ON A CHARACTERIZATION OF STOCHASTIC PROCESSES BY THE ABSOLUTE MOMENTS OF STOCHASTIC INTEGRALS*

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Abstract. A condition is given in terms of the absolute moments of stochastic integrals for two stochastic processes, continuous in probability with independent stationary symmetric increments, to be identical.

1. Introduction. Braverman [1] gave a condition for the equality of the distributions of two independent symmetric random variables in terms of some absolute moments. Here we extend the result to characterize stochastic processes with independent stationary and symmetric increments. We first state a few definitions. Interalia, we extend the result of Braverman [1] to random vectors.

DEFINITION 1. *A random variable X is said to be* symmetric *if X and $-X$ are identically distributed.*

DEFINITION 2. *A k-dimensional random vector $\mathbf{X}$ is said to be* symmetric *if $\boldsymbol{\lambda}^t \mathbf{X}$ is symmetric for every $\boldsymbol{\lambda} \in \mathbf{R}^k$.*

Suppose $\mathbf{X} \equiv \{X(t),\ 0 \leqq t \leqq T\}$ is a stochastic process continuous in probability and with stationary independent symmetric increments. It can be checked that, for any continuous function $\gamma(t)$ defined on $[0, T]$, the stochastic integral

$$S_{\mathbf{X}} \equiv \int_0^T \gamma(t)\, dX(t) \tag{1}$$

exists in probability and the characteristic function $\psi(\cdot)$ of $S_{\mathbf{X}}$ is given by the relation

$$\log \psi(u) = \int_0^T \varphi(u\gamma(t))\, dt,$$

where $\varphi(\cdot)$ is the log of the characteristic function of $X(t+1) - X(t)$. Note that the characteristic function of $X(t+1) - X(t)$ is a real-valued function by the symmetry of $X(t+1) - X(t)$, nonzero by the infinite divisibility of $X(t+1) - X(t)$, and does not depend on t by the stationarity of the increments of the process $\mathbf{X}$. In particular $\psi(\cdot)$ is a real-valued function. In other words, $S_{\mathbf{X}}$ has a symmetric distribution. For details, see [2], [4], or [5].

2. Characterization. We now state the main result of this paper.

THEOREM 1. *Suppose $\mathbf{X} \equiv \{X(t),\ 0 \leqq t \leqq T\}$ and $\mathbf{Y} \equiv \{Y(t),\ 0 \leqq t \leqq T\}$ are two stochastic processes with stationary independent symmetric increments, continuous in*

probability and $X(0) = Y(0) = 0$ a.s. Further suppose that there exists $p > 0$, $p \neq 2m$, $m = 1, 2, \ldots$, such that for some continuous function $\lambda(t)$ on $[0, T]$,

$$(2) \qquad \mathbf{E} \left| \int_0^T \lambda(t) \, dX(t) + \gamma U \right|^p = \mathbf{E} \left| \int_0^T \lambda(t) \, dY(t) + \gamma V \right|^p$$

for all γ real, where U and V are nonzero symmetric and identically distributed random variables with absolute moments of order p, U independent of $\mathbf{X}$ and V independent of $\mathbf{Y}$ with $(\mathbf{X}, U)$ independent of $(\mathbf{Y}, V)$. Suppose that the characteristic functions of $X(t+1) - X(t)$ and $Y(t+1) - Y(t)$ have power series expansions. Then the processes $\mathbf{X}$ and $\mathbf{Y}$ are identically distributed provided

$$(3) \qquad \int_0^T [\lambda(t)]^k dt \neq 0, \quad k \geqq 2.$$

Before we give a proof of this theorem, we now state a result due to Braverman [1].

THEOREM 2. *Suppose that p is a fixed positive number, $p \neq 2m$, $m = 1, 2, \ldots$, the pairs X, Z and Y, U consist of independent symmetric random variables and Z, U are identically distributed and nonzero. Further suppose that these random variables have finite absolute moments of order p such that*

$$\mathbf{E}|X + tZ|^p = \mathbf{E}|Y + tU|^p$$

for all t real. Then X and Y are identically distributed random variables.

Remark 1. Braverman [1] pointed out that the result is not true for $p = 2m$. It is easy to extend Theorem 2 to random vectors.

THEOREM 3. *Suppose that p is a fixed positive number, $p \neq 2m$, $m = 1, 2, \ldots$, the pairs $(\mathbf{X}, U)$ and $(\mathbf{Y}, V)$ consist of symmetric k-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$, U, V nonzero identically distributed random variables and $\mathbf{X}$ independent of U, $\mathbf{Y}$ independent of V with $(\mathbf{X}, U)$ independent of $(\mathbf{Y}, V)$. Further suppose that $(\mathbf{X}, U)$ and $(\mathbf{Y}, V)$ satisfy the condition*

$$\mathbf{E}|\boldsymbol{\lambda}^t \mathbf{X} + \beta U|^p = \mathbf{E}|\boldsymbol{\lambda}^t \mathbf{Y} + \beta V|^p$$

for all $\boldsymbol{\lambda} \in \mathbf{R}^k$ and $\beta \in \mathbf{R}$. Then $\mathbf{X}$ and $\mathbf{Y}$ are identically distributed.

Proof. Note that $\boldsymbol{\lambda}^t \mathbf{X}$ and $\boldsymbol{\lambda}^t \mathbf{Y}$ are symmetrically distributed by Definition 2. An application of Theorem 2 implies that $\boldsymbol{\lambda}^t \mathbf{X}$ and $\boldsymbol{\lambda}^t \mathbf{Y}$ are identically distributed for all $\boldsymbol{\lambda} \in \mathbf{R}^k$. Since this holds for all $\boldsymbol{\lambda} \in \mathbf{R}^k$, an application of the Cramer–Wold technique proves that $\mathbf{X}$ and $\mathbf{Y}$ are identically distributed.

Let us now give the proof of Theorem 1.

Proof. Let $g(\cdot)$ and $h(\cdot)$ be the log of the characteristic functions of $X(t+1) - X(t)$ and $Y(t+1) - Y(t)$, respectively. Note that $g(\cdot)$ and $h(\cdot)$ here are real-valued functions and do not depend on t. Let

$$S_{\mathbf{X}} = \int_0^T \lambda(t) \, dX(t) \quad \text{and} \quad S_{\mathbf{Y}} = \int_0^T \lambda(t) \, dY(t).$$

It has already been pointed out that $S_{\mathbf{X}}$ and $S_{\mathbf{Y}}$ have symmetric distributions and the pairs $(S_{\mathbf{X}}, U)$ are independent—and independent of $(S_{\mathbf{Y}}, V)$, which also are independent. Applying Theorem 2, it follows that $S_{\mathbf{X}}$ and $S_{\mathbf{Y}}$ are identically distributed. Hence

$$(4) \qquad \int_0^T g(u\lambda(t)) \, dt = \int_0^T h(u\lambda(t)) \, dt, \quad -\infty < u < \infty,$$

since the left side is the logarithm of the characteristic function of $S_{\mathbf{X}}$ and the right side is the logarithm of the characteristic function of $S_{\mathbf{Y}}$ (cf. [4]). The processes $\mathbf{X}$ and $\mathbf{Y}$ have moments of all orders and the terms of both the sides of (4) can be differentiated under the integral sign any number of times since the characteristic functions of $X(t+1) - X(t)$ and $Y(t+1) - Y(t)$ have power series expansions. We obtain that

$$\int_0^T (\lambda(t))^k g^{(k)}(u\lambda(t))\, dt = \int_0^T (\lambda(t))^k h^{(k)}(u\lambda(t))\, dt,$$

where $g^{(k)}(u)$ denotes the kth derivative of $g(\cdot)$ at u. Choosing $u = 0$, we have

$$(5) \qquad g^{(k)}(0) \int_0^T (\lambda(t))^k dt = h^{(k)}(0) \int_0^T (\lambda(t))^k dt.$$

It follows that $g^{(k)}(0) = h^{(k)}(0)$ for $k \geqq 2$ from (5) in view of condition (3). Observe that $g(\cdot)$ and $h(\cdot)$ are real-valued functions with $g(0) = h(0) = 0$ and $g(-u) = g(u)$. Expanding the functions g and h in Taylor series expansions in a neighborhood of zero, we obtain that $g(u) = h(u)$ in a neighborhood of zero. Since the characteristic functions $e^{g(u)}$ and $e^{e(u)}$ have power series expansions, it follows that

$$e^{g(u)} = e^{h(u)}, \quad -\infty < u < \infty.$$

Hence the distributions of $X(t+1) - X(t)$ and $Y(t+1) - Y(t)$ are the same. In other words, the processes $\mathbf{X}$ and $\mathbf{Y}$ are identically distributed since the processes $\mathbf{X}$ and $\mathbf{Y}$ are completely determined by the distributions of $X(t+1) - X(t)$ and $Y(t+1) - Y(t)$, respectively, under the hypotheses $X(0) = Y(0) = 0$ a.s. stated above.

   Remark 2. If $p = 2m$ for some integer $m \geqq 1$ in Theorem 1, it is easy to check that

$$\mathbf{E}[S_{\mathbf{X}}]^{2m} = \mathbf{E}[S_{\mathbf{Y}}]^{2m}$$

from (2), and $S_{\mathbf{X}}$ and $S_{\mathbf{Y}}$ are not necessarily identically distributed from Remark 1.

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