MATHEMATICAL FRAMEWORK TO SHOW THE EXISTENCE OF ATTRACTOR OF PARTITIONED ITERATIVE FUNCTION SYSTEMS

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Mathematical Framework to Show the Existence of Attractor of Partitioned Iterative Function Systems

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ABSTRACT

The technique of image compression using Iterative Function System (IFS) is known as fractal image compression. An extension of IFS theory is Partitioned or local Iterative Function System (PIFS) for coding the gray level images. Several techniques of PIFS based image compression have already been proposed by many researchers. The theory of PIFS appears to be different from the theory of IFS in the sense of application domain. The present article discusses some basic differences between IFS and PIFS and provides a separate mathematical formulation for the existence of attractor of partitioned IFS. In particular, it has been shown that the attractor exists and it is an approximation of the given target image. The experimental results have also been presented in support of the theory. The experimental results have been obtained by using a GA based PIFS technique proposed by Mitra et al [1].

key words : Image Compression, Iterative Fuction System (IFS), Partitioned Iterative Function System (PIFS), Attractor, Isometry.

1 Introduction

The theory of fractal based image compression using Iterative Function System (IFS) was proposed by Barnsley [2, 3]. He modeled real life images by means of deterministic fractal objects *i.e.*, by the attractor evolved through iterations of a set of contractive affine transformations. Once the set of contractive affine transformations $\mathcal{F}$ (say) is obtained the rest is an iterative sequence of the form $\{\mathcal{F}^N(O)\}_{N>0}$, where "$O$" is an initial object to start the iterative sequence. The set of contractive affine transformations $\mathcal{F}$ is called IFS. In particular at the Nth iteration, the object O_N is used as input to the IFS, where O_N is the output object obtained from the $(N-1)th$ iteration. The detailed mathematical description of the IFS theory and other relevant results are available in [2, 3, 4, 5, 6].

Image compression using IFS can be looked upon as an inverse problem of iterative transformation theory [7]. The basic problem here is to find appropriate contractive affine transformations whose attractor is an approximation of the given image. Thus for the purpose of image compression it is enough to store the relevant parameters of the said transformations instead of the whole image. This technique reduces the memory requirement to a great extent. But the question is how to construct the transformations for a given image. A fully automated fractal based image compression technique of digital monochrome image was first proposed by Jacquin [7, 8, 9]. This technique is known as partitioned [10] or local [3] iterative function system. The partitioned/local IFS is an IFS where the domain of application of the contractive affine transformations is restricted to the small portions of the image (subimages) instead of the whole image as in the case of IFS. In PIFS the given image is first partitioned into non overlapping square blocks. In the encoding process, separate transformations for each square block is then found out on the basis of its similarity with other square blocks, located any where within the image support. As the transformations are applied partition wise the scheme is known as partitioned IFS or local IFS. Different schemes, using PIFS, have been proposed by several other researchers [1, 10, 11, 12].

The theory of partitioned/local IFS appears to be different from the theory of IFS in the sense of restriction of the application domain for the contractive affine transformations. So, the questions are, how PIFS produces an attractor and how it becomes a close approximation of the given target image. Generally, PIFS is considered to be a simple extension of IFS and it is assumed that the theoretical foundation of PIFS is same as that of IFS. In reality IFS and PIFS techniques differ widely. The present article discusses some basic differences between IFS and PIFS in the context of image compression and provides a separate mathematical formulation for the existence of attractor of partitioned IFS. In particular, firstly we have shown that the transformations, in the PIFS scheme, give rise to a fixed point (attractor). Secondly, it has been shown that the transformations, though not exactly contractive, are eventually contractive. Finally, we have proved the attractor and the given image are very close to each other in the sense of a chosen distance measure.

In the next section we have described the theory of image coding using IFS. Section 3 consists of basic features of constructing PIFS codes for a given image. Section 4 deals with the basic difference between IFS and PIFS techniques. The proposed mathematical formulation of PIFS has been discussed in Section 5. The experimental results have been presented in Section 6 and the conclusions are drawn in Section 7.

2 Theoretical Foundation of IFS

The salient features of IFS theory and image coding through IFS are given below. An elaborate description of this methodology is available in [2].

Let (X, d) be a complete metric space, where X is a set and d is a metric. Let f be a contractive map defined on the metric space (X, d) such that $f : X \to X$ and $d(f(x_1), f(x_2)) \leq s\ d(x_1, x_2)$; $\forall x_1, x_2 \in X$, where $0 \leq s < 1$ is called contractivity factor of the map f. Note that, $\lim_{N\to\infty} f^N(x) = a$, $\forall x \in X$, and also $f(a) = a$. "a" is called fixed point of f. Here $f^N(x)$ is defined as

$$f^N(x) \ = \ f(\ f^{N-1}(x)\ ), \text{ with } f^1(x) = f(x), \ \forall x \in X.$$

Now let $\mathcal{H}(X)$ be the space of all non empty compact subsets of X. Let D denote Hausdorff metric defined on $\mathcal{H}(X)$. It can be shown that $(\mathcal{H}(X), D)$ is a complete metric space.

Let f be a contractive map on $(\mathcal{H}(X), D)$. Then also $\lim_{N\to\infty} f^N(B) = A$, $\forall B \in \mathcal{H}(X)$ and $f(A) = A$. Here also $f^N(B)$ is defined as

$$f^N(B) \ = \ f(\ f^{N-1}(B)\ ), \text{ with } f^1(B) = f(B), \ \forall B \in \mathcal{H}(X). \qquad \spadesuit$$

Now, let us consider n contractive maps $f_1, f_2, \cdots, f_n$ with contractivity factors $s_1, s_2, \cdots, s_n$, on $(\mathcal{H}(X), D)$. Let for any $B \in \mathcal{H}(X)$,

$$B_1 \ = \ \bigcup_{i=1}^{n} f_i(B)$$

and

$$B_N \ = \ \bigcup_{i=1}^{n} f_i(B_{N-1}) \quad \forall N \geq 2.$$

Then it can be shown that there exists a set $C \in \mathcal{H}(X)$ such that

$$\bigcup_{i=1}^{n} f_i(C) \ = \ C \ \textit{and} \ \lim_{N\to\infty} B_N = C, \forall B \in \mathcal{H}(X).$$

C is called the attractor of the IFS $(\mathcal{H}(X); f_1, f_2, \cdots, f_n)$. $\spadesuit$

Let for any $B \in \mathcal{H}(X)$,

$$\mathcal{W}(B) \ = \ \bigcup_{i=1}^{n} f_i(B),$$

and

$$\mathcal{W}^N(B) \ = \ \mathcal{W}(\ \mathcal{W}^{N-1}(B)\ ), \ \forall N \geq 2 \text{ where } \mathcal{W}^1(B) = \mathcal{W}(B), \ \forall B \in \mathcal{H}(X).$$

Note that

$$D(\mathcal{W}(B_1), \mathcal{W}(B_2)) \leq s\ D(B_1, B_2); \ \forall B_1, B_2 \in \mathcal{H}(X) \quad \text{where} \quad 0 \leq s < 1. \qquad (1)$$

and $s = Max\{s_1, s_2, \cdots, s_n\}$.

Collage Theorem [2] :- Let $I \in \mathcal{H}(X)$ and $\epsilon > 0$. Let there exists n contractive maps $f_1, f_2, \cdots, f_n$, with contractivity factors $s_1, s_2, \cdots, s_n$, from $\mathcal{H}(X)$ to $\mathcal{H}(X)$ such that

$$D(I, \mathcal{W}(I)) \leq \epsilon. \tag{2}$$

Then

$$D(I, C) \leq \frac{\epsilon}{1-s} \tag{3}$$

where $s = \max\{s_1, s_2, \cdots, s_n\}$ and C is the attractor of $\{\mathcal{H}(X); f_1, f_2, \cdots, f_n\}$. ♠

The above theory is given for contractive maps defined on complete metric spaces. The space X under consideration in this article is a subset of the three dimensional Euclidian space $\mathbb{R}^3$. One can define affine contractive maps $f_1, f_2, \cdots, f_n$ on this X and these maps become contractive maps on $(\mathcal{H}(X), D)$. In this article, hence forth, we shall be dealing with affine contractive maps.

Now, let $I \in \mathcal{H}$ be a given set and our intention here is to find a set $\mathcal{W}$ of affine contractive maps in such a way that the distance between the given set and the attractor of $\mathcal{W}$ is very small. Thus, by Collage theorem, the given set I can be approximated by the attractor of $\mathcal{W}$.

From (3) it is clear that, after a sufficiently large number (N) of iterations, the set of affine contractive maps $\mathcal{W}$ produces a set C belonging to $\mathcal{H}(X)$ and it is very close to the given original set I. Here, $(\mathcal{H}(X), \mathcal{W})$ is called iterative function system and $\mathcal{W}$ is called the set of fractal codes for the given set I.

In the context of image coding, the given set I is the given digital monochrome image. Note that any digital image I with w rows, w columns and gray level value as $g(i,j)$'s ($g(i,j)$ is the gray level values of the ith row and jth column pixel) can be represented by $I = \{(i, j, g(i,j)) \ : \ i = 1, 2, \cdots, w; j = 1, 2, \cdots, w\}$. Thus any image I is a subset of three dimensional Euclidian space $\mathbb{R}^3$ and hence the theory stated above is applicable for images.

In the context of digital monochrome image coding, the scheme suggested by Jacquin [8] is called partitioned or local Iterative Function System (PIFS). In the next section, the construction of PIFS codes has been described.

3 Construction of PIFS Codes

The structure of PIFS codes are almost same as that of IFS codes. The only difference is that PIFS codes are obtained and applied to a particular portion of the image instead of the whole image.

Let I be a given digital image having size $w \times w$ and the range of gray level values be $\{0, 1, 2, \cdots, l-1\}$. Thus the given image I can be expressed as a matrix $((g(i,j)))_{w \times w}$, where i and j stand for row number and column number respectively and $g(i,j)$ represents the gray level value for the position (i,j). The image is partitioned into n non overlapping squares of size, say $b \times b$, and let this partition be represented by $\mathcal{N} = \{\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_n\}$. Each $\mathcal{R}_i$ is named as range block. Note that the number of range blocks $n = \frac{w}{b} \times \frac{w}{b}$. Let $\mathcal{M}$ be the collection of all possible blocks of size $2b \times 2b$ in the image. Let $\mathcal{M} = \{\mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_m\}$. Here $m = (w - 2b) \times (w - 2b)$ and $\mathcal{D}_j$'s are named as "domain blocks".

Now, let us define,

$$\mathcal{A} \;=\; \{1, 2, \cdots, w\} \times \{1, 2, \cdots, w\} \times \{0, 1, 2, \cdots, l-1\}.$$

Here $\mathcal{A} \subset \mathbb{R}^3$. Note that any image I is a subset of $\mathcal{A}$ but any subset of $\mathcal{A}$ is not necessarily an image. Also $\mathcal{R}_i \subset \mathcal{A}; \forall i$ and $\mathcal{D}_j \subset \mathcal{A}; \forall j$.

Let, for a range block $\mathcal{R}_i$,

$$\mathcal{F}_j = \{f : \mathcal{D}_j \to \mathcal{A} \;\; ; \;\; f \;\; \text{is an affine contractive map}\}.$$

Let, $f_{i|j} \in \mathcal{F}_j$ be such that

$$\rho(\mathcal{R}_i,\; f_{i|j}(\mathcal{D}_j)) \;\leq\; \rho(\mathcal{R}_i,\; f(\mathcal{D}_j)) \;\; \forall f \in \mathcal{F}_j, \; \forall j.$$

Here ρ is a suitably chosen distance measure. (A detailed description of ρ is given at the end of this section).

Now let k be such that

$$\rho(\mathcal{R}_i,\; f_{i|k}(\mathcal{D}_k)) \;=\; \min_j\{\; \rho(\mathcal{R}_i,\; f_{i|j}(\mathcal{D}_j)) \;\}. \tag{4}$$

Also, let $\quad f_{i|k}(\mathcal{D}_k) = \widehat{\mathcal{R}}_{i|k}$. We shall denote $f_{i|k}$ by $f_{i|\bullet}$.

The aim here is to find $f_{i|k}(\mathcal{D}_k)$ for each $i \in \{1, 2, \cdots, n\}$. In other words, for every range block $\mathcal{R}_i$, one needs to find an appropriately matched domain block $\mathcal{D}_k$ as well as an appropriate transformation $f_{i|k}$. The set of maps $\mathcal{F} = \{f_{1|\bullet}, f_{2|\bullet}, \cdots, f_{n|\bullet}\}$ thus obtained is called the partitioned or local IFS or fractal codes of image I.

To find the best matched domain block as well as the best matched transformation, all possible domain blocks as well as all possible transformations are to be searched with the help of equation (4). The problem of searching for an appropriately matched domain block and transformation for a range block has already been solved by enumerative search [8] and by using *Genetic Algorithms* [1].

The affine contractive transformation $f_{i|\bullet}$ is constructed using the fact that the gray values of the range block are a scaled, translated and rotated version of the gray values of domain block. The contractive affine transformation $f_{i|j}$ is defined in such a way that $f_{i|j}(\mathcal{D}_j)$ is close to $\mathcal{R}_i$. Also $f_{i|j}$ can be separated into two parts, one for spatial information and the other for information of gray values. The first part indicates which pixel of the domain block corresponds to which pixel of range block. The second part is to find the scale and shift parameters for the set of pixels of the domain blocks to the range blocks.

The first part is shuffling the pixel positions of the domain block. Generally, eight transformations (isometries) are considered for this purpose [1, 11]. On the other hand, second part is the estimation of a set of values (gray values) of range blocks from the set of values of the transformed domain blocks. These estimates can be obtained by using the least square analysis of two sets of values [1, 11].

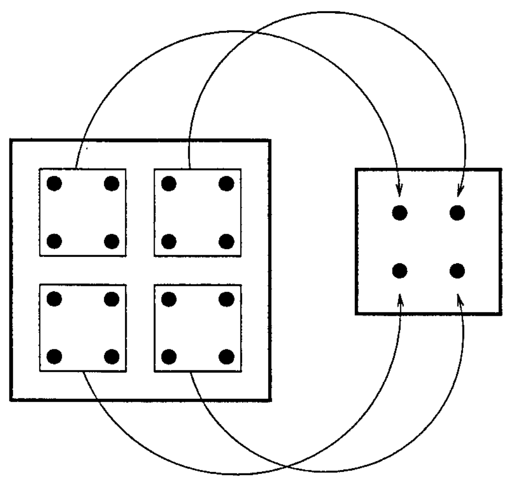

Figure 1: Construction of contracted domain block : Scheme 1

The second part is obtained using least square analysis of two sets of gray values once the first part is fixed. Moreover the size of a domain block is double that of any range block. But, the method of least squares (straight line fitting) needs point to point correspondence. To overcome this, one has to construct contracted domain block such that the number of pixels in a contracted domain block becomes equal to that of any range block. The contracted domain block is obtained by adopting any one of the following two techniques. In the first technique, as shown in Figure 1, for a 4×4 domain block, the average values (integers) of four pixel values in 2×2 non overlapping squares within the domain block are considered as the pixel values of the contracted domain block. In this scheme, row number and column number corresponding to each pixel value of the contracted domain block are equal to the row number and column number of the topmost pixel value in every 2×2 square considered within the domain block [8]. In the other scheme, as shown in Figure 2, for a 4×4 domain block, contracted domain block is constructed by taking pixel values along with the row number and column number from every alternative rows and columns of the domain block [1, 11].

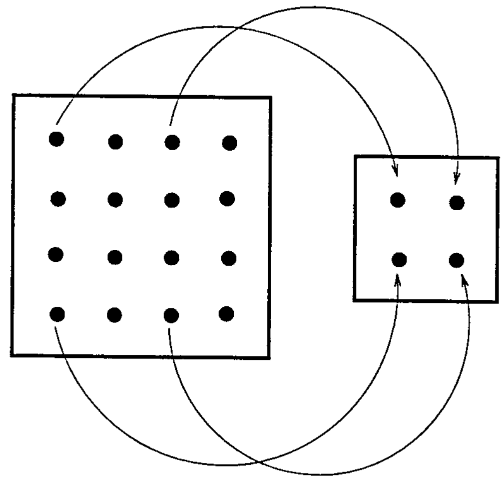

Figure 2: Construction of contracted domain block : Scheme 2

Now to select an appropriately matched domain block ($\mathcal{D}_k$) and appropriately matched transformation ($f_{i|k}$) for a range block ($\mathcal{R}_i$), the distance measure "ρ" plays an important role. The distance measure "ρ" [used in equation (4)] is taken to be the simple Root Mean Square Error (RMSE) between the original set of gray values and the obtained set of gray values of the concerned range block. The map from $\mathcal{D}_k$ to $\mathcal{R}_i$ is constructed in such a way that the pixel positions of $\mathcal{R}_i$ and $\widehat{\mathcal{R}}_{i|k}$ are same. The difference between $\mathcal{R}_i$ and $\widehat{\mathcal{R}}_{i|k}$ are found only in the gray level values of the pixels. Let $\mathcal{R}_i(p,q)$ and $\widehat{\mathcal{R}}_{i|k}(p,q)$ be respectively the original and the obtained values of the $(p,q)^{th}$ pixel for the range block $\mathcal{R}_i$ of size $b \times b$. The distance measure ρ is then computed as

$$\rho(\mathcal{R}_i,\ \widehat{\mathcal{R}}_{i|k}) = \sqrt{\frac{1}{b}\sum_p \frac{1}{b}\sum_q \{\mathcal{R}_{i|k}(p,q) - \widehat{\mathcal{R}}_i(p,q)\}^2}.$$

The RMSE is not a metric though it serves the purpose of a distance measure. Note that the same measure had been used in several articles [1, 8, 10, 11, 12].

4 How PIFS technique differs from IFS

Extension of the iterative function system concept results in the partitioned iterative function system. PIFS mainly differs from IFS in the domain of application of their respective transformations. In PIFS the transformations are not applied to the whole image, as in the case of IFS, but rather have restricted domains. In every PIFS, the information on every transformation f_i should contain the location of the domain block to which f_i is applied.

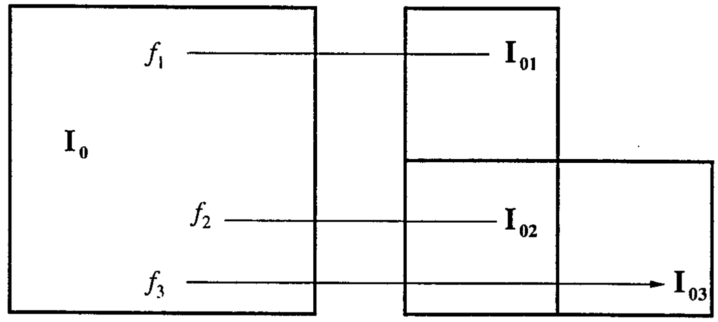

Figure 3: Mapping for an IFS scheme

The difference in the domain of application of the two techniques is shown in Figure 3 and Figure 4. In Figure 4, three affine contractive transformations are applied on the image I_0 to result in an image which consists of three parts I_{01}, I_{02} and I_{03}. These three transformations are then applied sequentially to result in a fixed point. The set of transformations $\{f_1, f_2, f_3\}$ is called the IFS, and the fixed point of this set of transformations in this case is *Sierspinski gasket* [2].

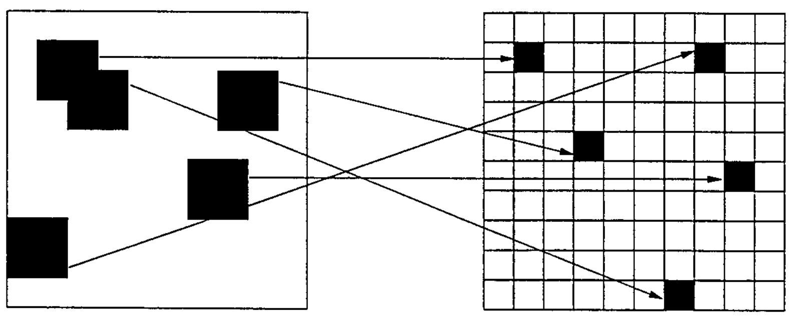

Figure 4: Mapping from domain blocks to range blocks in PIFS scheme

In PIFS, contrary to the above, as shown in Fig. 2(b) the map $f_{i|j}$ is applied to the domain $\mathcal{D}_j$ to result in $\widehat{\mathcal{R}}_i$, which is an estimate of $\mathcal{R}_i$. In the next iteration, this estimate ($\widehat{\mathcal{R}}_i$) is not used as the domain to the map $f_{i|j}$. In particular in the next iteration an estimate of $\mathcal{D}_j$ is used as the domain to obtain improved estimate, $\widehat{\widehat{\mathcal{R}}}_i$, of $\mathcal{R}_i$. Note that a domain block includes many other range blocks or part of them (Figure 4). So, the estimate of $\mathcal{D}_j$ consists of several other estimated range blocks or part of them.

PIFS differs from IFS not only in the application domain but also in the case of selecting distance measure. In, IFS, the Hausdorff distance, which is a metric, is considered as the distance measure. But in PIFS, RMSE is considered as the distance measure and RMSE is not a metric. Even then RMSE serves the purpose of selecting an appropriately matched map and domain block for a range block. As our purpose is to measure the closeness of the set of pixel values of the concerned range block and the scaled, translated and rotated version of the set of pixel values of appropriately matched domain block, selection of RMSE as distance measure is sufficient. But it needs to be examine whether the attractor, if it exists, of PIFS is close to the given image, considering RMSE as distance measure.

Another important and significant difference of PIFS and IFS lies in the context of contractivity factor of the transformations. For an IFS with an expansive map f_i, the set of maps will not converge to a compact fixed point. The expansive part will cause the limit set to be infinitely stretched along some direction. This is not necessarily true for a PIFS. PIFS can contain expansive transformations and still have a bounded attractor. So, it is not necessary, in PIFS, to impose any contractivity condition on all the transformations. A sufficient contractivity requirement is that the set of transformations $\mathcal{F}$ be *eventually contractive* [10]. Fisher et al [10] have shown experimentally that maximum allowable value of s (contractivity factor) can be 1.5 (> 1). Also they have shown that this maximum value of s, for a particular image, yields minimum distortion between the original image and the attractor evolved through the iterative process of the eventually contractive transformations.

In the next section we have described the mathematical formulation of attractor of PIFS, where the meaning of eventually contractive maps has been defined.

5 Mathematical Formulation of PIFS

In this section we have proposed a mathematical formulation of PIFS. To make it convenient we have divided our tasks into three stages. Firstly, it has been shown that the PIFS code ($\mathcal{F}$) possesses a fixed point or attractor in iterative sequence. Secondly, the eventual contractivity of the maps in PIFS setup has been proved. Finally, it has been shown that the given image and the attractor are very close to each other in the sense of a chosen distortion measure which is root mean square (RMSE).

Let, I be a given image having size $w \times w$ and the range of gray level values be $\{0, 1, 2, \cdots, l-1\}$. For this given image we can construct a vector $\underline{x}$ whose elements are the pixel values of the given image I. Note that there are w^2 pixel values of I. Thus,

$$\underline{x} = (x_1, \quad x_2, \quad x_3, \quad \ldots \quad , x_{w^2})'$$

is the given image where x_1 is the pixel value corresponding to the $(1,1)th$ position of I. Likewise, let x_r be the pixel value corresponding to the $(i,j)th$ position of I, where, $r = (i-1)\,w + j, \quad 1 \leq i, \; j \leq w.$

In this setup PIFS can be viewed as following. There exists an affine (linear), not necessarily strictly contractive, map for each element of $\underline{x}$ and this map is called forward map of the element. In the process of iteration, the input to a forward map will be any one of the w^2 elements of $\underline{x}$ and the map is called backward map for this input element. Thus for each element of $\underline{x}$ there exists a forward map and an element of $\underline{x}$ can have one or more or no backward map(s). The set $\mathcal{F}$, of forward maps, is called the PIFS codes of I.

Let us define a set $\mathcal{P}$ as

$$\mathcal{P} = \{0, 1, 2, \cdots, l-1\}$$

Now consider the set S where,

$$S \; = \; \{ \; \underline{x} \; | \; \underline{x} \; = \; (x_1, \quad x_2, \quad x_3, \quad \ldots \quad , x_{w^2})' \; , \qquad x_i \in \mathcal{P} \; \}.$$

S is the set of all possible images. The given image I is surely an element of S *i.e.* $I \in S$. The PIFS codes $\mathcal{F}$ can be looked upon as

$$\mathcal{F} \; : \; S \; \rightarrow \; S \; .$$

The attractor of $\mathcal{F}$, $\underline{a}$ (say), if exists, will also belongs to S. So, the first task is to show the existence of $\underline{a}$.

Let f_1 be the forward map for a particular element x_{r_1} , where $r_1 = (i_1 - 1)w + j_1$. Also let this element be mapped from the element x_{r_2} ($r_2 = (i_2 - 1)\, w + j_2$) . Thus f_1 is the backward map for xr_2 . Again x_{r_2} is being mapped from x_{r_3}, $(r_3 = (i_3 - 1)w + j_3)$ with a forward map f_2 . Thus we have a sequence of maps for the element x_{r_1} as following.

$$(\, i_1, \, j_1 \,) \; \stackrel{f_1}{\leftarrow} \; (\, i_2, \, j_2 \,) \; \stackrel{f_2}{\leftarrow} \; (\, i_3, \, j_3 \,) \; \stackrel{f_3}{\leftarrow} \; \cdots \; \stackrel{f_{m-1}}{\leftarrow} \; (\, i_m, \, j_m \,); \quad m \; \leq \; (w^2 \; - \; 1) \, . \quad (5)$$

The above sequence will be stopped at $(\, i_m, \, j_m \,)$ if

$$(\, i_{m+1}, \, j_{m+1} \,) \quad = \quad (\, i_k, \, j_k \,); \quad \text{for } k \; = \; 0 \text{ or } 1 \text{ or } 2 \text{ or } \cdots \text{ or } m. \qquad (6)$$

The stopping phenomenon of this sequence is mandatory as there are finite number (w^2) of elements in $\underline{x}$. Moreover all the elements of $\underline{x}$ possess same type of sequence in PIFS codes. Thus it is enough to show that the element x_{r_1} has got a fixed point in the process of iteration and this will lead to prove the existence of $\underline{a}$ (attractor of $\underline{x}$).

It is clear from the sequence (5) that during the iterative process the element x_{r_1} will have a fixed point once the element x_{r_2} is fixed. Again the convergence (to a fixed point) of the element x_{r_3} confirms the convergence of the element x_{r_2} and likewise for the rest of the elements. Thus convergence of the last element of the sequence implies the convergence of the rest of the elements. The convergence of the last element of the sequence is possible in four different ways according to the stopping condition (6).

An important point to be noted in this context is the problem of discretization. To get the decoded image in an iterative process using PIFS codes one need to discretize the output. This can be done in two ways. One is discretization of the output in each iteration. Another is discretization at the end of the iterative process. The iterative process is stopped whenever there is no change in gray values in two successive iterations. To prove the convergence of the elements in four different ways we have used the discretization of the second type.

Case 1 : $m = 1$.
Here $(i_2, j_2) = (i_1, j_1)$.
It implies that (i_1, j_1) is mapped into itself with a map f_1.
Here $f_1 = a_1 x + b_1 ; \; x \in \mathcal{P}$ and $0 \leq a_1 < 1$.
Note that in this case the affine map f_1 should necessarily be a strictly contractive map otherwise the element will not converge to a fixed point.

If we start with any value ($x \in \mathcal{P}$) of (i_1, j_1), the element will converge to the fixed point $\frac{b_1}{1 - a_1}$.

Case 2 : $m > 0$ and $k = m$
Here $(i_{m+1}, j_{m+1}) = (i_m, j_m)$.
It implies that (i_m, j_m) is mapped into itself with a map $f_m = a_m x + b_m$; $x \in \mathcal{P}$ and $0 \leq a_m < 1$. Thus (i_m, j_m) will converge to $\frac{b_m}{1-a_m}$. Once (i_m, j_m) is fixed at $\frac{b_m}{1-a_m}$, the element (i_{m-1}, j_{m-1}) will be fixed at

$$\frac{a_{m-1}\, b_m}{1 - a_m} + b_{m-1} = \frac{a_{m-1}\, b_m - a_m\, b_{m-1} + b_{m-1}}{1 - a_m}.$$

In this case the forward map is $f_{m-1} = a_{m-1}\, x + b_{m-1}$; $0 \leq x \leq g$. Again (i_{m-1}, j_{m-1}) is fixed implies convergence of (i_{m-2}, j_{m-2}) with forward map $f_{m-2} = a_{m-2}\, x + b_{m-2}$; $x \in \mathcal{P}$, at

$$\frac{a_{m-2}\, a_{m-1}\, b_m - a_{m-2}\, a_m\, b_{m-1} + a_{m-2}\, b_{m-1} - a_m\, b_{m-2} + b_{m-2}}{1 - a_m}.$$

Proceeding in this way, the fixed point of (i_1, j_1) is found out to be

$$\frac{a_1\, a_2 \cdots a_{m-1}\, b_m + (a_1\, a_2 \cdots a_{m-2}\, b_{m-1} + a_1\, a_2 \cdots a_{m-3}\, b_{m-2} + \cdots + a_1\, a_2\, b_3 + a_1\, b_2 + b_1)(1 - a_m)}{1 - a_m}.$$

Note that in this case the affine map f_m should necessarily be contractive in strict sense. But the rest of the maps need not be strictly contractive. The eventual contractivity, associated with the element $x_{r_1} = (i_1, j_1)$, will be $s_{r_1} = \prod_{i=1}^{m} a_i$.

Case 3 : $m > 0$ and $k = 1$
Here $(i_{m+1}, j_{m+1}) = (i_1, j_1)$.
It implies that the starting and the last element of the sequence (5) is same. This can be looked as a complete loop for the sequence. This case has been solved stepwise. First of all

the case is solved for $m = 2$, and $m = 3$. Then on the basis of these the fixed point for the case of general m is solved.

Case 3(a) : $m = 2$

Here we have only two elements viz. (i_1, j_1) and (i_2, j_2). The element (i_1, j_1) is being mapped from the element (i_2, j_2) by the affine map $f_1 = a_1 x + b_1$; $x \in \mathcal{P}$. On the other hand the element (i_2, j_2) is being mapped from (i_1, j_1) by the affine map $f_2 = a_2 x + b_2$; $x \in \mathcal{P}$.

$$(i_1, j_1) \xleftarrow{f_1} (i_2, j_2) \xleftarrow{f_2} (i_1, j_1).$$

Let x be the starting value of (i_1, j_1) and y be the starting value of (i_2, j_2). After first iteration the values of (i_1, j_1) and (i_2, j_2) will be $a_1 y + b_1$ and $a_2 x + b_2$ respectively. Again after second iteration these will be $a_1 a_2 x + a_1 b_2 + b_1$ and $a_2 a_1 y + a_2 b_1 + b_2$ respectively. Proceeding this way after infinite (practically large but finite) number of iterations, the fixed point of (i_1, j_1) and (i_2, j_2) will be independent of x and y. The Coefficients of x and y after N (even) iterations will be $(a_1 a_2)^{N/2}$ which tends to zero as N tends to infinity. The fixed points of (i_1, j_1) thus will be

$$\frac{a_1 b_2 + b_1}{1 - a_1 a_2}.$$

The same for the element (i_2, j_2) will be

$$\frac{a_2 b_1 + b_2}{1 - a_1 a_2}.$$

Note that both the maps need not be contractive. Moreover the eventual contractivity associated with the element x_{r_1} is $a_1 a_2$ which should be less than one.

Case 3(b) : $m = 3$

Here we have three elements viz. (i_1, j_1), (i_2, j_2) and (i_3, j_3). These three elements are making a complete loop in the sequence. The sequence of forward and backward maps is as follows;

$$(i_1, j_1) \xleftarrow{f_1} (i_2, j_2) \xleftarrow{f_2} (i_3, j_3) \xleftarrow{f_3} (i_1, j_1).$$

Taking the starting values of three elements as x, y and z and proceeding as *case 3(a)* we have the following results.

The fixed point of (i_1, j_1) will be

$$\frac{a_1 (a_2 b_3 + b_2) + b_1}{1 - a_1 a_2 a_3}.$$

The element (i_2, j_2) will converge to

$$\frac{a_2 (a_3 b_1 + b_3) + b_2}{1 - a_1 a_2 a_3}.$$

The fixed point of (i_3, j_3) will be

$$\frac{a_3 (a_1 b_2 + b_1) + b_3}{1 - a_1 a_2 a_3}.$$

Here also the maps need not be contractive in the strict sense. The eventual contractivity will be $a_1 a_2 a_3$ in this case.

Case 3(c) : General m

Here we have m elements which are making a complete loop of sequence. It is clear from case *3(a)* and *case 3(b)* that all the elements of this sequence will have a fixed point after a large but finite number of iteration. Also the affine maps which are used, need not to be contractive. In particular, in this case the element (i_1, j_1) will converge to

$$\frac{a_1 (a_2 (\dots (a_{m-2} (a_{m-1} b_m + b_{m-1}) + b_{m-2}) + \dots) + b_2) + b_1}{1 - a_1 a_2 \dots a_m}.$$

Also the eventual contractivity for the element is $s_{r_1} = \prod_{i=1}^{m} a_i$.

Case 4 : $m > 0$ and $0 < k < m$

Here $(i_{m+1}, j_{m+1}) = (i_k, j_k)$, where $k = 2$ or 3 or $\dots$ or $m-2$.

Without loss of generality say, $1 < k = m_0 < m-1$.

This case can be viewed as mixture two cases. Taking (i_{m_0}, j_{m_0}) as the starting element, a complete loop of sequence can be formed with rest of the elements. Thus, one can find the fixed point of this element as it is nothing but *case 3*. Once the element (i_{m_0}, j_{m_0}) is fixed then the fixed point of the original starting element (i_1, j_1) can be found out by using *case 2*. Like all the previous cases the eventual contractivity, in this case, will be $s_{r_1} = \prod_{i=1}^{m} a_i$.

The above stated four cases provide the fixed point of the PIFS codes $\mathcal{F}$. Thus for a very large positive number N, we have

$$\mathcal{F}^N(\underline{o}) \rightarrow \underline{a} \;\forall\, \underline{o} \in S . \tag{7}$$

Note that for each element there will be a sequence of the form (5). This sequence will follow any one of the above mentioned four cases. Thus for each element there will be a sequence of forward maps. The contractivity factor associated with this element will be the product of all the scaled parameters (a_i) of the forward maps.

The next task is to define, mathematically, the eventual contractivity of the PIFS codes. In this context we are stating the following theorem.

Theorem 1 : Let $\mathcal{F}$ be the PIFS codes and S be the set of all possible images. For every $\underline{x}$ and $\underline{y}$, $\underline{x} \neq \underline{y} \in S \;\exists\; N > 0$ and $0 \leq s < 1$ such that

$$| \mathcal{F}^p(\underline{x}) - \mathcal{F}^q(\underline{y}) | \leq s \,| \underline{x} - \underline{y} | ; \quad \forall\, p, q > N .$$

Proof : Using equation (7) we have for a very small positive number $\epsilon > 0$, $\exists$ a large positive number N_1 such that

$$p > N_1 \Rightarrow | \mathcal{F}^p(\underline{x}) - \underline{a} | < \frac{\epsilon}{2} ; \; \forall \underline{x} \in S.$$

Also, $\exists$ another large positive number N_2 such that

$$q > N_2 \Rightarrow | \mathcal{F}^q(\underline{y}) - \underline{a} | < \frac{\epsilon}{2} ; \; \forall \underline{y} \in S.$$

Thus for $\underline{x} \neq \underline{y} \in S$,

$$\begin{aligned} | \mathcal{F}^p(\underline{x}) - \mathcal{F}^q(\underline{y}) | &= | \mathcal{F}^p(\underline{x}) - \underline{a} + \underline{a} - \mathcal{F}^q(\underline{y}) |. \\ &\leq | \mathcal{F}^p(\underline{x}) - \underline{a} | + | \mathcal{F}^q(\underline{y}) - \underline{a}|. \\ &< \epsilon ; \text{ where , } p, q > N = \text{Max } (N_1, N_2). \end{aligned}$$

Thus, for $\underline{x} \neq \underline{y}$, $\exists$ a large number N and $0 \leq s < 1$ such that

$$p, q > N \Rightarrow | \mathcal{F}^p(\underline{x}) - \mathcal{F}^q(\underline{y}) | \leq s | \underline{x} - \underline{y} | . \quad Q.E.D.$$

Once the eventual contractivity of the PIFS codes $\mathcal{F}$ has been proved the last task is to show that the given image and the fixed point of $\mathcal{F}$ are very close to each other.

Theorem 2 : Let $\underline{a}$ be the fixed point of the PIFS codes $\mathcal{F}$ and $\underline{x}$ be the given image. Also let "ρ" be the given distortion measure. Under this setup, if

$$\rho(\underline{x} , \mathcal{F}(\underline{x})) \leq \alpha$$

then

$$\rho(\underline{x} , \underline{a}) \leq \frac{\alpha}{1 - s_{max}}.$$

Where $s_{max} = Max \{s_1, s_2, \ldots, s_{w^2}\}$, s_i being the eventual contractivity of the *ith* element of $\underline{x}$.

Proof : Let

$$\begin{aligned} \underline{x} &= (x_1, x_2, \ldots, x_{w^2})' \\ \mathcal{F}(\underline{x}) &= (\widehat{x_1}, \widehat{x_2}, \ldots, \widehat{x_{w^2}})' \\ \mathcal{F}^2(\underline{x}) &= (\widehat{\widehat{x_1}}, \widehat{\widehat{x_2}}, \ldots, \widehat{\widehat{x_{w^2}}})' \\ \mathcal{F}^3(\underline{x}) &= (\widehat{\widehat{\widehat{x_1}}}, \widehat{\widehat{\widehat{x_2}}}, \ldots, \widehat{\widehat{\widehat{x_{w^2}}}})'. \end{aligned}$$

In the PIFS scheme, the given image is partitioned into range blocks $\mathcal{R}_i$ of sizes $b \times b$. So, there are $n = (\frac{w}{b})^2$ range blocks each having b^2 pixel values. Pixel values are nothing but the elements of $\underline{x}$. Also the distortion measure "ρ" is Root Mean Square Error (RMSE). "ρ" , defined on S, is as follows

$$\rho(\underline{u} , \underline{v}) = \sqrt{\frac{1}{w^2} \sum_{i=1}^{w^2} \beta(u_i , v_i)} \quad \forall \; \underline{u} , \underline{v} \in S.$$

Where, $\beta (u_i , v_i) = |u_i - v_i|^2$.
Now,

$$\begin{aligned}
\rho(\underline{x} , \mathcal{F}(\underline{x})) &= \sqrt{\frac{1}{w^2}\sum_{i=1}^{w^2} \beta (x_i , f_i(x_j));} \\
&\quad [x_i \text{ is being mapped from } x_j \text{ with forward map } f_i.] \\
&= \sqrt{\frac{1}{w^2}\sum_{i=1}^{w^2} \beta (x_i , \widehat{x_i}).} \\
&\le \sqrt{\frac{1}{w^2}\sum_{i=1}^{w^2} Max_i|x_i - \widehat{x_i}|^2.} \\
&= \sqrt{\frac{1}{w^2}\sum_{i=1}^{w^2} (\alpha)^2;} \quad \text{where, } \alpha = Max_{i\in\{1,2,\dots,w^2\}} |x_i - \hat{x_i}|. \\
&\le \alpha .
\end{aligned}$$

Again,

$$\rho(\underline{x} , \mathcal{F}(\underline{x})) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{b^2}\sum_{j=1}^{b^2} \beta(x_{j|i} , f_{j|i}(x_{k|l}))} , \quad [\text{as } w^2 = nb^2]$$

where, $x_{j|i}$ is the jth pixel value of ith range block and $f_{j|i}$ is the forward map associated with $x_{j|i}$ which is being mapped from $x_{k|l}$, the k^{th} element of l^{th} range block.

Thus,

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{b^2}\sum_{j=1}^{b^2} \beta(x_{j|i} , f_{j|i}(x_{k|l}))} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{b^2}\sum_{j=1}^{b^2} \beta(x_{j|i} , \widehat{x_{j|i}})}.$$

It implies that

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{b^2}\sum_{j=1}^{b^2} \beta(x_{j|i} , \widehat{x_{j|i}})} \le \alpha \tag{8}$$

Again,

$$\rho(\mathcal{F}(\underline{x}) , \mathcal{F}^2(\underline{x})) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{b^2}\sum_{j=1}^{b^2} \beta(f_{j|i}(x_{k|l}) , f_{j|i}(\widehat{x_{k|l}}))} .$$

Note that the size of the range block and the contracted domain block [from where this range block is being mapped] is same (Section 3). Moreover the number of range blocks and the number of matched contracted domain blocks is same as there is only one matched contracted domain block for each range block. Also for each element there is an eventual contractivity factor and s_{max} is the maximum of these factors.

So,

$$\begin{aligned}
\rho(\mathcal{F}(\underline{x})\ ,\ \mathcal{F}^2(\underline{x})\ ) &\leq s_{max} \sqrt{\frac{1}{n} \sum_{l=1}^{n} \frac{1}{b^2} \sum_{k=1}^{b^2} \beta(\ x_{k|l}\ ,\ \widehat{x_{k|l}})}\ . \\
&\leq s_{max}\ \alpha\ . \qquad \text{[By (8)]}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\rho(\mathcal{F}^2(\underline{x})\ ,\ \mathcal{F}^3(\underline{x})\ ) &= \sqrt{\frac{1}{n} \sum_{i=1}^{n} \frac{1}{b^2} \sum_{j=1}^{b^2} \beta(\ f_{j|i}(\widehat{x_{k|l}})\ ,\ f_{j|i}(\widehat{\widehat{x_{k|l}}}))}\ . \\
&\leq s_{max} \sqrt{\frac{1}{n} \sum_{l=1}^{n} \frac{1}{b^2} \sum_{k=1}^{b^2} \beta(\ \widehat{x_{k|l}}\ ,\ \widehat{\widehat{x_{k|l}}})}\ . \\
&= s_{max} \sqrt{\frac{1}{n} \sum_{l=1}^{n} \frac{1}{b^2} \sum_{k=1}^{b^2} \beta(\ f_{k|l}(x_{p|q})\ ,\ f_{k|l}(\widehat{x_{p|q}}))}\ . \\
&\qquad \text{[where } x_{k|l} \text{ is being mapped from } x_{p|q} \\
&\qquad \text{with forward map } f_{k|l}\text{]} \\
&\leq s_{max}^2 \sqrt{\frac{1}{n} \sum_{q=1}^{n} \frac{1}{b^2} \sum_{p=1}^{b^2} \beta(\ x_{p|q}\ ,\ \widehat{x_{p|q}})}\ . \\
&\leq s_{max}^2\ \alpha\ . \qquad \text{[By (8)]}
\end{aligned}$$

So, finally we have

$$\begin{aligned}
\rho\ (\ \underline{x}\ ,\ \underline{a}\ ) &= \rho\ (\ \underline{x}\ ,\ \mathcal{F}^N(\underline{o})\ )\ ;\ \ \forall\ \underline{o} \in S \text{ and for a large } N. \text{ [By (7)]} \\
&= \rho\ (\ \underline{x}\ ,\ \mathcal{F}^N(\underline{x})\ )\ ; \\
&\leq \rho(\underline{x}\ ,\ \mathcal{F}(\underline{x})) + \rho(\mathcal{F}(\underline{x})\ ,\ \mathcal{F}^2(\underline{x})) + \rho(\mathcal{F}^2(\underline{x})\ ,\ \mathcal{F}^3(\underline{x})) + \ldots. \\
&= \alpha + s_{max}\,\alpha + s_{max}^2\,\alpha + \ \ldots\ . \\
&= \alpha\ (\ 1 + s_{max} + s_{max}^2 + \ \ldots\ )\ . \\
&= \frac{\alpha}{1 - s_{max}}\ . \qquad Q.\ E.\ D.
\end{aligned}$$

In the next section we have presented the experimental results in support of the mathematical formulation of PIFS.

6 Experimental Results

In the context of PIFS, a technique for fractal image compression using Genetic Algorithm (GA) has been proposed by Mitra et al [1, 11]. Using this technique, the PIFS codes for 256×256, 8 bits/pixel "Lena" image and "LFA"(Low Flying Aircraft) image are found. Both the images have been reconstructed iteratively starting from different images. In particular we have used "Blank" image (having all the gray values zero), "Seagull" image, "Lena" image and "LFA" image as starting images. At the time of reconstruction, RMSE (distortion measure) between two successive iterations has been computed. In all the cases, computed values of RMSE are gradually decreasing with the increasing iteration number. It has been found that the PIFS codes almost achieved a fixed point after ten (10) iterations in both the images. Finally, the distance (RMSE), as expected, between the given image and the attractor is found to be very small. The distances are found to be on an average 7.75 and 11.44 for "Lena"image and "LFA" image respectively.

The computed values of RMSE have been presented in tabulated form in Table 1 and Table 2. The notation $A_{i,j}$ used in the tables denotes the RMSE value between ith and jth iterations. Thus we have

$$A_{i,j} \ = \ \rho \ (\ \mathcal{F}^i(\underline{o}) \ , \ \mathcal{F}^j(\underline{o}) \ ) \ ,$$

where $\mathcal{F}$ is the PIFS codes used and $\underline{o}$ is the starting image. As we have stopped the process of iterations after 10 iterations, the value of $A_{0,10}$ provides the distance between the attractor and the given image. The value of A_{01} will provide an approximate value of α if the starting image is the given image itself. The approximate values of α are found to be 7.18 and 10.67 for "Lena" and "LFA" images respectively.

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Table 1: RMSE values between successive iterations using PIFS code of "Lena" image

Starting image	$A_{0,10}$	$A_{0,1}$	$A_{1,2}$	$A_{2,3}$	$A_{3,4}$	$A_{4,5}$	$A_{5,6}$	$A_{6,7}$	$A_{7,8}$	$A_{8,9}$	$A_{9,10}$
Lena	7.73	7.18	2.91	1.12	0.45	0.24	0.15	0.10	0.07	0.04	0.03
Blank	7.86	X	43.93	32.05	19.72	11.30	6.25	3.48	1.97	1.13	0.66
Seagull	7.71	89.24	60.40	39.31	17.16	8.89	3.91	1.62	0.87	0.41	0.26
LFA	7.74	66.47	41.59	23.07	11.26	4.97	2.13	0.97	0.48	0.27	0.17

Table 2: RMSE values between successive iterations using PIFS code of "LFA" image

Starting image	$A_{0,10}$	$A_{0,1}$	$A_{1,2}$	$A_{2,3}$	$A_{3,4}$	$A_{4,5}$	$A_{5,6}$	$A_{6,7}$	$A_{7,8}$	$A_{8,9}$	$A_{9,10}$
LFA	11.42	10.67	4.32	1.56	0.63	0.34	0.23	0.16	0.12	0.09	0.06
Blank	11.48	X	53.78	36.51	21.87	12.70	7.39	4.28	2.46	1.42	0.84
Seagull	11.41	141.92	99.69	44.12	21.14	9.91	4.11	2.14	1.12	0.78	0.54
Lena	11.43	64.95	45.50	27.23	13.38	5.66	2.36	1.09	0.60	0.38	0.26

To judge the validity of the PIFS technique, the original and reconstructed images have also been checked visually. Figure 5, Figure 6 and Figure 7 show the original images of "Lena", "LFA" and "Seagull" respectively. Figure 8 and Figure 9 show the reconstructed images of "Lena" and "LFA" respectively. In both the cases the starting image is the "Seagull" image. Both the reconstructed images are found to be of good quality. Also the compression ratios are found to be 10.5 and 5.5 for "Lena" and "LFA" images respectively, using GA based PIFS technique [1]. Note that the compression ratio depends upon several factors like image size, range block size, bits per pixel and the gray level variation present in the given image.

Figure 5: Original "Lena" image

The next section concludes the present article.

7 Conclusions

The present article provides an elaborate and direct proof for the existence of the attractor and the closeness of the attractor to the given image in the partitioned IFS scheme. The upper bound of the difference, between the original image and the attractor evolved through its PIFS code, is almost same as that in the IFS set up (Collage theorem [2]). The only distinction between these two scheme is the contractivity factor which is eventually contractive in the case of PIFS whereas it is strictly contractive for IFS.

In PIFS technique the estimates of all the range blocks are obtained by finding the self similarities present in the given image. The domain block which is most similar to a range block is named as appropriately matched domain block for that range block. The similarity between the range block and its estimate is measured by RMSE. Thus the efficiency of PIFS technique depends on two factors. The first one is the efficiency of the distortion measure. The second one is the extent of similarity present in the given image after choosing an "appropriate" distortion measure.

RMSE, being a global measure has its own limitations [13]. In this context a better and

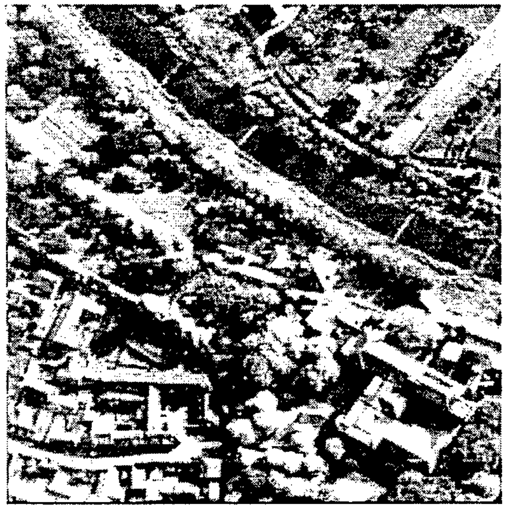

Figure 6: Original "LFA" image

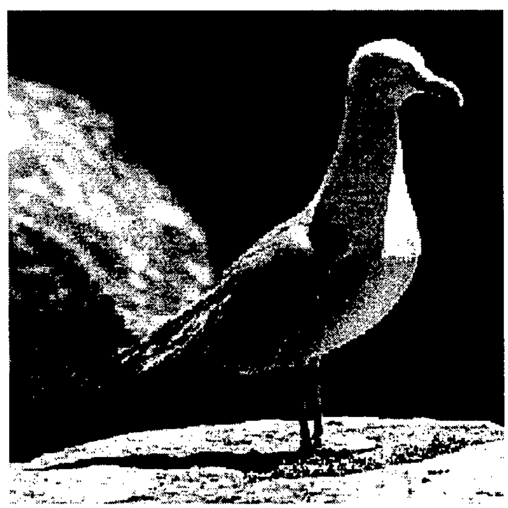

Figure 7: "Seagull" image

Figure 8: Decode "Lena" image

Figure 9: Decoded "LFA" image

reliable distortion measure can probably make the PIFS technique more efficient. Regarding the second factor, it may so happen that there is hardly any domain block which is appropriately matched with the concerned range block. In other words, the domain block closest to a range block in the sense of similarity, may provide a quantitatively large distortion. This may lead to inefficient coding. In such a case, many authors proposed subdivision of the concerned range block [8, 10] and coding of those divided blocks. The continuation of this process [10] will ultimately lead to a coding with less distortion, though sometimes it may lead to a coding with a very small compression ratio. Thus a proper choice of the size of the range block is important for obtaining good compression ratios.

The theory of PIFS technique is also applicable to one dimensional signals. In this context the technique has already been applied to code the fractal curves [7] and EEG signals [14]. But either case, computational time of the PIFS scheme for two dimensional signal (gray level image) and one dimensional signal (curve) is quite large. So, many attempts have been made to reduce the computational time [1, 15].

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13. ABSTRACT (Maximum 200 words)

The technique of image compression using Iterative Function System (IFS) is known as fractal image compression. An extension of IFS theory is Partitioned or local Iterative Function System (PIFS) for coding the gray level images. Several techniques of PIFS based image compression have already been proposed by many researchers. The theory of PIFS appears to be different from the theory of IFS in the sense of application domain. The present article discusses some basic differences between IFS and PIFS and provides a separate mathematical formulation for the existence of attractor of partitioned IFS. In particular, it has been shown that the attractor exists and it is an approximation of the given target image. The experimental results have also been presented in support of the theory. The experimental results have been obtained by using a GA based PIFS technique proposed by Mitra et al [1].

14. SUBJECT TERMS Image Compression, Iterative Function System (IFS) Partitioned Iterative Function System (PIFS), Attractor, Isometry

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