

Asymptotics of the Perron eigenvalue and eigenvector using max-algebra

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Abstract. We consider the asymptotics of the Perron eigenvalue and eigenvector of irreducible nonnegative matrices whose entries have a geometric dependence in a large parameter. The first term of the asymptotic expansion of these spectral elements is solution of a spectral problem in a semifield of jets, which generalizes the max-algebra. We state a “Perron-Frobenius theorem” in this semifield, which allows us to characterize the first term of this expansion in some non-singular cases. The general case involves an aggregation procedure à la Wentzell-Freidlin. © Académie des Sciences/Elsevier, Paris

Asymptotique de la valeur propre et du vecteur propre de Perron via l’algèbre max-plus

Résumé. On s’intéresse à l’asymptotique de la valeur propre et du vecteur propre de Perron de matrices à coefficients positifs ou nuls, dépendant géométriquement d’un grand paramètre. Le premier terme du développement asymptotique de ces éléments spectraux est solution d’un problème spectral sur un semi-corps de jets, qui généralise le semi-corps max-plus. Nous établissons un « théorème de Perron-Frobenius » pour les jets, qui nous permet de caractériser le premier terme de ce développement dans des cas non-singuliers. Le cas général requiert une procédure d’agrégation à la Wentzell-Freidlin. © Académie des Sciences/Elsevier, Paris

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Soit \mathcal{A}_p une matrice $n \times n$ à coefficients réels positifs ou nuls, dépendant d’un grand paramètre p . On considère le problème spectral (1) dans le cas où \mathcal{A}_p est irréductible : \mathcal{L}_p est unique et il existe un unique \mathcal{U}_p vérifiant $\sum_i (\mathcal{U}_p)_i = 1$ (voir par exemple [4], Ch. 2). On cherche à déterminer les asymptotiques de \mathcal{L}_p et \mathcal{U}_p à partir de celles de \mathcal{A}_p . En utilisant les résultats analogues au théorème

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de Perron–Frobenius dans le semi-corps $\mathbb{R}_{\max} = (\mathbb{R}^+, \max, \times, 0, 1)$, isomorphe au semi-corps max-plus (voir par exemple [12], [2], Th. 3.100, [6], §VI, [11], §3.7 et ses références), on obtient les asymptotiques de type grandes déviations de \mathcal{L}_p , et dans certains cas celles de \mathcal{U}_p .

THÉORÈME 1. – *Si les limites (2) existent et si $A = (A_{ij})$ est irréductible, alors $\lim_{p \rightarrow \infty} (\mathcal{L}_p)^{\frac{1}{p}}$ existe. Elle est égale à la valeur propre de A dans \mathbb{R}_{\max} , notée $\rho_{\max}(A)$, donnée par le second membre de (3).*

Un circuit $c = (i_1, \dots, i_k)$ est dit *critique* si $(A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_k i_1})^{\frac{1}{k}}$ réalise le maximum dans (3). On appelle *graphe critique* le graphe orienté formé des nœuds et arcs des circuits critiques. On appelle *classe critique* une composante fortement connexe du graphe critique. On pose $\tilde{A} = \rho_{\max}(A)^{-1} A$, et on note $(\tilde{A})^* = \bigoplus_{k=0}^{\infty} (\tilde{A})^k$ l'étoile de Kleene de \tilde{A} (la somme et les puissances sont dans \mathbb{R}_{\max}).

THÉORÈME 2. – *Si \mathcal{A}_p satisfait aux hypothèses du théorème 1, et si A n'a qu'une classe critique, alors $\lim_{p \rightarrow \infty} (\mathcal{U}_p)_i^{\frac{1}{p}} = U_i$ où $U = (U_i)$ est l'unique vecteur propre de A dans \mathbb{R}_{\max} vérifiant $\max_i U_i = 1$. Celui-ci est proportionnel à n'importe quelle colonne de $(\tilde{A})^*$ d'indice critique.*

Afin d'obtenir des asymptotiques plus précises, on utilise le semi-corps de jets \mathbb{J}_{\max} (introduit dans [8]) composé de l'ensemble des couples (b, B) , avec $b, B > 0$ ou $b = B = 0$, muni des lois (6). On dit que la fonction f de p admet un *premier max-jet* si $f(p) = bB^p + o(B^p)$ autour de $p = +\infty$. On note alors $f(p) \sim (b, B)$. On étend cette notation aux vecteurs et matrices (coordonnée par coordonnée).

THÉORÈME 3. – *Si $\mathcal{A}_p \sim \mathcal{A} = (a, A) \in (\mathbb{J}_{\max})^{n \times n}$, avec \mathcal{A} irréductible, alors $\mathcal{L}_p \sim \rho_{\mathcal{J}}(\mathcal{A})$ où $\rho_{\mathcal{J}}(\mathcal{A}) = (\rho(a^{\text{CG}(A)}), \rho_{\max}(A))$ est la valeur propre de \mathcal{A} dans \mathbb{J}_{\max} , $a^{\text{CG}(A)}$ est la matrice obtenue à partir de a en annulant les coefficients a_{ij} tels que l'arc (i, j) n'est pas dans le graphe critique, et où $\rho(\cdot)$ désigne la valeur propre de Perron.*

Si $a^{\text{CG}(A)}$ n'a qu'une classe basique, alors $\mathcal{U}_p \sim \mathcal{U}$, l'unique vecteur propre de \mathcal{A} dans \mathbb{J}_{\max} de somme 1. Celui-ci est de la forme (u, U) , où U est une colonne de $(\tilde{A})^$ d'indice basique et u est un vecteur propre positif de la matrice $a^{S(A, U)}$, obtenue en annulant les a_{ij} tels que $A_{ij} U_j < \rho_{\max}(A) U_i$.*

On appellera *classes basiques* de \mathcal{A} les classes basiques de $a^{\text{CG}(A)}$. Si \mathcal{A} admet les classes basiques B_1, \dots, B_s , alors tout vecteur propre \mathcal{U} de \mathcal{A} dans \mathbb{J}_{\max} s'écrit $\mathcal{U} = \mathcal{V} \mathcal{U}'$, où \mathcal{V} est donné par (7). Dans (7), $\mathcal{V}_1, \dots, \mathcal{V}_s$ sont des vecteurs propres de $\mathcal{A}_{B_1 B_1}, \dots, \mathcal{A}_{B_s B_s}$, respectivement, \mathcal{A}_{JK} désigne la $J \times K$ sous-matrice de \mathcal{A} , $B = \bigcup_{1 \leq i \leq s} B_i$, $N = \{1, \dots, n\} \setminus B$, et l'étoile dans \mathbb{J}_{\max} est définie par la même formule que dans \mathbb{R}_{\max} . Soit $\mathcal{M} = \text{diag}(\mathcal{M}_1, \dots, \mathcal{M}_s) [I \ 0]$, où $\mathcal{M}_1, \dots, \mathcal{M}_s$ désignent les vecteurs propres à gauche de $\mathcal{A}_{B_1 B_1}, \dots, \mathcal{A}_{B_s B_s}$, respectivement, vérifiant $\mathcal{M}_i \mathcal{V}_i = 1$.

THÉORÈME 4. – *Si $\mathcal{A}_p \sim \mathcal{A} \in (\mathbb{J}_{\max})^{n \times n}$ et $\mathcal{R}_p = \mathcal{A}_p - \mathcal{A} \sim \mathcal{R} \in (\mathbb{J}_{\max})^{n \times n}$, alors \mathcal{L}_p admet le développement asymptotique (8), où $\mathcal{A}' = \mathcal{M} \mathcal{R} \mathcal{V} \in (\mathbb{J}_{\max})^{s \times s}$.*

Si \mathcal{A}' n'a qu'une classe basique, alors $\mathcal{U}_p \sim \mathcal{U}$, l'unique élément de $(\mathbb{J}_{\max})^n$ de somme 1, de la forme $\mathcal{V} \mathcal{U}'$, où \mathcal{U}' est un vecteur propre de \mathcal{A}' , et \mathcal{V} est donné par (7).

1. Introduction

Let \mathcal{A}_p denote a $n \times n$ nonnegative matrix, depending on a large real parameter p . We consider the nonnegative spectral problem:

$$\mathcal{A}_p \mathcal{U}_p = \mathcal{L}_p \mathcal{U}_p, \quad \mathcal{U}_p \in (\mathbb{R}^+)^n \setminus 0, \quad \mathcal{L}_p \in \mathbb{R}^+, \quad (1)$$

where \mathbb{R}^+ denotes the set of nonnegative real numbers. When \mathcal{A}_p is irreducible, \mathcal{L}_p is unique, and it is called the *Perron eigenvalue* of \mathcal{A}_p (see e.g. [4], Ch. 2). We call *normalized Perron eigenvector* the unique \mathcal{U}_p that satisfies $\sum_i (\mathcal{U}_p)_i = 1$. In this note, we address the following problem: *can we determine the asymptotic behavior of \mathcal{L}_p and \mathcal{U}_p from that of \mathcal{A}_p ?* We begin with an elementary large deviation type result, which extends the result given in [10] for $\mathcal{A}_p = (A_{ij}^p)$.

THEOREM 1 (Large deviation of \mathcal{L}_p). – *If the limits*

$$A_{ij} \stackrel{\text{def}}{=} \lim_{p \rightarrow \infty} (\mathcal{A}_p)_{ij}^{\frac{1}{p}} \tag{2}$$

exist for $i, j = 1, \dots, n$, and if $A = (A_{ij})$ is irreducible, then

$$\lim_{p \rightarrow \infty} (\mathcal{L}_p)^{\frac{1}{p}} = \max_{1 \leq k \leq n} \max_{i_1 \dots i_k} (A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_k i_1})^{\frac{1}{k}}. \tag{3}$$

Indeed, $0 \leq (\mathcal{U}_p)_i \leq \sum_j (\mathcal{U}_p)_j = 1$. Hence, $(\mathcal{U}_p)_i^{\frac{1}{p}}$, which is bounded, has a limit point $0 \leq U_i \leq 1$, and $\max_j U_j = 1$. It follows from (1) that $(\mathcal{L}_p)^{\frac{1}{p}}$ also has a limit point Λ , which satisfies

$$\max_j A_{ij} U_j = \Lambda U_i, \text{ for } i = 1, \dots, n. \tag{4}$$

Now, it is convenient to introduce the *max-times semifield* $\mathbb{R}_{\max} = (\mathbb{R}^+, \max, \times, 0, 1)$. We recognize in (4) a spectral problem for the matrix A in the semifield \mathbb{R}_{\max} . The \mathbb{R}_{\max} analogue of the Perron–Frobenius theorem states that an irreducible matrix A has a unique eigenvalue, given by the *maximal circuit mean* $\rho_{\max}(A)$, which, by definition, is the right hand side of (3) (see e.g. [12], [2], Th. 3.100, [6], §VI, [11], §3.7 and the references therein). Thus, $\Lambda = \rho_{\max}(A)$ holds for all limit points Λ of $(\mathcal{L}_p)^{\frac{1}{p}}$. This proves Theorem 1.

The above argument does not guarantee the convergence of $(\mathcal{U}_p)^{\frac{1}{p}}$, except when all the eigenvectors of A are proportional: this simple case is dealt with in section 2. In section 3, we show that if the non-zero entries of \mathcal{A}_p have asymptotic expansions of the form

$$(\mathcal{A}_p)_{ij} \sim a_{ij} A_{ij}^p, \tag{5}$$

then \mathcal{L}_p has an asymptotic expansion of the same form. This expansion is the unique eigenvalue of $(a_{ij} A_{ij}^p)$, seen as a matrix with entries in a semifield of *jets*. When all the eigenvectors of the later matrix are proportional, the entries of \mathcal{U}_p also have asymptotic expansions of the form (5). However, in general, (5) need not imply the existence of the limits $U_i \stackrel{\text{def}}{=} \lim_{p \rightarrow \infty} (\mathcal{U}_p)_i^{\frac{1}{p}}$, as shown by the following counter example:

$$\mathcal{A}_p = \begin{bmatrix} 1 + \cos(p)e^{-p} & e^{-2p} \\ e^{-2p} & 1 \end{bmatrix}, \quad \liminf_{p \rightarrow \infty} \left(\frac{(\mathcal{U}_p)_2}{(\mathcal{U}_p)_1} \right)^{\frac{1}{p}} = e^{-1} < \limsup_{p \rightarrow \infty} \left(\frac{(\mathcal{U}_p)_2}{(\mathcal{U}_p)_1} \right)^{\frac{1}{p}} = e.$$

In [1], we prove via an extension of the Puiseux expansion theorem that when the entries of \mathcal{A}_p have *Dirichlet series* expansions (see [14]), \mathcal{L}_p and the entries of \mathcal{U}_p also have Dirichlet series expansions. Then, a fortiori, the limit $U = (U_i)$ exists. It can be computed using an aggregation procedure. In section 4, we only present the first step of this procedure, which is enough to determine U in some non-singular cases.

The problem of computing the limits Λ and U arises in particular in Statistical Mechanics, when using the transfer operator methods at small temperatures $T = 1/p$ (see e.g. [3], [5]). Some of the

results given below can be seen as partial extensions to the case of nonnegative matrices of the classical Freidlin–Wentzell singular perturbation results [9], Ch. 6, which deal with the special case of Markov matrices \mathcal{A}_p . Other max-algebra related (W.K.B. type) asymptotic results have been obtained in [7].

2. When max-times spectral theory determines the asymptotics

Let $(S, \oplus, \otimes, 0, 1)$ denote an arbitrary semiring. With a $n \times n$ matrix A with entries in S , we associate (as in conventional Perron–Frobenius theory) a digraph $G(A)$ with nodes $1, \dots, n$, and set of arcs $\{(i, j) \mid A_{ij} \neq 0\}$. We say that A is *irreducible* if $G(A)$ is strongly connected.

When the reflexive and transitive relation \preceq , defined by $a \preceq b \iff \exists c, b = a \oplus c$, is an order relation, and in particular, when $S = \mathbb{R}_{\max}$, we define the *Kleene star* a^* as the least upper bound of the monotone sequence $(\bigoplus_{1 \leq k \leq K} a^k)_{K \geq 1}$, when it exists.

When S is the \mathbb{R}_{\max} semiring, we say that a circuit $c = (i_1, \dots, i_k)$ is *critical* if its mean geometric weight $(A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_k i_1})^{\frac{1}{k}}$ attains the maximum in the right hand side of (3). The *critical graph* $CG(A)$ is the subgraph of $G(A)$, composed uniquely of the nodes and arcs in critical circuits. The strongly connected components of the critical graph are called *critical classes*. We set $\tilde{A} = \rho_{\max}(A)^{-1} A$. Then, $(\tilde{A})^*$ exists. A column of $(\tilde{A})^*$, whose index lies in a critical class, is called *critical*. The max-times spectral theorem (see e.g. [12], [2], Th. 3.100, [6], Th. VI.10, [11], §3.7) states that if we select (arbitrarily) one critical column per critical class, we obtain a minimal generating set of the eigenspace of an irreducible matrix A . As an application of this result, we obtain:

THEOREM 2 (Large deviation of \mathcal{U}_p). – *If \mathcal{A}_p satisfies the assumptions of Theorem 1, and if A has a unique critical class, then for all nodes j in this critical class:*

$$\lim_{p \rightarrow \infty} (\mathcal{U}_p)_i^{\frac{1}{p}} = \frac{(\tilde{A})_{ij}^*}{\bigoplus_k (\tilde{A})_{kj}^*}, \text{ for } i = 1, \dots, n.$$

Recall that $(\tilde{A})^*$ is equal to $\bigoplus_{0 \leq k \leq n-1} (\tilde{A})^k$, and that it can be computed in $O(n^3)$ time using semiring versions of Gauss algorithm (see [2], Th. 3.20 and [13], Ch. 3, Algo. 3, respectively).

3. When the spectral theory of max-jets determines the asymptotics

We denote by \mathbb{J}_{\max} the semifield with set of elements $\{(b, B) \mid b > 0, B > 0\} \cup \{(0, 0)\}$, equipped with the two laws

$$(b, B) \oplus (c, C) = \begin{cases} (b, B) & \text{if } B > C, \\ (c, C) & \text{if } B < C, \\ (b + c, B) & \text{if } B = C, \end{cases} \quad (b, B) \otimes (c, C) = (bc, BC). \quad (6)$$

The zero element $(0, 0)$ and the unit $(1, 1)$ will be denoted by $0, 1$, respectively. This semifield was introduced in [8]. It is isomorphic to the semifield of asymptotic expansions of the form $bB^p + o(B^p)$ around $p = \infty$, equipped with the usual addition and multiplication.

We will say that a nonnegative real valued function f of a large parameter p has a *first max-jet* (b, B) , and we will write $f(p) \sim (b, B)$, if $f(p) = bB^p + o(B^p)$ around $p = \infty$ (when $(b, B) = 0$, this means that $f(p) = 0$ for p large enough). The above definition and notation will be extended to matrices and vectors (entrywise).

If \mathcal{A}_p and \mathcal{U}_p have first max-jets $\mathcal{A} \in (\mathbb{J}_{\max})^{n \times n}$ and $\mathcal{U} \in (\mathbb{J}_{\max})^n$ respectively, it follows from (1) that \mathcal{L}_p has also a first max-jet $\mathcal{L} \in \mathbb{J}_{\max}$, which satisfies $\mathcal{A}\mathcal{U} = \mathcal{L}\mathcal{U}$. Thus, the max-jet \mathcal{U} of \mathcal{U}_p , if

it exists, will be characterized in the particular cases when all the eigenvectors of \mathcal{A} are proportional. We next state a \mathbb{J}_{\max} analogue of the Perron–Frobenius theorem.

For any subgraph C of the digraph associated with a matrix A with entries in any semiring, we denote by A^C the matrix with entries $A_{ij}^C = A_{ij}$ if (i, j) is an arc of C , and $A_{ij}^C = 0$ otherwise. Given an eigenvector $U \in (\mathbb{R}_{\max})^n$ of a matrix $A \in (\mathbb{R}_{\max})^{n \times n}$, the *saturation graph* $S(A, U)$ is the subgraph of $G(A)$ with set of arcs $\{(i, j) \mid A_{ij}U_j = \rho_{\max}(A)U_i\}$.

Let $\mathcal{A} = (a, A) \in (\mathbb{J}_{\max})^{n \times n}$. Clearly, \mathcal{A} has an eigenvector $\mathcal{U} = (u, U) \in (\mathbb{J}_{\max})^n$, with eigenvalue $\mathcal{L} = (\lambda, \Lambda)$, if and only if $AU = \Lambda U$, and $a^{S(A, U)}u = \lambda u$ (the first identity is a spectral problem in \mathbb{R}_{\max} , the second identity is an ordinary nonnegative spectral problem). The saturation graph in general depends on the particular choice of U , but when A is irreducible, for all eigenvectors U of A , $CG(A) \subset S(A, U)$, and any circuit of $S(A, U)$ is contained in $CG(A)$. The matrix $a^{CG(A)}$ is block diagonal, the blocks being exactly the critical classes. We call *basic classes* of $\mathcal{A} = (a, A)$ the basic classes of $a^{CG(A)}$ in the usual sense, i.e. the classes with maximal Perron eigenvalue. We denote by $\rho(b)$ the usual Perron eigenvalue of a matrix b .

THEOREM 3 (“Perron–Frobenius Theorem” for max-jets). – *An irreducible matrix $\mathcal{A} = (a, A) \in (\mathbb{J}_{\max})^{n \times n}$ admits the unique eigenvalue $\rho_{\mathcal{J}}(\mathcal{A}) \stackrel{\text{def}}{=} (\rho(a^{CG(A)}), \rho_{\max}(A))$.*

The characterization of the eigenspace is more subtle in \mathbb{J}_{\max} than in \mathbb{R}_{\max} . We will only need the following simple result.

THEOREM 4 (Geometric simplicity of the eigenvalue). – *An irreducible matrix $\mathcal{A} = (a, A) \in (\mathbb{J}_{\max})^{n \times n}$ has a unique eigenvector (up to a proportionality factor) if and only if it has a unique basic class. An eigenvector $\mathcal{U} = (u, U)$ is obtained as follows: U is a column of $(A)^*$, whose index belongs to the basic class; u is a positive eigenvector of $a^{S(A, U)}$.*

As a consequence of Theorems 3 and 4, we obtain:

THEOREM 5 (First order asymptotics). – *If \mathcal{A}_p has a first max-jet $\mathcal{A} \in (\mathbb{J}_{\max})^{n \times n}$, then $\mathcal{L}_p \sim \rho_{\mathcal{J}}(\mathcal{A})$. Moreover, if \mathcal{A} has a unique basic class, then \mathcal{U}_p has a first max-jet, which is the unique eigenvector \mathcal{U} of \mathcal{A} with sum 1.*

4. Aggregated matrix

When the matrix \mathcal{A} has several basic classes, the determination of the limit eigenvector relies on an aggregation procedure, the first step of which we next describe.

We denote by B_1, \dots, B_s the basic classes of \mathcal{A} . We set $B = \bigcup_{1 \leq i \leq s} B_i$ and $N = \{1, \dots, n\} \setminus B$. Let $\mathcal{V}_1, \dots, \mathcal{V}_s$ be eigenvectors of the submatrices $\mathcal{A}_{B_1 B_1}, \dots, \mathcal{A}_{B_s B_s}$, respectively (for all subsets $J, K \subset \{1, \dots, n\}$, \mathcal{A}_{JK} denotes the $J \times K$ submatrix of \mathcal{A}). The following key lemma is a consequence of the fact that $au \leq \rho(a)u$ implies $au = \rho(a)u$, for all irreducible nonnegative matrices a and nonnegative vectors u (see e.g. [4], Ch. 1, Th. 3.35).

LEMMA 6. – *Any eigenvector \mathcal{U} of an irreducible matrix $\mathcal{A} \in (\mathbb{J}_{\max})^{n \times n}$ is of the form $\mathcal{U} = \mathcal{V}\mathcal{U}'$, for some $\mathcal{U}' \in (\mathbb{J}_{\max})^s$, where*

$$\mathcal{V} \stackrel{\text{def}}{=} \begin{bmatrix} I \\ (\rho_{\mathcal{J}}(\mathcal{A})^{-1} \mathcal{A}_{NN})^* \rho_{\mathcal{J}}(\mathcal{A})^{-1} \mathcal{A}_{NB} \end{bmatrix} \text{diag}(\mathcal{V}_1, \dots, \mathcal{V}_s). \quad (7)$$

Here, I is the $B \times B$ identity matrix, and for all (possibly rectangular) matrices X_1, \dots, X_k , $\text{diag}(X_1, \dots, X_k)$ denotes the (possibly rectangular) block diagonal matrix with diagonal blocks X_1, \dots, X_k .

In the sequel, we will identify the max-jet (resp. the matrix of max-jets) (b, B) to the function $p \mapsto bB^p$ (resp. $p \mapsto (b_{ij}B_{ij}^p)$). This allows us to write $\mathcal{A}_p = \mathcal{A}^{\text{BG}(\mathcal{A})} + \mathcal{R}_p$, where $\text{BG}(\mathcal{A})$ denotes the subgraph of $\text{CG}(\mathcal{A})$ with set of nodes B . In general, the remainder matrix \mathcal{R}_p has negative entries.

Let $\mathcal{M}_1, \dots, \mathcal{M}_s$ denote the left eigenvectors of the submatrices $\mathcal{A}_{B_1 B_1}, \dots, \mathcal{A}_{B_s B_s}$, respectively, normalized by the condition $\mathcal{M}_i \mathcal{V}_i = 1$, for $i = 1, \dots, s$. We set $\mathcal{M} = \text{diag}(\mathcal{M}_1, \dots, \mathcal{M}_s)[I \ 0]$ (0 is the $B \times N$ zero matrix). Left multiplying $\mathcal{A}_p \mathcal{U}_p = \mathcal{L}_p \mathcal{U}_p$ by \mathcal{M} , we obtain $\mathcal{M} \mathcal{R}_p \mathcal{U}_p = (\mathcal{L}_p - \rho_J(\mathcal{A})) \mathcal{M} \mathcal{U}_p$. Using Lemma 6, we obtain the following result.

THEOREM 7 (Second order asymptotics). – *If \mathcal{A}_p and \mathcal{R}_p have first max-jets \mathcal{A} and \mathcal{R} , respectively, then*

$$\mathcal{L}_p = \rho_J(\mathcal{A}) + \rho_J(\mathcal{A}') + o(\rho_J(\mathcal{A}')), \tag{8}$$

where $\mathcal{A}' \stackrel{\text{def}}{=} \mathcal{M} \mathcal{R} \mathcal{V} \in (\mathbb{J}_{\text{max}})^{s \times s}$. Moreover, if \mathcal{A}' has a unique basic class, then \mathcal{U}_p has a first max-jet, which is the unique vector with sum 1 of the form $\mathcal{V} \mathcal{U}'$, \mathcal{U}' being an eigenvector of \mathcal{A}' , and \mathcal{V} being defined in (7).

Example 8. – Consider the transfer matrix of the simplest one dimensional Ising model [3], Ch. 2:

$$\mathcal{A}_{1/T} = \begin{bmatrix} \exp((J+H)/T) & \exp(-J/T) \\ \exp(-J/T) & \exp((J-H)/T) \end{bmatrix}, \text{ with } J > 0, \quad H \in \mathbb{R}.$$

Setting $K = \exp(J) > 1$, $L = \exp(H) > 0$, $p = 1/T$, we can write the first max-jet of \mathcal{A}_p as $\mathcal{A} = (a, A)$ with $a = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} KL & K^{-1} \\ K^{-1} & KL^{-1} \end{bmatrix}$. We have $\rho_J(\mathcal{A}) = (1, \max(KL, KL^{-1}))$. Thus, $\mathcal{L}_p \sim (\max(KL, KL^{-1}))^p$. When $H > 0$, there is a unique critical class, $C_1 = \{1\}$, and $\tilde{A} = \begin{bmatrix} 1 & K^{-2}L^{-1} \\ K^{-2}L^{-1} & 1 \end{bmatrix}$, $(\tilde{A})^* = \begin{bmatrix} 1 & K^{-2}L^{-1} \\ K^{-2}L^{-1} & 1 \end{bmatrix}$. By Theorem 5, $\mathcal{U}_p \sim [1 \ (K^{-2}L^{-1})^p]^T$ and by symmetry, $\mathcal{U}_p \sim [(K^{-2}L^{-1})^p \ 1]^T$ when $H < 0$: as is well known, the eigenvector at zero temperature is selected by the sign of the magnetic field H . When $H = 0$, \mathcal{A} has two distinct critical classes $C_1 = \{1\}, C_2 = \{2\}$, that are both basic. Theorem 7 allows us to determine the equivalent of \mathcal{U}_p . Indeed, $\mathcal{V} = \mathcal{M} = I$ and $\mathcal{A}^{\text{BG}(\mathcal{A})} = \begin{bmatrix} (1, K) & 0 \\ 0 & (1, K) \end{bmatrix}$. We obtain $\mathcal{R}_p = \mathcal{R} = \mathcal{A}' = \begin{bmatrix} 0 & (1, K^{-1}) \\ (1, K^{-1}) & 0 \end{bmatrix}$. Thus, $\rho_J(\mathcal{A}') = (1, K^{-1})$, $\mathcal{L}_p = K^p + K^{-p} + o(K^{-p})$, and $\mathcal{U}_p \sim [1/2 \ 1/2]^T$.

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