

# UNIFIED STUDY OF PLANAR FIELD THEORIES

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Abstract:

A "Master" gauge theory is constructed in 2+1-dimensions through which various gauge invariant and gauge non-invariant theories can be studied. In particular, Maxwell-Chern-Simons, Maxwell-Proca and Maxwell-Chern-Simons -Proca models are considered here. The Master theory in an enlarged phase space is constructed both in Lagrangian (Stuckelberg) and Hamiltonian (Batalin-Tyutin) frameworks, the latter being the more general one, which includes the former as a special case. Subsequently, BRST quantization of the latter is performed. Lastly, the master Lagrangian, constructed by Deser and Jackiw (Phys. Lett. B139, (1984) 371), to show the equivalence between the Maxwell-Chern-Simons and the self-dual model, is also reproduced from our Batalin-Tyutin extended model. Symplectic quantization procedure for constraint systems is adopted in the last demonstration.

## I: Introduction

The Maxwell-Chern-Simons (MCS) theory allows the presence of massive modes in the gauge field sector without breaking gauge invariance. This is possible because of the existence of the topological Chern-Simons (CS) term in 2+1-dimensions. This has been demonstrated long ago by Schonfeld [1] and by Deser, Jackiw and Templeton [2]. The two First Class Constraints (FCC) (according to the classification scheme of Dirac [3]) in the theory removes two degrees of freedom. The violation of parity, induced by the CS term, is manifested in the appearance of a single helicity component of the remaining excitation, the sign of which is correlated with that of the coefficient of the CS term.

However, in the MCS model, one can include a conventional vector field mass term (or Proca term) which explicitly breaks gauge invariance. The resulting model, known as the MCS-Proca (MCSP) model, was analysed by Pisarski and Rao at the perturbative one loop level and by Paul and Khare [4] in the Lagrangian framework. Later it was shown [5] that this model appears naturally, under certain approximations, in the recently studied higher dimensional bosonization [6].

The MCSP model was considered briefly in [7] in the Lagrangian framework. The detailed constraint analysis in the Hamiltonian framework was carried out in [8].

The effect of the Proca term is quite interesting. It breaks the gauge invariance resulting in a theory with two Second Class Constraints (SCC) [3], instead of the two FCCs present before. This change brings to life a second massive mode. The parity violation is seen in the fact that the above mentioned two modes of *distinct* masses carry spins  $\pm 1$ . This is similar to the Maxwell-Proca (MP) model, the difference being that here parity is not broken, which is reflected in the *same* masses of the opposite spins. Recently it was directly established [9] that MCSP theory is a free one, comprising of the modes discussed above, via a complicated set of canonical transformations.

Let us now put the present work in its proper perspective. From the above introduction, it is evident that both the gauge invariant and gauge non-invariant models in 2+1 dimensions have been studied mainly in the Lagrangian formalism, (except [9] where canonical transformations in a classical Hamiltonian framework has been used). However, *all* the above models can be conveniently discussed in a unified way, in a Hamiltonian formulation, where different theories, such as MP, MCS and MCSP, appear as special cases of an underlying "master" theory. This constitutes the main body of our work. The master theory constructed here is a gauge theory, which for different choices of parameters and gauge fixing conditions, leads to the different theories. Interestingly enough, we are also able to rederive the "master" Lagrangian proposed by Deser and Jackiw [10], which demonstrated the equivalence between the MCS and self-dual model, as a special case.

This underlying theory is obtained from two different viewpoints: (i) the Stuckelberg (Lagrangian) extension and (ii) BRST-BFV [11, 12] Hamiltonian scheme. In particular we follow a specific formulation of the BFV scheme, known as the Batalin- Tyutin (BT) [13] extension. Although in spirit, both the formalisms, (i.e. Lagrangian and Hamiltonian), introduce extra dynamical degrees of freedom in order to elevate a gauge non-invariant theory to a gauge invariant one, (in the enlarged phase space), the BFV and BT [12, 13] schemes are much more general and are applicable for any kind of gauge breaking term. The usefulness of this prescription, specially in non-linear theories has been demonstrated in [14]. Indeed, the Stuckelberg result follows as a special case of the BT construction, as we will discuss below. Also, being a gauge theory, these extended gauge models enjoy a wide range of freedom, in the form of the choice of gauge fixing fermion, in BRST quantization.

Recent applications of these field theoretic models in condensed matter systems, where the dynamics normal to a plane is severely restricted, have proved very fruitful [15].

The plan of the paper is the following: In section II the Stuckelberg construction is discussed. The existing results corresponding to the different models are rederived in a unified manner. The

quantization of the BT extended model, in the BRST framework. The connection with the Stuckelberg Lagrangian is also shown here. The emergence of the master lagrangian in [10] from our BT extension is demonstrated in section V. The Deser-Jackiw model is analysed in the Faddeev-Jackiw formalism [16], which is more convenient in the present case. The paper ends with a conclusion in section VI.

## II: Stuckelberg extension

The MCSP model, with the metric being  $g_{\mu\nu} = \text{diag}(+ - -)$ ,  $\epsilon_{12} = 1$ , is

$$\mathcal{L}_{MCSP} = -\frac{1}{4}A_{\mu\nu}A^{\mu\nu} + \frac{\mu}{4}\epsilon_{\mu\nu\lambda}A^{\mu\nu}A^\lambda + \frac{m^2}{2}A_\mu A^\mu, \quad (1)$$

where  $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . Taking  $m^2 = 0$  reproduces the MCS theory, which being a gauge theory is amenable to gauge fixing conditions. This simplifies the model considerably and makes the field content transparent. In order to discuss the MCS theory as well, let us now construct the master Lagrangian by converting the above gauge non-invariant theory to a gauge invariant one by the Stuckelberg prescription, For this purpose, we introduce the Stuckelberg field  $\theta$  and define the extended Lagrangian as,

$$\mathcal{L}_{St} = -\frac{1}{4}A_{\mu\nu}A^{\mu\nu} + \frac{\mu}{4}\epsilon_{\mu\nu\lambda}A^{\mu\nu}A^\lambda + \frac{m^2}{2}(A_\mu - \partial_\mu\theta)(A^\mu - \partial^\mu\theta). \quad (2)$$

In the ensuing gauge theory, we define the conjugate momenta [2] and the Poisson bracket algebra as,

$$\begin{aligned} \frac{\partial\mathcal{L}_{St}}{\partial\dot{A}^i} \equiv \Pi^i &= -\dot{A}_i + \partial_i A_0 - \frac{\mu}{2}\epsilon_{ij}A_j; & \frac{\partial\mathcal{L}_{St}}{\partial\dot{A}^0} \equiv \Pi^0 &= m^2\theta; & \frac{\partial\mathcal{L}_{St}}{\partial\dot{\theta}} \equiv \Pi_\theta &= m^2\dot{\theta}, \\ \{A_\mu(x), \Pi_\nu(y)\} &= -g_{\mu\nu}\delta(x-y), & \{\theta(x), \Pi_\theta(y)\} &= \delta(x-y). \end{aligned} \quad (3)$$

The Hamiltonian is

$$\begin{aligned} \mathcal{H}_{St} &= \Pi^\mu \dot{A}_\mu + \Pi_\theta \dot{\theta} - \mathcal{L}_{St} \\ &= \frac{1}{2}\Pi_i^2 + \frac{1}{4}A_{ij}A_{ij} + \left(\frac{m^2}{2} + \frac{\mu^2}{8}\right)A_i A_i - \frac{\mu}{2}\epsilon_{ij}\Pi_i A_j \\ &+ \frac{1}{2m^2}\Pi_\theta^2 + \frac{m^2}{2}\partial_i\theta\partial_i\theta + m^2(\partial_i A_i)\theta - A_0(\partial_i \Pi_i + \frac{\mu}{2}\epsilon_{ij}\partial_i A_j + \frac{m^2}{2}A_0), \end{aligned} \quad (4)$$

where a total derivative term has been dropped. The two FCCs in involution are,

$$\chi_1 \equiv \Pi_0 - m^2\theta, \quad \chi_2 \equiv \partial_i \Pi_i + \frac{\mu}{2}\epsilon_{ij}\partial_i A_j + m^2 A_0 + \Pi_\theta. \quad (5)$$

The unitary gauge,  $\eta^1 \equiv \Pi_\theta$ ;  $\eta^2 \equiv \theta$ , establishes gauge equivalence between the embedded model and the original MCSP model. This ensures that in the gauge invariant sector, results obtained in any convenient gauge will be true for the MCSP theory. We invoke the rotationally symmetric Coulomb gauge [2]

$$\eta^1 \equiv A_0; \quad \eta^2 \equiv \partial_i A_i. \quad (6)$$

The  $(\chi_i, \eta^j)$  system of four constraints are now second class, meaning that the constraint algebra matrix is invertible. The Dirac brackets, defined in the conventional way, are given below:

$$\{A_i(x), \Pi_j(y)\}^* = (\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2})\delta(x-y); \quad \{\Pi_i(x), \Pi_j(y)\}^* = -\frac{\mu}{2}\epsilon_{ij}\delta(x-y)$$

The remaining brackets are same as the Poisson brackets. The reduced Hamiltonian in Coulomb gauge is

$$\begin{aligned} \mathcal{H}_S = & \frac{1}{2}\Pi_i^2 + \frac{1}{2}\partial_i A_j \partial_i A_j + \left(\frac{m^2}{2} + \frac{\mu^2}{8}\right)A_i A_i - \frac{\mu}{2}\epsilon_{ij}\Pi_i A_j \\ & + \frac{1}{2m^2}\Pi_\theta^2 + \frac{m^2}{2}\partial_i \theta \partial_i \theta. \end{aligned} \quad (8)$$

Although somewhat tedious, it is straightforward to verify that the following combinations,  $\phi = ((\epsilon_{ij}\partial_i A_j), (\epsilon_{ij}\partial_i \Pi_j), \Pi_\theta, \theta)$  obey the higher derivative equation

$$(\square + M_1^2)(\square + M_2^2)\phi = 0; \quad M_1^2(M_2^2) = \frac{1}{2}[2m^2 + \mu^2 \pm \mu\sqrt{\mu^2 + 4m^2}]. \quad (9)$$

The spectra agrees with [4, 8]. Note that for  $\mu^2 = 0$ , the roots collapse to  $M_1^2 = M_2^2 = m^2$ , which is just the Maxwell-Proca model, whereas for  $m^2 = 0$ , in MCS theory, the roots are  $M_1^2 = \mu^2$ ,  $M_2^2 = 0$ , indicating the presence of only the topologically massive mode, since the Stuckelberg field  $\theta$  is no longer present.

Prior to fixing the  $\eta_2$  gauge, the gauge invariant sector is identified as,

$$E_i = -\Pi_i + \frac{\mu}{2}\epsilon_{ij}A_j; \quad B = -\epsilon_{ij}\partial_i A_j; \quad \Pi_\theta; \quad A_i + \partial_i \theta, \quad (10)$$

where  $E_i$  and  $B$  are the conventional electric and magnetic fields. In the reduced space, the Hamiltonian and spatial translation generators are gauge invariant,

$$\begin{aligned} \mathcal{H}_{St} = & \frac{1}{2}(E_i^2 + B^2 + \frac{\Pi_\theta^2}{m^2} + m^2(A_i + \partial_i \theta)^2), \\ \mathcal{P}_{St}^i = & -\epsilon_{ij}E_j B - \Pi_\theta(A_i + \partial_i \theta). \end{aligned} \quad (11)$$

Defining the boost transformation as  $M^{i0} = -t \int d^2x \mathcal{P}_{St}^i(x) + \int d^2x x^i \mathcal{H}_{St}(x)$ , the Dirac brackets with the gauge invariant variables are easily computed. They will contain non-canonical pieces in order to be consistent with the constraints. However, changing to a new set of variables by the following canonical transformations,

$$Q_1(Q_2) = \frac{1}{\sqrt{-2\nabla^2}}[\epsilon_{ij}\partial_i A_j \pm \frac{1}{m}\Pi_\theta]; \quad P_1(P_2) = [\frac{1}{\sqrt{-2\nabla^2}}\epsilon_{ij}\partial_i \Pi_j \mp \frac{m}{2}\sqrt{-2\nabla^2}\theta], \quad (12)$$

we can convert our system to a nearly decoupled one. Passing on to the quantum theory, the redefined variables satisfy the canonical algebra,

$$i[P_i, Q_j] = \delta_{ij}\delta(x-y); \quad [Q_i, Q_j] = [P_i, P_j] = 0; \quad i, j = 1, 2. \quad (13)$$

The electric and magnetic fields and the translation generators are rewritten as,

$$B = -\frac{\sqrt{-2\nabla^2}}{2}(Q_1 + Q_2); \quad E_i = -\frac{1}{\sqrt{-2\nabla^2}}[\epsilon_{ij}\partial_j(P_1 + P_2) + (\mu + m)\partial_i Q_1 + (\mu - m)\partial_i Q_2], \quad (14)$$

$$\begin{aligned} H_{St} = & \int d^2x [\frac{1}{2}(P_1^2 + \partial_i Q_1 \partial_i Q_1 + M_1^2 Q_1^2) + \frac{1}{2}(P_2^2 + \partial_i Q_2 \partial_i Q_2 + M_2^2 Q_2^2) + \frac{\mu^2}{2}Q_1 Q_2] \\ P_{St}^i = & \int d^2x [P_1 \partial^i Q_1 + P_2 \partial^i Q_2] \end{aligned} \quad (15)$$

It is worthwhile to consider the special cases,  $m^2 = 0$  or  $\mu^2 = 0$  corresponding to MCS and MP models respectively. In the former case, i.e. in the MCS theory, as we noted before,  $\theta$  field

is absent, which makes the  $(Q_1, P_1)$  pair identical to the  $(Q_2, P_2)$  pair, leading to the following relations, with  $i[p(x), q(y)] = \delta(x - y)$ ,

$$B = \sqrt{-\nabla^2}q, \quad E_i = \frac{1}{\sqrt{-\nabla^2}}(\epsilon_{ij}\partial_j p + \mu\partial_i q),$$

$$H = \int d^2x \frac{1}{2}(p^2 + \partial_i q \partial_i q + \mu^2 q^2), \quad P^i = \int d^2x (p\partial^i q). \quad (16)$$

This set of relations is identical to those in [2] and hence their result, that the spin is  $\pm\mu/|\mu|$ , will follow in a straightforward fashion. Due to parity violation, depending on the sign of  $\mu$ , only one component of spin is present.

The latter case,  $\mu^2 = 0$ , refers to the Proca model, where  $M_1^2 = M_2^2 = m^2$ , and we get,

$$B = -\frac{\sqrt{-2\nabla^2}}{2}(Q_1 + Q_2); \quad E_i = -\frac{1}{\sqrt{-2\nabla^2}}[\epsilon_{ij}\partial_j(P_1 + P_2) + m(\partial_i Q_1 - \partial_i Q_2)],$$

$$H = \int d^2x \left[ \frac{1}{2}(P_1^2 + \partial_i Q_1 \partial_i Q_1 + m_1^2 Q_1^2) + \frac{1}{2}(P_2^2 + \partial_i Q_2 \partial_i Q_2 + m_2^2 Q_2^2) \right],$$

$$P^i = \int d^2x [P_1 \partial^i Q_1 + P_2 \partial^i Q_2]. \quad (17)$$

Let us briefly outline the analysis of DJT [2] where the subtle interplay between Poincare invariance and an unambiguous determination of the spin of the excitations in a vector theory was revealed. It was shown that the correct space-time transformation of the gauge invariant observables, such as electric and magnetic fields, were induced by Poincare generators which obeyed an anomalous algebra among themselves. However, a phase redefinition of the creation and annihilation operators removed the commutator anomaly and yielded the spin contribution in a single stroke.

Following the prescription of DJT given in [2], the boost generator  $M^{i0}$  should be reinforced by the additional terms,

$$m\epsilon_{ij} \int d^2x \left( \frac{P_1 \partial_j Q_1}{-\nabla^2} - \frac{P_2 \partial_j Q_2}{-\nabla^2} \right),$$

such that the electromagnetic fields transform correctly. This addition, however, generates a zero momentum anomaly in the boost algebra,

$$i[M^{i0}, M^{j0}] = \epsilon^{ij}(M - \Delta), \quad \Delta = \frac{m^3}{4\pi} \left\{ \left( \int Q_1 \right)^2 - \left( \int Q_2 \right)^2 \right\} + \frac{m}{4\pi} \left\{ \left( \int P_1 \right)^2 - \left( \int P_2 \right)^2 \right\}, \quad (18)$$

where  $M$  is the rotation generator

$$M = - \int d^2x (P_1 \epsilon^{ij} x^i \partial_j Q_1 + P_2 \epsilon^{ij} x^i \partial_j Q_2).$$

Making the mode expansions,

$$Q_1(x)(Q_2(x)) = \int \frac{d^2k}{2\pi\sqrt{2\omega(k)}} [e^{-ikx} a(k)(b(k)) + e^{ikx} a^+(k)(b^+(k))], \quad (19)$$

and effecting the phase redefinitions,

$$a \rightarrow e^{i\frac{m}{|\mu|}\theta} a, \quad b \rightarrow e^{-i\frac{m}{|\mu|}\theta} b, \quad (20)$$

where  $\theta = \tan^{-1}k_2/k_1$ , one recovers the full angular momentum as

where the second term is the spin. This indicates parity non-violation as opposite spins are contributed by modes having the same mass.

Let us now come to MCSP model. In this case we choose a gauge of the form

$$\eta^1 \equiv A_0 ; \quad \eta^2 \equiv \theta. \quad (22)$$

We can now implement the four SCCs strongly in the Hamiltonian by using

$$A_0 = \theta = 0 ; \quad \Pi_\theta = -(\partial_i \Pi_i + \frac{\mu}{2} \epsilon_{ij} \partial_i A_j).$$

The Hamiltonian simplifies to,

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} \Pi_i^2 + \frac{1}{4} A_{ij} A_{ij} + (\frac{m^2}{2} + \frac{\mu^2}{8}) A_i A_i \\ & - \frac{\mu}{2} \epsilon_{ij} \Pi_i A_j + \frac{1}{2m^2} (\partial_i \Pi_i + \frac{\mu}{2} \epsilon_{ij} \partial_i A_j)^2. \end{aligned} \quad (23)$$

Notice that this choice does not affect the symplectic structure of the remaining variables.  $\square$  The following canonical transformations discussed in  $\square$  are applicable here,

$$\begin{aligned} A_i = & \frac{2m}{\sqrt{4m^2 + \theta^2}} \epsilon_{ij} \frac{\partial_j (Q_1 + Q_2)}{\sqrt{-\nabla^2}} + \frac{1}{2m} \frac{\partial_i (P_1 - P_2)}{\sqrt{-\nabla^2}}, \\ \Pi_i = & \frac{\sqrt{4m^2 + \theta^2}}{4m} \epsilon_{ij} \frac{\partial_j (P_1 + P_2)}{\sqrt{-\nabla^2}} - m \frac{\partial_i (Q_1 - Q_2)}{\sqrt{-\nabla^2}}, \end{aligned} \quad (24)$$

and the decoupled Hamiltonian is

$$H = \int d^2x \left[ \frac{1}{2} (P_1^2 + \partial_i Q_1 \partial_i Q_1 + M_1^2 Q_1^2) + \frac{1}{2} (P_2^2 + \partial_i Q_2 \partial_i Q_2 + M_2^2 Q_2^2) \right], \quad (25)$$

(It should be noted that we have checked quite exhaustively to conclude that any Coulomb like gauge is *not* able to lead to this decoupling.) The resulting mass spectra and spin are discussed thoroughly in  $\square$  and is not repeated here. Substituting the transformations (24) in the boost generator, one can see  $\square$  that the anomalous term is a decoupled sum of '1' and '2' variables and incorporating phase redefinition as in (20) with  $m$  replaced by  $M_1$  and  $M_2$ , one can recover the analogue of (21) with  $M_1$  and  $M_2$  for  $a$  and  $b$  operators respectively. Now parity violation is manifest as the opposite helicities carry *different* masses.

### III: Batalin-Tyutin extension

In this section we will discuss the other (Hamiltonian) alternative, i.e. the BT extension, which is required to convert the SCCs to FCCs since quantization of a gauge theory is more familiar to us. (Also, in general, presence of SCCs can complicate the symplectic structure and the path integral measure.) The idea of enlarging the phase space in Hamiltonian framework, to be considered here, is similar in spirit to the Stueckelberg extension, which is in Lagrangian framework. However, the advantage of the former is that it can be applied  $\square$  in complicated non-linear SC systems as well. In fact, eventually we will show that, for simple systems, the latter appears as a special case of the former.

<sup>2</sup>Effectively this gauge is same as the conventional unitary gauge, (where one chooses BT variables to be zero as the gauge choice), as far as the present analysis is concerned.

The explicit expressions regarding the additional terms, depending on the BT degrees of freedom, required by the SCCs for conversion to FCCs are provided in [13]. Hence we simply show the results. From the original SCCs we derive the commuting FCCs as

$$\tilde{\chi}_1 \equiv \chi_1 - m^2\psi ; \quad \tilde{\chi}_2 \equiv \chi_2 + \Pi_\psi, \quad (26)$$

where the conjugate pair,  $\{\psi(x), \Pi_\psi(y)\} = \delta(x - y)$ , are the BT fields. One can also introduce a useful set of FC variables [13] that commute with  $\tilde{\chi}_i$  by construction. Once again exploiting the formulas given in [13], these are computed as,

$$\begin{aligned} \tilde{A}_0 &= A_0 + \frac{1}{m^2}\Pi_\psi ; \quad \tilde{A}_i = A_i + \partial_i\psi ; \\ \tilde{\Pi}_0 &= \Pi_0 - m^2\psi ; \quad \tilde{\Pi}_i = \Pi_i + \frac{\mu}{2}\epsilon_{ij}\partial_j\psi. \end{aligned} \quad (27)$$

Next the Hamiltonian is rewritten in terms of the FC fields as,

$$\begin{aligned} \tilde{\mathcal{H}} &= \frac{1}{2}(\Pi_i + \frac{\mu}{2}\epsilon_{ij}\partial_j\psi)^2 + \frac{1}{4}\tilde{A}_{ij}\tilde{A}_{ij} \\ &+ (\frac{m^2}{2} + \frac{\mu^2}{8})(A_i + \partial_i\psi)^2 - \frac{\mu}{2}\epsilon_{ij}(\Pi_i + \frac{\mu}{2}\epsilon_{ik}\partial_k\psi)(A_j + \partial_j\psi) + \frac{m^2}{2}(A_0 + \frac{1}{m^2}\Pi_\psi)^2 \\ &= \frac{1}{2}(\Pi_i)^2 + \frac{1}{4}A_{ij}A_{ij} + (\frac{m^2}{2} + \frac{\mu^2}{8})(A_i)^2 - \frac{\mu}{2}\epsilon_{ij}\Pi_i A_j + \frac{m^2}{2}A_0^2 \\ &+ \frac{1}{2m^2}\Pi_\psi^2 + A_0\Pi_\psi + \frac{m^2}{2}((\partial_i\psi)^2 + 2A_i\partial_i\psi). \end{aligned} \quad (28)$$

It can be proved [13] that  $\tilde{\mathcal{H}}$  commutes with  $\tilde{\chi}_i$ , by construction. Notice tht we have dropped the term proportional to  $\tilde{\chi}_2$  from the Hamiltonian since the "constraint terms" coupled to arbitrary multiplier fields will eventually appear in the action for BRST quantization. The original SCCs, written in terms of the FC variables are identical to the modified FCCs. This completes the BT extension.

#### IV: BRST quantization

The BRST quantization [11, 13, 14], in the enlarged phase space, proceeds in the conventional way since only FCCs are present. The phase space is further extended by introducing ghost, anti-ghost and multiplier fields. These are, the canonically conjugate fermionic ghost and anti-ghost pairs  $(C^i(x), \bar{P}_i(x))$  and  $(P^i(x), \bar{C}_i(x))$  respectively and the bosonic multipliers and their momenta  $(q^i(x), p_i(x))$ . The BRST charge  $Q_{BRST}$  is defined as

$$Q_{BRST} = \int d^2x(C^i\tilde{\chi}_i + P^i p_i) ; \quad \{Q, Q\} = 0 ; \quad \{H_{BRST}, Q_{BRST}\} = 0.$$

In the present instance  $\tilde{\mathcal{H}} = \mathcal{H}_{BRST}$  since the system is completely abelian with the FCCs and  $\tilde{\mathcal{H}}$  being strictly in involution. The unitary Hamiltonian,

$$H_U \equiv H_{BRST} - \{\Psi, Q_{BRST}\} = \tilde{H} - \{\Psi, Q_{BRST}\}$$

is obtained as,

The gauge fixing fermion operator  $\Psi$  is defined as,

$$\Psi = \int d^2x (\bar{C}_i \eta^i + \bar{P}_i q^i),$$

where  $\eta^i$  are arbitrary functions of the fields, which can be considered as conventional gauge fixing conditions. The Hamiltonian path integral is,

$$Z = \int \mathcal{D}[\alpha] \exp^{iS_{BRST}} ; \mathcal{D}[\alpha] = \mathcal{D}[A_\mu, \Pi_\mu, \psi, \Pi_\psi, C^i, \bar{P}_i, P^i, \bar{C}_i, q^i, p_i] ;$$

$$\mathcal{L}_{BRST} = \dot{A}_\mu \Pi^\mu + \dot{\psi} \Pi_\psi + \bar{P}_i \dot{C}^i + \bar{C}_i \dot{P}^i + q^i p_i - \mathcal{H}_U. \quad (30)$$

The sector of physical states is defined by

$$Q_{BRST} | Ph \rangle = 0.$$

With the Poisson brackets replaced by commutators or anti-commutators, this describes the BRST quantization. Now, according to our motivation, this path integral can be simplified further by choosing specific forms of the arbitrary functions  $\eta^i$ , which actually amounts to a gauge fixing in the space of physical degrees of freedom. The only restriction on this choice is that the total system of constraints  $(\bar{\chi}_i, \eta^j)$  must have a non-vanishing Poisson bracket.

In order to forge a connection with the Stueckelberg construction, (as stated before), let us consider a Coulomb like gauge,

$$\eta^1 \equiv A_0 ; \eta^2 \equiv -\partial_i A_i + \sigma(x), \quad (31)$$

where  $\sigma$  is an undetermined, scalar field function. Since

$$\int d^2x (p_1 \dot{q}^1 + \bar{C}_1 \dot{P}^1) = \{Q_{BRST}, \int d^2x \bar{C}_1 \dot{q}^1\}$$

the terms on the left hand side are removed from  $\mathcal{L}_{BRST}$  because  $\bar{C}_1 \dot{q}^1$  in the right hand side can be absorbed in the arbitrary  $\eta^1$  term. Integrating out  $P^1$ ,  $\bar{P}_1$  and  $p_1$  results in  $\delta(A_0)$  in the measure, which removes the  $A_0$  integral. Similarly,  $q^1$  and  $\Pi_0$  are trivially integrated out. Next,  $\bar{P}_2$  and  $P^2$  are integrated leading to the relation  $P^2 = \dot{C}^2$ . Hence we are left with

$$Z = \int \mathcal{D}[\alpha] \exp^{iS_{BRST}} ; \mathcal{D}[\alpha] = \mathcal{D}[A_\mu, \Pi_\mu, \psi, \Pi_\psi, C^i, \bar{C}_i, q^2, p_2],$$

$$\mathcal{L}_{BRST} = \dot{A}^i \Pi_i + \dot{\psi} \Pi_\psi + q^2 p_2 - p_2 \eta^2 - q^2 (\partial_i \Pi_i + \frac{\mu}{2} \epsilon_{ij} \partial_i A_j + \Pi_\psi)$$

$$+ \bar{C}_2 \dot{C}^2 - \bar{C}_1 \{\eta^1, \bar{\chi}_1\} C^1 - \bar{C}_2 \{\eta^2, \bar{\chi}_2\} C^2$$

$$- [\frac{1}{2} \Pi_i^2 + \frac{1}{4} A_{ij} A_{ij} + (\frac{m^2}{2} + \frac{\mu^2}{8}) A_i A_i - \frac{\mu}{2} \epsilon_{ij} \Pi_i A_j + \frac{1}{2m^2} \Pi_\psi^2 + \frac{m^2}{2} (\partial_i \psi \partial_i \psi + 2A_i \partial_i \psi)]. \quad (32)$$

For the gauge conditions that we have chosen, we get,

$$\{\eta^1(x), \bar{\chi}_1(y)\} = -\delta(x-y) ; \{\eta^2(x), \bar{\chi}_2(y)\} = \nabla^2 \delta(x-y) + \{\sigma(x), \bar{\chi}_2(y)\}.$$

Hence the ghost part of the Lagrangian becomes,

$$\bar{C}_1 C^1 - \partial_\mu \bar{C}_2 \partial^\mu C^2 - \bar{C}_2 \{\sigma, \bar{\chi}_2\} C^2.$$

The gaussian integrals corresponding to  $\Pi_i$  and  $\Pi_\psi$  contribute respectively,



in the action. Finally, identifying  $q^2 = -A_0$ , (since  $q^i$  were arbitrary anyway), and combining all the above terms we obtain,

$$\begin{aligned}
Z &= \int \mathcal{D}[\alpha] \exp^{iS} ; \mathcal{D}[\alpha] = \mathcal{D}[A_\mu, \psi, \Pi_\psi, \bar{C}, C] \delta(\partial_\mu A^\mu + \sigma), \\
\mathcal{L} &= -\frac{1}{4} A_{\mu\nu} A^{\mu\nu} + \frac{\mu}{4} \epsilon_{\mu\nu\lambda} A^{\mu\nu} A^\lambda + \frac{m^2}{2} A_\mu A^\mu \\
&+ \frac{m^2}{2} \partial_\mu \psi \partial^\mu \psi + m^2 A^\mu \partial_\mu \psi - \partial_\mu \bar{C} \partial^\mu C - \bar{C} \{ \sigma, \tilde{\chi}_2 \} C.
\end{aligned} \tag{33}$$

For  $\sigma = 0$ , this is nothing but the Stuckelberg extension of the MCSP model, in the Lorentz gauge. This completes the identification of the Stuckelberg extension with the Batalin-Tyutin extension for a specific choice of the gauge fixing fermion in the latter scheme. In the next section, we provide a different gauge fixing fermion which leads to another interesting model.

## V: Recovering the Deser-Jackiw "master" Lagrangian

The idea of constructing a "master" Lagrangian to show explicitly the equivalence between various theories was pioneered by Deser and Jackiw [10]. The "master" Lagrangian is an interacting model from which selective integration of some fields leads to the desired models and the latter models are termed as equivalent. It should be pointed out that in general, integration of a dynamical field requires that the quantum effects be considered. However, if the fields that are to be removed occur linearly or quadratically, classical equations of motion suffice. In this sense, the Deser-Jackiw model [10] demonstrated the equivalence between the abelian MCS theory and abelian self-dual theory. The "master" Lagrangian posited in [10] is,

$$\begin{aligned}
\mathcal{L}_{DJ} &= \frac{1}{2} f^\mu f_\mu - f_\mu \epsilon^{\mu\alpha\beta} \partial_\alpha A_\beta + \frac{\mu}{2} A_\mu \epsilon^{\mu\alpha\beta} \partial_\alpha A_\beta \\
&= \frac{1}{2} f^\mu f_\mu - \epsilon^{ij} (f^0 \partial^i A^j - f^i \partial^0 A^j + f^i \partial^j A^0) + \frac{\mu}{2} \epsilon^{ij} (2A^0 \partial^i A^j - A^i \partial^0 A^j).
\end{aligned} \tag{34}$$

Since the Lagrangian is first order (in time derivatives), all the canonical momenta lead to primary constraints. They are,

$$P^\mu \equiv \frac{\partial \mathcal{L}_{DJ}}{\partial \dot{f}^\mu} \approx 0 \quad , \quad \Pi^0 \equiv \frac{\partial \mathcal{L}_{DJ}}{\partial \dot{A}^0} \approx 0 \quad , \quad \Pi^j \equiv \frac{\partial \mathcal{L}_{DJ}}{\partial \dot{A}^j} = \epsilon^{ij} (f^i - \frac{\mu}{2} A^i). \tag{35}$$

Clearly the proliferation of SCCs and the first order nature of the Lagrangian suggests that the Faddeev-Jackiw symplectic quantization scheme [16] would be more appropriate in the present case. In the Faddeev-Jackiw [16] formalism, one is allowed to bypass the derivation and classification of all the constraints, isolation of the FCCs and invoking the SCCs via introduction of Dirac brackets. Instead, here a generic Lagrangian is expressed in the first order form as,

$$Ldt = a_i d\rho^i - V(\rho)dt, \tag{36}$$

in which the symplectic structure is provided by,

$$\omega_{ij} = \frac{\partial a_j}{\partial \rho^i} - \frac{\partial a_i}{\partial \rho^j} \quad , \quad \{ \rho^i(x), \rho^j(y) \} = \omega_{ij}^{-1}(x, y), \tag{37}$$

provided the matrix  $\omega_{ij}$  is invertible. The aim is to express the Lagrangian in the following form.

where  $\bar{\rho}^k$  are less in number than  $\rho^i$ . Some of the latter variables appear as multiplier fields  $\lambda_i$ .  $\Phi^I$ s are the constraints of the theory. The Lagrangian in (34) is rewritten as,

$$\begin{aligned}\mathcal{L}_{DJ} &= \epsilon^{ij} \left( \frac{\mu}{2} A^j - f^j \right) \dot{A}^i + \frac{1}{2} f^\mu f_\mu - \epsilon^{ij} [f^0 \partial^i A^j - A^0 (\mu \partial^i A^j + \partial^j f^i)] \\ &= \epsilon^{ij} \left( \frac{\mu}{2} A^j - f^j \right) \dot{A}^i - \frac{1}{2} f^i f^i - \frac{1}{2} (\epsilon^{ij} \partial^i A^j)^2 + A^0 (\mu \epsilon^{ij} \partial^i A^j - \epsilon^{ij} \partial^i f^j),\end{aligned}\quad (39)$$

where the equation of motion for  $f^0$ , i.e.  $f^0 - \epsilon^{ij} \partial^i A^j = 0$ , has been used to eliminate  $f^0$ .  $A^0 (\equiv \lambda)$  is simply a multiplier field attached to the single FCC,

$$\Phi \equiv \mu \epsilon^{ij} \partial^i A^j - \epsilon^{ij} \partial^i f^j \approx 0. \quad (40)$$

In order to obtain the symplectic structure of the remaining fields, we rewrite the kinetic part of (39) as

$$\mathcal{L}_{SD}^{symp} = \epsilon^{ij} \left( \frac{\mu}{2} A^j - f^j \right) \dot{A}^i \equiv a_i \dot{\bar{\rho}}^i = a_3 \dot{\bar{\rho}}^3 + a_4 \dot{\bar{\rho}}^4, \quad (41)$$

where the following identifications are made,

$$\begin{aligned}\bar{\rho}^1 &\equiv f^1, \quad \bar{\rho}^2 \equiv f^2, \quad \bar{\rho}^3 \equiv A^1, \quad \bar{\rho}^4 \equiv A^2, \\ a_1 &= a_2 = 0, \quad a_3 = -\bar{\rho}^2 + \frac{\mu}{2} \bar{\rho}^4, \quad a_4 = -\bar{\rho}^1 + \frac{\mu}{2} \bar{\rho}^3.\end{aligned}$$

The symplectic two form matrix  $\omega_{ij}$  is computed as,

$$\begin{pmatrix} 0 & 0 & 0 & \delta(x-y) \\ 0 & 0 & -\delta(x-y) & 0 \\ 0 & \delta(x-y) & 0 & -\mu\delta(x-y) \\ -\delta(x-y) & 0 & -\mu\delta(x-y) & 0 \end{pmatrix}$$

As the above matrix is invertible, one can directly read of the non-canonical symplectic structure from the definition,

$$\{\bar{\rho}^i(x), \bar{\rho}^j(y)\} = \omega_{ij}^{-1}(x, y), \quad (42)$$

where the inverse matrix  $\omega_{ij}^{-1}$  is,

$$\begin{pmatrix} 0 & -\mu\delta(x-y) & 0 & \delta(x-y) \\ \mu\delta(x-y) & 0 & \delta(x-y) & 0 \\ 0 & -\delta(x-y) & 0 & 0 \\ \delta(x-y) & 0 & 0 & 0 \end{pmatrix}$$

Returning to the original definition of variables, the new symplectic structure is,

$$\{f^i(x), f^j(y)\} = -\mu \epsilon^{ij} \delta(x-y), \quad \{f^i(x), A^j(y)\} = -\epsilon^{ij} \delta(x-y), \quad \{A^i(x), A^j(y)\} = 0. \quad (43)$$

Let us now go back to the  $m^2 = 0$  limit of our BT extended model in (28),

$$\begin{aligned}\mathcal{L}_{BRST} &= \dot{A}^i \Pi^i - \left[ \frac{1}{2} \Pi_i^2 + \frac{1}{4} A_{ij} A_{ij} + \frac{\mu^2}{8} A_i A_i - \frac{\mu}{2} \epsilon_{ij} \Pi_i A_j \right] \\ &- p_2 \eta^2 - q^2 (\partial_i \Pi_i + \frac{\mu}{2} \epsilon_{ij} \partial_i A_j) - \bar{C}_1 \{\eta^1, \bar{\chi}_1\} C^1 - \bar{C}_2 \{\eta^2, \bar{\chi}_2\} C^2.\end{aligned}\quad (44)$$

The condition  $m^2 = 0$  removes some of the terms in (32) directly. The term  $\frac{\Pi_\psi^2}{2m^2}$  dominates over rest of the  $\Pi_\psi$  terms and is decoupled. Also the constraints are appropriately modified. Notice that now we have used both

$$\int d^3x(p_2\dot{q}^2 + \bar{C}_2\dot{P}^2) = \{Q_{BRST}, \int d^3x \bar{C}_2\dot{q}^2\},$$

to remove the left hand side terms from the action. The gauge  $\eta^1 \equiv A_0$  is retained but  $\eta^2$  is kept arbitrary.

Clearly our first order Lagrangian now has the desired structure identical to (38) where the multipliers  $\lambda_i$ 's are to be identified with  $q_2$  and  $p^2$  and  $\Phi_{IS}$  are  $\eta^2 \approx 0$  and  $\partial_i\Pi_i + \frac{\mu}{2}\epsilon_{ij}\partial_i A_j \approx 0$ . So far there is no change in the symplectic structure since the kinetic part of the Lagrangian has retained its canonical form. According to the Faddeev-Jackiw procedure [16] one can now use the "true" constraints  $\Phi_i$ 's in the theory to reduce the number of degrees of freedom. However, the corresponding changes in the kinetic energy part of the action can induce a modification in the symplectic structure, to be computed as before. Notice that we have already done this job partially in using the constraints  $A_0 \approx 0$  and  $\Pi^0 \approx 0$  strongly, which however does not lead to any change in the remaining brackets.

Indeed, according to [16], it is not imperative to classify the constraints  $\Phi_i$  according to FCC or SCC, or to compute the appropriate Dirac brackets. But from the BRST formalism we know that here  $\Phi_i \equiv (\chi_2, \eta^2)$  constitute an SCC pair since  $\eta^2$  is present in the gauge fixing fermion. Hence, although  $\eta^2$  is arbitrary, we have to choose it such that  $\{\chi_2, \eta^2\}$  is non-vanishing.

Let us now fix the gauge  $\eta^2$  as,

$$\eta^2 \equiv \partial_i\Pi_i - \frac{\mu}{2}\epsilon_{ij}\partial_i A_j + \epsilon_{ij}\partial_i h_j, \quad (45)$$

where  $h_j$  has a representation such that the combination  $\epsilon_{ij}\partial_i h_j$  gives a non-trivial contribution in  $\{\chi_2, \eta^2\}$ . This choice reduces the action of our model to,

$$\begin{aligned} \mathcal{L} = & \epsilon^{ij}\left(\frac{\mu}{2}A^j - h^j\right)\dot{A}^i - \frac{1}{2}h^i h^i - \frac{1}{2}(\epsilon^{ij}\partial^i A^j)^2 - q^2(\mu\epsilon^{ij}\partial^i A^j - \epsilon^{ij}\partial^i h^j) \\ & - \bar{C}_1\{\eta^1, \bar{\chi}_1\}C^1 - \bar{C}_2\{\eta^2, \bar{\chi}_2\}C^2. \end{aligned} \quad (46)$$

It should be mentioned that in the action (44) we are actually using  $\Pi^i = \epsilon^{ij}\left(\frac{\mu}{2}A^j - f^j\right)$  as a solution of (45) and have not considered terms of the form  $\epsilon^{ij}\partial^j \alpha$ . This is slightly more restrictive than the gauge choice (45). The above action is immediately recognisable as the master Lagrangian in [10] once the fields are identified as  $h_i \equiv f_i$  and  $-q^2 \equiv A_0$ . The ghost contribution remains decoupled so long as the gauge choices are linear. The non-canonical symplectic structure also follows accordingly. The FCC removes one degree of freedom leading to the single helicity mode of the MCS theory. This completes the reduction of our system to that of the master Lagrangian constructed by Deser and Jackiw [10].

## VI: Conclusion

The power of the Batalin-Tyutin (BT) quantization scheme has been amply demonstrated in the present work, where vector field theories in 2+1-dimensions have been considered. The models discussed primarily are Maxwell-Proca (MP), Maxwell-Chern-Simons (MCS) and Maxwell-Chern-Simons-Proca (MCSP) models. Of this group, the second one is a gauge theory and parity is broken in the last two models due to the Chern-Simons term. The parity violation is reflected by the fact that MCS theory generates a *single* helicity mode and MCSP consists of *two* opposite helicity modes with *unequal* masses. The extra mode appears due to the absence of gauge symmetry. This can be compared with MP model, where the two helicities carry the *same* mass. The subtleties in computing the spin content was discussed in [10] for MCS theory. Similar analysis were carried out in [17], (an

In the enlarged phase space, the BT extension of MCSP model has two FCCs in involution. This system has been used in the (Hamiltonian) BRST quantization. The importance of the gauge choice has been demonstrated in relating various models. We have explicitly shown that our BT extended model is equivalent to the conventional Stuckelberg extension in a particular gauge, whereas in another gauge it relates to the master Lagrangian discovered by Deser and Jackiw [10] to show the equivalence between self-dual model and MCS theory. In our study of the latter connection, we have used the symplectic quantization approach, initiated by Faddeev and Jackiw [16]. All of the above discussion underlines the significance of the master Lagrangian technique, which connects apparently different models and brings greater insight from their equivalence.

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