1}$$

$$= \frac{1}{l}\sum_{j=1}^{l}\int_{|x|=r}\int_{\mathbb{R}^n} e^{ix.\xi}|u_j(\xi)|^2\, d\xi d\mu_r^{n-1}.$$

The proposition follows from the well known fact that

$$\int_{|x|=r} e^{ix.\xi}\, d\mu_r^{n-1} = c_n \frac{J_{\frac{n}{2}-1}(r|\xi|)}{(r|\xi|)^{\frac{n}{2}-1}}.$$

The above proposition is crucial for our study of the spherical averages. As good estimates for the Bessel function that appears in the proposition are known, we can obtain estimates for the averages of the spherical functions.

3 Maximal functions and local ergodic theorems

In this section we prove our main results on the ball and sphere averages of the $K-$spherical measures $\mu_{K.x}$. First we consider the maximal function M_ν. In order to prove the boundedness of this maximal function we will use Birkhoff's ergodic theorem for the actions of the group $\mathbb{R}$ of reals. This method has turned out to be very useful in establishing maximal theorems for uniform averages of singular measures, see for example Nevo [14], Nevo-Stein [16] and Nevo-Thangavelu [17]. In order to bring in the Birkhoff averages we have to pass to a bigger group, namely the Heisenberg motion group.

Given a Gelfand pair (H^n, K) consider $G_K = K \times H^n$, the semidirect product of K and H^n whose group law is given by

$$(k, z, t)(k', z', t') = (kk', z + kz', t + t' + \frac{1}{2}Im(kz'.\bar{z})).$$

The inverse of the element $(k, z, t) \in G_K$ is $(k^{-1}, -k^{-1}z, -t)$ and the identity element is $(I, 0, 0)$ where I is the $n \times n$ identity matrix. The group K is then isomorphic to a subgroup of G_K and so is H^n. The Haar measure on G_K is just the measure $dkdzdt$ and we can form the Lebesgue spaces with respect to this measure. It is easy to check that (H^n, K) is a Gelfand pair if and only is the subspace of $K-$bi-invariant functions in $L^1(G)$ forms a commutative subalgebra under convolution. Actually this is the traditional definition of a Gelfand pair.

Now, let us write m_K for the Haar measure on K and define

$$Pf(z, t) = \int_K f(k, z, t) dm_K$$

for a function f on G_K. This projection takes functions on G_K into functions on H^n. Note that any function f on H^n can be identified with a function on

G_K which is independent of k and for such functions $Pf = f$. Let δ_g be the Dirac point mass at $g \in G_K$. An easy calculation shows that the measure $m_K * \delta_g * m_K$ with $g = (k, w, s)$ where the convolution is taken on the group G_K is independent of k and depends only on s and the $K-$orbit through w. In fact,

$$f * m_K * \delta_g * m_K(z,t) = \int_K Pf(z - kw, t + s - \frac{1}{2} Im(kw.\bar{z}))dk.$$

In the above equation if we take $g(w) = (I, w, 0)$ then it follows that

$$f * m_K * \delta_{g(w)} * m_K(z,t) = Pf * \mu_{K.w}(z,t) \tag{3.1}$$

where the convolution on the right is on the Heisenberg group. Given a unit vector $\omega \in \mathbb{C}^n$ the set $A_\omega = \{(I, r\omega, 0) : r \in \mathbb{R}\}$ becomes a subgroup of G_K which is isomorphic to $\mathbb{R}$.

Proof of Theorem 1.2. Recall that

$$f * \nu_r = \frac{1}{cr^n} \int_{|x| \le r} f * \mu_{K.x} \, dx$$

which can be written, in view of (3.1) as

$$f * \nu_r(z,t) = \frac{1}{cr^n} \int_0^r \int_{|x'|=1} f * m_K * \delta_{g(sx')} * m_K(z,t) s^{n-1} d\mu_1^{n-1} ds.$$

Therefore, we have

$$|f * \nu_r(z,t)| \le C \int_{|x'|=1} m_K * \left(\frac{1}{r} \int_0^r |f| * \delta_{g(sx')} ds \right) * m_K(z,t) \, d\mu_1^{n-1}.$$

Now,

$$\frac{1}{r} \int_0^r |f| * \delta_{g(sx')} ds$$

are the Birkhoff averages over the group $A_{x'}$ which is bounded on L^p for $1 < p < \infty$ with a bound independent of x'. As convolution with the Haar measure m_K is bounded, Theorem 1.2 is proved.

We now turn our attention towards the proof of Theorem 1.3. In order to use square function arguments we need a usable expression for the measures σ_r. By abuse of notation let us write $\varphi_\alpha^\lambda(w) = \varphi_\alpha^\lambda(w, 0)$. Let $d_\alpha = dim(P_\alpha)$ be the dimension of P_α.

Proposition 3.1 *For $w \in \mathbb{C}^n$ and $f \in L^1 \cap L^2(H^n)$ we have the expansion*

$$\begin{aligned} &f * \mu_{K.w}(z,t) \\ &= (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{i\lambda t} \left(\sum_\alpha d_\alpha^{-1} \varphi_\alpha^\lambda(w) f^\lambda *_\lambda \varphi_\alpha^\lambda(z) \right) |\lambda|^n d\lambda. \end{aligned}$$

Proof. As f can be expanded in terms of the spherical functions e_α^λ it is enough to show that

$$e_\alpha^\lambda * \mu_{K.w}(z,t) = e^{i\lambda t} e_\alpha^\lambda(w,0) e_\alpha^\lambda(z,0).$$

But this follows from the definition of $\mu_{K.w}$ and the fact that e_α^λ are K-spherical functions so that they verify the identity (2.1).

Taking $w = x \in \mathbb{R}^n$ and integrating over $|x| = r$ we obtain

$$f * \sigma_r(z,t) = (2\pi)^{-n-1} \int_{-\infty}^{\infty} e^{i\lambda t} \left(\sum_\alpha \varphi_\alpha^\lambda(r) f^\lambda *_\lambda \varphi_\alpha^\lambda(z) \right) |\lambda|^n \, d\lambda$$

where we have written

$$\varphi_\alpha^\lambda(r) = \int_{|x'|=1} e_\alpha^\lambda(rx',0) d\mu_1^{n-1}.$$

In view of Proposition 2.2 we have

$$\varphi_\alpha^\lambda(r) = 2^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right) \int_{\mathbb{R}^n} \frac{J_{\frac{n}{2}-1}(r|\xi|)}{(r|\xi|)^{\frac{n}{2}-1}} u_\alpha(\xi) \, d\xi$$

where $u_\alpha(\xi)$ is a nonnegative function whose integral is one.

Once we have the above expansion for $f * \sigma_r$ and the Plancherel theorem for expansions in terms of φ_α^λ we can closely follow the arguments of Stein and Wainger [19] to prove the maximal theorem. In what follows we sketch the proof referring to [19] for details.

Proof of Theorem 1.3. Let

$$\varphi_\alpha^\lambda(r,\gamma) = 2^{\frac{n}{2}+\gamma-1} \Gamma\left(\frac{n}{2}+\gamma\right) \int_{\mathbb{R}^n} \frac{J_{\frac{n}{2}+\gamma+1}(r|\xi|)}{(r|\xi|)^{\frac{n}{2}+\gamma-1}} u_\alpha(\xi) d\xi$$

where γ is complex and define a family of operators M_r^γ by

$$M_r^\gamma f(z,t) = \int_{-\infty}^{\infty} e^{i\lambda t} \left(\sum_\alpha \varphi_\alpha^\lambda(r,\gamma) f^\lambda *_\lambda \varphi_\alpha^\lambda(z) \right) |\lambda|^n \, d\lambda.$$

We note that

$$\varphi_\alpha^\lambda(r,\gamma) = \frac{\Gamma\left(\frac{n}{2}+\gamma\right)}{\pi^{\frac{n}{2}} \Gamma(\gamma) r^n} \int_{\mathbb{R}^n} \int_{|x|\le r} (1 - \frac{|x|^2}{r^2})^{\gamma-1} e^{ix.\xi} u_\alpha(\xi) dx d\xi$$

and so we have the formula

$$M_r^\gamma f(z,t) = \frac{\Gamma\left(\frac{n}{2}+\gamma\right)}{\pi^{\frac{n}{2}} \Gamma(\gamma) r^n} \int_{|x|\le r} (1 - \frac{|x|^2}{r^2})^{\gamma-1} f * \mu_{K.x}(z,t) dx.$$

Let $M^\gamma f(z,t) = \sup_{r>0} |M_r^\gamma f(z,t)|$ be the maximal function associated to the family M_r^γ. Stein's argument involves the following three steps: (i) An L^2 estimate for $M^\gamma f$ when $Re(\gamma) > 1 - \frac{n}{2}$. (ii) The end point estimates for $Re(\gamma) > 0$ and $Re(\gamma) \geq 1$. (iii) Analytic interpolation. It is obvious from the formula above that for $Re(\gamma) > 0$ the maximal operator M^γ is bounded on L^∞. In view of Theorem 1.2 it also follows that $M^{1+i\gamma} f$ is bounded on L^p for all $p > 1$. It remains to show that M^γ is bounded on L^2 for all $Re(\gamma) > 1 - \frac{n}{2}$. Analytic interpolation will then complete the proof of the theorem.

In order to prove the L^2 estimate we introduce, as in [19], the square function

$$M^* f(z,t) = \sup_{r>0} \left(\frac{1}{r} \int_0^r |M_s^\gamma f(z,t)|^2 ds \right)^{\frac{1}{2}}.$$

It is enough to prove the inequality

$$||M^* f||_2 \leq C_\gamma ||f||_2, \quad Re(\gamma) > \frac{1}{2} - \frac{n}{2}$$

with the constant C_γ bounded on any compact subinterval of $\left(\frac{1}{2} - \frac{n}{2}, \infty\right)$. It then follows, as in [19], that M^γ is bounded on L^2 for $Re(\gamma) > 1 - \frac{n}{2}$.

Finally, since $M^1 f$ is bounded on L^2 it is enough to show that the $g-$function

$$(g_\gamma(f)(z,t))^2 = \int_0^\infty |M_r^\gamma f(z,t) - M_r^1 f(z,t)|^2 \frac{dr}{r}$$

is bounded on L^2 for $Re(\gamma) > \frac{1}{2} - \frac{n}{2}$. But this will follow from the Plancherel theorem once we show that

$$\int_0^\infty |\varphi_\alpha^\lambda(r,\gamma) - \varphi_\alpha^\lambda(r,1)|^2 \frac{dr}{r} \leq C_\gamma$$

which is a consequence of the estimate

$$\int_0^\infty |2^{\frac{n}{2}+\gamma-1} \Gamma\left(\frac{n}{2}+\gamma\right) \frac{J_{\frac{n}{2}+\gamma-1}(r)}{r^{\frac{n}{2}+\gamma-1}} - 2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \frac{J_{\frac{n}{2}}(r)}{r^{\frac{n}{2}}}|^2 \frac{dr}{r} \leq C_\gamma.$$

The last estimate follows for $Re(\gamma) > \frac{1}{2} - \frac{n}{2}$ once we use the asymptotic properties of the Bessel functions.

4 The annulus averages

In this section we prove Theorems 1.5, 1.6 and Corollary 1.7. We first consider Theorem 1.5. This theorem follows by a slight improvement of an estimate used in the proof of Theorem 3.6.5 in [22]. Therefore, we will only sketch the proof referring to [17] or [22] for details.

The Gelfand spectrum Σ of the commutative Banach algebra of radial functions on the Heisenberg group is the union of the Laguerre spectrum Σ_L and the Bessel spectrum Σ_B. Recall that the Laguerre spectrum is given by

$$\Sigma_L = \{(\lambda, k) : \lambda \neq 0, k \in \mathbb{N}\}.$$

For $\zeta \in \Sigma$ let φ_ζ be the associated spherical function. Note that when $\zeta = (\lambda, k)$,

$$\varphi_\zeta(z,t) = \frac{k!(n-1)!}{(k+n-1)!} e^{i\lambda t} \varphi_k^\lambda(z)$$

where

$$\varphi_k^\lambda(z) = L_k^{n-1}\left(\frac{1}{2}|\lambda||z|^2\right) e^{-\frac{1}{4}|\lambda||z|^2}.$$

For any function f on H^n the spherical means $f * \mu_r$ has the expansion

$$f * \mu_r(z,t) = \int_{-\infty}^{\infty} e^{i\lambda t}\left(\sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_k^\lambda(r) f^\lambda *_\lambda \varphi_k^\lambda(z)\right)|\lambda|^n d\lambda.$$

Similarly the annulus averages $A_r f$ defined by

$$A_r f(z,t) = \int_r^{r+1} f * \mu_s(z,t) ds$$

has the expansion

$$A_r f(z,t) = \int_{-\infty}^{\infty} e^{i\lambda t}(\sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \psi_k^\lambda(r) f^\lambda *_\lambda \varphi_k^\lambda(z))|\lambda|^n d\lambda$$

where

$$\psi_k^\lambda(r) = \int_r^{r+1} \varphi_k^\lambda(s) ds.$$

In the case of the reduced Heisenberg group the Laguerre part of the Gelfand spectrum of the algebra of radial functions consists precisely of the points (j,k) where j is a non-zero integer. Therefore, for functions on H_{red}^n with zero mean value we have the expansion

$$f * \mu_r(z,t) = \sum_{|j|=1}^{\infty} \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} e^{ijt} \varphi_k^j(r) f^j *_j \varphi_k^j(z)|j|^n.$$

Similarly, the annulus averages are given by the expansion

$$A_r f(z,t) = \sum_{|j|=1}^{\infty} \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} e^{ijt} \psi_k^j(r) f^j *_j \varphi_k^j(z) |j|^n.$$

We will make use of these expansions in the proof of Theorem 1.5.

Proceeding as in the proof of Theorem 3.6.5 in [22] we embed A_r in an analytic family of operators M_r^α by means of Riemann- Liouville fractional integrals. Let

$$S_\alpha^* f(z,t) = sup_{r>0} |M_r^\alpha f(z,t)|$$

be the associated maximal function. Recall that Theorem 3.6.5 in [22] was proved by analytic interpolation of the estimates

$$||S_{1+ib}^* f||_p \leq C e^{\pi |b|} ||f||_p$$

valid for $1 < p < \infty$ and

$$||S_{a+ib}^* f||_2 \leq C e^{\pi |b|} ||f||_2$$

valid for $n \geq 2$ and $a > -n + \frac{3}{2}$. The latter estimate followed easily from

$$||S_{-n+2}^* f||_2 \leq C ||f||_2.$$

In order to prove the above estimate we made use of the spectral theory and g-functions. What was really needed is the estimate

$$sup_{\zeta \in \Sigma} \int_0^\infty |\frac{d^m}{dr^m} \varphi_\zeta(r)|^2 r^{2m-1} dr \leq C_m$$

for all $1 \leq m \leq (n-1)$. In the case of annulus averages for mean-zero functions on the reduced Heisenberg group we need the estimates

$$sup_{\zeta \in \Sigma_L} \int_0^\infty |\frac{d^m}{dr^m} \psi_\zeta(r)| r^{2m-1} dr \leq C_m.$$

Note that $\psi_\zeta(r)$ is the integral of $\varphi_\zeta(s)$ over the interval $(r, r+1)$ and so we gain an extra derivative.

Recalling the definition of $\psi_\zeta(r)$ and making a change of variables we see that we need the uniform estimates

$$\left(\frac{k!(n-1)!}{(k+n-1)!} \right)^2 |j|^{-1} \int_0^\infty |\frac{d^m}{dr^m} \varphi_k(r)|^2 r^{2m+1} dr \leq C_m.$$

As we are considering only functions with mean value zero, in the above j is a non-zero integer. So it is enough to prove the above estimate with $j = 1$. As in the proof of Proposition 3.3.7 in [22] we can make use of the estimates on Laguerre functions. But now as we have gained an extra derivative the above estimates are valid for $1 \leq m \leq n$.

Therefore, if $M_r^\alpha f$ and $S_\alpha^* f$ are the Riemann- Liouville fractional integrals and the maximal function associated to $A_r f$ on H_{red}^n then for functions f with mean value zero we have the estimates

$$||S_{a+ib}^* f||_2 \leq Ce^{\pi|b|}||f||_2$$

for all $a > -n + \frac{1}{2}$. Analytic interpolation leads to the estimate

$$||Af||_p \leq C||f||_p$$

for all $p > \frac{2n+1}{2n}$. This completes te proof of Theorem 1.5.

Now we consider Theorem 1.6 and Corollary 1.7. Recall that when $n = 1$ the spherical mean value operator defined by

$$f * \mu_r(z,t) = \frac{1}{2\pi} \int_0^{2\pi} f((z,t)(re^{i\theta},0)^{-1})\, d\theta$$

has the expansion

$$f * \mu_r(z,t) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} e^{-i\lambda t} \psi_k(\sqrt{|\lambda|}r) f^\lambda *_\lambda \varphi_k^\lambda(z)|\lambda| d\lambda$$

where

$$\psi_k(t) = L_k\left(\frac{1}{2}t^2\right) e^{-\frac{1}{4}t^2}$$

are Laguerre functions of type zero. (See [22] for a proof of this fact.) If we define $T_r f(z,t) = f * \mu_{\sqrt{r}}(z,t)$ for $r > 0$ and 0 otherwise then the maximal function $M_\varphi f$ is given by

$$M_\varphi f(z,t) = \sup_{r>0} |\int_{-\infty}^{\infty} \varphi(r-s) T_s f(z,t) ds|.$$

Taking the Fourier transform in the r variable we see that

$$M_\varphi f(z,t) \leq \int_{-\infty}^{\infty} |\hat{\varphi}(r) \hat{T}_r f(z,t)| dr$$

where $\hat{T}_r f(z,t)$ is the Fourier transform of $T_r f(z,t)$ in the r variable. By Minkowski's integral inequality, we have

$$||M_\varphi f||_2 \leq \int_{-\infty}^{\infty} |\hat{\varphi}(r)| ||\hat{T}_r f||_2 dr.$$

Therefore, if we can show that the operator $\hat{T}_r f$ is bounded on L^2 with the bound

$$||\hat{T}_r f||_2 \leq C|r|^{-1}||f||_2$$

then by the hypothesis on φ we can conclude that the maximal function $M_\varphi f$ is bounded on L^2. As it is clearly bounded on L^∞ we get Theorem 1.5.

In order to show that $\hat{T}_r f$ is bounded on L^2 we make use of the following result a proof of which can be found in [18].

Lemma 4.1 *For $t > 0$ and $k = 0, 1, 2, ..$ we have*

$$L_k(t)e^{-\frac{1}{2}t} = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-its}\frac{1}{2is-1}\left(\frac{2is+1}{2is-1}\right)^k ds.$$

In view of this Lemma we have

$$\psi_k(\sqrt{|\lambda|}r) = \frac{2}{\pi}\int_{-\infty}^{\infty} e^{-irs}e_k(\lambda, s)ds$$

where we have written

$$e_k(\lambda, s) = \frac{1}{4is-1}\left(\frac{4is+\lambda}{4is-\lambda}\right)^k.$$

Therefore,

$$\hat{T}_r f(z,t) = \frac{2}{\pi}\int_{-\infty}^{\infty}(\sum_{k=0}^{\infty} e^{-i\lambda t}e_k(\lambda, r)f^\lambda *_\lambda \varphi_k^\lambda(z))|\lambda|d\lambda.$$

Appealing to the Plancherel theorem for the Fourier transform in the t variable we see that

$$||\hat{T}_r f||_2^2 = C\int_{-\infty}^{\infty}\int_{\mathbb{C}}|\sum_{k=0}^{\infty} e_k(\lambda, r)f^\lambda *_\lambda \varphi_k^\lambda(z)|^2|\lambda|^2 dzd\lambda$$

where C is a constant. As the functions $\left(\frac{4ir+|\lambda|}{4ir-|\lambda|}\right)^k$ have absolute value one, using the orthogonality properties of the special Hermite functions we obtain

$$||\hat{T}_r f||_2^2 = C'\int_{-\infty}^{\infty}(4r^2+\lambda^2)^{-1}\left(\sum_{k=0}^{\infty}\int_{\mathbb{C}}|f^\lambda *_\lambda \varphi_k^\lambda(z)|^2 dz\right)\lambda^2\, d\lambda.$$

Appealing to the Plancherel theorem for the special Hermite series we obtain

$$||\hat{T}_r f||_2^2 \le Cr^{-2}\int_{-\infty}^{\infty}\int_{\mathbb{C}}|f^\lambda(z)|^2 dzd\lambda$$

which proves the claim that

$$||\hat{T}_r f||_2 \le C|r|^{-1}||f||_2.$$

This completes the proof of Theorem 1.6.

We now turn to the proof of the corollary to Theorem 1.6. Let us recall the definition of the reduced Heisenberg group H^n_{red} which is $\mathbb{C}^n \times T$ with the group law

$$(z, e^{it})(w, e^{is}) = (z + w, e^{i(t+s+\frac{1}{2}Im(z.\bar{w}))}).$$

We usually denote the elements of H^n_{red} by (z, t) instead of (z, e^{it}) with the obvious identification. The spherical means $f * \mu_r$ are defined as in the case of the full Heisenberg group except that now we integrate over the reduced Heisenberg group. The spherical means are given by the following expansion:

$$f * \mu_r(z, t) = f^0 * \mu_r(z, t) + \sum_{k=0}^{\infty} \sum_{j \neq 0} e^{-ijt} \psi_k(\sqrt{|j|}r) f^j *_j \varphi_k^j(z)|j|$$

where now f^j are the Fourier coefficients of the function $f(z, t)$ in the t variable and ψ_k is as in the proof of Theorem 1.6.

Suppose now $\int_0^{2\pi} f(z, t)dt = 0$. Then the first term in the above expansion will be absent and proceeding as in the proof of Theorem 1.5 we get

$$\hat{T}_r f(z, t) = \frac{1}{2\pi} \sum_{j \neq 0} \sum_{k=0}^{\infty} e^{-ijt} e_k(j, r) f^j *_j \varphi_k^j(z)|j|.$$

As j is different from zero, we have

$$|e_k(j, r)| \leq C(1 + r^2)^{-\frac{1}{2}}$$

and therefore the first part of the corollary will follow once we have

$$\int_{-\infty}^{\infty} |\hat{\varphi}(r)|(1 + r^2)^{-\frac{1}{2}} dr < \infty.$$

This is the case when we take $\varphi(t) = \chi_{(-1,1)}(t)$. Thus the maximal function (1.1) is bounded on $L^2(H^1_{red})$ when f has mean value zero in the t variable. When this condition is not satisfied we have one extra term which is the usual spherical means on $\mathbb{R}^2$. The associated maximal function is bounded on $L^p(\mathbb{R}^2)$ for all $p > 2$ by Bourgain's theorem [1]. Thus Corollary 1.7 is completely proved.

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