

Phase properties of a new nonlinear coherent state

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Abstract

We study phase properties of a displacement operator type nonlinear coherent state. In particular we evaluate the Pegg-Barnett phase distribution and compare it with phase distributions associated with the Husimi Q function and the Wigner function. We also examine number-phase squeezing of this state.

1 Introduction

Coherent states are important in different branches of physics particularly in quantum optics. Historically coherent state of the harmonic oscillator (which is the coherent state corresponding to the Heisenberg-Weyl algebra) was first constructed by Schrödinger [1]. Subsequently coherent states corresponding to various Lie algebras like $Su(1,1)$, $Su(2)$ etc. have also been constructed and has been shown to play important roles in the description of various quantum optical processes [2, 3].

Recently another type of coherent states called the nonlinear coherent states [4] or the f-coherent states [5] have been constructed. In contrast to the coherent states mentioned

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above these are coherent states corresponding to nonlinear algebras. However nonlinear coherent states are not mere mathematical objects. It has been shown [4] that nonlinear coherent states are useful in connection with the motion of a trapped ion.

We note that unlike Lie algebras, the commutators of the generators of nonlinear algebras is a nonlinear function of the generators. As a consequence it is difficult to apply the BCH disentangling theorem [6] to construct displacement operator coherent states corresponding to nonlinear algebras. To avoid this difficulty nonlinear coherent states were constructed as eigenstates of a generalised annihilation operator [4, 5]. However, nonlinear coherent states can still be constructed using a displacement operator, *albeit*, a modified one [7, 8] and it has been shown that such states exhibit nonclassical behaviour [9]. In the present paper we shall study phase properties of such a nonlinear coherent state. In particular we shall obtain the Pegg-Barnett phase distribution and compare it with the phase distributions associated with Q-function and the Wigner function. We shall also evaluate the number-phase uncertainty relation and examine number-phase squeezing of the nonlinear coherent state. The organisation of the paper is as follows: in section 2 we derive phase distributions and number-phase uncertainty relation for displacement operator nonlinear coherent states; in section 3 we discuss numerical results obtained using the results of section 2; finally section 4 is devoted to a conclusion.

2 New nonlinear coherent states and their phase distributions

To begin with we note that the generalised creation and annihilation operators associated with nonlinear coherent states are of the form [4, 5]

$$A^\dagger = f(N)a^\dagger \quad , \quad A = af(N) \quad , \quad N = a^\dagger a \quad (1)$$

where a^\dagger and a are standard harmonic oscillator creation and annihilation operators and $f(x)$ is a reasonably well behaved real function, called the nonlinearity function. From the relations (1) it follows that A , A^\dagger and N satisfy the following closed algebraic relations:

$$[A, A^\dagger] = f^2(N)(N + 1) - f^2(N - 1)N \quad , \quad [N, A] = -A \quad , \quad [N, A^\dagger] = A^\dagger \quad (2)$$

Thus (2) represent a deformed Heisenberg algebra whose nature of deformation depends on the nonlinearity function $f(n)$. Clearly for $f(n) = 1$ we regain the Heisenberg algebra.

Nonlinear coherent states $|\chi\rangle$ are then defined as right eigenstates of the generalised annihilation operator A and in a number state basis is given by [4, 5]:

$$|\chi\rangle = C \sum_{n=0}^{\infty} \frac{d_n}{\sqrt{n!}} \chi^n |n\rangle \quad (3)$$

where C is a normalisation constant and the coefficients d_n are given by

$$d_0 = 1 \quad , \quad d_n = [\prod_{i=1}^n f(i)]^{-1} \quad (4)$$

We note that the canonical conjugate of the generalised annihilation and creation operator A and A^\dagger are given by [9]

$$B = a \frac{1}{f(N)} \quad , \quad B^\dagger = \frac{1}{f(N)} a^\dagger \quad (5)$$

Thus A and B^\dagger and their hermitian conjugates satisfy the algebras

$$[A, B^\dagger] = 1 \quad , \quad [B, A^\dagger] = 1 \quad (6)$$

Now following ref [9] we consider the operators ($\alpha = |\alpha|e^{i\phi}$)

$$\begin{aligned} D_1(\alpha) &= \exp(\alpha B^\dagger - \alpha^* A) \\ D(\alpha) &= \exp(\alpha A^\dagger - \alpha^* B) \end{aligned} \quad (7)$$

and note that for two operators X and Y such that $[X, Y] = 1$ the BCH disentangling theorem [6] is of the form

$$\exp(\alpha X - \alpha^* Y) = \exp\left(-\frac{\alpha\alpha^*}{2}\right) \exp(\alpha X) \exp(-\alpha^* Y) \quad (8)$$

Then the nonlinear coherent state corresponding to the first of the two algebras in (6) is defined as $|\alpha\rangle_1 = D_1(\alpha)|0\rangle$. Now applying (8) we obtain

$$|\alpha\rangle_1 = c_1 \sum_{n=0}^{\infty} \frac{d_n}{\sqrt{n!}} \alpha^n |n\rangle \quad (9)$$

Comparing this with the nonlinear coherent state $|\chi\rangle$ (see (3)) we find that both are exactly the same (provided of course we use the same nonlinearity function in both the cases).

The new nonlinear coherent state is then defined as $|\alpha\rangle = D(\alpha)|0\rangle$ i.e, it is the coherent state corresponding to the second algebra in (6). As before using the relation (8) we obtain

$$|\alpha\rangle = c \sum_{n=0}^{\infty} \frac{d_n^{-1}}{\sqrt{n!}} \alpha^n |n\rangle \quad (10)$$

where c is a normalisation constant which can be determined from the condition $\langle \alpha | \alpha \rangle = 1$ and is given by

$$c^2 = \left[\sum_{n=0}^{\infty} \frac{d_n^{-2}}{n!} (\alpha^* \alpha)^n \right]^{-1} \quad (11)$$

We now consider the phase probability distributions for the new nonlinear coherent state (10). According to the Pegg-Barnett formalism [10] a complete set of $(s+1)$ orthonormal phase states θ_p are defined by

$$|\theta_p\rangle = \frac{1}{\sqrt{(s+1)}} \sum_{n=0}^s \exp(in\theta_p) |n\rangle \quad (12)$$

where $|n\rangle$ are the number states which spans the $(s+1)$ dimensional state space and θ_p are given by

$$\theta_p = \theta_0 + \frac{2\pi p}{s+1}, \quad p = 0, 1, 2, \dots, s \quad (13)$$

In (13) θ_0 is arbitrary and indicates a particular basis in the phase space. The hermitian phase operator is then defined as

$$\Phi_\theta = \sum_{p=0}^s \theta_p |\theta_p\rangle \langle \theta_p| \quad (14)$$

The expectation value of the phase operator with respect to the new nonlinear state $|\alpha\rangle$ is given by

$$\langle \alpha | \Phi_\theta | \alpha \rangle = \sum_{p=0}^s \theta_p |\langle \theta_p | \alpha \rangle|^2 \quad (15)$$

where $|\langle \theta_p | \alpha \rangle|^2$ is the probability of being in the state $|\theta_p\rangle$. Then in the limit $s \rightarrow \infty$ we get from (15)

$$\langle \alpha | \Phi_\theta | \alpha \rangle = \int_{\theta_0}^{\theta_0+2\pi} \theta P(\theta) d\theta \quad (16)$$

where the continuous probability distribution $P(\theta)$ is given by

$$P(\theta) = \lim_{s \rightarrow \infty} \frac{s+1}{2\pi} |\langle \theta_p | \alpha \rangle|^2 \quad (17)$$

Now choosing θ_0 as

$$\theta_0 = \phi - \frac{\pi s}{s+1} \quad (18)$$

and using (17) we obtain the Pegg-Barnett phase probability distribution for the new nonlinear coherent states (10):

$$P_{PB}(\theta) = \frac{1}{2\pi} \left[1 + 2c^2 \sum_{n>k} \frac{d_n^{-1} d_k^{-1}}{\sqrt{n!k!}} \cos[(n-k)\theta] \right] , \quad -\pi \leq \theta \leq \pi \quad (19)$$

With the phase probability distribution known various quantum mechanical averages in the phase space can be obtained using this function. For example the phase variance is given by

$$\langle (\Delta\Phi_\theta)^2 \rangle = \int_{-\pi}^{\pi} \theta^2 P(\theta) d\theta = \frac{\pi^2}{3} + 4c^2 \sum_{n>k} \frac{d_n^{-1} d_k^{-1} (-1)^{n-k}}{\sqrt{n!k!} (n-k)^2} \quad (20)$$

It may be noted that since N and Φ_θ are canonically conjugate operators they obey the uncertainty relation

$$\langle (\Delta N)^2 \rangle \langle (\Delta\Phi_\theta)^2 \rangle \geq \frac{1}{4} | \langle [N, \Phi_\theta] \rangle |^2 \quad (21)$$

where $\langle (\Delta X)^2 \rangle = \langle X^2 \rangle - \langle X \rangle^2$ and the right hand side of (21) is given by

$$[N, \Phi_\theta] = i[1 - 2\pi P(\theta_0)] \quad (22)$$

Now to examine number-phase squeezing we introduce the following squeezing parameters:

$$S_N = \frac{2 \langle (\Delta N)^2 \rangle}{| \langle [N, \Phi_\theta] \rangle |} - 1 , \quad S_\Phi = \frac{2 \langle (\Delta\Phi_\theta)^2 \rangle}{| \langle [N, \Phi_\theta] \rangle |} - 1 \quad (23)$$

If $S_N < 0$ ($S_\Phi < 0$) then the nonlinear coherent state is number(phase) squeezed.

We note that the phase quasiprobability distributions $P_{Q,W}(\theta)$ associated with the Husimi Q-function and the Wigner function can be obtained by integrating these functions over the radial variable $|\beta|$. The forms of these distributions are given by

$$P_{Q,W}(\theta) = \frac{1}{2\pi} \left[1 + 2c^2 \sum_{n>k} \frac{d_n^{-1} d_k^{-1}}{\sqrt{n!k!}} \cos[(n-k)\theta] F(n, k) \right] , \quad -\pi \leq \theta \leq \pi \quad (24)$$

where the coefficients $F(n, k)$ in the case of Q-function are given by [11, 12]

$$F(n, k) = \frac{\Gamma(\frac{n+k}{2} + 1)}{\sqrt{n!k!}} \quad (25)$$

while in the case of Wigner function they are given by [12, 13]

$$\begin{aligned}
F(n, k) &= 2^{(n-k)/2} \sqrt{\frac{k! \Gamma(\frac{n}{2} + 1)}{n! (\frac{k}{2})!}} \quad , n \text{ even} \\
&= 2^{(n-k)/2} \sqrt{\frac{k! \Gamma(\frac{n+1}{2})}{n! (\frac{k-1}{2})!}} \quad , n \text{ odd}
\end{aligned} \tag{26}$$

3 Phase properties

We shall now analyse various phase distributions for the nonlinear coherent state. However, before we do this it is necessary to specify a nonlinearity function $f(n)$. It is clear that for different choices of the nonlinearity function we shall get different nonlinear coherent states. In the present case we choose a nonlinearity function which has been used in the description of the motion of a trapped ion [4]:

$$f(n) = L_n^1(\eta^2)[(n+1)L_n^0(\eta^2)]^{-1} \tag{27}$$

where η is known as the Lamb-Dicke parameter and $L_n^m(x)$ are generalised Laguerre polynomials. We shall now evaluate the distribution functions (19) and (24) with the nonlinearity function given by (27).

In figure 1 we plot Pegg-Barnett phase distribution $P_{PB}(\theta)$ against θ keeping α fixed and using different values of η for the three curves. From figure 1 we find that for lower values of η the distribution is broad at the top. However as η increases peaks begin to develop slowly and for a reasonably large value of η there are two well developed peaks at $\theta = \pm\pi/2$. The appearance of the peaks is an indication of quantum interference.

In figure 1a we plot the Pegg-Barnett phase distribution keeping η fixed at .8 and varying α . From the figures it is seen that the qualitative features of the distribution remains essentially the same when one of the parameters is kept fixed while the other varies. However it may be noted that as α increases the peak structure becomes more and more prominent. Interestingly for $\alpha = .37$ (it is the value where the phase variance is minimum, see below) the distribution shows practically no bifurcation.

In figure 2 we plot the three distributions $P_{PB}(\theta)$ and $P_{Q,W}(\theta)$ for the same values of the parameters. From the figure we find that although the distributions are of the same form they are not quite the same. It is seen that P_{PB} is roughly intermediate between

P_Q and P_W . Also the Wigner distribution P_W has the sharpest peaks while the Husimi Q distribution P_Q is the broadest. However in the present case P_W does not assume any negative value.

In figure 3 we have plotted phase variance against α for $\eta = .8$. From the figure we find that phase variance decreases upto a certain value of α and then again starts increasing. From the figure we find that phase variance assumes the minimum value at $\alpha = .37$. Thus for $\alpha = .37$ and around this value of α the best measurement of phase is possible. We note that the parameter values are not special but the phase variance shows the same pattern for other parameter values too.

In figure 4 we plot the squeezing parameters S_N and S_Φ . From the figure we find that $S_N < 0$ for a considerable range of α . This implies that the nonlinear coherent state exhibits squeezing in the N component. However, S_Φ is always positive implying absence of squeezing in the Φ component.

We note that it is interesting to compare the squeezing behaviour of the new coherent state (10) and the one given by (3). In figure 5 we plot the graphs of S_N corresponding to these states. From the figure it is seen that while for low values of α the coherent state (3) is more squeezed than (10), for larger values as well as a larger range of α , the coherent state (10) remains squeezed while (3) does not remain so.

Now we compare the phase squeezing of the nonlinear coherent states (3) and (10). From fig 6 we find that $S_\Phi < 0$ for (3) while $S_\Phi > 0$ for (10). Thus from (23) it follows that the nonlinear coherent state (3) exhibits phase squeezing while (10) does not.

Finally to examine the number-phase uncertainty relation (21), we consider the quantity $F(\alpha) = \sqrt{\langle (\Delta N)^2 \rangle \langle (\Delta \Phi_\theta)^2 \rangle} - \frac{1}{2} | \langle [N, \Phi_\theta] \rangle |$. Thus $F(\alpha) \geq 0$ and $F(\alpha) = 0$ would imply that the state is an intelligent state. On the other hand a nonzero value of $F(\alpha)$ is a measure which indicates how much the state is away from being an intelligent state. In figure 7 we plot $F(\alpha)$ against α for $\eta = .8$. From the figure it is seen that $F(\alpha)$ is nonzero and has an increasing trend. The maximum in fig 7 indicates how much the nonlinear coherent state (10) can be different from an intelligent state. Thus we conclude that the nonlinear coherent state (10) is not an intelligent state.

4 Conclusion

In this article we have considered a class of nonlinear coherent states constructed using an operator similar to the displacement operator. We have examined a number of their phase properties. In particular we have computed various phase distributions and compared them. Also it has been shown that the states (10) exhibit number squeezing.

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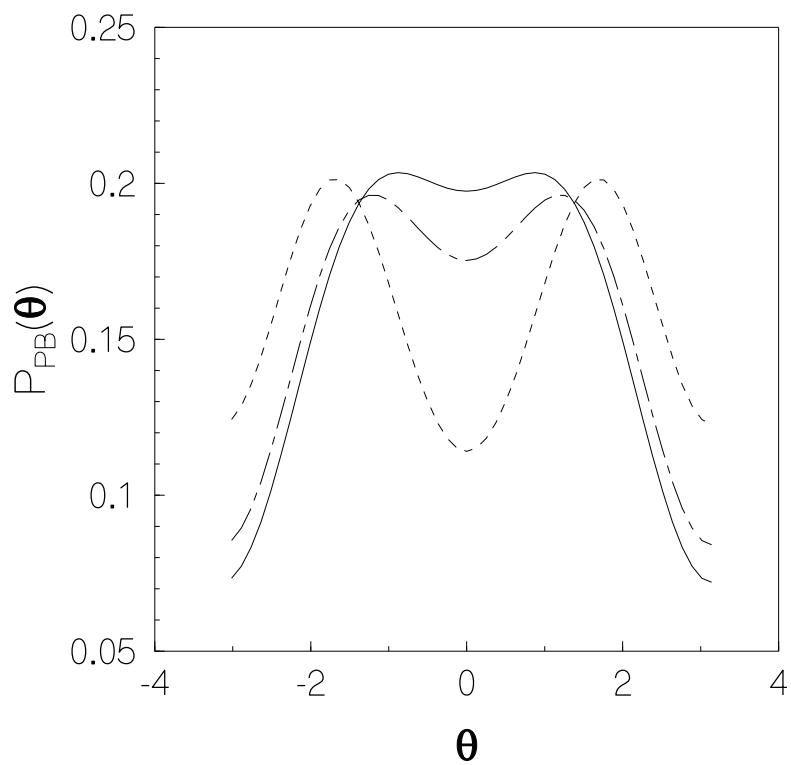


Fig 1. Pegg-Barnett phase distribution for $\alpha=.8$ and $\eta=.4$ (solid curve), $.6$ (broken curve), $.8$ (dotted curve)

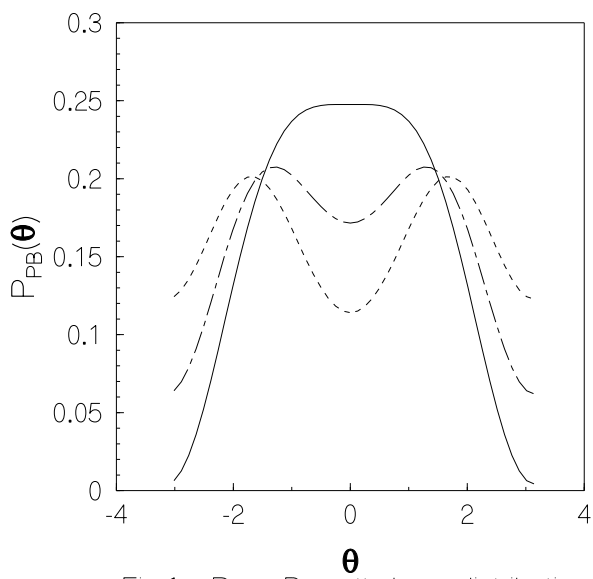


Fig 1a. Pegg-Barnett phase distribution for $\eta = .8$ and $\alpha = .8$ (dotted curve), $.6$ (broken curve) and $.37$ (solid curve)

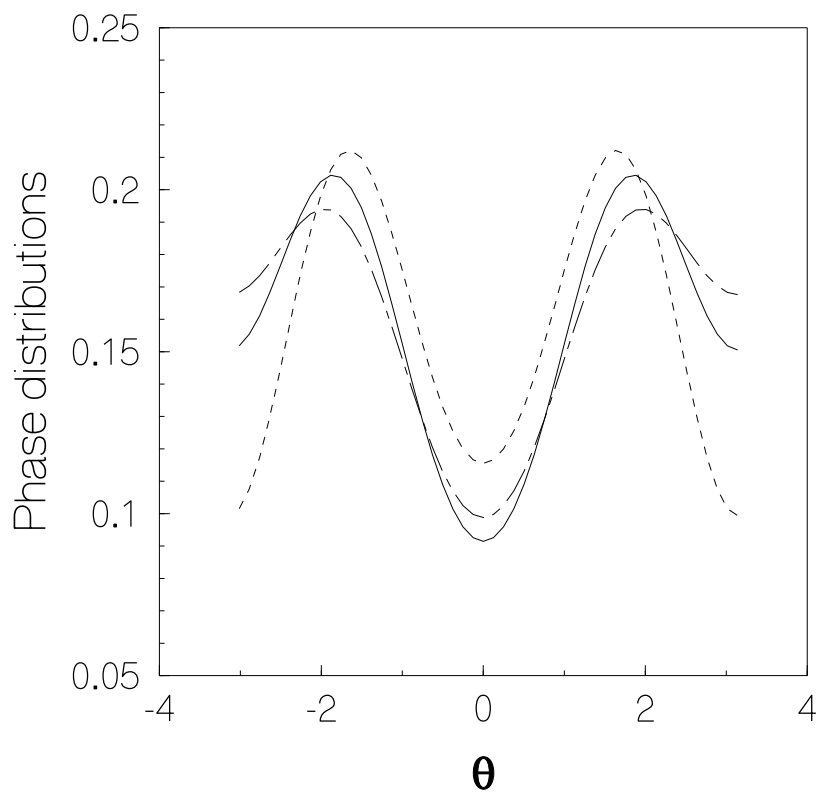


Fig 2. Phase distributions for $\eta = .8$ and $\alpha = .9$. Solid curve(P_{PB}),Broken curve(P_Q) and Dotted curve(P_W).

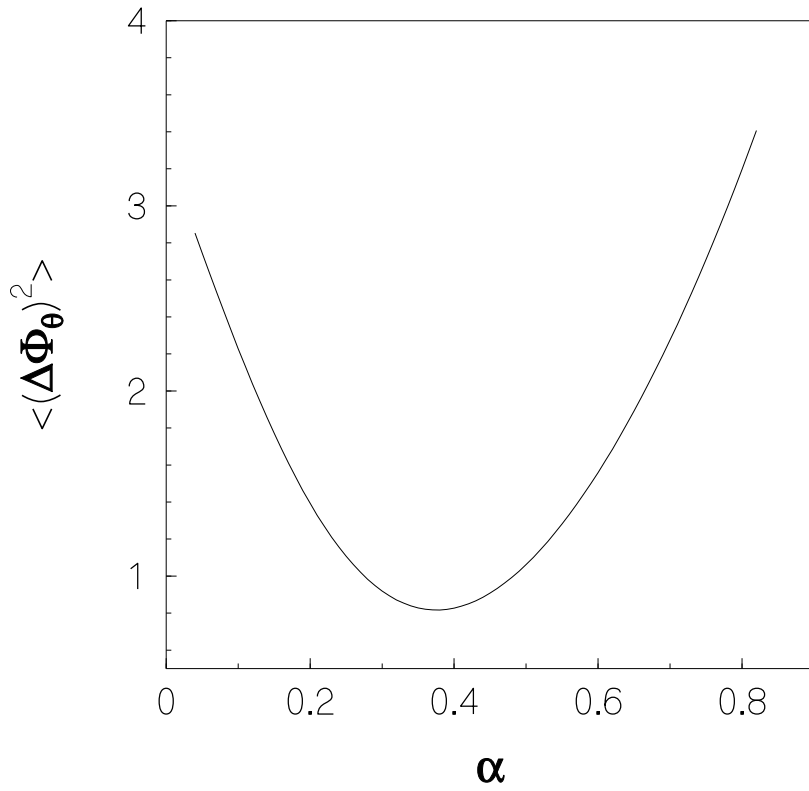


Fig 3. Phase variance against α for $\eta=.8$

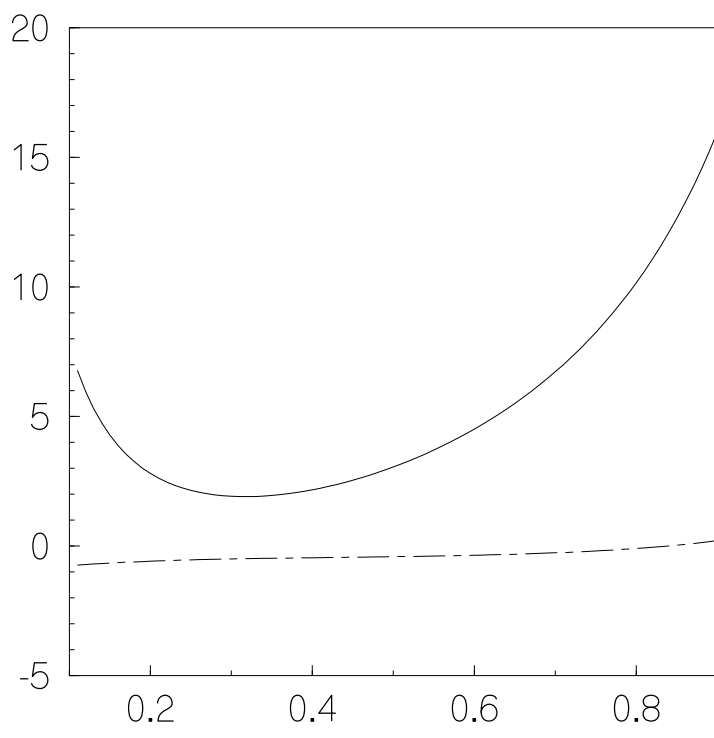


Fig 4. Plot of S_N (broken curve) and S_Φ (solid curve) for $\eta=.6$

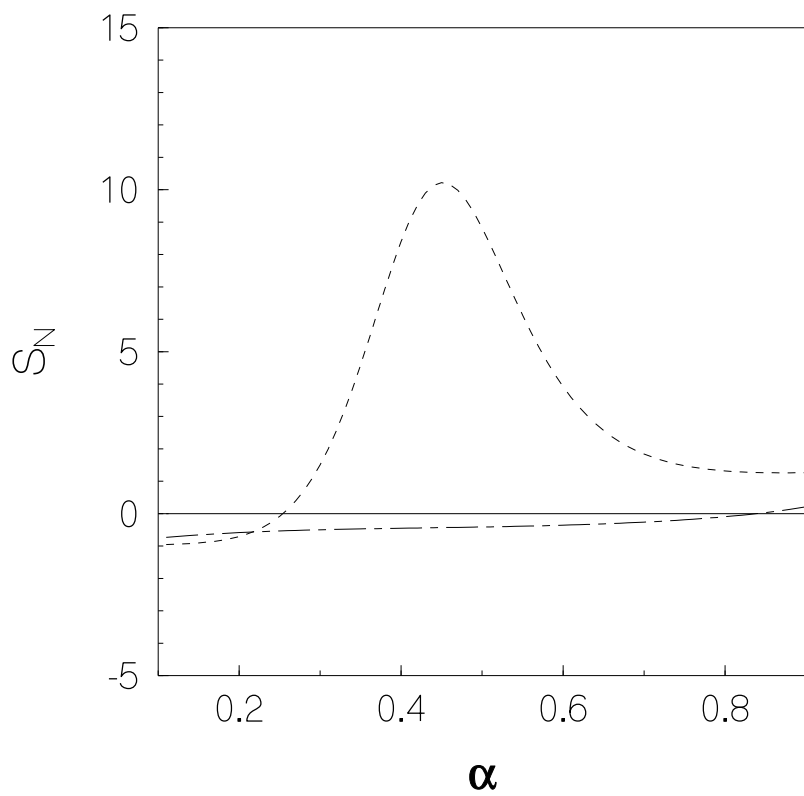


Fig 5. Plot of S_N for coherent states (3) (dotted curve) and (10)(broken curve) for $\eta=.6$

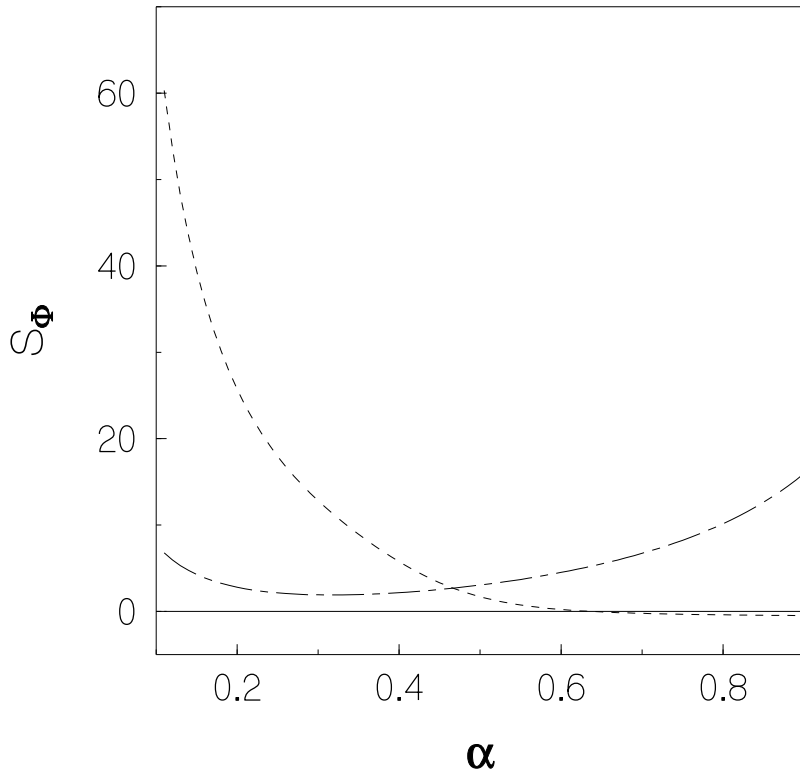


Fig 6. Plot of S_Φ for coherent states (3) (dotted curve) and (10)(broken curve) for $\eta=.6$

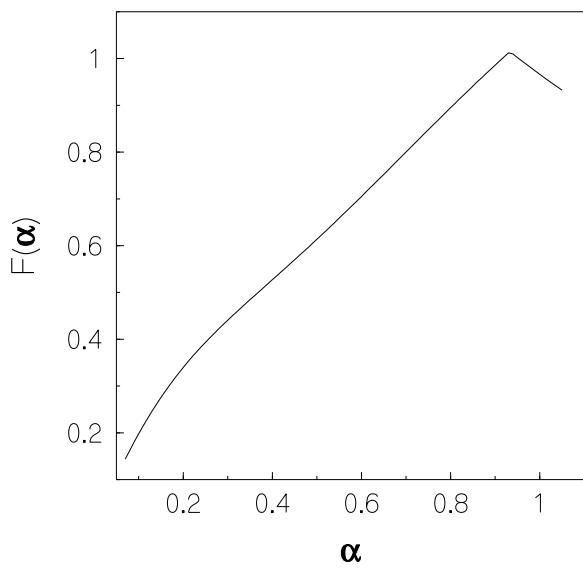


Fig 7. Plot of $F(\alpha)$ for $\eta=0.8$