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Maps on matrices that preserve the spectral radius distance

by

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Abstract. Let ϕ be a surjective map on the space of $n \times n$ complex matrices such that $r(\phi(A) - \phi(B)) = r(A - B)$ for all A, B, where $r(X)$ is the spectral radius of X. We show that ϕ must be a composition of five types of maps: translation, multiplication by a scalar of modulus one, complex conjugation, taking transpose and (simultaneous) similarity. In particular, ϕ is real linear up to a translation.

1. Introduction. One of the earliest theorems about maps between normed spaces is the Mazur–Ulam Theorem [4]. It asserts that if ϕ is a map between two normed spaces that is surjective, maps 0 to 0, and is isometric, then ϕ is real linear. The problem considered in the present paper is motivated by this theorem.

The spectral radius of an element a of a complex Banach algebra is denoted by $r(a)$. Let ϕ be a map between two Banach algebras $\mathcal{A}$ and $\mathcal{B}$. We say that ϕ *preserves the spectral radius distance* if

$$r(\phi(a) - \phi(b)) = r(a - b)$$

for all $a, b \in \mathcal{A}$.

We note that in semisimple commutative Banach algebras, the spectral radius is a norm. Therefore, it follows from the Mazur–Ulam Theorem that if $\mathcal{A}$ and $\mathcal{B}$ are semisimple commutative Banach algebras and $\phi : \mathcal{A} \to \mathcal{B}$ is a map that is surjective, maps 0 to 0, and preserves the spectral radius distance, then ϕ is real linear. It is natural to ask whether this assertion is valid for noncommutative Banach algebras as well.

We answer this question in the simplest case where $\mathcal{A} = \mathcal{B} = M_n$, the algebra of all $n \times n$ matrices. The answer, in this case, is in the affirmative. Further, we obtain a complete characterisation of all maps ϕ that satisfy the three given conditions. We prove the following theorem.

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THEOREM 1.1. *Let $\phi : M_n \to M_n$ be a surjective map that preserves the spectral radius distance. Then there exists a unimodular $\lambda \in \mathbb{C}$, $S \in M_n$, and an invertible $T \in M_n$ such that one of the following formulas holds for all $A \in M_n$:*

$$\phi(A) = \lambda TAT^{-1} + S,$$
$$\phi(A) = \lambda TA^*T^{-1} + S,$$
$$\phi(A) = \lambda TA^{\mathrm{tr}}T^{-1} + S,$$
$$\phi(A) = \lambda T\bar{A}T^{-1} + S,$$

REMARK. In general Banach algebras it is essential to impose some restrictions as the following examples demonstrate.

1. Let $\mathcal{A}$ be any radical algebra and let ϕ be any nonlinear map of $\mathcal{A}$ onto $\mathcal{A}$ that fixes 0. Let $\mathcal{A}_1$ be the algebra obtained by adjoining a unit $\mathbf{1}$ to $\mathcal{A}$. Extend ϕ to $\mathcal{A}_1$ by putting $\phi(\lambda\mathbf{1} + a) = \lambda\mathbf{1} + \phi(a)$. Then ϕ is surjective, $\phi(0) = 0$, and ϕ preserves the spectral radius distance, but ϕ is evidently not real linear. In this case, the algebra $\mathcal{A}_1$ is not semisimple.

2. Let $\mathcal{A} = \mathbb{C}$ and let $\mathcal{B}$ be the algebra of all 2×2 diagonal matrices. For each z, let

$$\phi(z) = \begin{bmatrix} z & 0 \\ 0 & \sin(\mathrm{Re}\, z) \end{bmatrix}.$$

Then $\phi(0) = 0$ and $r(\phi(z) - \phi(z')) = |z - z'|$ for all z, z', but ϕ is not real linear. In this case, the range of ϕ is not an algebra.

In view of the above, the natural restriction to impose on $\mathcal{A}$ and $\mathcal{B}$ is semisimplicity and the question to be raised is whether every surjective map ϕ between semisimple Banach algebras that maps 0 to 0 and preserves the spectral radius distance is real linear.

Let us fix the notation. For any pair of vectors $x, y \in \mathbb{C}^n$ we denote their inner product by y^*x. Every rank one matrix can be written as xy^*. Such a matrix is idempotent if $y^*x = 1$, and square-zero if $y^*x = 0$. We denote by $\{e_1, \ldots, e_n\}$ the standard basis of $\mathbb{C}^n$. Then E_{ij}, the matrix having 1 in the (i,j)th position and zeros elsewhere, equals $e_ie_j^*$. If A is an $n \times n$ matrix, then we write $\sigma(A)$, $\mathrm{tr}\, A$, A_{ij}, A^{tr}, and $\bar{A}$ for the spectrum of A, the trace of A, the (i,j)th entry of A, the transpose of A, and the matrix obtained from A by entrywise complex conjugation, respectively. Throughout this paper the spectrum of A is understood to be the set of all eigenvalues of A (so, we do not count their multiplicities). By $\mathcal{N}_n \subset M_n$ we denote the subset of all nilpotent matrices.

A special class of rank one nilpotents plays an important role in our proofs. These are the matrices xy^* with the property that there exists a subset $J \subset \{1, \ldots, n\}$ such that x belongs to the linear span of $\{e_i : i \in J\}$ while y belongs to the linear span of $\{e_i : i \notin J\}$. These are called *special rank one nilpotents* and are denoted by $\mathcal{M}$.

The set of all diagonal matrices is denoted by $\mathcal{D}$. By $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ we mean the diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_n$. A particular diagonal matrix D_0 is also often used:.

$$D_0 := \mathrm{diag}(1, \omega, \omega^2, \ldots, \omega^{n-1}) \tag{1.1}$$

where $\omega = \exp(2\pi i/n)$.

2. Preliminary results. We first prove that every map on matrices that preserves the spectral radius distance is injective. Then we show that under the additional assumption of surjectivity such maps are continuous.

PROPOSITION 2.1. *Let $\phi : M_n \to M_n$ be a map that preserves the spectral radius distance. Then ϕ is injective.*

Proof. Let $\phi(A) = \phi(B)$. Then $r(A - B) = r(\phi(A) - \phi(B)) = 0$, and consequently, $B = A + N$ with N nilpotent. For every $T \in M_n$ we have $r(A - T) = r(\phi(A) - \phi(T)) = r(\phi(B) - \phi(T)) = r(B - T)$. Replacing T by $A - S$ we get $r(S) = r(N + S)$ for every $S \in M_n$. This clearly implies that $N = 0$, or equivalently, $A = B$.

LEMMA 2.2. *Suppose $\{A_k\}$ is a sequence of matrices such that $r(A_k + T) \to r(A + T)$ for all T. Then $A_k \to A$.*

Proof. It is enough to prove that if

$$r(A_k + T) \to r(T) \quad \text{for all } T, \tag{2.1}$$

then $A_k \to 0$.

First note that condition (2.1) implies that $r(A_k) \to 0$. Hence, all the nonleading coefficients of the characteristic polynomial of A_k approach zero as $k \to \infty$. In particular, for each $\varepsilon > 0$, there exists an N such that

$$\Big|\sum_{i<j} M_{ij}(A_k)\Big| < \varepsilon \quad \text{for } n \geq N, \tag{2.2}$$

where $M_{ij}(X)$ stands for the following 2×2 minor of a matrix X:

$$M_{ij}(X) = \det \begin{bmatrix} x_{ii} & x_{ij} \\ x_{ji} & x_{jj} \end{bmatrix}.$$

We claim that the sequence $\{A_k\}$ is bounded. If this were not the case, then there would be at least one pair of indices i, j such that the entries $\{A_k(i,j)\}$ are not bounded. If $i = j$ for all such pairs of indices, then, by Gershgorin's Theorem, $\{r(A_k)\}$ is not bounded. This is not possible. On the other hand, if there exists a pair of indices i, j, $i \neq j$, such that the entries $\{A_k(i,j)\}$ are not bounded, choose $T = E_{ji}$. Then, by inequality (2.2), we deduce that $\sum_{i<j} M_{ij}(A_k + T)$ is not bounded, and hence it is not possible

for $\{r(A_k+T)\}$ to be bounded. This contradicts hypothesis (2.1). So, $\{A_k\}$ is bounded.

Let $\{A_m\}$ be any convergent subsequence of A_k. Suppose $A_m \to A$. By hypothesis (2.1), we have $r(A_m) \to r(0)$. This together with continuity of the spectrum implies that A is nilpotent. Again hypothesis (2.1) implies that $r(A_m \pm A^*) \to r(\pm A^*) = 0$. Therefore $A \pm A^* = 0$, and so $A = 0$. This is true for every convergent subsequence, hence $A_k \to 0$.

COROLLARY 2.3. *Let $\phi : M_n \to M_n$ be a surjective map that preserves the spectral radius distance. Then ϕ is continuous.*

Proof. Let $A_k \to A$ and let T be any element of M_n, $T = \phi(S)$. Then

$$\begin{aligned} r(\phi(A_k) - T) &= r(\phi(A_k) - \phi(S)) = r(A_k - S) \to r(A - S) \\ &= r(\phi(A) - \phi(S)) = r(\phi(A) - T). \end{aligned}$$

Hence, by Lemma 2.2, $\phi(A_k) \to \phi(A)$.

LEMMA 2.4. *Let a be a nonnegative real number and let $S \in M_n$ satisfy $r(N + S) = a$ for every nilpotent $N \in M_n$. Then $S = \lambda I$ for some complex number λ with $|\lambda| = a$.*

Proof. Suppose $S_{ij} \neq 0$ for some $i < j$. Let N be the matrix whose 2×2 principal submatrix corresponding to the ith and jth rows and columns is $\left[\begin{smallmatrix} 1 & 1/K \\ -K & -1 \end{smallmatrix}\right]$, and all other entries are zero. Then N is nilpotent and the minor $M_{ij}(S+N)$ can be made arbitrarily large by choosing K large enough. But then $r(S+N)$ would be arbitrarily large as well. Hence we must have $S_{ij} = 0$ for $i < j$. Similarly, $S_{ij} = 0$ for $i > j$ and so S is diagonal. If $S_{ii} \neq S_{jj}$ for some $i \neq j$, choose N with the 2×2 principal submatrix $\left[\begin{smallmatrix} K & K \\ -K & -K \end{smallmatrix}\right]$ instead of the one chosen earlier to obtain a contradiction.

The following two lemmas were proved in [5].

LEMMA 2.5 (see [5]). *Let $N \in \mathcal{N}_n$, $N \neq 0$. Then the following are equivalent:*

(i) $\operatorname{rank} N = 1$,

(ii) *for every $A \in \mathcal{N}_n$ satisfying $A + N \notin \mathcal{N}_n$ we have $A + \alpha N \notin \mathcal{N}_n$ for every nonzero $\alpha \in \mathbb{C}$.*

LEMMA 2.6 (see [5]). *Let $A, B \in \mathcal{N}_n$. Suppose that A is not of rank one. Assume that for every $N \in \mathcal{N}_n$ the matrix $A + N \in \mathcal{N}_n$ if and only if $B + N \in \mathcal{N}_n$. Then $A = B$.*

COROLLARY 2.7. *Let $A, B \in \mathcal{N}_n$. Assume that for every $N \in \mathcal{N}_n$ the matrix $A + N \in \mathcal{N}_n$ if and only if $B + N \in \mathcal{N}_n$. Then either $A = B$, or there exists a nonzero $\beta \in \mathbb{C}$ such that $B = \beta A$ and* $\operatorname{rank} A = 1$.

Proof. If $A \neq B$, then by Lemma 2.6 we have $\operatorname{rank} A = \operatorname{rank} B = 1$. Without loss of generality we can assume $A = E_{12}$. Clearly, $E_{12} + \alpha E_{ij}$ is nilpotent for every $\alpha \in \mathbb{C}$ if $i \leq 2$ and $j \geq 3$. The same holds true if $j \leq 2$ and $i \geq 3$. So, for such pairs i, j the matrix $B + \alpha E_{ij}$ must be nilpotent for every $\alpha \in \mathbb{C}$. Hence, the sum of all 2×2 principal minors of $B + \alpha E_{ij}$ must be zero. It follows that B has the block diagonal form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

with B_1 a 2×2 matrix. It is now easy to complete the proof.

The next result was proved in [2].

LEMMA 2.8 (see [2]). *Let $A \in M_n$, let $\lambda \notin \sigma(A)$, and let $x, y \in \mathbb{C}^n$. Then $\lambda \in \sigma(A + xy^*)$ if and only if $y^*(\lambda - A)^{-1}x = 1$.*

We recall that D_0 is the matrix defined by (1.1), and that $\mathcal{M}$ is the set of "special rank one nilpotent" matrices defined in §1.

COROLLARY 2.9. *Assume that $N = xy^*$ is a nilpotent of rank one. Then the following are equivalent:*

(a) $r(D_0 + \alpha N) = 1$ *for every* $\alpha \in \mathbb{C}$,

(b) $N \in \mathcal{M}$.

Proof. Assume first that (b) is satisfied. Applying a permutation similarity we can transform N into a strict upper triangular form while D_0 remains diagonal. Then (a) follows trivially.

To prove the converse, assume that N satisfies condition (a). For $|\lambda| > 1$ we have $\lambda \notin \sigma(D_0 + \alpha xy^*)$ for every $\alpha \in \mathbb{C}$. By Lemma 2.8, this is equivalent to $\alpha y^*(\lambda - D_0)^{-1}x \neq 1$ for every $\alpha \in \mathbb{C}$. Hence, $y^*(\lambda - D_0)^{-1}x = 0$ for $|\lambda| > 1$. If p is any polynomial, then

$$y^* p(D_0)x = y^*\Big(\int_C p(\lambda)(\lambda - D_0)^{-1}\,d\lambda\Big)x = 0,$$

where C is a counterclockwise oriented circle with center at 0 and radius greater than 1. In particular, we have $y^* E_{ii} x = 0$, $i = 1, \ldots, n$, which is equivalent to $xy^* \in \mathcal{M}$. This completes the proof.

LEMMA 2.10. *Let i, j, k, l be arbitrary indices, $\alpha \in \mathbb{C} \setminus \{0\}$, and $A, B \in M_n$. If $\sigma(A + \gamma E_{ij}) = \sigma(B + \alpha\gamma E_{kl})$ for every $\gamma \in \mathbb{C}$, then*

$$A_{ji} = \alpha B_{lk}.$$

Proof. For $|\lambda| > R := r(A) + r(B)$, Lemma 2.8 shows that $\gamma e_j^*(\lambda - A)^{-1} e_i = 1$ if and only if $\alpha\gamma e_l^*(\lambda - B)^{-1} e_k = 1$. Thus,

$$e_j^*(\lambda - A)^{-1} e_i = \alpha e_l^*(\lambda - B)^{-1} e_k$$

and hence

$$e_j^*(I-\mu A)^{-1}e_i = \alpha e_l^*(I-\mu B)^{-1}e_k$$

for $|\mu| < 1/R$. Upon taking the derivative at $\mu = 0$, we get the desired relation $e_j^*Ae_i = \alpha e_l^* Be_k$.

LEMMA 2.11. *Let $C, G \in M_n$. If $r(C+\lambda I) = r(G+\lambda I)$ for every $\lambda \in \mathbb{C}$, then the convex hull of $\sigma(C)$ coincides with the convex hull of $\sigma(G)$.*

Proof. Let μ be an extreme point of the convex hull of $\sigma(C)$. Then, after a rotation and translation, we may assume that $\mu = r(C)$ and that $\sigma(C)$ and $\sigma(G)$ are contained in the right half-plane. If $\mu \notin \sigma(G)$, then $\sigma(G) \subset \{z \in \mathbb{C} : 0 \le \operatorname{Re} z < \mu\}$ since $\mu = r(G)$. It follows that

$$r(G+I) < \mu + 1 = r(C+I),$$

a contradiction. Thus, the set of all extreme points of the convex hull of $\sigma(C)$ is a subset of the set of all extreme points of the convex hull of $\sigma(G)$. The reverse inclusion follows by symmetry. The equality of the two sets of extreme points is equivalent to the assertion of the lemma.

LEMMA 2.12. *Let E_{ij} and E_{kl} be matrix units such that either both are diagonal or both are nondiagonal. Let α_{kl} be a nonzero complex number. In the case where the matrix units are diagonal let $\alpha_{kk} = 1$. Let A and B be two rank one matrices or any 3×3 matrices. If*

$$r(A + \gamma E_{ij} + \lambda I) = r(B + \gamma\alpha_{kl}E_{kl} + \lambda I)$$

for every $\lambda, \gamma \in \mathbb{C}$, and if $\operatorname{tr} A = \operatorname{tr} B$, then $\alpha_{kl}B_{lk} = A_{ji}$.

Proof. In view of Lemma 2.10, it suffices to prove that $\sigma(A + \gamma E_{ij}) = \sigma(B + \gamma\alpha_{kl}E_{kl})$ for every $\gamma \in \mathbb{C}$. Each of these two spectra contains at most three distinct points. By Lemma 2.11, the set of all extreme points of the convex hull of $\sigma(A + \gamma E_{ij})$ coincides with the set of all extreme points of the convex hull of $\sigma(B + \gamma\alpha_{kl}E_{kl})$. This trivially implies that $\sigma(A+\gamma E_{ij}) = \sigma(B + \gamma\alpha_{kl}E_{kl})$ except when at least one of the spectra has three distinct eigenvalues and both spectra lie on the same line. In this case at least two distinct eigenvalues of $A + \gamma E_{ij}$ are also in $\sigma(B + \gamma\alpha_{kl}E_{kl})$. If $n = 3$, this and the trace condition prove equality of the spectra. The same holds in the rank one case since the multiplicity of the nonzero eigenvalues is at most two.

From now on till the end of this section we assume that $\phi : M_n \to M_n$ is a surjective map that maps 0 to 0 and preserves the spectral radius distance.

LEMMA 2.13. *There exists a unimodular complex number z such that either $\phi(\alpha I) = z\alpha I$ for all $\alpha \in \mathbb{C}$, or $\phi(\alpha I) = z\overline{\alpha} I$ for all $\alpha \in \mathbb{C}$.*

Proof. It follows from $\phi(0) = 0$ that ϕ maps $\mathcal{N}_n$ onto $\mathcal{N}_n$. We have $r(N - \alpha I) = |\alpha|$ for every $N \in \mathcal{N}_n$, and consequently, $r(N - \phi(\alpha I)) = |\alpha|$ for every $N \in \mathcal{N}_n$. Using Lemma 2.4, we now conclude that ϕ maps the set of scalar matrices into itself. The function $\varphi : \mathbb{C} \to \mathbb{C}$ given by $\phi(\alpha I) = \varphi(\alpha)I$ is an isometry of $\mathbb{C} = \mathbb{R}^2$. By the Mazur–Ulam Theorem, φ is real linear. It is well known that every linear isometry of $\mathbb{R}^2$ is either a rotation or a reflection in a line through the origin. This implies the desired conclusion.

LEMMA 2.14. *If $N \in M_n$ is a nilpotent of rank one then $\phi(N)$ is a nilpotent of rank one and ϕ maps the linear span of N onto the linear span of $\phi(N)$.*

Proof. Let N be a nilpotent of rank one and let $\alpha \in \mathbb{C}\setminus\{0,1\}$. According to Proposition 2.1, ϕ is bijective, and so $\phi(N) \ne \phi(\alpha N)$. Using Lemma 2.5 we see that for every nilpotent M the matrix $N - M$ is nilpotent if and only if $\alpha N - M$ is nilpotent. Hence, $\phi(N) - \phi(M)$ belongs to $\mathcal{N}_n$ if and only if $\phi(\alpha N) - \phi(M) \in \mathcal{N}_n$. Applying Corollary 2.7 we conclude that $\phi(N)$ is a nilpotent of rank one and $\phi(\operatorname{span} N) \subset \operatorname{span}\phi(N)$. Applying the same argument for ϕ^{-1} we get the reverse inclusion.

Until the end of this section we assume that $\phi : M_n \to M_n$ satisfies the additional assumptions that $\phi(\alpha I) = \alpha I$ for every $\alpha \in \mathbb{C}$, and $\phi(D_0) = D_0$, where D_0 is the matrix defined by (1.1).

LEMMA 2.15. $\phi(\mathcal{M}) = \mathcal{M}$.

Proof. Let $N \in \mathcal{M}$. By Corollary 2.9 we have $r(D_0 + \alpha N) = 1$ for every $\alpha \in \mathbb{C}$, and so, $r(\phi(D_0) - \phi(-\alpha N)) = 1$. Applying Lemma 2.14 we see that $r(D_0 + \alpha\phi(N)) = 1$ for every complex number α. Using Corollary 2.9 once again we see that $\phi(N) \in \mathcal{M}$. Hence, $\phi(\mathcal{M}) \subset \mathcal{M}$. The same holds for ϕ^{-1}, and consequently, $\phi(\mathcal{M}) = \mathcal{M}$.

LEMMA 2.16. *$\phi(A) = A$ for every $A \in \mathcal{D}$.*

Proof. We divide the proof into two steps. In the first step we prove that ϕ maps the set $\mathcal{D}$ of all diagonal matrices onto itself. Once again, it is enough to prove that $\phi(\mathcal{D}) \subset \mathcal{D}$. Assume to the contrary that there exist $A \in \mathcal{D}$ and $i, j \in \{1, \ldots, n\}$, $i \ne j$, such that $e_j^*\phi(A)e_i \ne 0$. Then

$$\lambda \mapsto e_j^*(\lambda - \phi(A))^{-1}e_i, \quad |\lambda| > r(\phi(A)) = r(A),$$

is a nonzero function. It follows that there exists λ_0, $|\lambda_0| > r(A)$, such that $e_j^*(\lambda_0 - \phi(A))^{-1}e_i \ne 0$. By Lemma 2.8, $\lambda_0 \in \sigma(\phi(A) + \beta E_{ij})$ for a nonzero scalar β. Since $\phi(\mathcal{M}) = \mathcal{M}$, we have $-\beta E_{ij} = \phi(N)$ with $N \in \mathcal{M}$. So, $r(A - N) = r(A) < |\lambda_0| \le r(\phi(A) - \phi(N))$, a contradiction. Hence, $\phi(\mathcal{D}) = \mathcal{D}$.

Now, the restriction $\phi_{|\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is a surjective isometry, and therefore, it is real linear. The space $\mathcal{D}$ is isomorphic to the real linear space $\mathbb{C}^n$ with the sup norm. The real linear isometries of the latter space are well known.

For example, it follows from [3] that

$$\phi(\operatorname{diag}(\lambda_1,\ldots,\lambda_n)) = \operatorname{diag}(\tau_1(\lambda_{\pi(1)})e^{i\varphi_1},\ldots,\tau_n(\lambda_{\pi(n)})e^{i\varphi_n}),$$

where $\varphi_1,\ldots,\varphi_n \in \mathbb{R}$, π is any permutation of the set $\{1,\ldots,n\}$, and τ_i is either the identity or the complex conjugation, $i = 1,\ldots,n$. Because of $\phi(I) = I$ we have $\exp(i\varphi_1) = \ldots = \exp(i\varphi_n) = 1$. From $\phi(\alpha I) = \alpha I$ we get $\tau_i(\alpha) = \alpha$, $i = 1,\ldots,n$. Finally, $\phi(D_0) = D_0$ implies that $\pi(i) = i$ for every $i \in \{1,\ldots,n\}$. This completes the proof.

LEMMA 2.17. *For every invertible matrix T there exists an invertible matrix S such that $\phi(TAT^{-1}) = SAS^{-1}$ for every $A \in \mathcal{D}$.*

Proof. The map $\phi' : M_n \to M_n$ defined by $\phi'(A) = \phi(TAT^{-1})$ is bijective, maps every scalar matrix to itself and preserves the spectral radius distance. By Lemma 2.11, we see that the convex hull of $\sigma(\phi'(D_0))$ coincides with the convex hull of $\sigma(D_0)$. This easily implies that $\sigma(D_0) = \sigma(\phi'(D_0))$ and that there exists an invertible $S \in M_n$ such that $S^{-1}\phi'(D_0)S = D_0$. The map

$$A \mapsto S^{-1}\phi(TAT^{-1})S$$

is bijective, maps every scalar matrix to itself, preserves the spectral radius distance, and maps D_0 to D_0. By the previous lemma, it maps every diagonal matrix to itself. This completes the proof.

LEMMA 2.18. *For every polynomial p and every $B \in M_n$ we have $\phi(p(B)) = p(\phi(B))$.*

Proof. Assume first that B is diagonalisable. Then there exists an invertible $T \in M_n$ such that $B = TAT^{-1}$ with $A \in \mathcal{D}$. According to the previous lemma we have

$$\phi(B) = \phi(TAT^{-1}) = SAS^{-1}$$

for some invertible S. But $p(A)$ is a diagonal matrix as well, and so, $\phi(p(B)) = \phi(Tp(A)T^{-1}) = Sp(A)S^{-1} = p(\phi(B))$. As the set of all diagonalisable matrices is dense in M_n, the general case follows by the continuity of ϕ.

LEMMA 2.19. *Let $E \in M_n$ be an idempotent. If $A, B \in M_n$ satisfy $A = EAE$ and $B = (I-E)B(I-E)$ then $\phi(A)\phi(B) = \phi(B)\phi(A) = 0$.*

Proof. It is easy to see that there exist sequences $\{A_j\}$ and $\{B_j\}$ such that $A_j \to A$, $B_j \to B$, A_j and B_j are simultaneously diagonalisable, $A_j = EA_jE$ and $B_j = (I-E)B_j(I-E)$ for every j. Therefore, it is enough to prove the statement under the additional assumption that A and B are simultaneously diagonalisable. Thus there exists an invertible $T \in M_n$ such that $A = TA'T^{-1}$ and $B = TB'T^{-1}$ with A' and B' diagonal matrices satisfying $A'B' = 0$. Applying Lemma 2.17 we complete the proof.

3. Proof of the main result.

Our main result will follow easily from the previous lemmas and the following result, proved by induction.

PROPOSITION 3.1. *Let $\phi : M_n \to M_n$ be a bijective map preserving the spectral radius distance. Assume also that ϕ has the following properties:*

(a) *$\phi(A) = A$ for every $A \in \mathcal{D}$,*
(b) *$\phi(\mathcal{M}) = \mathcal{M}$,*
(c) *if $A, B \in M_n$ satisfy $A = EAE$ and $B = (I-E)B(I-E)$ for some idempotent $E \in M_n$, then $\phi(A)\phi(B) = \phi(B)\phi(A) = 0$ and $\phi^{-1}(A)\phi^{-1}(B) = \phi^{-1}(B)\phi^{-1}(A) = 0$, and*
(d) *$\phi(p(A)) = p(\phi(A))$ for every $A \in M_n$ and every polynomial p.*

Then there exists an invertible diagonal matrix $T \in M_n$ such that either

$$\phi(A) = TAT^{-1}, \qquad A \in M_n,$$

or

$$\phi(A) = TA^{\mathrm{tr}}T^{-1}, \qquad A \in M_n.$$

Proof. Let us first observe some simple consequences of our assumptions that will be used without further explanation in the sequel. It follows from condition (d) that the minimal polynomial of A is the same as the minimal polynomial of $\phi(A)$. This implies that ϕ preserves the spectrum, that is, $\sigma(A) = \sigma(\phi(A))$ for every $A \in M_n$. In particular, $\operatorname{tr} A = \operatorname{tr}\phi(A)$ for every matrix having n distinct eigenvalues. By continuity, ϕ preserves the trace. The fact that ϕ preserves the spectrum and the trace implies that ϕ maps every idempotent of rank one to an idempotent of rank one. This, together with Lemma 2.14, implies that ϕ maps every operator of rank one to an operator of rank one. Finally, we note that (d) implies that

$$\phi(\alpha A + \beta I) = \alpha\phi(A) + \beta I$$

for every $A \in M_n$ and for all scalars α, β. In particular, ϕ is homogeneous.

As already mentioned we prove the statement by induction on n. There is nothing to prove in the case $n = 1$. For $n = 2$ we apply condition (b) to conclude that ϕ maps E_{12} either to a scalar multiple of E_{12}, or to a scalar multiple of E_{21}. Applying an appropriate diagonal similarity transformation and composing ϕ with the transpose mapping if necessary, we may assume that ϕ satisfies all the assumptions of our proposition and that $\phi(E_{12}) = E_{12}$. Applying (b) once again and bijectivity, we conclude that $\phi(E_{21}) = \alpha E_{21}$ for some nonzero $\alpha \in \mathbb{C}$. Using Lemma 2.12 with $B = \phi(A)$ we see that

$$\phi(A) = \phi\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & \alpha^{-1}a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

whenever A has rank one. Since $\phi(A)$ must also be of rank one, it is easy to see that $\alpha = 1$. This completes the proof for $n = 2$.

Now let $n \geq 3$ and assume that the assertion is true for maps on M_m with $m < n$. Let $E = I - E_{nn}$ and $F = I - E_{11}$. We have $\phi(E_{nn}) = E_{nn}$. If A satisfies $EAE = A$, then using (c) we get $\phi(A)E_{nn} = E_{nn}\phi(A) = 0$, hence $E\phi(A)E = \phi(A)$. Also ϕ^{-1} satisfies the same condition. Therefore, the restriction of ϕ to the invariant subalgebra of all matrices satisfying $EAE = A$ may be viewed as a map on M_{n-1} which satisfies all the assumptions of the proposition. By the induction hypothesis there exists an invertible diagonal matrix $S \in M_n$ such that either $\phi(A) = SAS^{-1}$ or $\phi(A) = SA^{\mathrm{tr}}S^{-1}$ for every $A \in M_n$ satisfying $A = EAE$. We may replace ϕ by ϕ' defined by $\phi'(A) = S^{-1}\phi(A)S$ and we may compose ϕ' with the transpose mapping if necessary. Therefore we may assume that $\phi(A) = A$ for every such A.

Similarly, $\phi(B) = S^{-1}BS$ or $S^{-1}B^{\mathrm{tr}}S$ for every B satisfying $B = FBF$ where S is a diagonal matrix, say $S = \mathrm{diag}(1, s_2, \ldots, s_{n-1}, s_n)$. There is no loss of generality in assuming that $s_2 = 1$. As $\phi(B) = B$ for all matrices satisfying $B = EFBEF$, we have $s_2 = \ldots = s_{n-1} = 1$. Moreover, after applying an appropriate similarity transformation we can assume that $S = I$ without affecting any of the previous assumptions. It follows that $\phi(A) = A$ when $A = EAE$ and $\phi(B) = B$ or B^{tr} when $B = FBF$.

Next, we show that it is not possible to have $\phi(B) = B^{\mathrm{tr}}$ for $B = FBF$. This is obvious for $n \geq 4$ by applying ϕ to EFM_nEF. The case $n = 3$ requires a special argument.

Assume that $n = 3$ and that $\phi(A) = A$ and $\phi(B) = B^{\mathrm{tr}}$ when $A = EAE$ and $B = FBF$. Let $G = I - E_{22}$. As before, ϕ maps GM_3G onto itself and since $\mathcal{M} \cap GM_3G$ consists of scalar multiples of E_{13} and scalar multiples of E_{31}, we have either

$$\phi(E_{13}) = \alpha E_{13} \quad \text{and} \quad \phi(E_{31}) = \beta E_{31},$$

or

$$\phi(E_{13}) = \alpha E_{31} \quad \text{and} \quad \phi(E_{31}) = \beta E_{13}$$

for some nonzero scalars α, β. We may assume that ϕ satisfies the former condition, since we may permute e_1 and e_3 and compose ϕ with the transpose mapping.

Now, let $A \in M_3$ and $B = \phi(A)$. Then by Lemma 2.12, $B_{ij} = A_{ij}$ for $\{i,j\} \subset \{1,2\}$ and for $i = j = 3$, $B_{23} = A_{32}$, $B_{32} = A_{23}$, $B_{13} = \beta^{-1}A_{13}$, and $B_{31} = \alpha^{-1}A_{31}$. In particular, if

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix},$$

then

$$\phi(A) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & -1 \\ \alpha^{-1} & 0 & 0 \end{bmatrix}.$$

But $r(A) = 0$ and $r(\phi(A)) = |\alpha|^{-1/3} \neq 0$. This contradiction ends the special argument for $n = 3$.

We have now established that there is no loss of generality in assuming that $\phi(A) = A$ when $A = EAE$ or $A = FAF$. Let $G = E_{11} + E_{nn}$. By a similar argument, if $C = GCG$, then $\phi(C) = S^{-1}CS$ for a diagonal matrix S. Hence, there exists a nonzero scalar α such that

$$\phi(E_{1n}) = \alpha E_{1n} \quad \text{and} \quad \phi(E_{n1}) = \alpha^{-1}E_{n1}.$$

If A is a rank one idempotent and $B = \phi(A)$, then B is also a rank one idempotent. Applying Lemma 2.12 we see that $B_{1n} = \alpha A_{1n}$, $B_{n1} = \alpha^{-1}A_{n1}$, and $B_{ij} = A_{ij}$ otherwise. In particular, if $A_{ij} = 1/n$ for every i, j, then A is an idempotent of rank one, and we must have $\alpha = 1$ in order that B be also of rank one. So, we have established that $\phi(A) = A$ for every rank one idempotent A.

Now, if A is a matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then $A = \sum_{j=1}^{n} \lambda_j P_j$ where P_j are idempotents and $P_j = p_j(A)$ with p_j the Lagrange interpolation polynomials. If $B = \phi(A)$, then $p_j(B) = \phi(p_j(A)) = \phi(P_j) = P_j$. Also, $\sigma(A) = \sigma(B)$. Thus, $B = \sum_{j=1}^{n} \lambda_j p_j(B) = A$. Hence, $\phi(A) = A$. Since the set of matrices with n distinct eigenvalues is dense in M_n and since ϕ is continuous, it follows that ϕ is the identity mapping. This completes the proof.

Proof of Theorem 1.1. The map $\phi'(A) = \phi(A) - \phi(0)$ is a surjective spectral radius distance preserving map that fixes 0. It is bijective by Proposition 2.1. Applying Lemma 2.13 we see that there is no loss of generality in assuming that $\phi(\alpha I) = \alpha I$, $\alpha \in \mathbb{C}$. It follows from Lemma 2.11 that $\sigma(\phi(D_0)) = \sigma(D_0)$. Hence, D_0 and $\phi(D_0)$ are similar, and so we may assume, after applying a similarity transformation if necessary, that $\phi(D_0) = D_0$. Using Lemmas 2.15, 2.16, 2.18, and 2.19 we see that all the assumptions of Proposition 3.1 are satisfied, and hence ϕ satisfies its conclusion. This ends the proof.

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Perturbation theorems for Hermitian elements in Banach algebras

by

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Abstract. Two well-known theorems for Hermitian elements in C^*-algebras are extended to Banach algebras. The first concerns the solution of the equation $ax - xb = y$, and the second gives sharp bounds for the distance between spectra of a and b when a, b are Hermitian.

1. Introduction. Let $\mathcal{A}$ be a complex unital Banach algebra. An element a of $\mathcal{A}$ is said to be *Hermitian* (or *conservative*) if $\|e^{ita}\| = 1$ for all real numbers t. This notion is a natural generalization of self-adjoint elements in a C^*-algebra, and has been of considerable interest in the theory of Banach algebras. See, e.g., [7].

Several properties of self-adjoint elements of C^*-algebras remain true for Hermitian elements of Banach algebras, while many others do not. For example, if a is Hermitian then $\|a\| = r(a)$, the spectral radius of a. This was proved, almost at the same time, by Browder [8], Katsnelson [11] and Sinclair [16]. All the three proofs depended on Bernstein's inequality for entire functions; in fact this theorem about Hermitian elements is *equivalent* to Bernstein's inequality [13]. Among the properties that are strikingly different from the corresponding fact in C^*-algebras is the following. If a is Hermitian and invertible, then a^{-1} need not be Hermitian. In this case, an interesting inequality has been proved by Partington [15]: if a is Hermitian and invertible, then

$$\|a^{-1}\| \le \frac{\pi}{2} r(a^{-1}), \tag{1}$$

and the inequality is sharp. Partington's proof used Kolmogorov's inequalities [12] for derivatives of functions. A different proof and a generalization were given by Haagerup and Zsidó [10].

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