ON THE RECOVERY OF INTER-GROUP INFORMATION IN ONE- AND TWO-WAY DESIGNS

BY

K. R. SHAH

Research and Training School
INDIAN STATISTICAL INSTITUTE
CALCUTTA
1964

P R E F A C E

This thesis is being submitted to the Indian Statistical Institute in support of the author's application for the degree of Doctor of Philosophy. The thesis embodies research carried out by the author during the period 1960-1963 under the supervision of Dr. J. Roy, Professor of Statistics at the Indian Statistical Institute, Calcutta.

Some aspects of recovery of inter-group information are examined in this thesis. The main results are concerned with the variance of the combined inter and intra-block estimators in incomplete block designs; recovery of inter-row and inter-column information in two-way designs laid out in rows and columns, and recovery of inter-block information in a second experiment when residual effects from the first are present. A short description of the major findings will be found in the Introduction.

The thesis consists of seven chapters and a list of references. The detailed plan of each chapter is given in the table of contents.

A few of the results in this thesis have already been published in Sankhyā, the Indian Journal of Statistics and in the Annals of Mathematical Statistics. A list of these publications of the author and some other papers awaiting publication is given at the end of the thesis..

Foremost thanks are due to Dr. J. Roy for his guidance and advice on the method of presentation of the material in this thesis, and to Dr. C. Radhakrishna Rao for his constant encouragement. The author wishes to express his gratitude to his colleagues Dr. S. K. Mitra and Dr. S. John, for several helpful discussions. The author also records his gratefulness to the Research and Training School of the Indian Statistical Institute for providing facilities for research. Thanks are due to Sri G. M. Das for his efficient typing of the thesis.

C O N T E N T S

Chapter I

INTRODUCTION

This thesis examines problems in the recovery of inter-group information in experiments with one-way or orthogonal two-way grouping of experimental material. The main contributions of the thesis are as follows: (i) An exact expression is obtained for the variance of a combined intra- and inter-block estimator for any treatment contrast in an incomplete block design. This result is valid for any incomplete block design and for a general class of procedures for combined estimation which includes Yates' procedure. (ii) A combined estimator obtained by any of the above procedures is shown to have variance smaller than that of the corresponding intra-block estimator, if the ratio of inter- to intra-block variance does not exceed two. (iii) Instances are obtained where recovery of inter-block information by the traditional procedure results in loss of efficiency. (iv) For some important incomplete block designs, a method of recovery is obtained which gives, for <u>all</u> treatment contrasts, combined estimators that have uniformly smaller variance than corresponding intra-block estimators. (v) Methods are developed for the recovery of inter-row and inter-column information in two-way designs laid out in orthogonal rows and columns. (vi) A procedure is obtained for recovery of inter-block information in a second experiment when residual effects from the first are present.

1.1. Recovery of inter-group information.

The science of design and analysis of planned experiments has grown out of the pioneering contributions of R. A. Fisher. One of his principal contributions there is the elegant technique of arranging heterogeneous experimental material into several homogeneous groups so that without any sacrifice of the inductive scope of the experiment, treatment effects can be estimated with high precision. In the basic designs (the randomised block design and the Latin square design) developed by him each group contains all the treatments exactly once and hence no information on treatment effects can be obtained from comparisons between the total yields of the groups.

Since the groups are to be homogeneous, the size of a group would depend on the nature of the experimental material and may at times fall short of the number of treatments to be compared. For this reason, Yates (1936) introduced designs with incomplete groups, i.e. designs where each group does not contain all the treatments. In such a design different groups contain different sets of treatments. Thus total yields of the groups can also be used to compare treatment effects.

In the case of one-way designs, Yates (1939) pointed out that when the formation of groups has not succeeded in controlling the within-group heterogeneity, it would be worthwhile to recover the information contained in the between-group comparisons and gave a procedure for doing this for three dimensional lattice designs. This procedure involves use of certain weights which are estimated from the observations. In this thesis consequences of such a recovery procedure in an incomplete block design are critically examined and similar procedures are obtained for recovery of information (if any) from between-row and between-column comparisons in an experiment laid out two-way in orthogonal rows and columns.

1.2. Incomplete block designs.

Since the introduction of the balanced incomplete block (BIB) designs by Yates (1936), a number of useful incomplete block designs have been contributed by various authors including Yates himself. The precision of the estimated treatment differences in one-way designs is related to the within-block variance and the use of incomplete block designs is motivated by the reduction in the within-block variance that is usually attained through the use of smaller blocks.

However, if the formation of blocks is not successful, i.e., if the inter-block variance is nearly the same as the intra-block variance, use of an incomplete block design would result in a loss in efficiency which might be severe for designs with low efficiency factors. To prevent this loss in efficiency, Yates (1939, 1940) suggested the use of information available from inter-block comparisons. The procedure given by Yates for three dimensional lattice designs (1939) and for BIB designs (1940) was adopted by Nair (1944) for PBIB designs and was later generalised by Rao (1947) for use with any incomplete block design.

The procedure is called recovery of inter-block information and consists of the following stages. The method of least squares is applied to both intra and inter-block contrasts, assuming that the value of ρ, the ratio of the inter-block variance to the intra-block variance is known. This gives the so called 'normal' equations for combined estimation. The equations involve ρ which is estimated from the observations by equating the error sum of squares (intra-block) and the adjusted block sum of squares in the standard analysis of variance to their respective expected values. This estimate is substituted for ρ in the normal equations and the combined estimates are obtained by solving these equations. A priori, the inter-block variance is expected to be larger than the intra-block variance and hence it is customary to use the above estimator of ρ, truncated at unity.

The error sum of squares in the inter-block analysis has at times been used in place of the adjusted block sum of squares in the above analysis (Yates (1936) for a cubic lattice design, Graybill and Deal (1959) for a BIB design).

We note that all attempts so far, at the study of the properties of combined estimators of the treatment contrasts have been made under the assumption that the block effects and the errors of observation are independently Normally distributed (Yates 1939, Graybill and Deal 1959, Graybill and Weeks 1959, Graybill and Seshadri 1960). In the absence of any assumption on the distribution the problem is very complex. In chapters II to V we shall assume Normality.

We first consider unbiased quadratic estimators of inter and intra-block variances. Variances of these estimators turn out to be quadratic in ρ. Since ρ is expected to be large, one might search for an estimator of the inter (intra)-block variance for which the term involving ρ^2 in the expression for variance is minimised. It turns out that the customary estimator of the intra-block variance has this property but the same is not true for the customary estimator of the inter-block variance. An estimator of the inter-block variance with the above property is put forth in section 2.4.

It is shown in section 2.5 that the ratio of inter to intra-block variance estimates does not provide an unbiased estimator of ρ. A simple correction is obtained which eliminates this bias. The problem of constructing an unbiased estimator of ρ is examined. For the class of estimators considered, the variance turns out to be again a quadratic expression in ρ. As before, we obtain an unbiased estimator which minimises the term in ρ^2 in the expression for variance.

Information limit for the variance of any unbiased estimator of ρ is obtained in section 2.6. The method of maximum likelihood for estimating the parameters gives rise to a somewhat complicated equation for estimation. A computational procedure for solving the equation by iteration is presented.

If ρ were known, the combined estimators would have all the good properties of least-squares estimates. Since only an estimate of ρ is used, the properties of the combined estimates have to be critically examined. One would expect these to depend on the type of estimator of ρ used. To use the combined estimator of a treatment contrast with confidence one would like to know if it is unbiased and if its variance is smaller than that of the corresponding intra-block estimator, uniformly in ρ.

The question of unbiasedness has been examined by some authors under the assumption of Normality. Graybill and Weeks (1959) showed that for a BIB design, the combined estimator of a treatment contrast based on the Yates' estimator of ρ in its untruncated form is unbiased. Graybill and Seshadri (1960) proved the same with the Yates estimator of ρ in its usual truncated form, again for BIB designs.

We show here that for any incomplete block design, if the estimator of ρ is the ratio of quadratic forms of a special type, the corresponding combined estimators of treatment contrasts are unbiased. It is also shown that the customary estimator of ρ (as given by Yates (1939) and Rao (1947)) is of the above type and hence gives rise to unbiased combined estimators.

The variance of the combined estimators has also been examined by some authors, again under the customary assumption of Normality. Yates (1939) used the method of numerical integration to show that for a cubic lattice design with twentyseven treatments and with six replications or more the combined estimator of a treatment contrast has variance smaller than that of the intra-block estimator, uniformly in ρ. Graybill and Deal (1959) used the exact expression for the variance to establish this property of the combined estimators for a BIB design for which the number of blocks exceeds the number of treatments by ten or more (or by nine if in addition, the number of degrees of freedom for the intra-block error mean square exceeds eighteen). In both the cases, the estimator of ρ is based on the inter-block error mean square and is not the usual one based on the adjusted block sum of squares.

In this thesis we present a number of results concerning the variance of the combined estimators.

In the first place, we derive an expression for the variance of the combined estimator of a treatment contrast based on any estimator of ρ belonging to the class described above. Though in general this expression is not easy to evaluate in terms of well-known functions, numerical quadrature methods can be applied for evaluation. A comparison with the variance of the intra-block estimator shows that the combined estimator of any treatment contrast in any incomplete block design has variance smaller than that of the intra-block estimator if ρ does not exceed 2.

The question that now arises is whether a combined estimator for a treatment contrast can be constructed which is 'uniformly better' than the intra-block estimator, in the sense of having a smaller variance for all values of ρ. It is shown in section 4.3 that for a linked block (LB) design with 4 or 5 blocks, recovery of inter-block information by the Yates-Rao procedure may even result in loss of efficiency for large values of ρ.

We present a method of constructing a certain estimator of ρ, applicable to any incomplete block design for which the association matrix has a non-zero latent root of multiplicity $p > 2$. For any treatment contrast belonging to a sub-space associated with the multiple latent root, the combined estimator based on this estimator of ϱ is shown to be uniformly better than the intra-block estimator if and only if $(p-4)(e_o - 2) \geq 8$, where e_o is the number of degrees of freedom for error (intra-block). For almost all well-known designs, the association matrix has multiple latent roots and this method can therefore be applied to many of the standard designs, at least for some of the treatment contrasts.

We note that, in general, this estimator of ϱ is different from the customary one given by Yates (1939) and Rao (1947). For LB designs however, this estimator of ϱ coincides with the customary one. We show that for a LB design the usual procedure of recovery of inter-block information gives uniformly better combined estimators for all treatment contrasts if the number of blocks exceeds five. As we have pointed out before, if the number of blocks is four or five and if ϱ is large, recovery of inter-block information by the usual procedure results in loss of efficiency.

Using the above method, we obtain an estimator of ϱ which produces a combined estimator uniformly better than the intra-block estimator for any treatment contrast for the following designs: (i) a BIB design with more than five treatments (ii) a simple lattice design with sixteen treatments or more and (iii) a triple lattice design with nine treatments or more. Applications to some other two-associate partially balanced incomplete block designs and to inter and intra-group balanced designs have also been worked out. A computational procedure for obtaining the estimate of ρ has been given for each case.

Table 2.2 of chapter 5 shows actual gain in efficiency due to recovery of inter-block information by the above method for some selected designs for ρ = 1, 2, 4 and 8. The results appear to indicate that even for designs for which combined estimators are uniformly better than the intra-block estimators the gain in efficiency is slight except when ρ is small. This indicates that recovery of inter-block information may be worthwhile only if the formation of blocks has not been effective in controlling the heterogeneity.

The criterion of efficiency used in the above discussion is the efficiency factor of an incomplete block design as defined by Yates (1936). This is known to be proportional to the harmonic mean of the latent roots of C, the matrix of co-efficients in the intra-block normal equations (Kempthorne 1956, Roy 1958). Two other criteria by Wald (1943) are given in terms of the latent roots of the matrix C, one being the minimum root and the other the geometric mean of the latent roots. With any of these measures, the designs for which the latent roots of C are not widely separated, will have high efficiency.

This suggests the use of a fourth criterion based on the dispersion of these roots.

By a slight modification of these criteria one can compare designs with the same number of treatments and the same block size but using unequal amounts of experimental material. Values of the four criteria evaluated for ten two-associate PBIB designs from the list prepared by Bose, Clatworthy and Shrikhande (1954) happen to give similar orderings of these designs. The fourth criterion based on the dispersion of the latent roots of C-matrix can be expressed in terms of the sum of squares of the elements of C-matrix and is easy to compute.

Kshirsagar (1958), Roy (1958) and Kiefer (1958) have proved the optimality of a BIB design (with the first three criteria) among designs with the same number of treatments, the same block size and the same number of blocks. It is shown that even in the wider class of designs where the number of blocks is unequal a BIB design is optimal with any of the four criteria described above.

1.2. Two-way designs with orthogonal grouping.

Designs with two-way elimination of heterogeneity have been considered by various authors (Latin squares by Fisher (1935), quasi-Latin squares by Yates (1937), Youden squares by Youden (1937), partially balanced Youden squares by Bose and Kishen (1939), Y_1 class of designs by Shrikhande (1951). A general method of analysis of such designs given by Shrikhande (1951) is based on the so called fixed effects Normal model and does not utilise the information available from differences of row or column totals.

We derive an analysis of such designs using only the assumption of additivity of plot and treatment effects and the distribution induced by the randomisation procedure. The equations for estimation in the interaction analysis where one uses only the contrasts orthogonal to rows and columns turn out to be the same as obtained by Shrikhande (1951) under the Normal model. When one also uses the information available from between-row and between-column comparisons the equations for estimation involve three variances ; the between-row variance, the between-column variance and the interaction-variance. These are usually unknown and have to be estimated from the observations themselves.

An estimate of the interaction-variance is provided by the error mean square in the interaction analysis. We obtain here two estimators of the between-row variance. One is based on the mean square of rows adjusted for columns and treatments. To obtain this one has to carry out a separate analysis of variance where the classification by rows is ignored. This is an extension of the traditional procedure of Yates (1939) and Rao (1947) for estimating the between-block variance in an incomplete block design. This involves rather heavy computations in the general case and hence the following procedure is recommended. We first consider the row-totals of the corrected yields where from the Yield of each plot the estimate of the treatment parameter (as given by the interaction analysis) is subtracted. The sum of squares of deviations of these row-totals from their mean is used in estimating the between-row variance. This obviates the need for an additional analysis of variance to be performed.

Conditions under which a two-way design compares favourably with the corresponding one-way designs are examined in section 6.5 and the relative efficiency factors are worked out.

1.3. One-way design with two sets of treatments.

An illustration of a one-way design with two sets of treatments is given by the following. Suppose we have experimental material consisting of p_0 blocks having r_0 plots each on which an experiment involving p_1 treatment was performed in the recent past. If the same material is to be used for comparing some p_2 treatments the yield on each plot might be affected by a) the treatment received by it in the previous experiment and b) the treatment applied in the current experiment. One may assume the effects of these two treatments to be additive. A similar situation is obtained when we have a one-way design involving two factors and when the interactions are known to be absent.

Designs of these type are considered by various authors (Pearce and Taylor 1948, Hoblyn, Pearce and Freeman 1954, Freeman 1957a, 1957b, 1958, 1961, Pothoff 1962).

For the special designs considered by them, Freeman (1957a) and Pothoff (1962) gave a method of intra-block analysis based on the Normal model.

In chapter seven, we derive a method of analysis for any such design with experimental units arranged one-way in blocks, and involving two sets of treatments whose effects are additive. We use only the distribution induced by the usual two-stage (within and between blocks) randomisation and the assumption of additivity of plot and treatment effects. In addition to the intra-block analysis we give a procedure for recovery of inter-block information. The estimating equations in this case involve the intra and inter-block variances. A procedure for estimating these unknown variances is given in section 7.3. The results are obtained in symbolic forms, in terms of pseudo-inverses of certain matrices.

When the design is not chosen properly, even the intra-block analysis is somewhat laborious for manual computations. An illustration given in section 7.4 serves to demonstrate that with a careful choice of the design, the analysis including recovery of inter=block information does not involve unduly heavy computations.

Chapter II

ESTIMATION OF INTER TO INTRA-BLOCK VARIANCE RATIO

2.1 Motivation and summary.

The key role played by the estimator of ρ, the ratio of inter to intra-block variance in the process of recovery of inter-block information was brought out in the previous chapter. In the present chapter we shall be concerned with the problem of estimating ρ.

In the study of this problem it is much simpler to deal with the canonical form of the observations rather than the observations themselves. Thus, in section 3 an orthogonal transformation of the following type is made. From the original observations one transforms to a set of mutually orthogonal normalised inter-block contrasts, a set of mutually orthogonal normalised intra-block contrasts and a constant times the grand mean of all observations. A corresponding transformation is made of the parameters measuring the effects of the v treatments, one being the average and the other (v-1) mutually orthogonal parametric contrasts. The whole transformation is so chosen that each of the first (v-1) intra-block contrasts of observations has expectation proportional to one of these transformed parametric contrasts and all other intra-block contrasts of observations have expectation zero. Similarly, of the inter-block contrasts of observations those whose expected values are not identically zero, again have expectation proportional to some of these same set of parametric contrasts. This provides a suitable framework for developing the methods of this and the succeeding chapter. Of course, in every case the numerical procedure for the actual application of the method does not necessitate the transformation to be physically carried out. Canonical form is used only to simplify the derivations.

Using the canonical form, a set of minimal sufficient statistics for the treatment contrasts and the inter and intra-block variances is derived for any incomplete block design. A set of minimal sufficient statistics for a BIB design under the variance components model was obtained by Weeks and Graybill (1961).

Unbiased quadratic estimators of inter and intra-block variances are considered in section 4. The expressions for the variances of these estimators turn out to be quadratic in ρ. When homogeneity within blocks is achieved, ρ is likely to be large and hence to compare two estimators t and t', one may take $\lim_{\rho \to \infty} V(t')/V(t)$ as a criterion.

It turns out that with this criterion of all unbiased quadratic estimators of intra-block variance, the customary estimator is 'best'. The same is not true for the customary estimator of inter-block variance. An estimator of inter-block variance which is 'best' in this sense is obtained in section 4. Computational procedure to obtain this estimate turns out to be fairly simple. It would be desirable to compare this estimator of inter-block variance with the customary estimator when ρ is moderately large. For asymmetrical BIB designs listed by Fisher and

Yates (1957) it is found that the customary estimator has larger variance for values of ϱ exceeding five. For symmetrical BIB designs, the two estimators are identical.

It is customary to take as an estimator of ϱ the ratio of inter to intra-block variance estimates. In section 5, we show that when these variances are estimated from the analysis of variance, the ratio does not provide an unbiased estimator of ϱ. A simple correction is obtained which eliminates this bias. The problem of constructing an unbiased estimator of ϱ is examined. For the class of estimators considered, the variance turns out to be again a quadratic expression in ϱ. As before, an unbiased estimator of ϱ is obtained, which is 'best' according to our criterion.

Information limit for the variance of any unbiased estimator of ϱ is obtained in section 6. The method of maximum likelihood for estimating the parameters is considered. This gives rise to a somewhat complicated equation for estimation. A computational procedure for solving the equation by iteration is presented. This procedure is illustrated with the help of a numerical example in section 7.

2.2. Preliminaries and notations.

Consider an experiment in which v treatments are applied on bk experimental units or plots, themselves divided into b blocks of k plots each. Only one treatment is applied on every plot, the actual allocation being done in the following manner.

First, we consider a design, that is an arrangement of v symbols (one corresponding to each treatment) in b rows each having k cells. The arrangement is characterised by the numbers m_{jiu}, $j = 1,2,\ldots, v$; $i = 1,2,\ldots, b$; $u = 1,2,\ldots, k$ where $m_{jiu} = 1$ or 0 according as the j^{th} symbol (treatment) occurs on the u^{th} cell of the i^{th} row or not.

It is customary to perform randomisation as follows. The blocks are numbered 1, 2,..., b at random and the plots in a block are numbered 1, 2,..., k again at random and independently for different blocks. The u^{th} plot in the i^{th} block then receives the treatment corresponding to the symbol which occurs in the u^{th} cell of the i^{th} row of the design. Let y_{iu} denote the yield on this plot. Let further $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_v)$ denote the row-vector of real valued parameters measuring the effects of the treatments.

A linear function $\sum\limits_{i,u} a_{iu} y_{iu}$ is said to be a contrast if $\sum\limits_{i,u} a_{iu} = 0$. A contrast is said to belong to blocks, or simply called an inter-block contrast if $a_{i1} = a_{i2} = \ldots = a_{ik}$ holds for all i. A contrast is said to be an intra-block contrast if $\sum\limits_{u} a_{iu} = 0$ holds for all i. A linear function

$\sum_{i,u} a_{iu} y_{iu}$ is said to be normalised if $\sum_{i,u} a_{iu}^2 = 1$. Two linear functions $\sum_{i,u} a_{iu} y_{iu}$ and $\sum_{i,u} b_{iu} y_{iu}$ are said to be orthogonal if $\sum_{i,u} a_{iu} b_{iu} = 0$. It is easy to see that any inter-block contrast and any intra-block contrast are mutually orthogonal. The rank of the vector space generated by all inter-block contrasts is $(b-1)$ and of that generated by all intra-block contrasts is $b(k-1)$.

We shall construct $(b-1)$ mutually orthogonal normalised inter-block contrasts and $b(k-1)$ mutually orthogonal normalised intra-block contrasts. For these contrasts we shall assume that (a) the expected value of any contrast is obtained by replacing in the contrast every observation by the corresponding treatment parameter, (b) the variance of any intra (inter)-block contrast is σ_o^2 (σ_1^2) and (c) the covariances are all zero. Thus we may call σ_o^2 (σ_1^2) intra (inter)-block variance per plot. Evidently, if these assumptions hold for one set of mutually orthogonal inter and intra-block contrasts they will hold for any other. We shall define

$$\rho = \sigma_1^2 / \sigma_o^2 . \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

We shall assume that $\rho \geq 1$.

It is implied in Rao (1959) that the above holds when one assumes the additivity of plot and treatment effects and considers the distribution induced by randomisation. In this case σ_o^2 and σ_1^2 are respectively the mean squares (of plot effects) within and between blocks.

It is easy to see that the above assumptions hold good under the so called 'Normal' model where in addition, the joint distribution of these contrasts is multivariate Normal. From section 4 onwards we shall further assume Normality.

Let $\sum_{u=1}^{k} m_{jiu} = n_{ji}$, the number of times the j-th treatment occurs on plots in the i-th block. Thus $n_{ji} = 1$ or 0 and $\sum_{j=1}^{v} n_{ji} = k$, $\sum_{i=1}^{b} n_{ji} = r$. The $v \times b$ matrix $N = ((n_{ji}))$ is called the incidence matrix of the design.

We shall denote by E_{mn} a matrix of the form $m \times n$ each element of which is unity. The matrices

$$C = r\,I - \frac{1}{k} N N' \quad \text{and} \quad C_1 = \frac{1}{k} N N' - \frac{r^2}{bk} E_{vv} \tag{2.2}$$

play important roles in the analysis. We shall assume that the matrix C is of rank $(v-1)$: this is equivalent to the assumption that the experimental design is connected.

Let B_i denote the total yield for the i-th block and T_j that for the j-th treatment and let G be the grand total, so that

$$B_i = \sum_u y_{iu}, \; T_j = \sum_{i,u} y_{iu}\, m_{jiu} \quad \text{and} \quad G = \sum_{i,u} y_{iu} \, . \tag{2.3}$$

We shall use the row-vectors $\underline{B} = (B_1, B_2, \ldots, B_b)$ and $\underline{T} = (T_1, T_2, \ldots, T_v)$. The adjusted totals for the treatments are defined as

$$\underline{Q} = \underline{T} - \frac{1}{k} \underline{B} N' . \tag{2.4}$$

Let, further

$$\underline{Q}_1 = \frac{1}{k} \underline{B} N' - \frac{rG}{bk} E_{1v} . \tag{2.5}$$

It can be seen that the elements of $\underline{Q}$ are intra-block contrasts and those of $\underline{Q}_1$ are inter-block contrasts.

It is known (see, for example, Rao 1947) that minimum variance unbiased linear estimates of the treatment effects $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_v)$, based on intra-block contrasts only, are obtained from the equations

$$\underline{\theta} C = \underline{Q} . \tag{2.6}$$

These shall be called the intra-block normal equations for estimation or simply the intra-block equations. We shall write $\underline{\theta}^*$ for the solution of these equations. If the ratio $\rho = \sigma_1^2 / \sigma_0^2$ is known, both intra-block and inter-block contrasts can be used together, and minimum variance linear unbiased estimates in this case are obtained from the equation

$$\underline{\theta}(C + \frac{1}{\rho} C_1) = \underline{Q} + \frac{1}{\rho} \underline{Q}_1 . \tag{2.7}$$

These shall be called the combined normal equations for estimation or simply the combined equations. The solution of these equations will be denoted by $\bar{\underline{\theta}}(\rho)$. When ρ is not known, an estimate ρ^* for ρ is substituted in (2.7) and $\bar{\underline{\theta}}(\rho^*)$ is taken as an estimate for $\underline{\theta}$.

For estimating ρ, the following procedure is generally recommended (See Yates, 1939, 1940 or for a general treatment, Rao, 1947). First the following table of analysis of variance is prepared.

Table 2.1: Analysis of Variance

Source	d.f.	S.S.
blocks (unadjusted)	b-1	$SS_B^* = \frac{1}{k} \underline{B}\underline{B}' - G^2/bk$
treatments (adjusted)	v-1	$SS_{tr} = \underline{Q}\underline{Q}^{*\prime}$
error	$e_o = bk - b - v + 1$	$SS_E = SS_T - SS_B^* - SS_{tr}$
total	bk-1	$SS_T = \sum_{i,u} y_{iu}^2 - G^2/bk$

The adjusted sum of squares due to blocks is then computed as

$$SS_B = SS_B^* + SS_{tr} - \left(\frac{1}{r} \underline{T}\,\underline{T}' - G^2/bk \right). \qquad (2.8)$$

Then s_o^2 and s_1^2 defined by

$$s_o^2 = SS_E / e_o, \quad v(r-1)s_1^2 = k\, SS_B - (v-k)s_o^2 \qquad (2.9)$$

provide unbiased estimators of σ_o^2 and σ_1^2 respectively; and as an estimate of ρ one takes

$$R = s_1^2 / s_o^2 . \qquad (2.10)$$

Since the blocks are formed so as to achieve homogeneity within blocks, we have assumed $\rho \geq 1$; but depending on fluctuations of sampling, R may not satisfy this inequality. For this reason, a modified estimate

R' (which we shall call the truncated form of R) given by

$$R' = \begin{cases} 1 & \text{if} \quad R \leq 1 \\ R & \text{if} \quad R \geq 1 \end{cases} \qquad (2.11)$$

is usually recommended.

2.3. Canonical form.

The assumption that the rank of C is (v-1) implies that there is exactly one latent root of the matrix NN' which is equal to rk, and all other latent roots are strictly smaller than rk. Let ξ_s s = 1, 2,..., q be a set of orthonormal latent vectors of NN', corresponding to the q positive latent roots ϕ_s, all smaller than rk. Let ξ_s , s = q+1,..., v-1, be a set of (v-1)-q orthonormal $1 \times v$ vectors, each orthogonal to $\xi_1, \xi_2, \ldots \ldots \xi_q$ and also to E_{1v}. We then define (v-1) intra-block contrasts x_{os} ; s = 1,2,..., v-1 as follows:

$$x_{os} = \begin{cases} k^{\frac{1}{2}} (rk - \phi_s)^{-\frac{1}{2}} \, Q \xi_s' & \text{for} \quad s = 1,2,\ldots, q \\ r^{-\frac{1}{2}} \, Q \xi_s' & \text{for} \quad s = q+1,\ldots, v-1. \end{cases} \qquad (3.1)$$

Since the rank of the vector-space generated by all intra-block contrasts is b(k-1), we can find $e_o = b(k-1) - (v-1)$ mutually orthogonal normalised intra-block contrasts, call them z_{os} ; s = 1,2,..., e_o, each orthogonal to $x_{01}, x_{02}, \ldots, x_{0,v-1}$.

Next we define q inter-block contrasts

$$x_{1s} = (k\,\phi_s)^{-\frac{1}{2}} \, B N' \xi_s' \qquad s = 1, 2, \ldots, q. \qquad (3.2)$$

Since the rank of the vector-space generated by inter-block contrasts is $(b-1)$, we can find $e_1 = (b-1)-q$ mutually orthogonal normalised inter-block contrasts, call them z_{1s} ; $s = 1,2,\ldots, e_1$, each orthogonal to $x_{11}, x_{12},\ldots, x_{1q}$. Finally, let

$$G^* = (bk)^{-\frac{1}{2}}\, G \cdot \qquad (3.3)$$

We shall also consider an orthogonal transformation from $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_v)$ to $\tau_1, \tau_2, \ldots, \tau_v$ defined by

$$\tau_s = \begin{cases} \underline{\theta}\, \underline{\xi}_s' & \text{for } s = 1,2,\ldots, v-1 \\ \underline{\theta}\, \underline{E}_{v1} / \sqrt{v} & \text{for } s = v\,. \end{cases} \qquad (3.4)$$

Since τ_v is not estimable in our set-up, we can assign to it an arbitrary value. For computational convenience we shall always take $\tau_v = 0$.

By straightforward algebra, one can easily verify the following results.

2.3.1. The linear transformation from y_{iu}'s to G^*, x_{os} $(s = 1,2,\ldots, v-1)$, x_{1s} $(s = 1,2,\ldots, q)$, z_{os} $(s = 1,2,\ldots, e_o)$ and z_{1s} $(s = 1,2,\ldots, e_1)$ is normalised orthogonal.

2.3.2. The transformed variables are mutually uncorrelated and their expectations and variances are :

$$E(x_{os}) = a_{os}\tau_s \text{ where } a_{os} = \begin{cases} (r-\phi_s/k)^{\frac{1}{2}} & \text{for } s=1,2,\ldots,q \\ r^{\frac{1}{2}} & \text{for } s=q+1,q+2,\ldots,v-1 \end{cases} \quad \ldots (3.5)$$

$$E(x_{1s}) = a_{1s}\tau_s \text{ where } a_{1s} = (\phi_s/k)^{\frac{1}{2}} \text{ for } s=1,2,\ldots,q \quad \ldots (3.6)$$

$$E(z_{os}) = 0 \text{ for } s = 1,2,\ldots,e_o,\ E(z_{1s}) = 0 \text{ for } s = 1,2,\ldots,e_1.$$

$$V(x_{os}) = \sigma_o^2 \text{ for } s = 1,2,\ldots,v-1,\ V(z_{os}) = \sigma_o^2 \text{ for } s=1,2,\ldots,e_o$$

$$V(x_{1s}) = \sigma_1^2 \text{ for } s = 1,2,\ldots,q\ \ V(z_{1s}) = \sigma_1^2 \text{ for } s = 1,2,\ldots,e_1. \quad \ldots (3.7)$$

2.3.3. The equations (2.6) are equivalent to $\tau_s = t_s$ where

$$t_s = x_{os}/a_{os} \qquad \text{for } s = 1,2,\ldots,v-1. \qquad (3.8)$$

2.3.4. The equations (2.7) are equivalent to $\tau_s = \bar{t}_s(\varrho)$ where

$$\bar{t}_s(\varrho) = \begin{cases} (\varrho a_{os}x_{os}+a_{1s}x_{1s})/(\varrho a_{os}^2 + a_{1s}^2), & \text{for } s=1,2,\ldots,q \\ x_{os}/a_{os} & \text{for } s=q+1,q+2,\ldots,v-1. \end{cases} \quad (3.9)$$

2.3.5. The error sum of squares in the table may be expressed as

$$SS_E = \sum_{s=1}^{e_o} z_{os}^2 = S_o, \text{ say}. \qquad (3.10)$$

2.3.6. The adjusted sum of squares due to blocks defined by (2.8) may be expressed as

$$SS_B = S_1 + \sum_{s=1}^{q} \phi_s\, \delta_s^2/rk \qquad (3.11)$$

where

$$S_1 = \sum_{s=1}^{q} z_{1s}^2 \qquad (3.12)$$

and

$$z_s = x_{os} - a_{os}\, x_{1s} / a_{1s} \quad \text{for } s = 1,2,\ldots, q . \qquad (3.13)$$

2.3.7. Let

$$w_{iu}(\underline{\theta}) = y_{iu} - \sum_j m_{jiu}\, \theta_j , \qquad (3.14)$$

$$B_i(\underline{\theta}) = \sum_u w_{iu}(\underline{\theta}) . \qquad (3.15)$$

Then

$$S_1 + \sum_{s=1}^{q} (x_{1s} - a_{1s}\, \tau_s)^2 = \frac{1}{k} \sum_i B_i^2(\underline{\theta}) - G^2/bk \qquad (3.16)$$

and

$$S_o + \sum_{s=1}^{v-1} (x_{os} - a_{os}\tau_s)^2 = \sum_{i,u} w_{iu}^2(\underline{\theta}) - \frac{1}{k} \sum_i B_i^2(\underline{\theta})$$

$$= \sum_{i,u} y_{iu}^2 - 2 \sum_j \theta_j T_j + r \sum_j \theta_j^2 - \frac{1}{k} \sum_i B_i^2(\underline{\theta}) . \qquad \ldots (3.17)$$

2.3.8. If the joint distribution of the contrasts of observations is in addition multivariate normal, a minimal set of sufficient statistics for the parameters τ_s $(s = 1,2,\ldots, v-1)$, σ_o^2 and σ_1^2 is provided by x_{os} $(s = 1,2,\ldots, v-1)$, x_{1s} $(s = 1,2,\ldots, q)$, S_o and S_1. If ϱ is given, $\bar{t}_s(\varrho)$ $(s = 1,2,\ldots, v-1)$ and V are complete sufficient, where

$$V = S_o + \frac{S_1}{\varrho} + \sum_{s=1}^{q} \frac{z_s^2}{1 + \varrho\, a_{os}^2 / a_{1s}^2} \cdot \qquad (3.18)$$

When ϱ is known $\bar{t}_s(\varrho)$ as defined by (3.9) is the unbiased minimum variance estimator of τ_s and its variance is given by

$$V\left\{\bar{t}_s(\varrho)\right\} = \begin{cases} \varrho\sigma_o^2/(\varrho a_{os}^2 + a_{1s}^2) & \text{for } s = 1,2,\ldots, q \\ \sigma_o^2 / a_{os}^2 & \text{for } s = q+1, q+2,\ldots, v-1 \end{cases} \qquad (3.19)$$

2.4. Quadratic estimators of inter and intra-block variances.

In this section we shall consider quadratic estimators of σ_o^2 and σ_1^2. We notice that the variables z_{os} $(s = 1,2,\ldots, e_o)$, z_{1s} $(s = 1, 2,\ldots, e_1)$ and z_s $(s = 1,2, \ldots, q)$ defined by (3.13) each have expectation zero and they are mutually uncorrelated. The variances of z_{os} and z_{1s}'s are given by (3.7) and the variance of z_s is given by

$$V(z_s) = \sigma_o^2 + c_s\, \sigma_1^2 \qquad (4.1)$$

where

$$c_s = a_{os}^2/a_{1s}^2 = (rk - \phi_s)/\phi_s \cdot \qquad (4.2)$$

We shall consider only the quadratic forms of the type

$$W = b_o S_o + b_1 S_1 + \sum_{s=1}^{q} a_s\, z_s^2 \qquad (4.3)$$

where b_o, b_1 and a_s $(s = 1,2,\ldots, q)$ are the coefficients to be determined. For any quadratic form in these variables, we can find one of the above type having the same expected value and having smaller variance. The expectation of W is

$$E(W) = (b_o e_o + \sum_{s=1}^{q} a_s)\, \sigma_o^2 + (b_1 e_1 + \sum_{s=1}^{q} a_s c_s)\sigma_1^2 . \qquad (4.4)$$

The variance of W, under the assumption of normality stated earlier, is

$$V(W) = 2[(b_o^2 e_o + \sum_{s=1}^{q} a_s^2) + 2\,\rho \sum_{s=1}^{q} a_s^2 c_s + \rho^2 (b_1^2 e_1 + \sum_{s=1}^{q} a_s^2\, c_s^2)]\sigma_o^4 . \qquad (4.5)$$

It is therefore possible to choose b_o, b_1 and a_s $(s = 1,2,\ldots, q)$ so as to make W an unbiased estimator of σ_o^2 (or of σ_1^2) with a variance which is minimum for a given value of $\rho = \sigma_1^2/\sigma_o^2$. This gives for estimating σ_o^2

$$b_o = (e_1 + B_1)/\Delta, \quad b_1 = - B_o/\Delta \qquad (4.6)$$

and for estimating σ_1^2

$$b_o = - A_1/\Delta, \; b_1 = (e_o + A_o)/\Delta . \qquad (4.7)$$

In either case

$$a_s = (b_o + \rho^2 b_1 c_s)/(1 + \rho\, c_s)^2 \qquad (4.8)$$

where

$$\Lambda_o = \sum_{s=1}^{q} (1+\varrho c_s)^{-2}, \quad \Lambda_1 = \varrho^2 \sum_{s=1}^{q} c_s(1+\varrho c_s)^{-2}$$

$$B_o = \sum_{s=1}^{q} c_s(1+\varrho c_s)^{-2}, \quad B_1 = \varrho^2 \sum_{s=1}^{q} c_s^2 (1+\varrho c_s)^{-2} \tag{4.9}$$

and $\Delta = (e_o + \Lambda_o)(e_1 + B_1) - B_o \Lambda_1$.

The case where ϱ is large is of special interest. In such a case the term involving ϱ^2 in $V(W)$ would be dominant, and we may like to minimise this term. The optimum unbiased estimates of σ_o^2 and σ_1^2 in this sense are given by

$$v_o = S_o / e_o \tag{4.10}$$

$$v_1 = \frac{1}{(b-1)} \left[S_1 + \sum_{s=1}^{q} \frac{z_s^2}{c_s}\right] - \frac{S_o}{e_o(b-1)} \sum_{s=1}^{q} \frac{1}{c_s} \tag{4.11}$$

respectively. In actual computation we may make use of the fact that

$$S_1 + \sum \frac{z_s^2}{c_s} = \frac{1}{k} \sum_i B_i^2(\underset{\sim}{\theta}^*) - \frac{G^2}{bk} \tag{4.12}$$

where $B_i(\underset{\sim}{\theta})$ is defined by (3.15) and $\underset{\sim}{\theta}^*$ is the intra-block estimate of $\underset{\sim}{\theta}$ obtained from (2.6). Thus, to compute v_1 one may use the formula

$$v_1 = \frac{1}{(b-1)}\left[SS_B^* + SS_{tr} - (2\underset{\sim}{T} - r\underset{\sim}{\theta}^*)\underset{\sim}{\theta}^{*\prime} - (v-1)(E^{-1}-1)s_o^2\right] \tag{4.13}$$

where E is the efficiency factor of the design (Kempthorne, 1956; Roy, 1958) given by

$$E = \frac{(v-1)}{rk} / \left[\frac{v-1-q}{rk} + \sum_{s=1}^{q} \frac{1}{rk - \phi_s} \right] . \tag{4.14}$$

Once, the intra-block estimates $\underline{\theta}^*$ are obtained, this is easily computed.

The variance of v_1 is

$$V(v_1) = \frac{2\,\sigma_o^4}{(b-1)^2} \left[(b-1)\rho^2 + 2(\alpha_{-1} - q)\rho + \alpha_{-2} - 2\alpha_{-1} \right.$$

$$\left. + q + (\alpha_{-1} - q)^2/e_o \right] \tag{4.15}$$

where $$\alpha_t = \sum_{s=1}^{q} \left(1 - \frac{\phi_s}{rk}\right)^t . \tag{4.16}$$

We may compare this with the customary estimator s_1^2 of σ_1^2 as defined by (2.9). The variance of this estimator is

$$V(s_1^2) = \frac{2k^2\,\sigma_o^4}{v^2(r-1)^2} \left[(e_1 + \alpha_2)\rho^2 + 2(\alpha_1 - \alpha_2)\rho \right.$$

$$\left. + 1 - 2\alpha_1 + \alpha_2 + \left(\frac{v}{k} - 1\right)^2/e_o \right] . \tag{4.17}$$

As one would naturally expect, $V(s_1^2) - V(v_1)$ is positive for somewhat large values of ρ. By ρ_o, we denote the value of ρ such that for $\rho > \rho_o$, $V(s_1^2) > V(v_1)$. The values of ρ_o are given in the table below for all asymmetrical BIB designs listed by Fisher and

Yates (1957). For each of these designs ρ_o happens to lie between 4 and 5.

Table 4.1 Values of ρ_o for all BIB designs listed by Fisher and Yates (Other than symmetrical designs)

k	r	b	v	ρ_o	k	r	b	v	ρ_o
3	6	10	5	4.3429	5	10	18	9	4.1795
3	5	10	6	4.3828	5	9	18	10	4.1875
3	4	12	9	4.4179	5	6	30	25	4.1900
3	6	26	13	4.6270	5	10	82	41	4.3435
3	9	30	10	4.6801	6	8	12	9	4.0822
3	7	35	15	4.7080	6	9	15	10	4.1062
3	9	57	19	4.8091	6	9	24	16	4.1584
3	10	70	21	4.8463	6	9	69	46	4.2142
4	7	14	8	4.2520	6	10	85	51	4.2406
4	10	15	6	4.2044	7	10	30	21	4.1304
4	6	15	10	4.2640	7	9	36	28	4.1290
4	8	18	9	4.3016	7	8	56	49	4.1157
4	5	20	16	4.2584	8	9	72	64	4.0976
4	8	50	25	4.4464	9	10	90	81	4.0672
4	9	63	28	4.4832					

It is easily seen that for symmetrical BIB designs in both v_1 and s_1^2, $b_1 = 0$ and $a_s = \frac{1}{(b-1)}$. Hence these two estimators of σ_1^2 are identical is this case.

2.5. Unbiased estimators of the variance ratio.

As a convenient unbiased estimator of ρ we may consider a statistic of the form

$$P = \frac{a\ S_1 + \sum b_s\ z_s^2}{S_o} + d \qquad (5.1)$$

where a, $b_s (s = 1,2,\ldots, q)$ and d are constants to be suitably determined . Since for $e_o > 2$

$$E(P) = \frac{a\ e_1\ \varrho + \sum b_s (1 + \varrho\ c_s)}{(e_o - 2)} + d, \qquad (5.2)$$

to make P an unbiased estimator of ϱ, we must have

$$\frac{\sum b_s}{e_o - 2} + d = 0$$

and $a\ e_1 + \sum b_s\ c_s = e_o - 2.$ (5.3)

If $e_o > 4$, the variance of such an unbiased estimator turns out to be

$$V(P) = D_o + D_1\ \varrho + D_2\ \varrho^2 \qquad (5.4)$$

where

$$D_o = 3 \sum b_s^2 + 2a\ e_1 \sum b_s - \left(\frac{\sum b_s}{e_o - 2} \right)^2$$

$$D_1 = \frac{6 \sum b_s^2\ c_s + 2ae_1 \sum b_s c_s}{(e_o - 2)(e_o - 4)} - \frac{2 \sum b_s}{(e_o - 2)}$$

$$D_2 = \frac{a^2 e_1 (e_1 + 2) + 3 \sum b_s^2\ c_s^2}{(e_o - 2)(e_o - 4)} - 1.$$

If we like to minimise D_2, the coefficient of ϱ^2 in (5.4), we have to take

$$a = \frac{3(e_o - 2)}{3e_1 + (e_1 + 2)q}, \quad b_s = \frac{(e_1 + 2)a}{3c_s} . \qquad (5.5)$$

It can be seen that R given by (2.10) is <u>not</u> an unbiased estimator of ρ, but a simple correction can be applied to make it unbiased. We thus get

$$\left(1 - \frac{2}{e_o}\right) R - \frac{2(v-k)}{e_o v(r-1)} \qquad (5.6)$$

as an unbiased estimator of ρ.

Similarly, if we start with v_o and v_1 defined by (4.10) and (4.11) as estimators of σ_o^2 and σ_1^2 respectively, we get, as another unbiased estimator of ρ:

$$\left(1 - \frac{2}{e_o}\right) \frac{v_1}{v_o} - \frac{2(v-1)}{e_o(b-1)} \left(\frac{1}{E} - 1\right) \qquad (5.7)$$

where, as before, E is the efficiency-factor of the design.

Since with positive probability these unbiased estimators of ρ may turn out to be less than unity, we may use their truncated forms instead, as indicated by (2.11). Let x be an unbiased estimator of ρ and x' its truncated form defined by $x' = 1$ if $x \leq 1$ and $x' = x$ otherwise. Then, even though x' is generally a biased estimator of ρ, it can be easily seen that its mean square error can never exceed that for x,

$$E(x' - \rho)^2 \leq E(x - \rho)^2 .$$

2.6. Maximum likelihood estimates.

Under the assumption of normality stated in section 2 it follows that the likelihood function L is given

$$\text{Log}_e \, L = \text{const} - \frac{1}{2} \Big[(b-1)\log_e \sigma_1^2 + b(k-1)\log_e \sigma_o^2$$

$$+ \frac{1}{\sigma_1^2} \Big\{ \sum_{s=1}^{q} (x_{1s} - a_{1s} \tau_s)^2 + S_1 \Big\}$$

$$+ \frac{1}{\sigma_o^2} \Big\{ \sum_{s=1}^{v-1} (x_{os} - a_{os} \tau_s)^2 + S_o \Big\} \Big] \qquad (6.1)$$

where S_o and S_1 are defined by (3.10) and (3.12) respectively.

The likelihood equations obtained by equating to zero the partial derivatives of $\text{Log}_e L$ with respect to the parameters, turn out to be

$$\tau_s = \bar{t}_s(\rho) \qquad \text{for } s = 1,2,\ldots, v-1 \qquad (6.2)$$

$$b(k-1)\sigma_o^2 = S_o + \sum_{s=1}^{v-1} (x_{os} - a_{os} \tau_s)^2 \qquad (6.3)$$

$$(b-1)\sigma_1^2 = S_1 + \sum_{s=1}^{q} (x_{1s} - a_{1s} \tau_s)^2 \,. \qquad (6.4)$$

The diagonal elements of the information matrix are

$$I_{\tau_s, \tau_s} = \begin{cases} a_{os}^2 \sigma_o^{-2} + a_{1s}^2 \sigma_1^{-2} & \text{for } s = 1,2,\ldots, q \\ a_{os}^2 \sigma_o^{-2} & \text{for } s=q+1,q+2,\ldots, v-1 \end{cases} \qquad (6.5)$$

$$I\,(\sigma_o^2, \sigma_o^2) = \frac{1}{2}\, b(k-1)\sigma_o^{-4} \qquad (6.6)$$

$$I\,(\sigma_1^2, \sigma_1^2) = \frac{1}{2}\, (b-1)\sigma_1^{-4} \qquad (6.7)$$

and all non-diagonal elements vanish.

We thus see that the maximum likelihood estimate of τ_s is $\hat{\tau}_s = \bar{t}_s(\hat{\varrho})$ where $\hat{\varrho}$ is the maximum likelihood estimate of ϱ. To compute $\hat{\varrho}$ we note that it can be expressed as

$$\hat{\varrho} = \frac{b(k-1)[S_1 + \sum_{s=1}^{q} (x_{1s} - a_{1s}\hat{\tau}_s)^2]}{(b-1)[S_o + \sum_{s=1}^{v-1} (x_{os} - a_{os}\hat{\tau}_s)^2]} . \tag{6.8}$$

We therefore use an iterative procedure. Starting with some suitable approximation for τ_s, we obtain a first approximation for $\hat{\varrho}$ using (6.8). This value of $\hat{\varrho}$ is used to obtain improved values of $\hat{\tau}_s$, which, in turn, when used in (6.8) provide a second better approximation for $\hat{\varrho}$. This iterative procedure is continued till one gets stable values for $\hat{\tau}_s$ and $\hat{\varrho}$.

In actual computation, we do not work with the canonical variables, but make use of the result 2.3.7 in section 3. The iteration formula then is

$$\varrho^{(n)} = \frac{b(k-1)\left[\frac{1}{k}\sum_i B_i^2(\underline{\theta}^{(n)}) - \frac{G^2}{bk}\right]}{(b-1)\left[\sum_{i,u} y_{iu}^2 - 2\sum_j \theta_j^{(n)} T_j + r\sum_j [\theta_j^{(n)}]^2 - \frac{1}{k}\sum_i B_i^2(\underline{\theta}^{(n)})\right]} \tag{6.9}$$

where $\underline{\theta}^{(n)} = (\theta_1^{(n)}, \theta_2^{(n)}, \ldots, \theta_v^{(n)})$ is the n-th approximation for $\underline{\theta}$, obtained by solving the equations

$$\underline{\theta}(C + \frac{1}{\rho(n-1)} C_1) = \underline{Q} + \frac{1}{\rho(n-1)} \underline{Q}_1. \qquad (6.10)$$

As a first approximation for $\underline{\theta}$ we may take its intra-block estimate.

The asymptotic variance of $\hat{\rho}$ obtained from the information matrix is:

$$V(\hat{\rho}) \doteq \frac{2k}{b(k-1)} \rho^2 . \qquad (6.11)$$

The right side of (6.11) serves as a lower bound for the variance of any unbiased estimator of ρ.

2.7. Numerical example.

The following data reproduced from Davies (1960) relates to an experiment on four tyres. The tyres were built up each in three parts using three of the treatments, one for each part.

Table 7.1. Wear values of tyres.

Treatments	Tyres (blocks)				Total
	1	2	3	4	
A	238	196	254	-	688
B	238	213	-	312	763
C	279	-	334	421	1034
D	-	308	367	412	1087
Total	755	717	955	1145	3572

The parameters of this BIB design are given by $b=v=4$, $r=k=3$, $\lambda = 2$.

For the intra-block equations (2.6) in $\underline{\theta} = (\theta_A, \theta_B, \theta_C, \theta_D)$ a solution is given by $\underline{\theta}^* = (-45.375, -41.000, 30.875, 55.500)$

Analysis of variance is given in Table 7.2.

Table 7.2. Analysis of variance.

Source	d.f.	S.S.	S.S.	d.f.	Source
Blocks (unadjusted)	3	$SS_B^* = 39122.67$	$SS_B = 21037.75$	3	Blocks (adjusted)
Treatments (adjusted)	3	$SS_{tr} = 20729.08$	$SS_{tr}^* = 38814.00$	3	Treatments (unadjusted)
Error	$e_o=5$	$SS_E = 1750.92$	$SS_E = 1750.92$	5	Error
Total	11	$SS_T = 61602.67$	$SS_T = 61602.67$	11	Total

Customary estimates of σ_o^2 and σ_1^2 as defined by (2.9) are found to be

$$s_o^2 = \frac{SS_E}{e_o} = 350.187$$

and

$$s_1^2 = \frac{kSS_B - (v-k)s_o^2}{v(r-1)} = 7845.383$$

respectively. This gives

$$R = \frac{s_1^2}{s_o^2} = 22.404$$

as the value of the customary estimator of ρ.

Computational details of estimation of ρ, the ratio of variances and of the treatment contrasts by the method of maximum likelihood are given in Table 7.3.

The purpose here is only to illustrate the nature of computations. The number of blocks appears to be too small to recommend use of inter-block information.

Table 7.3. Computational lay-out for estimation by the method of maximum likelihood.

stage n	$\theta_A^{(n)}$	$\theta_B^{(n)}$	$\theta_C^{(n)}$	$\theta_D^{(n)}$	$B_1(\theta^{(n)})$	$B_2(\theta^{(n)})$	$B_3(\theta^{(n)})$	$B_4(\theta^{(n)})$	ϱ
1	- 45.375	- 41.000	30.875	55.500	810.500	747.875	914.000	1099.625	36.046
2	- 46.089	- 41.073	31.377	55.785	810.785	748.377	913.927	1098.911	35.751
3	- 46.095	- 41.073	31.381	55.787	810.787	748.381	913.927	1098.905	35.748
4	- 46.095	- 41.073	31.381	55.787	(we have obtained stable values $\hat{\underline{\theta}}$ and $\hat{\varrho}$.)				

$$\underline{\theta}^{(n)} = \begin{cases} \underline{\theta}^* & \text{for } n = 1 \\ \dfrac{3\underline{Q}_1 + \varrho^{(n-1)}(3\underline{Q})}{1 + 8\varrho^{(n-1)}} & \text{for } n \geq 2 \quad [\text{Ref.}(6.10)] \end{cases}$$

where $3\underline{Q} = (-363, -328, 247, 444)^{\dagger}$ and $3\underline{Q}_1 = (-252, -62, 176, 138)$

$$B_i(\underline{\theta}^{(n)}) = B_i - \sum_u \sum_j m_{jiu}\, \theta_j^{(n)}$$

$$\varrho^{(n)} = \frac{b(k-1)\left[\frac{1}{k}\sum_i B_i^2(\underline{\theta}^{(n)}) - \frac{G^2}{bk}\right]}{(b-1)\left[\sum_{i}\sum_{u} y_{iu}^2 - 2\sum_j \theta_j^{(n)} T_j + r\sum_j [\theta_j^{(n)}]^2 - \frac{1}{k}\sum_i B_i^2(\underline{\theta}^{(n)})\right]}$$

† The value of $\underline{Q}$ was computed earlier to obtain $\underline{\theta}^*$, the solutions of the intra-block equations.

Chapter III

UNBIASEDNESS AND VARIANCE OF COMBINED ESTIMATORS

3.1. Introduction and summary.

When in the weighted least squares equations (2.2.7) for estimation one substitutes for the variance-ratio ρ an estimate for it, one has to examine afresh whether all 'good properties' of the least squares method still hold or not. Conceivably these properties would depend on the kind of estimator of ρ used. In this chapter, we shall examine this point in some detail.

First, one would like to examine if the combined estimator of a treatment contrast is unbiased. This problem seems to have been first examined by Graybill and Weeks (1959) for special designs. They showed that for BIB designs Yates' estimator of ρ in its untruncated form gives rise to unbiased combined estimators of treatment contrasts. Later, Graybill and Seshadri (1960) proved the same result for Yates' estimator in the usual truncated form, again for BIB designs. In this chapter, we obtain in section 2, a very strong generalisation of this result applicable to all incomplete block designs and for a whole class of estimators of ρ. It is shown that for any incomplete block design, if the estimator of ρ is the ratio of quadratic forms of a special type, the corresponding combined estimators of treatment contrasts are unbiased. It is also shown that the customary estimator of ρ (as

---

(2.2.7) refers to equation (2.7) of chapter two. Similar notation will be adopted throughout the thesis.

given by Yates (1939) and Rao (1947)) belongs to the above class and hence gives rise to unbiased combined estimators.

Another point worth examination is whether the combined estimator of a treatment contrast has variance smaller than that of the intra-block estimator. In section 3, we derive an expression for the variance of the combined estimator of a treatment contrast based on any estimator of ρ belonging to the class considered in section 2. Though in general, this expression is not easy to evaluate in terms of well-known functions, numerical quadrature methods can be applied for evaluation. A comparison with the variance of the intra-block estimator is made in section 4 and it is shown that the combined estimator of any contrast in any incomplete block design has variance smaller than that of the intra-block estimator if $\rho \leq 2$.

It appears however that for large ρ the combined estimator may have a larger variance for some designs. This raises the problem of searching for a combined estimator with variance uniformly smaller than that of the intra-block estimator. Graybill and Deal (1959) have obtained such combined estimators for BIB designs satisfying (1) $b - v \geq 10$ or (2) $b - v = 9$, $e_0 \geq 18$. The estimator of ρ used by them belongs to the class considered in section 2 but is different from the customary one. A much stronger result in this direction is obtained here. Methods presented in section 5 are applicable to any incomplete block design for which the association matrix has a non-zero latent root of multiplicity $p > 2$. Using an estimator of ρ constructed as in section 5, a neat expression is obtained for the variance of the combined estimator of a treatment contrast if it belongs to a subspace associated with the multiple latent root of the association matrix. It is shown that this is uniformly smaller than the variance of the corresponding intra-block estimator provided that $(p - 4)(e_o - 2) \geq 8$. It is also shown that when $p > 2$ but $(p - 4)(e_o - 2) < 8$, for large values of ρ the variance of the combined estimator exceeds that of the intra-block estimator.

For almost all well-known designs, the association matrix has multiple latent roots. Applications of the methods of section 5 to such designs are dealt with in the next chapter.

3.2. Unbiased combined estimators.

We saw in the previous chapter (result 2.3.8) that when ρ is given, the unbiased minimum variance estimators of treatment effects are obtained from the equations $T_s = \bar{t}_s(\rho)$, $s = 1,2,\ldots, v-1$, where the right-hand side is given by (2.3.9). In this section we shall show that when ρ is replaced by an estimate ρ^* of a certain type, the combined estimators are unbiased for their respective parameter values. For typographical simplicity, we shall write

$$\bar{t}_s = \bar{t}_s(\varrho), \qquad \bar{t}^*_s = \bar{t}_s(\varrho^*).$$

Let
$$w_s = \frac{(\varrho^* - \varrho)\, z_s}{1 + \varrho^* c_s}, \quad s = 1,2,\ldots, q. \qquad (2.1)$$

It is easy to verify that

$$\bar{t}^*_s = \bar{t}_s + \frac{c_s}{a_{os}\,(1 + \varrho\, c_s)}\, w_s\,. \qquad (2.2)$$

Since $\bar{t}_s$ is unbiased for τ_s, unbiasedness of $\bar{t}^*_s$ is equivalent to w_s having expectation identically zero. Let P be any statistic of the form (2.5.1) and let ϱ^* be defined as

$$\varrho^* = \begin{cases} P & \text{if } P \geq 1 \\ 1 & \text{otherwise.} \end{cases} \qquad (2.3)$$

It can be easily seen that ϱ^* is an even function of z_{os} $(s = 1,2,\ldots, e_o)$, z_{1s} $(s = 1,2,\ldots, e_1)$ and z_s $(s = 1,2,\ldots, q)$ and consequently, w_s is an odd function of these variables. Since the z's are mutually independent random variables each having a normal distribution with mean zero it follows that $E(w_s) = 0$ and consequently, $E(\bar{t}^*_s) = \tau_s$. This is only an extension of Graybill and Weeks (1959) argument for BIB designs to the general case of incomplete block design. The class of estimators of ϱ defined by (2.3) appears to be very wide. It seems to include all truncated estimators of ϱ considered in the literature as also all truncated estimators of ϱ considered in the previous chapter. In particular, it includes the customary estimator

R' defined by (2.2.11).

3.3. Variance of combined estimators.

When an estimator of ρ of the form (2.3) is used, we have shown that $E(w_s) = 0$. It can be easily checked that in this case, $E(w_s^2)$ is finite and hence w_s is a zero function. Also, when ρ is given, $\bar{t}_s$ is the unbiased minimum variance estimator of τ_s. By Stein's theorem (1950) $\bar{t}_s$ and w_s are uncorrelated and we have

$$V(\bar{t}_s^*) = V(\bar{t}_s) + \frac{c_s^2}{a_{os}^2(1 + \rho\, c_s)^2} V(w_s). \qquad (3.1)$$

In fact, this holds for all estimators ρ^* satisfying the conditions

$$E(w_s) = 0, \quad V(w_s) < \infty . \qquad (3.2)$$

The second term in (3.1) may be called the additional variance due to the sampling fluctuation in ρ^*.

An argument similar to the one employed in the previous section gives

$$E(w_s\, w_{s'}) = 0$$

for $s \neq s' = 1, 2, \ldots, q$. Since $\bar{t}_s$ and $\bar{t}_{s'}$ are independent, it follows that $\bar{t}_s^*$ and $\bar{t}_{s'}^*$ are uncorrelated for $s \neq s' = 1,2,\ldots,v-1$.

Now, any treatment contrast τ can be expressed as $\tau = \sum_{s=1}^{q} \ell_s \tau_s$ where ℓ_s (s = 1,2,..., v-1) are some constants. The minimum variance unbiased estimator of τ when ρ is known is $\bar{t} = \sum_{s=1}^{q} \ell_s \bar{t}_s$. When ρ is not known, for a combined estimator of τ, we take $\bar{t}^* = \sum_{s=1}^{q} \ell_s \bar{t}^*_s$ by substituting a suitable estimator ρ^* for ρ. If ρ^* satisfies the conditions of (3.2), $\bar{t}^*$ is an unbiased estimator of τ and its variance is given by

$$V(\bar{t}^*) = V(\bar{t}) + \sum_{s=1}^{q} \frac{c_s^2 \ell_s^2 V(w_s)}{a_{os}^2 (1+\rho c_s)^2} . \qquad (3.3)$$

3.4. Comparison of variances of the intra-block estimator and the combined estimator.

It is easy to check that the variance of t_s defined in (2.3.8) can be expressed in the form

$$V(t_s) = V(\bar{t}_s) + \frac{c_s^2}{a_{os}^2 (1 + \rho c_s)^2} E\left(\frac{z_s^2}{c_s^2} \right). \qquad (4.1)$$

This exceeds $V(\bar{t}^*_s)$ given by (3.1) if

$$E \left\{ \frac{z_s^2}{c_s^2} - \frac{(\rho^* - \rho)^2 z_s^2}{(1 + \rho^* c_s)^2} \right\} > 0 . \qquad (4.2)$$

It is readily seen that the term in the parenthesis is positive if

$$\rho < 2 \rho^* + \frac{1}{c_s} . \qquad (4.3)$$

Since ρ^* defined by (2.3) is truncated from above at unity (4.2) is satisfied provided that

$$\rho < 2 + \frac{1}{c_s}.$$

In view of (3.3), for an arbitrary treatment contrast T , a similar argument leads to the following result.

For any treatment contrast T , the combined estimator $\bar{t}(\rho^*)$ will have smaller variance than the intra-block estimator t provided that $\rho \leq 2$ and ρ^* is of the form (2.3).

3.5. Construction of combined estimators with uniformly smaller variance.

The variance of a combined estimator of T_s is given by (3.1). In this section we shall construct a suitable estimator ρ^* and evaluate this variance in terms of incomplete Beta functions.

Suppose the association matrix NN' has a latent root ϕ of multiplicity p $(p > 2)$. Without loss of generality we may say that the positive latent roots of NN' are rk , $\phi_1, \phi_2, \ldots, \phi_p, \phi_{p+1}, \ldots, \phi_q$ where, $\phi_s = \phi$ for $s \leq p$ and $\phi_s \neq \phi$ for $s > p$. Denote the common value of $a_{01}, a_{02}, \ldots, a_{op}$ by $\bar{a}_o$, of $a_{11}, a_{12}, \ldots, a_{1p}$ by $\bar{a}_1$ and of $c_1, c_2, \ldots, c_p$ by $\bar{c}$. Also let

$$\sum_{s=1}^{p} a_s^2 = Z. \qquad (5.1)$$

We take ρ^* as defined in (2.3) where, to obtain P we shall put

$$a = o$$

$$b_s = \begin{cases} \dfrac{e_o}{\bar{c}\ p} & \text{for } s = 1, 2, \ldots, p \\ 0 & \text{for } s = p+1, \ldots, q \end{cases} \tag{5.2}$$

and $$d = -\frac{1}{\bar{c}}$$

in the defining equation (2.5.1). This gives

$$\rho^* = \begin{cases} \dfrac{e_o\ Z}{\bar{c}\ p\ S_o} - \dfrac{1}{\bar{c}} & \text{if } S_o \leq K\ Z \\ 1 & \text{otherwise} \end{cases} \tag{5.3}$$

where

$$K = e_o / p(1 + \bar{c}) \tag{5.4}$$

and $\bar{c}$ is the common value of $c_1, c_2, \ldots, c_p$.

It can be easily seen that this gives us

$$w_s = \begin{cases} \dfrac{z_s}{c}\left(1 - p(1 + \bar{c}\ \rho)S_o / e_o\ Z\right) & \text{if } S_o \leq KZ \\ \dfrac{z_s}{c}\left(1 - p(1 + \rho)\ K/e_o\right) & \text{otherwise} \end{cases} \tag{5.5}$$

for $s = 1, 2, \ldots, p$.

Evidently, $w_1, w_2, \ldots, w_p$ are identically distributed and hence $E(w_1^2) = E(w_2^2) = \ldots, = E(w_p^2) = E(\frac{1}{p} \sum_{s=1}^{p} w_s^2)$. Using (5.5) one gets

$$\frac{1}{p}\sum_{s=1}^{p} w_s^2 = \begin{cases} \dfrac{1}{p\bar{c}^2}\, Z - \dfrac{2(1+\bar{c}\varrho)}{e_o\,\bar{c}^2}\, S_o + \dfrac{p(1+\bar{c}\varrho)^2}{e_o^2\,\bar{c}^2} \cdot \dfrac{S_o^2}{Z} & \text{if } S_o \le KZ \\[2ex] \dfrac{1}{p\,\bar{c}^2}\, Z - \dfrac{2(1+\bar{c}\varrho)}{e_o\,\bar{c}^2}\, K Z + \dfrac{p(1+\bar{c}\varrho)^2}{e_o^2\,\bar{c}^2}\, K^2 Z & \text{otherwise.} \end{cases} \qquad (5.6)$$

We shall now use the following lemma.

Lemma 3.5.1.

Let S and Z be two independent random variables, $\frac{S}{\sigma_s^2}$ being a χ^2 with e_s d.f. and $\frac{Z}{\sigma_z^2}$ being a χ^2 with e_z d.f. Let $m \le \frac{e_z}{2} + 1$ be a positive number and let $K > 0$ be a given constant. Consider a function $F(S, Z, m, K)$ defined by

$$F(S, Z, m, K) = \begin{cases} S^m Z^{1-m} & \text{if } S \le K Z \\ K^m Z & \text{otherwise.} \end{cases} \qquad (5.7)$$

The expectation of $F(S, Z, m, K)$ is given by

$$E\left\{F(S, Z, m, K)\right\} = \sigma_z^2 K^m e_z I_x\left(\frac{e_z}{2} + 1, \frac{e_s}{2}\right)$$

$$+ \frac{(e_z + e_s)\sigma_z^2}{(\sigma_z^2/\sigma_s^2)^m} \cdot \frac{B\left(\frac{e_s}{2} + m, \frac{e_z}{2} - m+1\right)}{B\left(\frac{e_s}{2}, \frac{e_z}{2}\right)} I_{1-x}\left(\frac{e_s}{2} + m, \frac{e_z}{2} - m+1\right) \qquad \ldots (5.8)$$

where $x = \frac{\sigma_z^2}{\sigma_s^2 + K\sigma_z^2}$, $B(p, q)$ denotes the Beta function with arguments p and q and $I_x(p, q)$ denotes the corresponding incomplete Beta function.

Proof :

The joint distribution of S and Z is given by

$$\Lambda .\ \exp\left\{ -(S/2\sigma_s^2) - (Z/2\sigma_z^2)\right\} S^{(e_s/2)-1} Z^{(e_z/2)-1}\ dSdZ$$

where $\frac{1}{\Lambda} = \Gamma(e_s/2)\ \Gamma(e_z/2)(2\sigma_s^2)^{e_s/2}\ (2\sigma_z^2)^{e_z/2}$.

Consider a transformation from S, Z to U, V given by

$$\frac{S}{Z} = U, \quad \frac{S}{2\sigma_s^2} + \frac{Z}{2\sigma_z^2} = V.$$

The Jacobian is given by

$$\frac{\partial(S,\ Z)}{\partial(U,\ V)} = V\left(\frac{U}{2\sigma_s^2} + \frac{1}{2\sigma_z^2}\right)^{-2}.$$

Hence the joint distribution of U and V is given by

$$\Lambda .\ e^{-V}\ V^{(e_s+e_z)/2-1}\ dV\ U^{(e_s/2)-1}\left(\frac{U}{2\sigma_s^2} + \frac{1}{2\sigma_z^2}\right)^{(e_s+e_z)/2} dU.$$

It is easily seen that

$$F(S,\ Z,\ m,\ K) = \begin{cases} V\ U^m\left(\frac{U}{2\sigma_s^2} + \frac{1}{2\sigma_z^2}\right)^{-1} & \text{if } U \leq K \\ V\ K^m\left(\frac{U}{2\sigma_s^2} + \frac{1}{2\sigma_z^2}\right)^{-1} & \text{otherwise.} \end{cases}$$

Hence we have

$$E\left\{F(S,Z,m,K)\right\} = \Lambda\left\{\int_{V=0}^{\infty} e^{-V}\ V^{(e_s+e_z)/2}\ dV\right\}$$

$$\left\{\int_{U=0}^{K} \frac{U^{(e_s/2)+m-1}\ dU}{\left(\frac{U}{2\sigma_s^2} + \frac{1}{2\sigma_z^2}\right)^{(e_s+e_z+2)/2}} + \int_{U=K}^{\infty} \frac{K^m\ U^{(e_s/2)-1}\ dU}{\left(\frac{U}{2\sigma_s^2} + \frac{1}{2\sigma_z^2}\right)^{(e_s+e_z+2)/2}}\right\}$$

The integral in V is $\Gamma\left(\frac{e_s + e_z + 2}{2}\right)$. Using the substitution $f = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_z^2 U}$ and integrating w.r.t f the result is readily obtained.

In the lemma, we set $S = S_o$, $Z = Z$, $\sigma_s^2 = \sigma_o^2$, $\sigma_z^2 = \sigma_o^2(1+\bar{c}\varrho)$, $e_s = e_o$ and $e_z = p$. Using (5.6) and (5.7) we get

$$\frac{1}{p}\sum_{s=1}^{p} w_s^2 = \frac{1}{p\bar{c}^2} F(S_o, Z, 0, K) - \frac{2(1+\bar{c}\varrho)}{e_o\ \bar{c}^2} F(S_o, Z, 1, K) + \frac{p(1+\bar{c}\varrho)^2}{e_o^2\ \bar{c}^2} F(S_o, Z, 2, K). \qquad \ldots \qquad \ldots (5.9)$$

An application of lemma 3.5.1, gives us

$$E\left(\frac{1}{p}\sum_{s=1}^{p} w_s^2\right) = \frac{(1+\bar{c}\varrho)\sigma_o^2}{\bar{c}^2}\left\{1 + X(X-2)I_x\left(\frac{p+2}{2}, \frac{e_o}{2}\right) - 2\,I_{1-x}\left(\frac{e_o+2}{2}, \frac{p}{2}\right) + \frac{p(e_o+2)}{e_o(p-2)} I_{1-x}\left(\frac{e_o+4}{2}, \frac{p-2}{2}\right)\right\} \quad \ldots (5.10)$$

where $X = \frac{1+\bar{c}\varrho}{1+\bar{c}}$ and

$$x = \frac{p}{p + e_o X}.$$

The following expression for variance follows from (3.1) and (5.10):

$$V(\bar{t}_s(\varrho^*)) = \frac{\sigma_o^2}{\bar{a}_o^2(1+\bar{c}\varrho)}\left\{1 + \bar{c}\varrho + X(X-2)I_x\left(\frac{p+2}{2}, \frac{e_o}{2}\right) - 2\,I_{1-x}\left(\frac{e_o+2}{2}, \frac{p}{2}\right) + \frac{p(e_o+2)}{e_o(p-2)} I_{1-x}\left(\frac{e_o+4}{2}, \frac{p-2}{2}\right)\right\} \quad \ldots (5.11)$$

for $s = 1, 2, \ldots, p$.

We note that as $\rho \to \infty$, $X \to \infty$ and $x \to 0$. It is easy to prove that

$$\lim_{X \to 0} V(\bar{t}_s(\rho^*)) = \sigma_o^2 / \bar{a}^2 \qquad \text{for } s = 1, 2, \ldots, p. \quad (5.12)$$

It can also be shown that for $s = 1, 2, \ldots, p$,

$$\frac{dV(\bar{t}_s(\rho^*))}{dx} = - \frac{\sigma_o^2}{\bar{a}_o^2(1+\bar{c})(1-x)^2} \left\{ \frac{p(1-x)^2}{e_o x^2} I_x \left(\frac{p+2}{2}, \frac{e_o}{2}\right) + \frac{2 e_o}{p} I_{1-x}\left(\frac{e_o+2}{2}, \frac{p}{2}\right) - \frac{e_o+2}{p-2} I_{1-x}\left(\frac{e_o+4}{2}, \frac{p-2}{2}\right) \right\} \quad \ldots (5.13)$$

We note that this is always negative if $2 e_o(p-2) \geq p(e_o + 2)$ or equivalently if $(p-4)(e_o - 2) \geq 8$. On the other hand if $(p-4)(e_o-2) < 8$, the derivative is positive for sufficiently small values of x. An examination of (5.12) and (5.13) leads to the following.

(1) If $(p-4)(e_o-2) \geq 8$, for $s = 1,2,\ldots, p$, $V(\bar{t}_s(\rho^*)) < V(t_s)$ uniformly in ρ. This is a consequence of the fact that $V(\bar{t}_s(\rho^*))$ increases with ρ and reaches the limit $V(t_s) = \sigma_o^2/\bar{a}_o^2$ as $\rho \to \infty$.

(2) If $p > 2$ and $(p-4)(e_o-2) < 8$, for $s=1,2,\ldots, p$, $V(\bar{t}_s(\rho^*)) > V(t_s)$ when ρ is sufficiently large; i.e. the combined estimator considered does not have uniformly smaller variance as compared with the intra-block estimator. This follows from the fact that for sufficiently large values of ρ, $V(t_s(\rho^*))$ decreases as ρ increases. Thus the limit as $\rho \to \infty$ (which coincides with the variance of the intra-block

estimator) is reached from above. We have already seen in the previous section that for values of ϱ not exceeding 2, the combined estimator has smaller variance.

In view of (3.3), it is clear that the above analysis holds for any treatment contrast $\boldsymbol{\tau}$ which is of the form $\boldsymbol{\tau} = \sum_{s=1}^{p} l_s \boldsymbol{\tau}_s$ where $l_1, l_2, \ldots l_p$ are some constants. It is clear that such treatment contrasts form a vector space, call it V, with the following properties:

i) rank $V = p$,

ii) variance of the intra-block estimator of any normalised contrast in V is the same,

iii) for any pair of mutually orthogonal treatment contrasts of which at least one belongs to V, their intra-block estimators are uncorrelated.

We thus have the following theorem:

Theorem 3.5.1.

Consider an incomplete block design for which the association matrix has a non-zero latent root (other than rk) of multiplicity p, let $\underline{l}$ be a latent vector associated with this root. Let ϱ^* be the estimator of ϱ constructed as in (5.3) based on this latent root. Let $\boldsymbol{\tau} = \underline{l}\,\underline{\varrho}'$, t its intra-block estimator, and $\bar{t}(\varrho^*)$ the combined estimator using ϱ^*. Then

$$V(\bar{t}\ (\rho^*)) < V(t) \quad \text{for all values of} \quad \rho \tag{5.14}$$

provided that

$$(p - 4)\ (e_o - 2) \geq 0 . \qquad \ldots \tag{5.15}$$

Further, if

$$p > 2 \quad \text{and} \quad (p - 4)\ (e_o - 2) < 0 \tag{5.16}$$

$$V(\bar{t}\ (\rho^*)) > V(t) \quad \text{for sufficiently large value of } \rho. \tag{5.17}$$

Chapter IV

UNIFORMLY BETTER COMBINED ESTIMATORS FOR STANDARD DESIGNS

4.1. Summary.

A general procedure for constructing a combined estimator of a treatment contrast with variance uniformly smaller than that of the intra-block estimator was developed in Chapter 3. In this chapter we shall discuss applications of this to some well-known designs.

Throughout this chapter, a combined estimator of a treatment contrast will be said to be 'uniformly better' if its variance is smaller than that of the intra-block estimator for all values of ρ. Further, any statement relating to combined estimators will apply only to those treatment contrasts on which inter-block information is available. The symbol e_o will always indicate the number of degrees of freedom for intra-block error.

A class of designs for which the association matrix has only one non-zero latent root (other than rk) is considered in section 2. The multiplicity of this root is denoted by q. For a design in this class satisfying $(q-4)(e_o-2) \geq 8$, uniformly better combined estimator is constructed for any treatment contrast. A computational procedure for obtaining ρ^* (an estimator of the variance-ratio ρ), to be used in building up the combined estimators is suggested for a design belonging to this class.

The BIB and the linked block (LB) designs belong to this class and are dealt with in section 3. Uniformly better combined estimators are obtained for all treatment contrasts in a BIB design with more than five treatments and for all treatment contrasts in a LB design with more than five blocks. It is shown that for a LB design the estimator of ρ used here coincides with the customary one. It is shown that for LB design with 4 or 5 blocks if ρ is large, the variance of the traditional combined estimator is actually larger than that of the intra-block estimator.

Applications to two-associate PBIB designs are given in section 4. A necessary and sufficient condition for the association matrix to have exactly one non-zero root is obtained in terms of the parameters of this type of design. A simple expression is obtained for this latent root and also for its multiplicity. It is shown that uniformly better combined estimator can be constructed for <u>any</u> treatment contrast in the following special cases: (1) Singular group divisible (GD) designs with $(m-5)(e_0-2) \geq 8$ and semi-regular GD designs with $(mn-m-4)(e_0-2) \geq 8$ where m denotes the number of groups and n the number of treatments in a group in a GD design (2). Simple lattice designs with sixteen treatments or more (3) Triple lattice designs with nine treatments or more.

Applications to designs for which the association matrix has two or more distinct non-zero latent roots are given in section 5.

In a inter and intra-group balanced design, uniformly better combined estimator is obtained for any intra-group treatment contrast provided that p, the number of treatments in that group satisfies $(p-5)(e_o-2) \geq 8$. In a regular GD design with $(mn-m-4)(e_o-2) \geq 8$, uniformly better combined estimators are obtained for intra-group treatment contrasts.

4.2. Construction of uniformly better combined estimator for any treatment contrast in a certain class of designs.

In what follows, we shall denote by D_1, the class of incomplete block designs for which the association matrix has only one non-zero latent root (other than rk). We shall use theorem 3.5.1 to construct a uniformly better combined estimator for <u>any</u> treatment contrast for any incomplete block design belonging to the class D_1.

For any design belonging to the class D_1, we shall denote by ϕ, the non-zero latent root of NN' (other than rk). Thus the multiplicity of ϕ is given by $q = (\text{rank NN}')-1$. As before, let $\tau_1, \tau_2, \ldots, \tau_q$ denote the canonical contrasts corresponding to ϕ. If $(q-4)(e_o-2) \geq 8$, we can apply theorem 3.5.1 to obtain a uniformly better combined estimator for a treatment contrast τ of the form $\tau = \sum_{s=1}^{q} l_s \tau_s$. As is evident from (2.3.9) for $\tau_{q+1}, \tau_{q+2}, \ldots, \tau_{v-1}$, the (v-1-q) canonical contrasts corresponding to the zero root of NN', no inter-block information is available. Hence, $\bar{t}_s(\rho^*) \equiv t_s$ for s= q+1, q+2,..., v-1. Now any treatment contrast τ is a linear combination of $\tau_1, \tau_2, \ldots, \tau_{v-1}$ and for ρ^* defined by (3.5.3), $\bar{t}_s(\rho^*)$ and $\bar{t}_{s'}(\rho^*)$ are uncorrelated for $s \neq s'$. It follows from theorem 3.5.1 that for any treatment contrast τ which admits of inter-block information, $V(\bar{t}(\rho^*)) < V(t)$ for all values of ρ.

To compute ρ^* defined by (3.5.3), we note that with the help of equations (2.3.11) and (2.4.12), $Z = \sum_{s=1}^{q} z_s^2$ may be expressed in the form

$$Z = \bar{c}(1 + \bar{c}) \left\{ \left(\frac{\underline{T}\,\underline{T}'}{r} - \frac{G^2}{bk}\right) - (2\,\underline{T} - r\,\underline{\theta}^*)\,\underline{\theta}^{*\prime} \right\} \qquad (2.1)$$

where as before $\bar{c} = \frac{rk}{\phi} - 1$. Thus, ρ^* may be written down as

$$\rho^* = \begin{cases} \dfrac{\phi}{rk-\phi}\left\{\dfrac{Z}{q\,s_o^2} - 1\right\} & \text{if } \dfrac{Z}{s_o^4} > \dfrac{rkq}{\phi} \\ 1 & \text{otherwise} \end{cases} \qquad (2.2)$$

where s_o^2 denotes the intra-block error mean square.

For a design in the class D_1, ϕ and q may be evaluated in the following manner. Since the association matrix is symmetric, the sum of the latent roots is equal to the sum of the diagonal elements, and the sum of squares of the latent roots is equal to the sum of squares of all the elements. Since only non-zero latent roots are rk and ϕ with multiplicities 1 and q respectively, we have

$$q\,\phi = r(v - k) \qquad (2.3)$$

and $q\rho^2 = v r^2 + \sum_{j \neq j'} \lambda_{jj'}^2 - r^2 k^2$ (2.4)

where $\lambda_{jj'}$ denotes the number of blocks containing both the treatments j and j'.

We thus have the following theorem.

Theorem 4.2.1.

Consider an incomplete block design belonging to the class D_1. When ρ^* as given by (2.2) is used, for <u>any</u> treatment contrast T

$$V(\bar{t}(\rho^*)) < V(t) \quad \text{for all values of } \rho \tag{2.5}$$

provided that

$$(q - 4)(e_o - 2) \geq 8 \tag{2.6}$$

where q is obtained from equations (2.3) and (2.4). Further, if $q > 2$ and $(q-4)(e_o-2) < 8$, (2.5) does not hold for all values of ρ.

Applications of this theorem to some well-known designs are given in the next two sections.

4.3. Applications to BIB designs and to LB designs.

It is implied in Bose (1949) that for a BIB design the association matrix is of full rank and has only one latent root other than rk. Thus any BIB design belongs to the class D_1. Since $q = v-1$, it follows that inter-block informatiin is available for all treatment contrasts and hence when (2.6) holds we get uniformly better combined estimators for all treatment contrasts.

For all BIB designs with more than 5 treatments, the condition (2.6) holds. It may be noted that the estimator of ρ given by (2.2) differs from the customary one proposed by Yates (1940).

Linked block (LB) designs were obtained by Youden (1951) by dualising the BIB designs. It is shown by Roy and Laha (1956) that for a LB design the association matrix has a non-zero latent root (other than rk) of multiplicity (b-1). Since rank NN' $\leq$ b, all other latent roots must be zero. Thus a LB design belongs to the class D_1.

Condition (2.6), which in this case amounts to $(b-5)(e_o-2) \geq 8$, holds for all LB designs with $b \geq 6$.

It is readily checked that R, the untruncated form of the customary estimator can be expressed as

$$R = \frac{e_o k \left\{ S_1 + \sum_{s=1}^{q} \phi_s z_s^2 / rk \right\}}{v(r-1) S_o} - \frac{v-k}{v(r-1)} . \tag{3.1}$$

For a LB design $e_1 = b-1-q = 0$ and $\phi_1 = \phi_2 = \ldots = \phi_q = \frac{r(v-k)}{b-1}$. Consequently $\bar{c} = (rk - \phi)/\phi = \frac{v(r-1)}{v-k}$. Substituting these in (3.1) it is readily seen that the customary estimator of ρ coincides with the one given by (2.2). It is also easily seen that linked blocks are the only designs for which these two estimators coincide.

It follows from theorem 4.2.1 that for a LB design the traditional combined estimators are uniformly better than the intra-block estimators if $b \geq 6$, but not so if $b = 4$ or 5.

4.4. Applications to PBIB designs with two associate classes.

In this section we shall search for PBIB designs with two associate classes belonging to the class D_1. We shall adopt the standard definition and notation for these designs as given in Bose and Connor (1952).

Connor and Clatworthy (1954) have shown that the association matrix of a PBIB design with two associate classes has exactly two distinct latent roots other than rk. From the values of these roots given there it can be deduced that a necessary and sufficient condition for one of the roots to be zero (i.e. for the design to belong to the class D_1) is:

$$\frac{(r-\lambda_1)(r-\lambda_2)}{(\lambda_1 - \lambda_2)} = p_{12}^1 (r-\lambda_2) - p_{12}^2(r-\lambda_1). \qquad (4.1)$$

Evidently if $b < v$, rank $NN' < v$. Consequently zero is a latent root.

For any two-associate PBIB design in D_1, ϕ and q obtained from (2.3) and (2.4) turn out to be

$$\begin{aligned} \phi &= \left\{ v(r^2 + n_1\lambda_1^2 + n_2\lambda_2^2) - r^2k^2 \right\} / r(v-k), \\ q &= r^2(v-k)^2 / \left\{ v(r^2 + n_1\lambda_1^2 + n_2\lambda_2^2) - r^2k^2 \right\} \end{aligned} \qquad (4.2)$$

Two-associate PBIB designs have been classified by Bose, Clatworthy and Shrikhande (1954) as (1) Group Divisible: (a) Singular

(b) Semi-regular, (c) Regular, (2) Triangular, (3) Latin Square Type, (4) Simple and (5) Cyclic.

(1) For a Group Divisible (GD) design Bose and Connor (1952) have shown that the two distinct latent roots of NN' are $(r-\lambda_1)$ and $(rk-v\lambda_2)$ with multiplicities $m(n-1)$ and $m-1$ respectively. Thus a GD design belongs to the class D_1 if either $r-\lambda_1 = 0$ i.e., if the design is singular or if $rk-v\lambda_2 = 0$ i.e., if the design is semi-regular. For a regular GD design $r > \lambda_1$ and $rk > v\lambda_2$ and hence no regular GD belongs to D_1. In a singular GD design uniformly better combined estimators are obtained if $(m-5)(e_o-2) \geq 8$. A corresponding condition in the case of a semi-regular GD is $(m(n-1)-4)(e_o-2) \geq 8$. For the next two types, use of (4.1) gives the following conditions on the parameters which ensure that they belong to the class D_1. In each case to apply theorem 4.2.1, condition (2.6) may be verified with the help of q given by equations (4.2).

(2) In a Triangular Design defined by Bose and Shimamoto (1952) $v = \frac{1}{2} n(n-1)$, $n_1 = 2(n-2)$, $n_2 = \frac{1}{2}(n-2)(n-3)$, $p^1_{12} = (n-2)$. A necessary and sufficient condition for a Triangular Design to belong to the class D_1 is that $r = 2\lambda_1-\lambda_2$ or $(n-3)\lambda_2 - (n-4)\lambda_1$.

(3) For a Latin Square Type design with i constraints $v = n^2$, $n_1 = i(n-1)$, $n_2 = (n-1)(n-i+1)$ and $p^1_{11} = i(i-3) + n$. A Latin Square Type Design belongs to the class D_1 if and only if $r = (i-n)(\lambda_1-\lambda_2)+\lambda_2$ or $i(\lambda_1 - \lambda_2) + \lambda_2$. In particular, the simple lattice is a two-

associate PBIB design of the Latin Square Type with $i = 2$. Since $\lambda_1 = 1$ and $\lambda_2 = 0$ the above condition is satisfied. Condition (2.6) holds for $n > 3$. Similarly the triple lattice is a two-associate PBIB design of the Latin Square Type with $i = 3$. Since $\lambda_1 = 1$, $\lambda_2 = 0$ the design again belongs to the class D_1. Condition (2.6) in this case is satisfied for $n > 2$.

4.5. Some other applications.

In this section we shall consider two applications of theorem 3.5.1 where uniformly better combined estimators will be constructed only for treatment contrasts of a certain type.

(A) Inter and intra group balanced (IIGB) designs: IIGB designs were first introduced by Nair and Rao (1942). An IIGB design with equal number of replications for all treatments may be defined as follows. In an incomplete block design let there be m groups of treatments, there being v_i treatments in the i-th group. Let each pair of treatments in the i-th group occur in λ_{ii} blocks and let each pair of treatments one of which belongs to the i-th group and the other to the j-th group occur in λ_{ij} blocks. Such a design is called an IIGB design.

The association matrix NN' is given by

$$NN' = \begin{bmatrix} (r-\lambda_{11})I_{v_1}+\lambda_{11}E_{v_1v_1} & \lambda_{12}E_{v_1v_2} & \cdots & \cdots \lambda_{1m}E_{v_1v_m} \\ \lambda_{12}E_{v_2v_1} & (r-\lambda_{22})I_{v_2}+\lambda_{22}E_{v_2v_2} & \cdots & \lambda_{2m}E_{v_2v_m} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{1m}E_{v_mv_1} & \lambda_{2m}E_{v_mv_2} & \cdots & (r-\lambda_{mm})I_{v_m}+\lambda_{mm}E_{v_mv_m} \end{bmatrix} \quad (5.1)$$

It is easily seen that the vector of co-efficients $\underset{\sim}{\ell}$ corresponding to any treatment contrast $T = \underset{\sim}{\ell}\,\underset{\sim}{\theta}'$ involving treatments from the ith group only is a latent vector of NN' corresponding to a latent root of multiplicity (v_i-1). By theorem 3.5.1, we can construct uniformly better combined estimator for any intra-group contrast involving treatments from the i-th group only provided that $(v_i-5)(e_o-2) \geq 8$ (we consider only those groups for which $r - \lambda_{ii} \neq 0$). An estimator of ρ as in (3.5.3) may be computed as follows. Let $\theta_1, \theta_2, \ldots, \theta_{v_i}$ denote the treatment effect parameters for the treatments in the i-th group. Let further $\theta_1^*, \ldots, \theta_{v_i}^*$ denote the solutions (corresponding to these treatments) of the intra-block equations and let $\theta_1', \theta_2', \ldots, \theta_{v_i}'$ the corresponding part of the solutions of the inter-block normal equations namely $\underset{\sim}{\theta}\, C_1 = \underset{\sim}{Q}_1$. We shall put $d_j = \theta_j^* - \theta_j'$. To obtain ρ^*, we substitute in (3.5.3)

$p = v_i - 1,$

$$Z = \frac{(rk - r + \lambda_{ii})}{k}\left\{\sum_{j=1}^{v_i} d_j^2 - \frac{(\sum d_j)^2}{v_i}\right\}$$

$$\text{and} \quad \bar{c} = \frac{rk - r + \lambda_{ii}}{r - \lambda_{ii}} . \tag{5.2}$$

(B) Regular GD designs : Bose and Connor have shown that a GD design is a special case of an IIGB design where, $\lambda_{ii} = \lambda_1$, $v_i = n$ and $\lambda_{ij} = \lambda_2$ for all i, j = 1,2,..., m $(i \neq j)$. The association matrix of a GD design is obtained by substituting these in the right hand side of (5.1).

It is easy to check that the vector of co-efficients $\underset{\sim}{\ell}$ corresponding to any treatment contrast $T = \underset{\sim}{\ell}\, \underset{\sim}{\theta}'$ involving treatments all from the same group is a latent vector of NN' corresponding to the root $(r - \lambda_1)$. Thus the vector space of treatment contrasts associated with $(r - \lambda_1)$ consists of all intra-group contrasts and this has rank $m(n-1)$. In this section we consider only regular GD designs so that $r - \lambda_1 \neq 0$.

If $(m(n-1) - 4)(e_o - 2) \geq 8$, by theorem 3.5.1, we can construct uniformly better combined estimator for any intra-group treatment contrast.

The following computational procedure may be adopted to obtain the estimate of ϱ as defined by (3.5.3). Let θ_{ij} denote the treatment effect parameter for the j-th treatment in the i-th group Let further θ^*_{ij} and θ'_{ij} ($i = 1,2,\ldots, m$; $j = 1,2,\ldots, n$) denote the respective solutions of the intra-block and the inter-block normal equations. Put $d_{ij} = \theta^*_{ij} - \theta'_{ij}$. The estimate ϱ^* is obtained by substituting in (3.5.3)

$$p = m(n - 1)$$

$$Z = \frac{rk - r + \lambda_1}{k} \left[\sum_i \sum_j d_{ij}^2 - \frac{\sum_i \left(\sum_j d_{ij} \right)^2}{n} \right], \quad (5.3)$$

and $$\bar{c} = \frac{rk - r + \lambda_1}{r - \lambda_1}.$$

When $(m - 5)(e_o - 4) \geq 8$, a similar procedure can be adopted to obtain uniformly better combined estimators of inter-group contrasts.

Chapter V

EFFICIENCY OF AN INCOMPLETE BLOCK DESIGN

5.1. Introduction and summary.

Yates (1936) introduced the concept of efficiency of an incomplete block design. His measure of efficiency is based on the average variance of intra-block estimators of paired treatment comparisons.

The following aspects of this efficiency measure are considered in this chapter. For a given design what is the improvement in efficiency when inter-block information is also utilised? In ordering designs from a given class (with the same number of treatments, the same block size etc.) how does this measure of efficiency compare with other optimality criteria given by Wald (1943)? Can one obtain other measures of efficiency which are simpler to compute, but result in similar orderings of a set of designs?

The efficiency factor of an incomplete block design, based on the variances of combined inter and intra-block estimators is defined in section 2. Computational procedure for evaluating this when the variance ratio ρ is estimated as in (4.2.2) is illustrated for a BIB design. An upper bound to this efficiency factor is obtained. Values of the efficiency factor with recovery are presented for some values of ρ for a few BIB designs and for the simple lattice design with sixteen treatments.

Kempthorne (1956) has shown that the efficiency measure of Yates is based on the harmonic mean of the positive latent roots of C, the matrix of co-efficients in the intra-block equations. Two other criteria by Wald (1943) are also given in terms of the latent roots of the matrix C, one being the minimum root and the other the geometric mean of the latent roots. Thus, with any of these measures the designs for which the latent roots of C are not widely separated, will have high efficiency. This suggests the use of dispersion of these roots in obtaining a measure of efficiency: this is our fourth criterion.

In section 3, the four criteria of efficiency based on the above considerations are made comparable by slight modification, so that one can compare designs with the same number of treatments and the same block size but using unequal amounts of experimental material. Values of the four efficiency criteria evaluated for ten two-associate PBIB designs from the list prepared by Bose, Clatworthy and Shrikhande (1954) happen to give similar orderings of these designs. The fourth criterion based on the dispersion of the latent roots of the C-matrix can be evaluated without finding the roots and is easy to compute. It is shown in section 4 that in the class $D_{v,k}$ of designs where v treatments are applied on blocks of k plots each, a BIB design (there is at least one in the class $D_{v,k}$) has maximum possible efficiency with any of the four criteria. For the first three criteria, this is an extension (to the class $D_{v,k}$) of the earlier results of Kshirsagar (1958), Roy (1958), and Kiefer (1958).

5.2. Improvement in efficiency due to recovery of inter-block information.

Let us denote by V the average of variances of the intra-block estimators of all paired contrasts of treatment effects (i.e. contrasts of the type $\theta_i - \theta_j$) so that

$$V = \frac{2\sigma_o^2}{rE} \qquad \ldots \qquad \ldots \qquad (2.1)$$

where E is the efficiency factor of the design, as introduced by Yates (1936). An expression for E was obtained by Kempthorne (1956) in terms of the latent roots of the association matrix of the design and is reproduced in our formula (2.4.14). From this, it is readily seen that $E < 1$ for any incomplete block design. In fact, the variance of the intra-block estimator of any normalised treatment contrast can not be less than σ_o^2/r.

In a similar manner, one can define the efficiency factor $\bar{E}$ of an incomplete block design, when inter-block information is also used in terms of the relationship.

$$\bar{V} = \frac{2\sigma_o^2}{r\bar{E}} \qquad \ldots \qquad \ldots \qquad (2.2)$$

where $\bar{V}$ denotes the average variance of all estimated paired contrasts (when inter-block information is used). To be more specific, if in the recovery process an estimator ϱ^* is used for ϱ, the efficiency factor with recovery will be denoted by $\bar{E}^*$. By $\bar{E}_o$ we shall denote the corresponding value when the combined estimators are based on the known values of ϱ.

One would intuitively expect the following inequality to hold for all designs and for all values of ϱ:

$$E < \bar{E}^* < \bar{E}_o \leq 1 . \quad \ldots \quad \ldots \quad (2.3)$$

It was shown in section 2.3 that when ϱ is known, the combined estimator of a treatment contrast has minimum variance among all unbiased estimators. Thus $\bar{E}_o$ serves as an upper bound to both E and $\bar{E}^*$. Using an approach similar to one adopted by Kempthorne (1956), we derive the following expression for $\bar{E}_o$:

$$\bar{E}_o = (v-1)/\left[\sum_{s=1}^{q} \left\{ 1 - \frac{\phi_s}{rk} \left(1 - \frac{1}{\varrho}\right) \right\}^{-1} + v-1-q \right] . \quad (2.4)$$

Since ϕ_s are all non-negative it follows that $\bar{E}_o \leq 1$ for all values of $\varrho \geq 1$. Thus (2.3) would hold if $E < \bar{E}^*$. It turns out however that for some designs $\bar{E}^*$ is actually smaller than E for large values of ϱ. This is shown in section 3.5 and will be illustrated later in this section by comparing actual values of E and $\bar{E}^*$ for a suitable design.

For the class of designs D_1 considered in section 4.2 and for the estimator of ϱ given there one can compute $\bar{E}^*$. To illustrate the nature of such computations we shall evaluate $\bar{E}^*$ for one BIB design, using values of incomplete Beta functions as tabulated by Karl Pearson (1934). For a BIB design, an expression for $\bar{E}^*$ obtained with the help of (3.5.11) and (2.2) turns out to be

$$\bar{E}^* = E(1 + \bar{c}\varrho)\left\{1+\bar{c}\varrho + X(X-2)\left[1 - I_{1-x}\left(\frac{e_o}{2}, \frac{v+1}{2}\right)\right] - 2\, I_{1-x}\left(\frac{e_o+2}{2}, \frac{v-1}{2}\right) + \frac{(v-1)(e_o+2)}{(v-3)e_o}\, I_{1-x}\left(\frac{e_o+4}{2}, \frac{v-3}{2}\right)\right\}^{-1} \quad (2.5)$$

where $\bar{c} = \frac{v(k-1)}{(v-k)}$, $X = \frac{1+\bar{c}\varrho}{1+\bar{c}}$, and $x = \frac{v-1}{(v-1)+e_oX}$.

It is also easily seen that for a BIB design, $\bar{E}_o = E + \frac{1}{\varrho}(1 - E)$. Details of computations are shown in Table 2.1.

Values of $\bar{E}^*$ and $\bar{E}_o$ for a few BIB designs and for the simple lattice design with 16 treatments are presented in Table 2.2 for ϱ = 1,2,4 and 8. For the simple lattice design we also give the corresponding values relating only to a set of mutually orthogonal normalised treatment contrasts on which inter-block information is available.

When $\varrho = 8$, for the first design in Table 2.2 ($b = v = 4$, $r=k=3$) $\bar{E}^* < E$ and for others $\bar{E}^* - E$ is very small. When $\varrho \leq 4$, gain in efficiency appears to be appreciable for designs with low values of E.

Table 2.1. Computational lay-out for evaluating $\bar{E}^*$ for a BIB design.

ϱ	$1+\bar{c}\varrho$	$E(1+\bar{c}\varrho)$	X	$X(X-2)$	$1-x$	$1-I_{1-x}(\frac{e_o}{2},\frac{v+1}{2})$	$I_{1-x}(\frac{e_o+2}{2},\frac{v+1}{2})$	$I_{1-x}(\frac{e_o+4}{2},\frac{v-3}{2})$	$\bar{E}^*$	$\bar{E}_o$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1	5	4.0	1.0	-1.00	0.75000	0.33835	0.38059	0.13259	0.9636	1.0000
2	9	7.2	1.8	-0.36	0.84375	0.11049	0.67969	0.34651	0.8722	0.9000
4	17	13.6	3.4	4.76	0.91071	0.02238	0.88569	0.61011	0.8249	0.8500
8	33	26.4	6.6	30.36	0.95192	0.00325	0.96931	0.81080	0.8075	0.8250

$b = 10,\ v = 6,\ k = 3,\ r = 5,\ \lambda = 2,$

$E = \dfrac{(1 - 1/k)}{(1 - 1/v)} = 0.80$, $e_o = bk - b - v+1 = 15,$ $\bar{c} = \dfrac{v(k-1)}{v-k} = 4$, $\dfrac{(v-1)(e_o+2)}{(v-3)e_o} = 1.88889$, $X = \dfrac{1+\bar{c}\varrho}{1+\bar{c}}$

$$\bar{E}^* = E(1+\bar{c}\varrho)\left\{1+\bar{c}\varrho + X(X-2)\left[1-I_{1-x}\left(\frac{e_o}{2},\frac{v+1}{2}\right)\right]-2\,I_{1-x}\left(\frac{e_o+2}{2},\frac{v-1}{2}\right)+\frac{(v-1)(e_o+2)}{(v-3)e_o}\,I_{1-x}\left(\frac{e_o+4}{2},\frac{v-3}{2}\right)\right\}^{-1}$$

$\bar{E}_o = E + \dfrac{1}{\varrho}(1 - E).$

Table 2.2. Values of $\bar{E}^*$ and $\bar{E}_o$ for some selected designs.

Design	ρ	$\bar{E}^*$	$\bar{E}_o$	E
BIB design with $b = v = 4$, $r = k = 3$ and $\lambda = 2$.	1	0.9691	1.0000	0.8889
	2	0.9225	0.9445	
	4	0.8958	0.9167	
	8	0.7156	0.9028	
BIB design with $b = 12$, $v = 9$, $k = 3$, $r = 4$ and $\lambda = 1$.	1	0.9641	1.0000	0.7500
	2	0.8351	0.8750	
	4	0.7929	0.8125	
	8	0.7642	0.7812	
BIB design with $b = 21$, $v = 15$, $k = 5$, $r = 7$ and $\lambda = 2$.	1	0.9901	1.0000	0.85714
	2	0.9177	0.9286	
	4	0.8857	0.8929	
	8	0.8714	0.8750	
Simple lattice design with 16 treatments (overall efficiency).	1	0.9396	1.0000	0.7143
	2	0.8398	0.8824	
	4	0.7732	0.8065	
	8	0.7327	0.7627	
Simple lattice design with 16 treatments (efficiency relating to a set of treatment contrasts on which inter-block information is available).	1	0.8616	1.0000	0.5000
	2	0.6771	0.7500	
	4	0.5770	0.6250	
	8	0.5231	0.5625	

5.3. Other criteria of efficiency.

Though Yates' measure of efficiency of a design defined by (2.1) was the first optimality criterion to be introduced, two other measures were proposed by Wald from the viewpoint of the power of the F-test for comparing treatment effects. In this section we shall examine these three criteria and propose a fourth new criterion which has got some operational advantages.

Let $\underline{\tau} = (\tau_1, \tau_2, \ldots, \tau_{v-1})$ denote the row-vector of the canonical treatment effects defined by (2.3.4) and let $\lambda_s = r - \phi_s/k$, $s = 1,2,\ldots, v-1$ denote the latent roots of the matrix C. It is known that (see for example Tang, 1938) for the customary F-test of the hypothesis $\underline{\tau} = \underline{0}$, the power is a monotonically increasing function of $\beta = \sum_{s=1}^{v-1} (\tau_s^2 \lambda_s)\sigma_0^2$. The three criteria may be described as follows.

(A) If we wish to minimise the average variance of all paired treatment contrasts, we should minimise $\sum \lambda_i^{-1}$ (Kempthorne, 1956, Kshirsagar, 1958, Roy 1958).

(B) Wald (1943) argues that it is not possible to maximise the power of the customary F-test for all values of $\underline{\tau}$. Hence we would minimise β subject to $\underline{\tau\tau}'/\sigma^2$ = const. This leads to maximising λ_{min} (Ehrenfeld 1955, Wald 1943).

(C) Wald (1943) further argues that from certain mathematical considerations it would be simpler to minimise $\prod_{s=1}^{v-1} \lambda_s^{-1}$. This minimises the generalised variance. Also, as Nandi (1951) has pointed out, this has the desirable effect of minimising the volume of equipower ellipsoid given by $\sum_{s=1}^{v-1} (\tau_s^2 \lambda_s)/\sigma^2$ = constant. It should also be noted that the design which minimises $\pi \lambda_s^{-1}$ gives certain optimum properties for the usual F test associated with it (Kiefer 1958).

Kempthorne (1956) and Kshirsagar (1958) considered only equi-replicate designs while Roy (1958) considered the more general case where the number of replications need not be the same for all treatments.

Since efficiency should relate to the manner of utilization of resources, in framing an efficiency criterion, it seems natural to take into account the amount of experimental material used. This would enable us to compare designs with different sizes. Hence, we consider the class of designs, $D_{v,k}$, for fixed values of v and k $(v > k)$, where v treatments are arranged in blocks of k plots each. Denote by r_i and $\bar{R}$, the number of replications for the i^{th} treatment and the average number of replications respectively. Further, let λ_s, $s = 1,2,\ldots, v-1$ denote the latent roots of C, the co-efficient matrix in the intra-block normal equations. The efficiency criteria, analogous to those in (A), (B) and (C) above would be

$$E_1=(v-1)/\bar{R} \sum \lambda_s^{-1},\ E_2 = \lambda_{min}/\bar{R},\ E_3 =(\pi \lambda_s)^{\frac{1}{v-1}}/\bar{R}. \qquad (3.1)$$

The above three criteria are based on different considerations and need not necessarily result in the same ordering of two given designs. Which criterion should be adopted depends upon our aim in conducting the experiment.

It should be noted that, for each of the three criteria, higher values are associated with lower dispersion in the λ's. In fact in the first and the third criteria, we are concerned with the harmonic and the geometric means subject to the arithmetic mean being constant. When the λ's are all equal, the three means coincide. This suggests the use of $\sum (\lambda_s - \bar{\lambda})^2/(v-1)$ with $\sum \lambda_s$ = constant as a criteria for optimality, i.e. among designs of given size, we should make $\sum \lambda_s^2$ as small as possible, subject to existence of a design. To eliminate the effect of the size of the design we note that $\sum \lambda_s$ = Trace C $=(k-1)v\bar{R}/k$ is linearly related to the total number of plots. Thus we define

$$E_4 = \bar{\lambda}^2/[\ \bar{R}(\ \sum \lambda_s^2\ /(v-1))^{\frac{1}{2}}] = (v-1)^{-\frac{3}{2}}(\ \sum \lambda_s)^2/[\bar{R}(\ \sum \lambda_s^2)^{\frac{1}{2}}]\ . \quad (3.2)$$

Though this criterion does not agree exactly with any of the three criteria given above, generally large (or small) values of this criterion will be associated with large (or small) values of the other three criteria. Though the contours of equal efficiency (in the space of λ's) are not identical with those of the other three criteria (which themselves are not identical), our criterion will be quite useful. For the points on the line given by $\lambda_1 = \lambda_2 = \dots = \lambda_{v-1}$ all give the same result and for the class of designs with higher efficiency, i.e. for λ's not too widely spread, they will not differ much. This is the region where our criterion will be quite effective. As shown below this criterion has the advantages of simplicity and practical usefulness. To compute E_4 we note that $\sum \lambda_s^2$ = Trace $C^2 = \sum_i \sum_j c_{ij}^2$. This gives us

$$E_4 = \frac{(k-1)^2 \, v^2 \, \bar{R}}{k^2 (v-1)^{3/2} \, \left(\sum_i \sum_j c_{ij}^2 \right)^{\frac{1}{2}}} . \tag{3.3}$$

A further simplification is obtained for PBIB and circulant designs, where $\sum_j c_{ij}^2$ is the same for all i.

For the other three criteria, elegant expressions are seldom available. Since E_4 follows directly from the C matrix we do not have to solve the normal equations or to evaluate the λ's.

To asess the comparability of the orderings (of the designs) obtained in accordance with these criteria, we consider the class of designs with $v = 9$ and $k = 3$ and take ten two-associate PBIB designs from the list prepared by Bose, Clatworthy and Shrikhande (1954). Values of E_1, E_2, E_3 and E_4 are obtained for each of these and the designs are ranked according to each of these criteria. The results are shown in Table 3.2.

Table 3.2. Values of E_1, E_2, E_3 and E_4 for ten PBIB designs.

Design No.	Design reference	E_1		E_2		E_3		E_4	
1	R 9	0.7385	(6)	0.6000	(8)	0.7445	(6)	0.7450	(5.5)
2	R 10	0.7143	(9)	0.5000	(9.5)	0.7334	(9)	0.7365	(8.5)
3	R 11	0.7453	(3)	0.7143	(1)	0.7476	(3)	0.7475	(3)
4	R 12	0.7467	(1.5)	0.6667	(5)	0.7484	(1)	0.7485	(1)
5	R 13	0.7412	(4)	0.7000	(2.5)	0.7454	(5)	0.7450	(5.5)
6	S R 12	0.7273	(8)	0.6667	(5)	0.7378	(8)	0.7365	(8.5)
7	L S 3	0.7407	(5)	0.6667	(5)	0.7455	(4)	0.7454	(4)
8	L S 4	0.7292	(7)	0.6250	(7)	0.7395	(7)	0.7398	(7)
9	L S 5	0.7467	(1.5)	0.7000	(2.5)	0.7483	(2)	0.7483	(2)
10	L S 6	0.6667	(10)	0.5000	(9.5)	0.7071	(10)	0.7115	(10)

For each criterion, figures in the brackets denote the ranks. It is seen that rankings in accordance with E_1, E_3 and E_4 do not differ appreciably.

5.4. Optimality of BIB designs.

It is easily seen that for a fixed $\bar{R}$, the theoretical maxima of E_1, E_2, E_3, E_4 are attained when $\lambda_1 = \lambda_2 = \dots = \lambda_{v-1}$. Since this maximising solution is independent of $\bar{R}$, it is also the unconditional maximising solution.

It is well known (Rao, 1958) that for any incomplete block design in which all blocks have the same number of plots, $\lambda_1 = \lambda_2 = \dots = \lambda_{v-1}$ if and only if the design is a balanced one. Now in the class of designs $D_{v,k}$ a BIB design always exists. Hence, judged by any of the four criteria, within the class of designs D_{vk} the BIB designs have maximum possible efficiency. It can be easily seen that for any BIB design in $D_{v,k}$, $E_1 = E_2 = E_3 = E_4 = (1 - \frac{1}{k})/(1 - \frac{1}{v})$. And as is to be expected, this maximum increases with k. In the limit when k = v, i.e., for randomised complete block designs, $E_1 = E_2 = E_3 = E_4 = 1$.

The above is an extension (to the class $D_{v,k}$) of the previous results of Kshirsagar (1958), Roy (1958) and Kiefer (1958) on the optimality of a BIB among the designs using the same number of plots. Roy proved this with E_1 as the criterion, Kshirsagar with both E_1 and E_3 while Kiefer proved this with each of the three criteria E_1, E_2 and E_3. Kshirsagar and Kiefer considered equi-replicate designs only.

Chapter VI

ANALYSIS OF TWO-WAY DESIGNS WITH ORTHOGONAL GROUPING

6.1. Introduction and summary.

In the previous chapters, we discussed situations where the experimental material is classified in one way into homogeneous groups called blocks, classification being made on the basis of some prior information about the nature of the experimental material. In some situations just one-way classification may not be adequate and one may with advantage use two, or even multi-way classification of the experimental units (eu's). Such designs have been considered by various authors (Latin squares by Fisher (1935), quasi-Latin squares by Yates (1937), Youden squares by Youden (1937), partially balanced Youden squares by Bose and Kishen (1939), Y_1 class of designs by Shrikhande (1951), Graeco-Latin squares by Dunlop (1933)).

In this chapter, we shall confine ourselves to designs where the eu's are classified in two ways such that there is exactly one unit which belongs to the ith class of one classification and the jth class of the other. For simplicity of description the first way of classification may be called rows, and the second columns. Shrikhande (1951) gave a general method of analysis of such designs using only the information available within rows and columns. This analysis is based on the so called fixed effects 'Normal' model and does not utilise the information available from differences of row or column totals.

The same analysis is obtained in section 3 of this chapter using only the assumption of additivity of plot and treatment effects and the distribution induced by the randomisation procedure. We use an orthogonal transformation from original observations to (i) a constant times the grand mean of all observations (ii) a set of mutually orthogonal normalised row-contrasts (iii) a set of mutually orthogonal normalised column-contrasts and (iv) a set of mutually orthogonal normalised interaction-contrasts. All the contrasts are mutually uncorrelated and contrasts belonging to the same set have the same variance but the contrasts belonging to different sets have different variances. To estimate treatment differences we can apply the method of least-squares to any of these sets of contrasts. If one uses the set of interaction-contrasts, the equations for estimation turn out to be the same as obtained by Shrikhande (1951) under the Normal model.

Since contrasts in different sets have different variances in order to use all the contrasts for estimating the treatment differences one may use the method of weighted least-squares. Equations for estimation are derived in section 4. As in the case of incomplete block designs the weights in the above equations are usually unknown and have to be estimated from the observations themselves.

If the weight for a (normalised) interaction-contrast is taken to be unity that for a row-contrast would be the ratio of the variance of an interaction contrast to that of a row-contrast. Estimate of the variance of a interaction-contrast is provided by the error mean square in the analysis of variance of interaction-contrasts. In section 4, we obtain two estimators of the variance of a row-contrast . One is based on the mean square of rows adjusted for columns and treatments. To obtain this one has to carry out a separate analysis of variance where the classification by rows is ignored. It may be noted that this procedures is an extension of the traditional procedure of Yates (1939) and Rao (1947) for estimating the inter-block variance in the case of one-way designs. However this involves rather heavy computations in the general case and hence the following procedure may be recommended. Consider the row totals of the corrected yields where from the yield of each plot the estimate of the treatment parameter (as given by the interaction analysis) is subtracted. The sum of squares of deviations of these row-totals from their mean may be used in estimating the variance of a row-contrast. This obviates the need for an additional analysis of variance to be performed.

Finally as an estimator of the ratio of variances we take the ratio of their respective estimators.

The weight for a column-contrast can be estimated in like manner.

Conditions under which a two-way design compares favourably with the corresponding one-way designs are examined is section 5 and the relative efficiency factors worked out. It is shown in section 6 that if the columns of the design, ignoring rows, form a BIB design, the analysis is much simpler. The special case where in addition the rows are partially balanced is discussed in full and a numerical example is worked out in section 8.

Some results which are used in sections 3 and 4 are derived in section 7.

6.2. Preliminaries.

Suppose there are mn plots or experimental units (eu's) on which a comparative trial involving v treatments is to be carried out. The eu's are arranged in a $m \times n$ two-way classification, so that each eu is determined by a pair of co-ordinates (t, u) $t = 1,2,\ldots, m$; $u = 1,2,\ldots, n$. With the (t, u)-th eu is associated a number x_{tu} to be called the plot effect and we assume that if the k-th treatment is applied on the (t, u)-th eu, the 'yield' would be $x_{tu} + \theta_k$ where the parameter θ_k is to be regarded as the effect of the k-th treatment; $k = 1,2,\ldots, v$. This is the so-called additive or no-interaction model. The purpose of the experiment is to compare the θ_k's.

We now define

$$\begin{aligned} \mu &= \sum \sum x_{tu} / mn, \text{ the general mean,} \\ \sigma_1^2 &= n \sum \alpha_t^2 / (m-1), \text{ the between-row variance,} \\ \sigma_2^2 &= m \sum \gamma_u^2 / (n-1), \text{ the between-column variance,} \\ \sigma_0^2 &= \sum \sum \eta_{tu}^2 / [(m-1)(n-1)] \text{ the interaction variance,} \end{aligned} \quad (2.1)$$

where $\alpha_t = \frac{1}{n} \sum_u x_{tu} - \mu$, $\gamma_u = \frac{1}{m} \sum_t x_{tu} - \mu$ and $\eta_{tu} = x_{tu} - \alpha_t - \gamma_u - \mu$.

We shall write

$$\rho_i = \sigma_i^2 / \sigma_o^2 \qquad i = 1,2, \qquad \ldots \quad (2.2)$$

for the ratios of the variances. It will be assumed that $\rho_i \geq 1$, for $i = 1,2$.

The treatments are allocated to the eu's in the following manner. First a two-way design, that is, an arrangement of the v treatments in m rows and n columns is taken. The design is thus completely characterized by the numbers ε_{ijk}, $i = 1,2,\ldots, m$; $j = 1,2,\ldots, n$; $k = 1,2,\ldots, v$ where $\varepsilon_{ijk} = 1$ if the k-th treatment occurs in the intersection of the i-th row and the j-th column of the design and $\varepsilon_{ijk} = 0$, otherwise. The k-th treatment thus occurs in $m_{ki} = \sum_{j=1}^{n} \varepsilon_{ijk}$ positions in the i-th row and in $n_{kj} = \sum_{i=1}^{m} \varepsilon_{ijk}$ positions in the j-th column. We shall restrict outselves to equi-replicate designs, that is to those designs, where each treatment occurs altogether in r positions. Thus $\sum_i m_{ki} = \sum_j n_{kj} = r$ and of course, $\sum_k m_{ki} = n$, $\sum_k n_{kj} = m$. We shall call $M = ((m_{ki}))$ and $N = ((n_{kj}))$ the row incidence-matrix and the column incidence-matrix respectively. The rows and the columns of the design are then allotted to the two ways of classification of the eu's independently and at random.

Let us denote the yield of the eu corresponding to the i-th row and the j-th column of the design by y_{ij}. The randomisation procedure ensures that

$$E(y_{ij}) = \mu + \sum_{k=1}^{v} \varepsilon_{ijk}\,\theta_k \qquad \ldots \quad (2.3)$$

and that

$$\mathrm{cov}(y_{ij}, y_{i'j'}) = (\delta_{ii'} - \frac{1}{m}) \frac{\sigma_1^2}{n} + (\delta_{jj'} - \frac{1}{n}) \frac{\sigma_2^2}{m} + (\delta_{ii'} - \frac{1}{m})(\delta_{jj'} - \frac{1}{n})\sigma_o^2 \quad (2\cdot4)$$

where $\delta_{ii'}$ is the Kronecker symbol, $\delta_{ii'} = 1(0)$ if $i = i'(i \neq i')$.

Since the y_{ij}'s are correlated, it is convenient to make a linear transformation and obtain uncorrelated random variables. For this purpose, we use the following definitions. A linear function of the form $l = \sum\sum l_{ij}\, y_{ij}$ is said to be a contrast if $\sum\sum l_{ij} = 0$. A contrast l is said to belong to rows, or simply called a row-contrast if $l_{i1} = l_{i2} = \dots = l_{in}$ holds for $i = 1,2,\dots, m$. Similarly, a constrast l is said to be a column-contrast if $l_{1j} = l_{2j} = \dots = l_{mj}$ holds for $j = 1,2,\dots, n$. A contrast l is said to belong to interaction or simply called an interaction-contrast if $\sum_i l_{ij} = 0$ for $j = 1,2,\dots, n$ and $\sum_j l_{ij} = 0$ for $i = 1,2,\dots, m$. A contrast l is said to be normalised if $\sum\sum l_{ij}^2 = 1$. Two contrasts l and $l' = \sum\sum l'_{ij}\, y_{ij}$ are said to be orthogonal if $\sum\sum l_{ij}\, l'_{ij} = 0$ holds.

If then we make a linear transformation from y_{ij}'s to (i) $G^* = G/\sqrt{mn}$, where $G = \sum\sum y_{ij}$ is the grand total, (ii) a set of (m-1) mutually orthogonal normalised row-contrasts (iii) a set (n-1) mutually orthogonal normalised column-contrasts, and (iv) a set of (m-1)(n-1) mutually orthogonal normalised interaction-contrasts, it can then be shown as in section 7 that the transformation is orthogonal and that these transformed variables are uncorrelated, the variance of any normalised row-contrast being σ_1^2, the variance of any normalised column-contrast being σ_2^2 and that of any normalised interaction-contrast being σ_o^2. Since the expectation of each contrast is a linear function of the θ_k's, the method of least squares can be used for purposes of estimation.

We shall write R_i for the total yield of the i-th row, C_j for that of the j-th column, and T_k for that of the k-th treatment, thus

$$R_i = \sum_{j=1}^{n} y_{ij},\ C_j = \sum_{i=1}^{m} y_{ij} \text{ and } T_k = \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_{ijk}\, y_{ij}.$$

We shall use the matrix notations: $\underline{R} = (R_1, R_2, \ldots, R_m)$, $\underline{C} = (C_1, C_2, \ldots, C_n)$, $\underline{T} = (T_1, T_2, \ldots, T_v)$ and $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_v)$. As before, a matrix of the form $p \times q$ with all elements unity will be denoted by E_{pq}.

If Λ is a positive semi-definite matrix of form $n \times n$ and rank b, it has b positive latent roots, say α_i, $i = 1, 2, \ldots, b$. Let x_i of the form $1 \times n$ be a latent vector of Λ corresponding to the latent root α_i, $i = 1, 2, \ldots, b$ such that $x_i x_i' = \delta_{ij}$. Then the matrix $\Lambda^* = \sum_{i=1}^{b} \frac{1}{\alpha_i} x_i' x_i$ will be called a pseudo-inverse of Λ, in the sense of Rao (1955).

6.3. Estimation of treatment effects from interaction contrasts.

Since row-contrasts, column-contrasts and interaction-contrasts have different variances, it is not convenient to use them simultaneously for estimation of treatment effects in an efficient way unless the relative magnitudes of these variances are known. We shall, therefore, consider first the problem of estimation from interaction-contrasts only. Recovery of information provided by row-contrasts and column-contrasts will be taken up in the next section.

As we have pointed out in section 2, any set of (m-1)x(n-1) mutually orthogonal normalised interaction-contrasts are mutually uncorrelated and each of them has the same variance σ_o^2. Also the expectation of each is a linear function of the θ_k's. Consequently, the method of least-squares can be used to derive linear unbiased estimators with minimum variance ('best' estimators) of linear functions of treatment effects. As will be shown later in section 7 the method of least-squares gives the equations:

$$\underline{\theta} K = \underline{Q} \quad \ldots \quad (3\cdot1)$$

where the elements of

$$\underline{Q} = \underline{T} - \frac{1}{n} \underline{R} M' - \frac{1}{m} \underline{C} N' + \frac{r G}{m n} E_{1v} \quad \ldots \quad (3\cdot2)$$

are called the adjusted yields of the treatments and

$$K = r I - \frac{1}{n} M M' - \frac{1}{m} N N' + \frac{r^2}{mn} E_{vv} \quad \ldots \quad (3\cdot3)$$

will be called the coefficient-matrix of the two-way design.

Since $K E_{v1} = 0$, rank $(K) \leq v-1$. A two-way design will be said to be doubly connected if its coefficient matrix K is of rank (v-1). In whatever follows, we shall assume that the two-way design is doubly connected.

It is well known from the theory of least-squares (see Rao, 1952) that any linear parametric function of the form $\bar{(H)} = \sum_{k=1}^{v} l_k \theta_k$ with $\sum_{k=1}^{v} l_k = 0$ admits linear unbiased estimators, and amongst them the one with minimum variance is $T = \sum l_k t_k$ where $\underline{t} = (t_1 \; t_2, \ldots, t_k)$ is any solution of (3.1). To obtain the variance of T express it in the alternative form $T = \sum_{k=1}^{v} m_k Q_k$ and then $V(T) = (\sum_{k=1}^{v} l_k m_k)\sigma_o^2$.

It may be noted that the equation (3.1) are the same as obtained by Shrikhande (1951) from the so-called 'Normal' model. The present approach demonstrates the robustness of this procedure. To estimate σ_o^2 and to carry out an omnibus test of significance of treatment differences, the analysis of variance of interaction-contrasts is to be done as shown in the following table.

Table : 3.1. Analysis of variance

source (1)	degrees of freedom (2)	sum of squares (3)
rows (unadjusted)	$m - 1$	$SS_R^* = \frac{1}{n} \sum_{i=1}^{m} R_i^2 - \frac{G^2}{mn}$
columns (unadjusted)	$n - 1$	$SS_C^* = \frac{1}{m} \sum_{j=1}^{n} C_j^2 - \frac{G^2}{mn}$
treatments (adjusted for rows and columns)	$v - 1$	$SS_{tr} = \sum_{k=1}^{v} Q_k t_k$
error	$\nu = (m-1)(n-1) - (v-1)$	$SS_o = SS_I - SS_{tr}$
interaction	$(m-1)(n-1)$	$SS_I = SS_T - SS_R^* - SS_C^*$
total	$mn - 1$	$SS_T = \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^2 - \frac{G^2}{mn}$

An unbiased estimator of σ_o^2 is provided by the error mean square $MS_o = SS_o / \nu$. To examine the significance of treatment differences, one may use the customary ratio of mean squares MS_{tr} / MS_o where $MS_{tr} = SS_{tr} / (v-1)$ is the treatment mean square. The sampling distribution of this statistic under randomisation, when the treatment effects are identical, is usually approximated by the Snedecor F-distribution with $(v-1)$ and ν degrees of freedom.

6.4. Recovery of information from row-contrasts and column-contrasts.

In the previous section, we had simply thrown away the row-contrasts and the column-contrasts. If the ratios $\varrho_i = \sigma_i^2 / \sigma_o^2$, i=1,2 are known, the method of weighted least squares can be applied on all the three sets of contrasts simultaneously. If the weight for the normalised interaction-contrasts is taken as unity, the weight for normalised row-contrasts will be $\frac{1}{\varrho_1}$ and that for normalised column-contrasts will be $\frac{1}{\varrho_2}$. As will be shown in section 7, the method of weighted least-squares now gives the equations:

$$\underline{\theta}\,\bar{K} = \underline{\bar{Q}} \qquad \ldots \qquad (4.1)$$

where

$$\underline{\bar{Q}} = \underline{T} - \frac{1}{n+\Delta_1}\,\underline{R}\,M' - \frac{1}{m+\Delta_2}\,\underline{C}\,N' + \frac{rG}{mn}\left(1-\frac{\Delta_1}{n+\Delta_1}-\frac{\Delta_2}{m+\Delta_2}\right)E_{1v} \quad \ldots (4.2)$$

$$\bar{K} = rI - \frac{1}{n+\Delta_1}\,MM' - \frac{1}{m+\Delta_2}\,NN' + \frac{r^2}{mn}\left(1-\frac{\Delta_1}{n+\Delta_1}+\frac{\Delta_2}{m+\Delta_2}\right)E_{vv} \quad \cdots (4.3)$$

and

$$\Delta_1 = \frac{n}{\varrho_1 - 1}, \quad \Delta_2 = \frac{m}{\varrho_2 - 1} . \tag{4.4}$$

If $\varrho_1 = 1$, the analysis should be performed ignoring the row-classification. Similarly when $\varrho_2 = 1$, we shall completely ignore the classification by columns.

The best estimator of $\bar{H} = \sum_{k=1}^{v} l_k \theta_k$ where $\sum_{k=1}^{v} l_k = 0$, is given by $\bar{T} = \sum_{k=1}^{v} l_k \bar{t}_k$ where $\bar{t} = (\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_k)$ is any solution of (4.1). The variance of $\bar{T}$ is most easily obtained by writing it in the form $\sum_{k=1}^{v} \bar{m}_k \bar{\lambda}_k$ and then $V(\bar{T}) = (\sum_{k=1}^{v} l_k \bar{m}_k) \sigma_o^2$.

Generally, however, the parameters Δ_1 and Δ_2 would be unknown and estimates D_1 and D_2 for them may have to be substituted. In this case, of course, the 'bestness' of $\bar{T}$ as an estimator of $\bar{H}$ would no longer hold, but if D_1, and D_2 are good estimators of Δ_1 and Δ_2, $\bar{T}$ might yet be better than T especially when ϱ_1 and ϱ_2 are not large.

We shall propose two methods for estimating the Δ's. First, we shall make use of SS_R, the adjusted row sum of squares. To compute SS_R one has to carry out another analysis of variance ignoring rows. Let $\underset{\sim}{t}_1$ be any solution of the following equations in $\underset{\sim}{\theta}$:

$$\underset{\sim}{\theta} K_1 = \underset{\sim}{Q}_1 \tag{4.5}$$

where

$$\underset{\sim}{Q}_1 = \underset{\sim}{T} - \frac{1}{m} \underset{\sim}{C} N' \text{ and } K_1 = r\, I - \frac{1}{m} NN' . \tag{4.6}$$

Then

$$SS_R = SS_R^* + SS_{tr} - \underset{\sim}{Q}_1 \, \underset{\sim}{t}_1' \, . \tag{4.7}$$

We shall show in section 7 that writing $MS_R = SS_R / (m-1)$ for the adjusted row mean square, the expectation of MS_R is given by

$$E(MS_R) = \sigma_1^2 \left(1 - \frac{a_1}{n}\right) + \frac{a_1}{n} \sigma_o^2 \tag{4.8}$$

where

$$a_1 = \text{tr } K_1^* \, M \, M' / \, (m-1) \tag{4.9}$$

in which tr denotes the trace of a matrix and K_1^* is a pseudo-inverse of the matrix K_1.

An estimate of σ_1^2 may be obtained from equation (4.8). Based on this, we can take

$$D_1 = \frac{(n-a_1) MS_o}{MS_R - MS_o} \tag{4.10}$$

as an estimator of Δ_1 in the sense that the ratio of the expectation of the numerator and the denominator of D_1 is equal to Δ_1. However, D_1 as defined in (4.10) shall be used only when $MS_R > MS_o$. When $MS_R \leq MS_o$, we shall decide to ignore the classification by rows and perform the analysis as in the case of one-way grouping (by columns only).

In an alternative procedure for estimation of σ_1^2 proposed below one does not need to perform a separate analysis ignoring rows. Let $y_{ij}(\underset{\sim}{\theta}) = y_{ij} - \sum_{k=1}^{v} \varepsilon_{ijk} \theta_k$, $R_i(\underset{\sim}{\theta}) = \sum_{j=1}^{n} y_{ij}(\underset{\sim}{\theta})$ and let further

$$V_R = \frac{1}{n(m-1)} R(\underset{\sim}{t})(I - \frac{1}{m} \; E_{mm}) \, R'(\underset{\sim}{t}) \tag{4.11}$$

where $\underset{\sim}{t}$ is a solution of (3.1).

It is easy to check that

$$E(V_R) = \sigma_1^2 + a_1^* \sigma_o^2 \qquad (4.12)$$

where

$$a_1^* = \text{tr } K^* \, MM'/n(m-1). \qquad (4.13)$$

Equation (4.12) may be used to provide an estimator of σ_1^2 and hence of Δ_1. This estimator of Δ_1 may be written as

$$D_1^* = \frac{n\, MS_o}{V_R - (1+ a_1^*)MS_o} \qquad (4.14)$$

Again, this estimate will be used only when $V_R > (1 + a_1^*)\, MS_o$.

Similar procedures may be adopted to obtain estimators D_2 and D_2^* of Δ_2.

It may be noted that the estimator of σ_1^2 based on MS_R corresponds to the traditional estimator of between block variance in an incomplete block design given in our formula (2.2.9) while the estimator of σ_1^2 based on V_R corresponds to the estimator of inter-block variance given by the formula (2.4.11).

It may be noted that the estimator of σ_1^2 obtained by any of the above two methods is not positive-definite. The following procedure gives positive-definite estimator of σ_1^2 for certain types of designs.

As will be shown in section 7, the least squares equations for estimation of θ_k's from row contrasts are

$$\underset{\sim}{\theta}(MM' - \frac{r^2}{m} E_{vv}) = (RM' - \frac{rG}{m} E_{1v}). \qquad (4.15)$$

If $\underline{t}^*$ is any solution of the above equations in $\underline{\theta}$ and if

$$p = \text{rank}\ (MM' - \frac{r^2}{m} E_{vv}) < m - 1 \tag{4.16}$$

the residual sum of squares in the analysis of variance of normalised row-contrasts can be used to estimate σ_1^2. Thus,

$$s_1^2 = \frac{(\sum R_i^2 - G^2/m) - \underline{R}M'\underline{t}^{*'}}{n(m - 1 - p)} \tag{4.17}$$

is a positive-definite unbiased estimator of σ_1^2. The corresponding estimator of Δ_1 is $n\,MS_o/(s_1^2 - MS_o)$.

6.5. Efficiency.

It is known [Kempthorne (1956), Roy (1958)] that the average variance of interaction estimators of differences of the type $\theta_k - \theta_{k'}$, is $2\,\sigma_o^2/h(K)$ where K is the coefficient-matrix of the design and $h(K)$ denotes the harmonic mean of the positive latent roots of K. If instead of the two-way design, a one-way design using columns as blocks were used, the average variance of intra-block estimates would then be $2[(1 - \frac{1}{n})\sigma_o^2 + \frac{\sigma_1^2}{n}]/h(K_1)$ where $K_1 = rI - \frac{1}{m} N N'$. As a measure of the efficiency of the two-way design in comparison with the one-way (column) design, we propose the ratio of the reciprocals of these average variances. This turns out to be

$$E = e\,\phi \tag{5.1}$$

where $e = h(K)/h(K_1)$ will be called the efficiency-factor of the two-way design relative to the one-way design using columns as blocks and

$\phi = 1 + \frac{1}{\Delta_1}$.

To prove that the relative efficiency factor e cannot exceed unity, we need the following result in matrix theory.

Lemma: If A and B are positive-definite matrices of the same order and $C = A - B$ is positive definite (semi-definite), then $D = B^{-1} - A^{-1}$ is positive definite (semi-definite).

Proof: We observe that if P and Q are symmetric matrices of the same order and P is positive definite, a necessary and sufficient condition for Q to be positive definite (semi-definite) is that the roots of the determinantal equation $| Q - \lambda P | = 0$ are all positive (non-negative). Now, the determinantal equation $| C - \lambda A | = 0$ is equivalent to the equation $| D - \lambda B^{-1} | = 0$. This follows by pre and post-multiplying the former equation by A^{-1} and B^{-1} respectively. Also C being positive definite (semi-definite) implies that the roots λ are all positive (non-negative) which in turn implies that D is positive definite (semi-definite).

Next consider a $(v-1) \times v$ matrix P satisfying $P P' = I$ and $P' P = I - \frac{E_{vv}}{v}$. Since our design is doubly connected it follows that $A = PK_1P'$ and $B = P K P'$ are both positive definite and $A - B = \frac{1}{n} P M M' P'$ is positive definite or semi-definite.

Now
$$H(K_1) - H(K) = H(A) - H(B) = (v-1)\left(\frac{1}{trA^{-1}} - \frac{1}{trB^{-1}}\right)$$
$$= \frac{(v-1)tr(B^{-1}-A^{-1})}{(trA^{-1})(trB^{-1})} \geq 0$$
from which it follows that $e \leq 1$.

6.6. Two-way designs with column balance.

A two-way design will be said to have column-balance if each treatment occurs in a column at most once, and any pair of treatments occurs together in the same number, say λ, of columns ; or in other words if the columns of the design regarded as blocks form a Balanced Incomplete Design. A column balanced design is said to be a Youden Square if the row incidence-matrix $M = E_{vm}$ and an extended Youden Square if $M = pE_{vm}$ where p is a positive integer, $p \geq 2$.

Shrikhande (1951) claims that all known column-balanced designs can be arranged in rows in such a way that (i) a partially balanced association scheme with two associate classes can be imposed on the treatments and (ii) the m_{ki}'s satisfy.

$$\sum_i m_{ki}\, m_{k'i} = \begin{cases} \mu_o & \text{if } k = k' \\ \mu_u & \text{if } k \neq k' \text{ are } u\text{-th associates ;} \end{cases} \quad u = 1, 2 . \qquad (6.1)$$

For a definition of a partially balanced association scheme, the reader is referred to Bose and Shimamoto (1952).

If $\mu_1 \neq \mu_2$, the designs satisfying (6.1) are said to belong to the class Y_1 (Shrikhande, 1951).

The analysis of designs belonging to the class Y_1 is particularly easy, being similar to that of partially balanced incomplete block designs with two associate classes. The analysis based on only the interaction-contrasts under the 'Normal' model is given by Shrikhande (1951): here we give the complete analysis including recovery of information from row-contrasts and column-contrasts.

For the parameters of the partially balanced association scheme, we shall use, the standard notations p^i_{jk}, n_i; i, j, k = 1,2.

Let now,

$$\begin{aligned} a &= \frac{\lambda v}{m} - \frac{1}{n}(\mu_o - \mu_2) \\ b &= \frac{1}{n}(\mu_1 - \mu_2) \\ c &= p^1_{11} - p^2_{11} \\ d &= n_1 - p^2_{11}. \end{aligned} \tag{6.2}$$

Then a solution of the equations (3.1) for a design of the class Y_1 turns out to be

$$t_k = [\Lambda Q_k + bS_1(Q_k)]/D \tag{6.3}$$

where S_1 denotes summation over first associates and

$$\begin{aligned} \Lambda &= a + bc \\ D &= a\Lambda - b^2 d. \end{aligned} \tag{6.4}$$

Since the variances of estimates of treatment differences are given by

$$V(t_k - t_{k'}) = \begin{cases} [2(\Lambda - b)/D]\sigma_o^2, & \text{if } k, k' \text{ are first associates} \\ [2\Lambda/D]\,\sigma_o^2, & \text{otherwise} \end{cases}$$

the relative efficiency factor of this design turns out to be

$$e = \frac{m(v-1)D}{\lambda v\left[(v-1)\Lambda - n_1 b\right]} . \tag{6.5}$$

For combined estimation, a solution of the equations (4.1) is

$$\bar{t}_k = [\bar{\Lambda}\,\bar{Q}_k + \bar{b}\,S_1(\bar{Q}_k)]\,/\bar{D} \tag{6.6}$$

where $\bar{\Lambda} = \bar{a} + \bar{b}c$

$$\bar{D} = \bar{a}\,\bar{\Lambda} - \bar{b}^2 d \tag{6.7}$$

and

$$\bar{a} = \frac{\lambda v + r\Delta_2}{m + \Delta_2} - \frac{1}{n + \Delta_1}(\mu_o - \mu_2)$$

$$\bar{b} = \frac{1}{n + \Delta_1}(\mu_1 - \mu_2). \tag{6.8}$$

To obtain D_1 and D_2 as estimators of Δ_1 and Δ_2 one may proceed as follows.

A solution of (4.5) turns out to be

$$t_{1k} = \frac{m}{\lambda v} Q_{1k} \tag{6.9}$$

where Q_{1k} is the k-th element of $\underline{Q}_1$ defined by (4.6). SS_R and hence MS_R can be easily obtained with the help of (4.7) and (6.9). Since in this case $K_1 = \frac{\lambda v}{m}(I - \frac{E}{v})$, it is easily seen that $K_1^* = \frac{m}{\lambda v}[I - \frac{E}{v}]$. Hence $K_1^* MM' = \frac{m}{\lambda v} MM' - \frac{mnr}{\lambda v^2} E$. Substituting this in (4.9) one gets

$$a_1 = \frac{m\mu_o - r^2}{\lambda(m-1)} . \tag{6.10}$$

Using MS_R and a_1 as obtained above D_1 is easily obtained.

To estimate Δ_2, one needs MS_C the adjusted column mean square given by

$$(n-1)MS_C = SS_C^* + SS_{tr} - \underline{Q}_2\underline{t}_2' \tag{6.11}$$

$\underset{\sim}{t}_2$ being any solution for $\underset{\sim}{\theta}$ in

$$\underset{\sim}{\theta} K_2 = \underset{\sim}{Q}_2 \qquad \ldots \qquad \ldots \qquad (6.12)$$

where

$$\underset{\sim}{Q}_2 = \underset{\sim}{T} - \frac{1}{n} \underset{\sim}{R} M' \text{ and } K_2 = r\, I - \frac{1}{n} MM'. \qquad (6.13)$$

In this case $\underset{\sim}{t}_2 = (t_{21}, \ldots, t_{2k})$ is given by

$$t_{2k} = [\Lambda' Q_{2k} + bS_1[Q_{2k}]] / D' \qquad \ldots \qquad (6.14)$$

where $\Lambda' = \Lambda + (r - \lambda)/m$ and $D' = [a + (r-\lambda)/m]\Lambda' - b^2 d$.

In this case since $NN' = (r-\lambda)I + \lambda E$, we have $\text{tr } K_2^* N N' = (r-\lambda)\text{tr} K_2^*$, Now if $\underset{\sim}{t} = \underset{\sim}{Q}_2 Y$ be any solution in $\underset{\sim}{\theta}$ of equations (6.12), it can be shown that $\text{tr } K_2^* = \text{tr}(Y) - \frac{1}{v}$ (sum of all elements of Y). Since (6.14) provides a solution of (6.12), it follows that

$$a_2 = \frac{\text{tr } K_2^* NN'}{(n-1)} = \frac{(r - \lambda)[(v - 1)\Lambda' - n_1 b]}{(n - 1)D'} \qquad \ldots \qquad (6.15)$$

D_2 may now be obtained using the formula

$$D_2 = \frac{(m - a_2)\, MS_o}{MS_C - MS_o}. \qquad \ldots \qquad \ldots \qquad (6.16)$$

To obtain D_1^* and D_2^* we shall need $\text{tr } K^* MM'$ and $\text{tr } K^* NN'$. From (6.3) it can be deduced that the (k,k')th element of K^* is given by

$$\gamma^{kk'} = \begin{cases} [(v-1)\Lambda - n_1 b] / vD & \text{if } k = k' \\ [-\Lambda + (v-n_1)b] / vD & \text{if the k-th and the k'th treatments are first associates} \\ [-\Lambda - n_1 b] / vD & \text{otherwise.} \end{cases} \qquad (6{\cdot}17)$$

where Λ, b and D are defined by (6.2) and (6.4). Trace of $K^* MM'$ is obtained by taking the sum of products of the corresponding elements of K^* and MM'. This gives

$$a_1^* = \frac{\text{tr } K^* MM'}{n(m-1)} = \frac{v(\Lambda\mu_0 + n_1\mu_1 b) - rn(\Lambda + n_1 b)}{Dn\ (m-1)} . \qquad (6.18)$$

Using a_1^* as given above D_1^* may be computed from formula (4.14).

Similarly to obtain D_2^* we use the formula

$$D_2^* = \frac{m\ MS_0}{V_0 - (1 + a_2^*)MS_0} \quad \ldots \quad \ldots \qquad (6.19)$$

where

$$V_C = C(t)(I - \frac{1}{m} E_{mm})\ C(t)'/m(n-1) \quad \ldots \qquad (6.20)$$

the j-th element of $C(t)$ being given by $C_j(t) = \sum_i y_{ij}(t)$ and

$$a_2^* = \frac{\text{tr } K^* NN'}{m(n-1)} = \frac{v(\Lambda r + \lambda n_1 b) - rm(\Lambda + n_1 b)}{D\ m\ (n-1)} . \qquad (6.21)$$

6.7. Derivation of results.

In this section we prove some of the results stated earlier. We shall use the following lemmas.

Lemma 6.7.1.

If $l_{ij}^{(\alpha\beta)}$, i, α = 1,2,..., m; j, β = 1,2,..., n are real numbers chosen to satisfy : (a) $\sum_{ij} l_{ij}^{(\alpha\beta)}\, l_{ij}^{(\alpha'\beta')} = \delta_{\alpha\alpha'}\, \delta_{\beta\beta'}$ where δ is the Kronecker symbol;

(b) $l_{ij}^{(mn)} = 1/\sqrt{mn}$, (c) $l_{ij}^{(\alpha m)} = l_{io}^{(\alpha m)}$, $\alpha = 1,2,\ldots, m-1$ and
(d) $l_{ij}^{(m\beta)} = l_{oj}^{(m\beta)}$, $\beta = 1,2,\ldots, n-1$ then we have:

$$\sum_{\alpha=1}^{m-1} l_{ij}^{(\alpha n)} l_{i'j'}^{(\alpha n)} = \frac{1}{n}\left(\delta_{ii'} - \frac{1}{m}\right)$$
$$\sum_{\beta=1}^{n-1} l_{ij}^{(m\beta)} l_{i'j'}^{(m\beta)} = \frac{1}{m}\left(\delta_{jj'} - \frac{1}{n}\right) \qquad (7.1)$$
$$\sum_{\alpha=1}^{m-1}\sum_{\beta=1}^{n-1} l_{ij}^{(\alpha\beta)} l_{i'j'}^{(\alpha\beta)} = \left(\delta_{ii'} - \frac{1}{m}\right)\left(\delta_{jj'} - \frac{1}{n}\right)$$

and

$$\sum_{ij}\sum_{i'j'} l_{ij}^{(\alpha\beta)} l_{i'j'}^{(\alpha'\beta')}\delta_{ii'} = \begin{cases} n\,\delta_{\alpha\alpha'} & \text{when } \beta = \beta' = n \\ 0 & \text{otherwise}\end{cases}$$
$$\sum_{ij}\sum_{i'j'} l_{ij}^{(\alpha\beta)} \cdot l_{i'j'}^{(\alpha'\beta')}\delta_{jj'} = \begin{cases} m\,\delta_{\beta\beta'} & \text{when } \alpha = \alpha' = m \\ 0 & \text{otherwise}\end{cases} \qquad (7.2)$$
$$\sum_{ij}\sum_{i'j'} l_{ij}^{(\alpha\beta)} l_{i'j'}^{(\alpha'\beta')}\delta_{ii'}\delta_{jj'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}.$$

These follow easily from the properties of orthogonal matrices.

Lemma 6.7.2.

Let $Z_{\alpha\beta} = \sum_{ij} l_{ij}^{(\alpha\beta)} y_{ij}$. Then

$$E(Z_{\alpha\beta}) = \begin{cases} \dfrac{1}{\sqrt{mn}}\left(mn\mu + r\sum_k \theta_k\right) & \text{when } \alpha = m,\ \beta = n \\ \sum_k a_k^{(\alpha\beta)}\theta_k & \text{otherwise}\end{cases} \qquad (7.3)$$

where

$$a_k^{(\alpha\beta)} = \sum_{ij} l_{ij}^{(\alpha\beta)}\varepsilon_{ijk}.$$

Also, these $Z_{\alpha\beta}$'s are mutually uncorrelated and they have variances given by

$$V(Z_{\alpha\beta}) = \begin{cases} 0 & \text{if } \alpha = m; \beta = n \\ \sigma_1^2 & \text{if } \beta = n; \alpha = 1,2,\ldots, m-1 \\ \sigma_2^2 & \text{if } \alpha = m; \beta = 1,2,\ldots, n-1 \\ \sigma_o^2 & \text{if } \alpha = 1,2,\ldots, m-1; \beta = 1,2,\ldots, n-1. \end{cases} \qquad (7.4)$$

These results are obtained by direct computation using the expectations and covariances of y_{ij}'s given by (2.3) and (2.4) and the properties of $l_{ij}^{(\alpha\beta)}$'s given by (7.2).

Lemma 6.7.3.

With $Z_{\alpha\beta}$ and $a_k^{(\alpha\beta)}$'s as defined in Lemma 6.7.2, we have

$$Q_k' = \sum_{\alpha=1}^{m-1} Z_{\alpha n}\, a_k^{(\alpha n)} = \frac{1}{n}\sum_{i=1}^{m} m_{ki} R_i - \frac{rG}{mn}$$

$$Q_k'' = \sum_{\beta=1}^{n-1} Z_{m\beta}\, a_k^{(m\beta)} = \frac{1}{m}\sum_{j=1}^{n} n_{kj} C_j - \frac{rG}{mn} \qquad (7.5)$$

$$Q_k = \sum_{\alpha=1}^{m-1}\sum_{\beta=1}^{n-1} Z_{\alpha\beta}\, a_k^{(\alpha\beta)} = T_k - \frac{1}{n}\sum_{i=1}^{m} m_{ki} R_i - \frac{1}{m}\sum_{j=1}^{n} n_{kj} C_n + \frac{rG}{mn}$$

$$\gamma_{kk'}' = \sum_{\alpha=1}^{m-1} a_k^{(\alpha n)} a_{k'}^{(\alpha n)} = \frac{1}{n}\sum_{i=1}^{m} m_{ki} m_{k'i} - \frac{r^2}{mn}$$

$$\gamma_{kk'}'' = \sum_{\beta=1}^{n-1} a_k^{(m\beta)} a_{k'}^{(m\beta)} = \frac{1}{m}\sum_{j=1}^{m} n_{kj} n_{k'j} - \frac{r^2}{mn} \qquad (7.6)$$

$$\gamma_{kk'} = \sum_{\alpha=1}^{m-1}\sum_{\beta=1}^{n-1} a_k^{(\alpha\beta)} a_{k'}^{(\alpha\beta)} = r\delta_{kk'} - \frac{1}{n}\sum_{i=1}^{m} m_{ki} m_{k'i} - \frac{1}{m}\sum_{j=1}^{n} n_{kj} n_{k'j} + \frac{r^2}{mn}.$$

These results are obtained by direct computation using (7.1) and (7.2)

We shall now derive the least square equations. According to the method of weighted least-squares, we have to minimise

$\Sigma \frac{1}{V(Z_{\alpha\beta})} [Z_{\alpha\beta} - E(Z_{\alpha\beta})]^2$, where the summation is over all values of α, β except $\alpha = m$, $\beta = n$. On multiplication by σ_0^2, this reduces to minimising

$$L = \sum_{\alpha=1}^{m-1} \sum_{\beta=1}^{n-1} [Z_{\alpha\beta} - E(Z_{\alpha\beta})]^2 + \frac{1}{\rho_1} \sum_{\alpha=1}^{m-1} [Z_{\alpha n} - E(Z_{\alpha n})]^2 + \frac{1}{\rho_2} \sum_{\beta=1}^{n-1} [Z_{m\beta} - E(Z_{m\beta})]^2.$$

Equating the partial derivative of L with respect to θ_k $(k = 1,2,\ldots,v)$ to zero, we get the equations:

$$\sum_{k'=1}^{v} \left(\gamma_{kk'} + \frac{1}{\rho_1} \gamma'_{kk'} + \frac{1}{\rho_2} \gamma''_{kk'} \right) \theta_{k'} = Q_k + \frac{1}{\rho_1} Q'_k + \frac{1}{\rho_2} Q''_k, \quad (7.7)$$

$$k = 1,2,\ldots, v.$$

where Q_k and $\gamma_{kk'}$, etc., are given by (7.5) and (7.6). This, in matrix notation is our equation (4.1) for combined estimation. If we want estimates from interaction-contrasts only, the equations would be $\sum_{k'=1}^{v} \gamma_{kk'} \theta_{k'} = Q_k$ which is the same as equation (3.1). Similarly if we want estimates from row-contrasts only, the equations would be $\sum_{k'=1}^{v} \gamma'_{kk'} \theta_{k'} = Q'_k$ which, in matrix notation, is our equation (4.15).

Expectation of the adjusted row sum of squares is obtained as follows. Since the adjusted row sum of squares SS_R defined by (4.7) is invariant under the transformation $y'_{ij} = y_{ij} - \sum_{k=1}^{v} \varepsilon_{ijk} \theta_k$, its distribution and therefore its expectation does not involve θ_k's. Consequently in computing $E(SS_R)$ we can ignore the terms involving θ_k's. Now, since

$$E(SS_R^*) = (m-1)\sigma_1^2 + \text{terms in } \theta_k\text{'s}$$

and (7.8)

$$E(SS_{tr}) = (v-1)\sigma_o^2 + \text{terms in } \theta_k\text{'s}$$

it follows from (4.7) that all that we need now to compute $E(SS_R)$ is $E(\underset{\sim}{Q}_1 \underset{\sim}{t}'_1)$ where $\underset{\sim}{Q}_1$, $\underset{\sim}{t}_1$ are defined by (4.5) and (4.6). Let K_1^* be a pseudo-inverse of the matrix K_1 defined by (4.8), so that a particular solution of (4.5) is $\underset{\sim}{t}_1 = \underset{\sim}{Q}_1 K_1^*$. Hence

$$\begin{aligned} E(\underset{\sim}{Q}_1 \underset{\sim}{t}'_1) &= E(\underset{\sim}{Q}_1 K_1^* \underset{\sim}{Q}'_1) = E(tr(K_1^* \underset{\sim}{Q}'_1 \underset{\sim}{Q}_1) \\ &= tr\,[K_1^* E(\underset{\sim}{Q}'_1 \underset{\sim}{Q}_1)] \\ &= tr\, K_1^*\; D\,(\underset{\sim}{Q}_1) + \text{terms in } \theta_k\text{'s} \end{aligned} \tag{7.9}$$

where $D\,(\underset{\sim}{Q}_1)$ stands for the dispersion matrix of $\underset{\sim}{Q}_1$. To compute $D\,(\underset{\sim}{Q}_1)$ we express $\underset{\sim}{Q}_1$ in the form $\underset{\sim}{Q}_1 = \underset{\sim}{Q} + \frac{1}{n}\underset{\sim}{R}\,(M' - \frac{rE_{mv}}{m})$. Since the elements of $\underset{\sim}{Q}$ are interaction-contrasts and those of $\frac{1}{n}\underset{\sim}{R}(M' - \frac{rE_{mv}}{m})$ are row-contrasts, these are uncorrelated, and therefore

$D\,(\underset{\sim}{Q}_1) = D\,(\underset{\sim}{Q}) + D\,[\frac{1}{n}\underset{\sim}{R}(M' - \frac{r\,E_{mv}}{n})]$. Since $D(\underset{\sim}{Q}) = K\,\sigma_o^2$ and $D\,(\underset{\sim}{R}) = (I - \frac{E}{m})\,n\,\sigma_1^2$; we get on simplification

$$D\,(\underset{\sim}{Q}_1) = (K_1 + \frac{r^2 E_{vv}}{m\,n})\sigma_o^2 + \frac{1}{n} MM'\,(\sigma_1^2 - \sigma_o^2). \tag{7.10}$$

Using (7.8), (7.9) and (7.10), we get finally

$$\begin{aligned} E(SS_R) &= E(SS_R^*) + E(SS_{tr}) - E(\underset{\sim}{Q}_1 \underset{\sim}{t}'_1) \\ &= [\,(m-1) - \frac{1}{n}\, tr\, K_1^*\, M\, M'\,]\;\sigma_1^2 + (tr\; K_1^*\, M\, M')\sigma_o^2\,/n \end{aligned}$$

from which (4.8) follows.

6.8. Numerical example.

Table 8.1 gives the yields and the lay-out of a design of the Y_1 class with treatments indicated by numbers within brackets.

Table 8.1. Yields and the lay-out at a design of the Y_1 class (with treatments indicated by numbers within brackets).

i/j	1	2	3	4	5	6	7	8	9	10	R_i
1	140.1 (2)	161.8 (5)	112.2 (1)	153.9 (4)	116.5 (4)	189.2 (6)	160.3 (2)	152.7 (3)	178.0 (3)	134.9 (1)	1499.6
2	102.6 (1)	129.2 (6)	89.5 (3)	97.4 (1)	103.9 (3)	142.5 (5)	138.8 (6)	106.9 (4)	133.3 (5)	87.9 (2)	1132.0
3	155.9 (6)	165.8 (3)	138.3 (6)	141.6 (5)	79.8 (2)	141.6 (4)	161.2 (4)	136.1 (1)	155.8 (2)	107.1 (5)	1383.2
C_j	398.6	456.8	340.0	392.9	300.2	473.3	460.3	395.7	467.1	329.9	4014.8

The parameters of the design are : $m = 3$, $n = 10$, $v = 6$, $r = 5$, $\lambda = 2$, $n_1 = 1$, $n_2 = 4$, $\mu_o = 9$, $\mu_1 = 9$, $\mu_2 = 8$, $p_{11}^1 = 0$, $p_{11}^2 = 0$.

$$a = \frac{\lambda v}{m} - \frac{1}{n}(\mu_o - \mu_2) = 3.9 \qquad b = \frac{1}{n}(\mu_1 - \mu_2) = 0.1$$

$$c = p_{11}^1 - p_{11}^2 = 0 \qquad d = n_1 - p_{11}^2 = 1$$

$$\Lambda = a + bc = 3.9 \qquad D = a\Lambda - b^2 d = 15.2$$

The computational details of estimation are given in Table 8.2.

If we want interaction estimates only we need proceed only upto column (7) in Table 8.2. The analysis of variance table may be prepared at this stage as shown in Table 8.3.

Table 8.2. Computational lay-out for interaction estimates and combined estimates.

treat-ments	1st associates of	T_k	$[R]_k$	$[C]_k$	mnQ_k	$mnDt_k$	mQ_{1k}	nQ_{2k}	$nD't_{2k}$	$\bar{Q}_k$	$\bar{D}\bar{t}_k$	$\bar{Q}^*_k$	$\bar{D}^*\bar{t}^*_k$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1	3	583.2	6646.4	1857.1	- 940.2	-3543.4	-107.5	-814.4	-3965.3	-37.12	-145.29	-37.25	-145.89
2	4	623.9	6897.6	1956.1	-1462.8	-5748.9	- 84.4	-658.6	-3236.8	-50.04	-202.24	-50.08	-202.54
3	1	689.9	6646.4	1959.8	1233.8	4717.8	109.9	252.6	1156.3	39.20	153.82	39.16	153.75
4	2	680.1	6897.6	2022.4	- 439.8	-1861.5	17.9	- 96.6	- 539.2	-13.45	- 58.85	-13.43	- 58.82
5	6	686.3	6530.0	2120.0	- 127.0	- 321.7	- 61.1	333.0	1730.1	- 0.51	3.93	- 0.42	4.32
6	5	751.4	6530.0	2129.0	1736.0	6757.7	125.2	984.0	4854.9	61.92	248.63	62.02	249.18
		4014.8	10G= 4014.8**	3G= 12044.4**	0**	0**	0**	0**	0**	0**	0**	0**	0**

** Denotes check.

$[R]_k = \sum m_{ki} R_i$, $[C]_k = \sum n_{kj} C_j$;

$mnQ_k = mnT_k - m[R]_k - n[C]_k + rG$;

$mnDt_k = \Lambda\, mnQ_k + bS_1\,(mnQ_k)$;

$mQ_{1k} = mT_k - [C]_k$;

$nQ_{2k} = nT_k - [R]_k$; $nD't_{2k} = \Lambda' nQ_{2k} + bS_1\,(nQ_{2k})$;

$$\bar{Q}_k = T_k - \frac{1}{n+D_1}[R]_k - \frac{1}{m+D_2}[C]_k + \frac{rG}{m\,n}\left[1 - \frac{D_1}{n+D_1} - \frac{D_2}{m+D_2}\right]$$

$$\bar{D}\bar{t}_k = \bar{\Lambda}\,\bar{Q}_k + \bar{b}S_1\,(\bar{Q}_k).$$

$$\bar{Q}^*_k = T_k - \frac{1}{n+D^*_1}[R]_k - \frac{1}{m+D^*_2}[C]_k + \frac{rG}{m\,n}\left[1 - \frac{D^*_1}{m+D^*_1} - \frac{D^*_2}{m+D^*_2}\right]$$

$$\bar{D}^*\,\bar{t}^*_k = \bar{\Lambda}^*\,\bar{Q}^*_k + \bar{b}^*\,S_1(\bar{Q}^*_k).$$

Table 8.3. Analysis of variance.

source	degrees of freedom	sum of squares	mean squares	F
rows (unadjusted)	2	$SS^*_R = 7059.34$		
columns (unadjusted)	9	$SS^*_C = 11753.55$		
treatments (adjusted for rows and columns)	5	$SS_{tr} = 2204.15$	$MS_{tr} = 440.83$	3.39
error	13	$SS_o = 1690.66$	$MS_o = 130.05$	
interaction	18	$SS_I = 3894.81$		
total	29	$SS_T = 22707.70$		

For recovery of information from row-contrasts and column-contrasts, if one uses D_1 and D_2 defined by (4.10) and (6.16) as estimates of Δ_1 and Δ_2 respectively the computations may be made as shown in the columns (8) to (12) of Table 8.2. The constants required are given below.

$$\Lambda' = \Lambda + (r - \lambda)/m = 4.9, \qquad D' = [a+(r-\lambda)/m]\Lambda' - b^2 d = 24.$$

$$a_1 = \frac{m\mu_o - r^2}{\lambda(m-1)} = 0.5, \qquad a_2 = \frac{(r-\lambda)[(v-1)\Lambda' - n_1 b]}{(n-1)D'} = 0.3389.$$

We obtain

$$\underline{Q}_1\underline{t}_1' = 1402.40 \qquad \underline{Q}_2\underline{t}_2' = 4607.75$$

$$SS_R = SS^*_R + SS_{tr} - \underline{Q}_1\underline{t}_1' = 7861.09, \qquad SS_C = SS^*_C + SS_{tr} - \underline{Q}_2\underline{t}_2' = 9349.95$$

$$D_1 = \frac{(n-a_1)MS_o}{MS_R - MS_o} = 0.3251 \qquad D_2 = \frac{(m-a_2)MS_o}{MS_C - MS_o} = 0.3808$$

$$\bar{a} = \frac{\lambda v + rD_2}{m+D_2} - \frac{1}{n+D_1}(\mu_o - \mu_2) = 4.01579; \quad \bar{b} = \frac{1}{n+D_1}(\mu_1 - \mu_2) = 0.09685$$

$$\bar{\Lambda} = \bar{a} + \bar{b}c = 4.01579 \qquad \bar{D} = \bar{a}\,\bar{\Lambda} - \bar{b}^2 d = 16.11719.$$

The procedure is somewhat simpler if one uses D_1^* and D_2^* defined by (4.14) and (6.19) as estimators of Δ_1 and Δ_2 respectively. Computations in this case may be made as shown in columns (13) and (14) of Table 8.2. The constants required are given below.

$$a_1^* = \frac{v(\Lambda\mu_o + n_1\mu_1 b) - rn(\Lambda + n_1 b)}{Dn(m-1)} = 0.05263$$

$$a_2^* = \frac{v(\Lambda r + \lambda n_1 b) - rm(\Lambda + n_1 b)}{Dm(n-1)} = 0.14181$$

$$D_1^* = \frac{n\,MS_o}{V_R - (1 + a_1^*)\,MS_o} = 0.3251,$$

$$D_2^* = \frac{m\,MS_o}{V_C - (1 + a_2^*)MS_o} = 0.3905,$$

$$\bar{a}^* = \frac{\lambda v + r D_2^*}{m + D_2^*} - \frac{1}{n + D_1^*}(\mu_o - \mu_2) = 4.01841$$

$$\bar{b}^* = \frac{1}{n + D_1^*}(\mu_1 - \mu_2) = 0.09685$$

$$\bar{\Lambda}^* = \bar{a}^* + \bar{b}^* c = 4.01841$$

$$\bar{D}^* = \bar{a}^*\,\bar{\Lambda}^* - \bar{b}^{*2} d = 16.13818\,.$$

The three sets of estimates are given below in Table 8.4.

Table 8.4: Estimates of treatment effects.

k	t_k	$\bar{t}_k$	$\bar{t}^*_k$
(1)	(2)	(3)	(4)
1	- 7.77	- 9.01	- 9.04
2	- 12.61	- 12.55	- 12.55
3	10.35	9.54	9.53
4	- 4.08	- 3.65	- 3.65
5	- 0.71	0.24	0.27
6	14.82	15.43	15.44

For interaction estimates $t_k - t_{k'}$,

$$\text{Est}\{V(t_k - t_{k'})\} = \begin{cases} \dfrac{2(\Lambda - b)}{D} \cdot MS_o = 65.03, & \text{if } k \text{ and } k' \text{ are first associates} \\ \dfrac{2\Lambda}{D} \cdot MS_o = 66.74, & \text{otherwise.} \end{cases}$$

If the combined estimates are obtained as shown in columns (8) to (12) of Table 8.2 and if one ignores the effect of the sampling fluctuations in D_1 and D_2 on the variance of $\bar{t}_k - \bar{t}_{k'}$,

$$\text{Est}\{V(\bar{t}_k - \bar{t}_{k'})\} = \begin{cases} \dfrac{2(\bar{\Lambda} - \bar{b})}{\bar{D}} \cdot MS_o = 63.24 & \text{if } k, k' \text{ are first associates} \\ \dfrac{2\bar{\Lambda}}{\bar{D}} \; MS_o = 64.81, & \text{otherwise.} \end{cases}$$

If the combined estimates are obtained as shown in columns (13) and (14) of Table 8.2 and if one ignores the effect of sampling fluctuations in D_1^* and D_2^* on the variance of $\bar{t}_k^* - \bar{t}_{k'}^*$,

$$\text{Est}\left\{V(\bar{t}_k^* - \bar{t}_{k'}^*)\right\} = \begin{cases} \dfrac{2(\bar{\Lambda}^* - \bar{b}^*)}{\bar{D}^*} MS_o = 63.20 & \text{if } k, k' \text{ are first associates} \\ \dfrac{2\ \bar{\Lambda}^*}{\bar{D}^*} MS_o = 64.76 & \text{otherwise.} \end{cases}$$

The above shows that in this case recovery of information from row-contrasts and column contrasts has not resulted in appreciable gain in precision.

Next we compare this design with the design formed by taking columns as blocks. The relative efficiency factor e turns out to be 0.97938. If Δ_1 is replaced by D_1 (D_1^*), $\phi = 1 + \dfrac{1}{\Delta_1} = 4.07598(4.07598)$. Hence $E = e \times \phi = 3.99$ (3.99) is the efficiency of the two-way design relative to the column-design, which shows that the gain in precision is appreciable.

Chapter VII

ONE-WAY DESIGNS WITH TWO SETS OF TREATMENTS

7.1. Introduction and summary.

Pearce and Taylor (1948) and Hoblyn, Pearce and Freeman (1954) considered the problem of designing an experiment which is to run over successive periods of experimentation. Treatments are applied to the experimental units in every period with the provision that the set of treatments may change from period to period. In such experiments a treatment applied on an experimental unit may affect the yield for that unit in succeeding periods. Such effects are called the residual effects of the treatments. Hoblyn, Pearce and Freeman (1954) considered the case where the residual effect is operative only in the period following the period of application. They further assumed that the residual effect and the direct effect (effect due to the treatment applied in the current period) are additive.

In this chapter we shall consider experiments for two periods only. Suppose we have p_1 treatments in the first period (to be called the first set of treatments) and p_2 treatments in the second period (to be called the second set of treatments) and suppose that the plots (experimental units) are grouped in one way into p_0 blocks. The design for such an experiment can be specified by the numbers n_{ijk}, $i = 1, 2, \ldots, p_0$; $j = 1,2,\ldots, p_1$; $k = 1,2,\ldots, p_2$; where

n_{ijk} denotes the number of times the j-th treatment of the first set and the k-th treatment of the second set occur together on a plot in the i-th block. We shall restrict ourselves to the most important class of binary designs (designs where every n_{ijk} can take only the values 0 or 1).

The three-dimensionally ordered set of values $(((n_{ijk})))$ will be called the <u>incidence cube</u> and will be denoted by N. Let $n_{*jk} = \sum_i n_{ijk}$, $n_{i*k} = \sum_j n_{ijk}$ and $n_{ij*} = \sum_k n_{ijk}$. Thus n_{ij*} denotes the number of plots in the i-th block on which the j-th treatment is applied. Following Pothoff (1962), the matrix $N_{01} = ((n_{ij*}))$ will be called a marginal matrix. Other marginal matrices are defined as $N_{12} = ((n_{*jk}))$ and $N_{02} = ((n_{i*k}))$. We shall also define $N_{10} = (N_{01})'$, $N_{20} = (N_{02})'$ and $N_{21} = (N_{12})'$.

The marginal matrices play a key role in the analysis of such designs. Though the estimating equations obtained in section 2 can always be solved in theory, unless the marginal matrices obey certain restrictions, the numerical procedure is likely to be unwieldy. The problem of construction of designs for which the estimating equations are easy to solve deserves some attention. Even if one succeeds in finding a set of matrices satisfying the easy solvability conditions on the marginal matrices, the existence of the incidence cube with these as the marginal matriaces does not follow.

Hoblyn, Pearce and Freeman (1954) classified such designs according to the properties of the marginal matrices taken separately. The construction of designs for which N_{10} is the incidence matrix of a randomised block design while N_{20} and N_{21} are incidence matrices of a PBIB design has been dealt with by Freeman (1957a, 1957b, 1958, 1961) in a series of papers.

Potthoff (1962) derived the analysis of designs for which (i) $N_{01} N_{12} = \mu E$, $N_{02} N_{21} = \nu E$ or (ii) $N_{01} N_{10} = \alpha I + \beta E$ where I is the identity matrix, E denotes a matrix with all elements unity and μ, ν, α, β are some constants. Potthoff noted that it is sufficient that the above conditions hold for some permutation of the indices 0, 1 and 2. He gave some illustrations for each of the two types of designs considered by him.

For the special designs mentioned above, Freeman (1957a) and Potthoff (1962) gave method of intra-block analysis based on the Normal model.

In this chapter, we derive the method of analysis for any such design with experimental units arranged one way in blocks, and involving two sets of treatments whose effects are additive. We use only the distribution induced by the usual two-stage (within and between blocks) randomisation and the assumption of additivity of plot and treatment effects. In addition to the intra-block analysis we give a procedure for recovery of inter-block information.

It is known that (Rao, 1959) for one-way designs the two-stage randomisation gives rise to two sets of contrasts such that all contrasts are uncorrelated and contrasts from the same set have the same variance. The expected value of each contrast is obtained by replacing every observation by the sum of the parameters corresponding to the two treatments (one from each set) associated with it. In section 2 we apply the method of weighted least-squares to these two sets of contrasts and obtain the normal equations for estimating the parametric contrasts for any one of the two sets of treatments. The procedure for estimating the unknown weights is given in section 3. As is to be expected, the results are obtained in symbolic forms, in terms of pseudo-inverses of certain matrices.

If the design is not chosen properly, even the intra-block analysis is somewhat laborious for manual computations. But with proper choice of design, the analysis including recovery of inter-block information does not involve unduly heavy computations. An illustration is given in section 4 with a particular design where both the sets contain the same number of treatments, the same BIB design is used for both the sets and pairing of treatments from the two sets is done in a special way.

7.2. Estimating equations for randomisation model.

Consider an experiment in which the experimental units or plots are divided into p_o blocks of r_o plots each. There are two sets of treatments and two treatments, one from each set, are applied on each plot. The first set consists of p_1 treatments each occurring r_1 times and the second set consists of p_2 treatments each occurring r_2 times. The actual allocation is done in the following manner.

First, we consider a design consisting of p_o rows having r_o cells each where, the u-th cell in the i-th block contains the j-th treatment symbol of the first set a_{jiu} times, $j = 1,2,\ldots, p_1$; $i = 1,2,\ldots, p_o$; $u = 1,2,\ldots, r_o$ and the k-th treatment symbol of the second set b_{kiu} times, $k = 1,2,\ldots, p_2$; $i = 1,2,\ldots, p_o$; $u = 1,2,\ldots, r_o$. Here $a_{jiu} = 1$ or 0 according as the j-th symbol (treatment) of the first set occurs on the u-th cell of the i-th row or not. Evidently $\sum_j a_{jiu} = 1$. Similar remarks will apply to b_{kiu}.

We shall perform the two-stage randomisation as usual, i.e., the blocks are numbered $1, 2,\ldots, p_o$ at random and the plots in a block are numbered $1, 2,\ldots, r_o$ again at random, independently for different blocks. The u-th plot in the i-th block then receives the treatments from the two sets corresponding to the two symbols which occur in the u-th cell of the i-th row of the design. Let y_{iu} denote the yield on this plot. Let, further, $\underline{\theta}_1 = (\theta_1^1, \theta_2^1,\ldots, \theta_{p_1}^1)$ and $\underline{\theta}_2 = (\theta_1^2, \theta_2^2, \ldots, \theta_{p_2}^2)$ denote the row vectors of real-valued parameters measuring the effects of the first and the second set of treatments respectively. We shall assume that

$$y_{iu} = x_{iu} + \sum_j a_{jiu}\,\theta^1_j + \sum_k b_{kiu}\,\theta^2_k \qquad \ldots \tag{2.1}$$

where x_{iu} denotes the effect of the plot which has been called the u-th plot in the i-th block. We shall write $\underline{T}_0 = (T_{01}, T_{02}, \ldots, T_{0p_0})$, $\underline{T}_1 = (T_{11}, T_{12}, \ldots, T_{1p_1})$, $\underline{T}_2 = (T_{21}, T_{22}, \ldots, T_{2p_2})$
where

$$T_{0i} = \sum_u y_{iu}, \quad T_{1j} = \sum_i \sum_u a_{jiu}\,y_{iu}, \quad T_{2k} = \sum_i \sum_u b_{kiu}\,y_{iu}\,. \tag{2.2}$$

Let, further, $G = \sum_i \sum_u y_{iu}$.
It is easily verified that

$$E(y_{iu}) = \mu + \sum_j a_{jiu}\,\theta^1_j + \sum_k b_{kiu}\,\theta^2_k$$

and

$$\text{cov}\,(y_{iu}, y_{i'u'}) = (\delta_{ii'} - \frac{1}{p_0})\,\frac{\sigma^2_1}{r_0} + (\delta_{uu'} - \frac{1}{r_0})\,\delta_{ii'}\,\sigma^2_0 \tag{2.3}$$

where

$$\mu = \frac{1}{p_0 r_0} \sum_i \sum_u x_{iu}\,,$$

$$\sigma^2_0 = \frac{1}{p_0(r_0-1)} \sum_i \sum_u \left(x_{iu} - \frac{\sum_u x_{iu}}{r_0}\right)^2, \tag{2.4}$$

$$\sigma^2_1 = \frac{r_0}{(p_0-1)} \sum_i \left(\frac{\sum_u x_{iu}}{r_0} - \frac{\sum_i \sum_u x_{iu}}{p_0\,r_0}\right)^2,$$

and $\delta_{ss'}$ is the Kronecker symbol, $\delta_{ss'} = 1(0)$ if $s = s'(s \neq s')$.

Let $a_{\alpha u}$, $\alpha = 1,2,\ldots, r_o-1$, $u = 1,2,\ldots, r_o$, be real numbers satisfying the conditions

$$\sum_u a_{\alpha u} = 0, \quad \sum_u a_{\alpha u} a_{\alpha' u} = \delta_{\alpha\alpha'} \quad \ldots \qquad (2.5)$$

Let, further, $b_{\beta i}$, $\beta = 1,2,\ldots, p_o-1$, $i = 1,2,\ldots, p_o$ be real numbers satisfying the equations

$$\sum_i b_{\beta i} = 0, \quad \sum_i b_{\beta i} b_{\beta' i} = \delta_{\beta\beta'} \quad \ldots \qquad (2.6)$$

Define $z_{i\alpha} = \sum_u a_{\alpha u} y_{iu}$, $i = 1,2,\ldots, p_o$; $\alpha = 1,2,\ldots, r_o-1$.

It can be shown that

$$E(z_{i\alpha}) = \sum_u \sum_j a_{\alpha u} a_{jiu} \theta_j^1 + \sum_u \sum_k a_{\alpha u} b_{kiu} \theta_k^2$$

$$\text{and} \quad \operatorname{cov}(z_{i\alpha}, z_{i'\alpha'}) = \delta_{ii'} \delta_{\alpha\alpha'} \sigma_o^2 . \quad \ldots \qquad (2.7)$$

We shall also define $z_\beta = \sum_i \sum_u b_{\beta i} y_{iu}$, $\beta = 1,2,\ldots, p_o-1$.

It can be shown that

$$E(z_\beta) = \sum_i \sum_u \sum_j b_{\beta i} a_{jiu} \theta_j^1 + \sum_i \sum_u \sum_k b_{\beta i} b_{kiu} \theta_k^2$$

$$\text{and} \quad \operatorname{cov}(z_\beta, z_{\beta'}) = \delta_{\beta\beta'} \sigma_1^2 . \qquad (2.8)$$

The method of weighted least-squares can now be applied to the two sets of variables, $z_{i\alpha}$, $i = 1,2,\ldots, p_o$; $\alpha = 1,2,\ldots, r_o-1$ (intra-block contrasts) and z_β, $\beta = 1,2,\ldots, p_o-1$ (inter-block contrasts).

The weights for contrasts from these two sets are given by $w_o = \frac{1}{\sigma_o^2}$ and $w_1 = \frac{1}{\sigma_1^2}$ respectively. Putting $\rho = \sigma_1^2/\sigma_o^2$, we shall minimise

$$\phi = \sum_i \sum_\alpha [z_{i\alpha} - E(z_{i\alpha})]^2 + \frac{1}{\rho} \sum_\beta [z_\beta - E(z_\beta)]^2 \qquad (2.9)$$

w.r.t. $\underline{\theta}_1$ and $\underline{\theta}_2$. The normal equations turn out to be

$$\begin{aligned} \bar{\underline{Q}}_1 &= \underline{\theta}_1 \bar{C}_{11} + \underline{\theta}_2 \bar{C}_{21} \\ \bar{\underline{Q}}_2 &= \underline{\theta}_1 \bar{C}_{12} + \underline{\theta}_2 \bar{C}_{22} \end{aligned} \qquad \ldots \qquad (2.10)$$

where

$$\bar{\underline{Q}}_1 = \underline{T}_1 - \frac{1}{r_o} \underline{T}_o N_{o1} + \frac{1}{\rho r_o} (\underline{T}_o N_{o1} - \frac{r_1 G}{p_o} E_{1p_1})$$

$$\bar{\underline{C}}_2 = \underline{T}_2 - \frac{1}{r_o} \underline{T}_o N_{o2} + \frac{1}{\rho r_o} (\underline{T}_o N_{o2} - \frac{r_2 G}{p_o} E_{1p_2})$$

$$\bar{C}_{11} = r_1 I - \frac{1}{r_o} N_{1o} N_{o1} + \frac{1}{\rho r_o} (N_{1o} N_{o1} - \frac{r_1^2}{p_o} E_{p_1 p_1}) \qquad (2.11)$$

$$\bar{C}_{12} = N_{12} - \frac{1}{r_o} N_{1o} N_{o2} + \frac{1}{\rho r_o} (N_{1o} N_{o2} - \frac{r_1 r_2}{p_o} E_{p_1 p_2}), \bar{C}_{21} = (\bar{C}_{12})'$$

and $$\bar{C}_{22} = r_2 I - \frac{1}{r_o} N_{2o} N_{o2} + \frac{1}{\rho r_o} (N_{2o} N_{o2} - \frac{r_2^2}{p_o} E_{p_2 p_2}).$$

Here E_{mn} denotes a $m \times n$ matrix with all elements unity.

Eliminating $\underline{\theta}_1$ we get

$$\bar{\underline{Q}}_{2.1} = \underline{\theta}_2 \bar{K}_{2.1} \qquad \ldots \qquad (2.12)$$

where

$$\bar{Q}_{2.1} = \bar{Q}_2 - \bar{Q}_1 \bar{C}^*_{11} \bar{C}_{12}$$
$$\bar{K}_{2.1} = \bar{C}_{22} - \bar{C}_{21} \bar{C}^*_{11} \bar{C}_{12} \quad \ldots \quad (2.13)$$

and for a symmetrix Λ, Λ^* denotes its pseudo-inverse. (We shall adopt the same definition of pseudo-inverse as given in section 6.3).

To compute $\bar{Q}_{2.1}$ and $\bar{K}_{2.1}$ one needs the knowledge of σ_0^2 and σ_1^2. As in the case of incomplete block designs, we shall replace σ_0^2 and σ_1^2 by their respective estimates obtained from the observations themselves. A procedure for estimating σ_0^2 and σ_1^2 is given in the next section.

To obtain estimates based on intra-block contrasts, we minimise only the first term in the expression for ϕ given by (2.9). In this case, the normal equations are

$$Q_1 = \theta_1 C_{11} + \theta_2 C_{21}$$
$$Q_2 = \theta_1 C_{12} + \theta_2 C_{22} \qquad (2.14)$$

where

$$Q_1 = T_1 - \frac{1}{r_0} T_0 N_{01}$$
$$Q_2 = T_2 - \frac{1}{r_0} T_0 N_{02}$$
$$C_{11} = r_1 I - \frac{1}{r_0} N_{10} N_{01} \qquad (2.15)$$
$$C_{12} = N_{12} - \frac{1}{r_0} N_{10} N_{02}, \quad C_{21} = C'_{12}$$

and

$$C_{22} = r_2 I - \frac{1}{r_0} N_{20} N_{02}.$$

Eliminating $\underset{\sim}{\theta}_1$ we get

$$\underset{\sim}{Q}_{2.1} = \underset{\sim}{\theta}_2 K_{2.1} \quad \ldots \quad (2.16)$$

where

$$\underset{\sim}{Q}_{2.1} = \underset{\sim}{Q}_2 - \underset{\sim}{Q}_1 C^*_{11} C_{12}$$

and $\ldots$ (2.17)

$$K_{2.1} = C_{22} - C_{21} C^*_{11} C_{12} \, .$$

If $\hat{\underset{\sim}{\theta}}_2$ is a solution of equation (2.16), a solution of equation (2.14) is given by $(\underset{\sim}{Q}_1 C^*_{11} - \hat{\underset{\sim}{\theta}}_2 C_{21} C^*_{11}, \hat{\theta}_2)$.

For each of the matrices C_{11}, C_{12} and C_{22}, the sum of the elements in any row (column) is zero. From this it can be easily deduced that

$$\text{rank} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \leq p_1 + p_2 - 2 \, . \quad (2.18)$$

We shall assume that the left hand side of (2.18) equals $p_1 + p_2 - 2$. This implies ;

$$\text{rank } C_{11} = p_1 - 1, \quad \text{rank } C_{22} = p_2 - 1. \quad (2.19)$$

It also implies the estimability of any treatment contrast involving treatments from the first (second) set only. A design with this property will be called a fully connected design.

7.3. Estimation of variances.

Let $(\hat{\underline{\theta}}_1, \hat{\underline{\theta}}_2)$ denote a solution of equations (2.14). An intra-block analysis of variance may be made as follows.

Table 7.1: Analysis of variance.

source	degrees of freedom	sum of squares
(1)	(2)	(3)
blocks	p_C-1	$SS^*_B = \underline{T}_C\ \underline{T}'_C - G^2/a_C p_C$
first set of treatments adjusted for blocks,	p_1-1	$SS^1_{tr} = \underline{Q}_1 \hat{\underline{\theta}}'_1 + \underline{Q}_2\ \hat{\underline{\theta}}'_2 - \underline{Q}_{2.1}\ \hat{\underline{\theta}}'_2$
second set of treatments adjusted for blocks and the first set of treatments,	p_2-1	$SS^{2.1}_{tr} = \underline{Q}_{2.1}\ \hat{\underline{\theta}}'_2$
error	$e_C=p_Ca_C-p_C-p_1-p_2+2$	$SS_E = SS_T - SS^*_B - SS^1_{tr} - SS^{2.1}_{tr}$
total	$p_C\ a_C - 1$	$SS_T = \sum\sum y^2_{iu} - G^2/a_C p_C$

An estimate of σ^2_C is provided by $\hat{\sigma}^2_C = \dfrac{SS_E}{e_C}$. To estimate σ^2_1, we first compute $\underline{T}^*_C = (T^*_{C1}, T^*_{C2}, \ldots, T^*_{Cp_C})$ defined by

$$\underline{T}^*_C = \underline{T}_C - \hat{\underline{\theta}}_1\ N_{1C} - \hat{\underline{\theta}}_2\ N_{2C}. \tag{3.1}$$

An estimator of σ^2_1 is given by

$$\hat{\sigma}^2_1 = \frac{\Psi - b\ \hat{\sigma}^2_C}{a} \tag{3.2}$$

where

$$\psi = \sum_{i=1}^{p_0} (T^*_{0i})^2 - \frac{(\Sigma T^*_{0i})^2}{p_0} \tag{3.3}$$

$$a = r_0 (p_0 - 1) \tag{3.4}$$

and $$b = \text{trace}\{K^*_{2.1} N_{20} N_{02} + (C^*_{11} + C^*_{11} C_{12} K^*_{2.1} C_{21} C^*_{11}) N_{10} N_{01} - 2C^*_{11} C_{12} K^*_{2.1} N_{20} N_{01}\}. \tag{3.5}$$

Once C^*_{11} and $K^*_{2.1}$ are obtained, $\hat{\sigma}^2_1$ is easy to compute. It may be noted that C^*_{11} and $K^*_{2.1}$ are either computed for the intra-block analysis or can be easily constructed from a solution of (2.14), the intra-block normal equations.

As an estimate of ϱ we take

$$\hat{\varrho} = \hat{\sigma}^2_1 / \hat{\sigma}^2_0 . \tag{3.6}$$

7.4. Illustrative example.

An example discussed here will serve to demonstrate that with a careful choice of the design the above analysis does not involve very heavy computations.

We shall consider the case where both the sets contain the same number of treatments. Suppose we have a BIB design for the first set of treatments. We shall further assume that for the second set, treatments bearing the same numbers as the treatments from the first set are put in a block and that the pairing of treatments in the two sets is done in such a way that each of the matrices $N_{12} + N_{21}$ and $N_{21} N_{12}$ is of the form

$$\begin{bmatrix} l\, I_\alpha + m\, E_{\alpha\alpha} & n\, E_{\alpha\beta} \\ n\, E_{\beta\alpha} & p\, I_\beta + q\, E_{\beta\beta} \end{bmatrix}.$$

Direct computations show that for such a design both the matrices $K_{2.1}$ and $\bar{K}_{2.1}$ are of the above form. To solve the estimating equations for such a design the following lemma may be used.

Lemma 7.4.1.

Let M be a matrix given by

$$M = \begin{bmatrix} l\, I_\alpha + m\, E_{\alpha\alpha} & n\, E_{\alpha\beta} \\ n\, E_{\beta\alpha} & p\, I_\beta + q\, E_{\beta\beta} \end{bmatrix} \qquad (4.1)$$

satisfying the conditions $ME_{\alpha+\beta,1} = 0$, $n \neq 0$. The pseudo-inverse of M is given by

$$M^* = \begin{bmatrix} l^*\, I_\alpha + m^*\, E_{\alpha\alpha} & n^*\, E_{\alpha\beta} \\ n^*\, E_{\beta\alpha} & p^*\, I_\beta + q^*\, E_{\beta\beta} \end{bmatrix} \qquad (4.2)$$

where

$$l^* = \frac{1}{l}\ , \quad m^* = -\ \frac{(m - n)(\alpha + \beta) + l}{l(\alpha + \beta)(l + \overline{m-n}\ \alpha)}$$

$$n^* = -\ \frac{1}{(\alpha+\beta)(l + \overline{m-n}\ \alpha)}, \quad p^* = \frac{1}{p} \quad \text{and} \quad q^* = -\ \frac{(q-n)(\alpha+\beta) + p}{p(\alpha+\beta)(l+\overline{m-n}\ \alpha)} .$$

The proof consists in verifying that $MM^* = M^*M = I - \frac{E}{\alpha + \beta}$.

We shall consider here the following design.

Block	Treatments		
1	(1,3)	(2,1)	(3,2)
2	(2,3)	(3,4)	(4,2)
3	(3,5)	(4,3)	(5,4)
4	(4,1)	(5,4)	(1,5)
5	(5,1)	(1,2)	(2,5)
6	(1,6)	(3,1)	(6,3)
7	(2,4)	(4,6)	(6,2)
8	(3,6)	(5,3)	(6,5)
9	(4,6)	(1,4)	(6,1)
10	(5,2)	(2,6)	(6,5)

The first symbol refers to the treatment from the first set, the second refers to the treatment from the second set. We have $p_0 = 10$, $p_1 = p_2 = 6$, $r_0 = 3$, $r_1 = r_2 = 5$.

It is easily seen that

$$N_{10} = N_{20},$$

$$N_{10}\, N_{01} = N_{20}\, N_{02} = 3\,I + 2\,E$$

$$N_{12} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 2 & 0 \end{bmatrix}$$

which gives

$$N_{21}\, N_{12} = \begin{bmatrix} I_3 + 4\,E_{33} & 4\,E_{33} \\ 4\,E_{33} & 4\,I_3 + 3\,E_{33} \end{bmatrix}, \quad N_{21} + N_{12} = 2(E_{66} - I_6).$$

Thus the design belongs to the type considered above.

For intra-block analysis, we compute

$$C_{11} = r_1 I - \frac{1}{r_0} N_{10}\, N_{01} = 4\,I_6 - \frac{2}{3}\,E_{66}, \quad C_{11}^{*} = \frac{1}{4}\,I_6 - \frac{1}{24}\,E_{66},$$

$$C_{21}\, C_{11}^{*}\, C_{12} = \frac{1}{4}\,(I_6 + N_{21}\, N_{12} - N_{21} - N_{12} - \frac{8}{3}\,E_{66}).$$

This gives

$$K_{2.1} = \frac{1}{4} \begin{bmatrix} 12\,I_3 - 2\,E_{33} & -2\,E_{33} \\ 2\,E_{33} & 9\,I_3 - E_{33} \end{bmatrix}.$$

An application of lemma 7.4.1 gives us

$$K^*_{2.1} = \frac{1}{54} \begin{bmatrix} 18\, I_3 - 3\, E_{33} & -3\, E_{33} \\ -3\, E_{33} & 24\, I_3 - 5\, E_{33} \end{bmatrix},$$

A solution of (2.14) is given by $(\hat{\underline{\theta}}_1, \hat{\underline{\theta}}_2)$

where

$$\hat{\underline{\theta}}_1 = (\underline{Q}_1 - \underline{Q}_{2.1} K^*_{2.1} C_{21}) C^*_{11}$$

and

$$\hat{\underline{\theta}}_2 = \underline{Q}_{2.1} K^*_{2.1}.$$

For recovery of inter-block information we compute

$$a = r_0 (p_0 - 1) = 27$$

$$b = \text{tr} \left\{ K^*_{2.1} N_{20} N_{02} + (C^*_{11} + C^*_{11} C_{12} K^*_{2.1} C_{21} C^*_{11}) N_{10} N_{01} - 2\, C^*_{11} C_{12} K^*_{2.1} N_{20} N_{01} \right\} = 17.$$

$\hat{\sigma}^2_0$ is obtained from the analysis of variance table and $\hat{\sigma}^2_1$ can be obtained with the help of $\hat{\underline{\theta}}_1$, $\hat{\underline{\theta}}_2$, a and b given above. An estimate of ϱ is provided by $\hat{\varrho} = \hat{\sigma}^2_1 / \hat{\sigma}^2_0$.

To write down equation (2.12), we compute

$$\bar{C}_{11} = (4 + \varrho^{-1}) I_6 - \frac{4 + \varrho^{-1}}{6} E_{66} ,$$

$$\bar{C}^*_{11} = \frac{1}{4 + \varrho^{-1}} I_6 - \frac{1}{6(4+\varrho^{-1})} E_{66} ,$$

$$\bar{C}_{21} \bar{C}^*_{11} \bar{C}_{12} = \frac{1}{4 + \varrho^{-1}} \left[N_{21} N_{12} - (1 - \varrho^{-1})(N_{21} + N_{12}) + (1 - \varrho^{-1})^2 I_6 - \frac{(4 + \varrho^{-1})^2}{6} E_{66} \right]$$

and

$$\bar{K}_{2.1} = \frac{1}{4 + \rho^{-1}} \begin{bmatrix} 12(1+\rho^{-1})I_3 - 2(1+\rho^{-1})E_{33} & -2(1+\rho^{-1})E_{33} \\ -2(1+\rho^{-1})E_{33} & (9 + 12\rho^{-1})I_3 - (1+2\rho^{-1})E_{33} \end{bmatrix} .$$

ρ may now be replaced by the computed value of $\hat{\rho}$ and Lemma 7.4.1 may be applied to obtain $\bar{K}^*_{2.1}$. A solution of equations (2.10) is given by $(\underline{\theta}^*_1, \underline{\theta}^*_2)$

where

$$\underline{\theta}^*_1 = (\bar{\underline{Q}}_1 - \bar{\underline{Q}}_{2.1} \bar{K}^*_{2.1} \bar{C}_{21})\bar{C}^*_{11} ,$$

and

$$\underline{\theta}^*_2 = \bar{\underline{Q}}_{2.1} \bar{K}^*_{2.1} .$$

LIST OF REFERENCES

[1] Bose, R. C. (1949): A note on Fisher's inequality for balanced incomplete block designs, Ann. Math. Stat., 20, 619-620.

[2] Bose, R. C., Clatworthy, W.H. and Shrikhande, S.S. (1954): Tables of partially balanced designs with two associate classes, North Car. Ag. Expt. Sta. Tech. Bull. 107.

[3] Bose, R.C. and Connor, W.S. (1952): Combinatorial properties of group divisible incomplete block designs, Ann. Math. Stat., 23, 367-383.

[4] Bose, R.C. and Kishen, K. (1939): On partially balanced Youden Squares, Science and Culture, 5, 136-137.

[5] Bose, R.C. and Shimamoto, T. (1952): Classification and analysis of partially balanced incomplete block designs with two associate classes, Jour. Amer. Stat. Ass., 47, 151-184.

[6] Connor, W.S. and Clatworthy, W.H. (1954): Some theorems for partially balanced designs, Ann. Math. Stat., 25, 100-112.

[7] Davies, O.L. Ed. (1960): Design and analysis of industrial experiments, 2nd ed., Imperial Chemical Industries, London.

[8] Dunlop, G. (1933): Methods of experimentation in animal nutrition, Jour. Agri. Science, 23, 580-614.

[9] Ehrenfeld, S. (1955): On the efficiency of experimental designs, Ann. Math. Stat., Vol. 26, 247-255.

[10] Fisher, R.A. (1935): The designs of experiments, Oliver and Boyd, Edinburgh.

[11] Fisher, R.A. and Yates, F. (1957): Statistical tables for biological, Agricultural and Medical Research, 5th ed., Oliver and Boyd, London.

[12] Freeman, G.H. (1957a): Some experimental designs of use in changing from one set of treatments to another - Part I, Jour. Roy. Stat. Soc. Se. B, 19, 154-162.

[13] Freeman, G.H. (1957b) : Some experimental designs of use in changing from one set of treatments to another - Part II, Existence of the designs, Jour. Roy. Stat. Soc. Se.B, 19, 163-165.

[14] Freeman, G.H. (1958) : Families of designs for two successive experiments, Ann. Math. Stat., 29, 1063-1078.

[15] Freeman, G.H. (1961) : Some further designs of type O:PP, Ann. Math. Stat., 32, 1187-1190.

[16] Graybill, F.A. and Deal, R.B. (1959) : Combining unbiased estimators, Biometrics, 15, 543-550.

[17] Graybill, F.A. and Seshadri, V. (1960) : On the unbiasedness of Yates' method of estimation using inter-block information, Ann. Math. Stat., 31, 786-787.

[18] Graybill, F. A. and Weeks, D.L. (1959) : Combining inter-block and intra-block information in balanced incomplete blocks, Ann. Math. Stat., 30, 799-805.

[19] Hoblyn, T.N., Pearce, S.C. and Freeman, G.H. (1954) : Some considerations in the design of successive experiments in fruit plantations, Biometrices, 10, 503-515.

[20] Kempthorne, O. (1956) : The efficiency factor of an incomplete block design, Ann. Math. Stat., 27, 846-849.

[21] Kiefer, J. (1958) : On the non-randomised optimality and randomised non-optimality of symmetric designs. Ann. Math. Stat., 29, 675-699.

[22] Kshirsagar, A.M. (1958) : A note on incomplete block designs, Ann. Math. Stat., 29, 907-910.

[23] Nair, K. R. (1944) : The recovery of inter-block information in incomplete designs, Sankhyā, 6, 383-390.

[24] Nair, K.R. and Rao, C.R. (1942) : Incomplete block designs for experiments involving several groups of varieties, Science and Culture, 7, 625.

[25] Nandi, H.K. (1951) : On the efficiency of experimental designs, Calcutta Stat. Assoc. Bull., 3, 167-171.

[26] Pearce, S.C. and Taylor, J. (1948): The changing of treatments in a long term trial, Jour. Ag. Sci., 38, 402-410.

[27] Pearson, Karl ed. (1934): Tables of the incomplete Beta-function, Proprietors of Biometrika, London.

[28] Pothoff, R.F. (1962): Three-factor additive designs more general than the Latin Square, Technometrics, 4, 187-208.

[29] Rao, C. R. (1947): General methods of analysis for incomplete book designs, J.Amer. Stat., Ass., 42, 541-561.

[30] Rao, C.R. (1952): Advanced statistical methods in biometric research, John Wiley and Sons, New York.

[31] Rao, C. R. (1955): Analysis of dispersion for multiply classified data with unequal numbers in cells, Sankhyā, 15, 253-280.

[32] Rao, C. R. (1959): Expected values of mean squares in the analysis of incomplete block experiments and some comments based on them, Sankhyā, 21, 327-336.

[33] Rao, V.A. (1958): A note on balanced designs, Ann. Math. Stat., 29, 290-294.

[34] Roy, J. (1958): On the efficiency factor of block designs, Sankhyā, 19, 181-188.

[35] Roy, J. and Laha, R.G. (1958). Classification and analysis of linked block designs, Sankhyā, 17, 115-132.

[36] Shrikhande, S.S. (1951): Designs for two way elimination of heterogeneity, Ann. Math. Stat., 22, 235-247.

[37] Stein, C. (1950): Unbiased estimates with minimum variance, Ann. Math. Stat., 21, 406-415.

[38] Tang, P. C. (1938): The power function of the analysis of variance tests with tables and illustrations of their use, Statistical Research Memoirs, 2, 126-149.

[39] Wald, A. (1943) : On the efficient design of statistical investigations. Ann. Math. Stat., 14, 134-140.

[40] Weeks, D.L. and Graybill, F.A. (1961) : Minimal sufficient statistics for the balanced incomplete block design under an Eisenhart model II, Sankhyā, Series A, 23, 261-268.

[41] Yates, F. (1936) : Incomplete randomized blocks, Ann. Eugen., 7, 121-140.

[42] Yates, F. (1937) : The design and analysis of factorial experiments, Imp. Bur. Soil. Sci. Tech. Comm. 25.

[43] Yates, F. (1939) : The recovery of inter-block information in varietal trials arranged in three dimensional lattices, Ann. Eugen., 9, 136-156.

[44] Yates, F. (1940) : The recovery of inter-block information in balanced incomplete block designs, Ann. Eugen., 10, 317-325.

[45] Youden, W.J. (1937) : Use of incomplete block replications in estimating tobacco-mosaic virus, Contr. Boyce Thompson Inst., 9, 41-48.

[46] Youden, W.J. (1951) : Linked blocks : a new class of incomplete block designs (Abstract), Biometrics, 7, 124.

A list of author's papers :

1] (1959) : Optimality criteria for incomplete block designs, Ann. Math. Stat., 31, 791-794.

2] (1961) : Analysis of two-way designs, Sankhyā, Series A, 23, 129-144 (jointly with J. Roy).

3] (1962) : An estimate of inter-group variance in one and two-way designs, Sankhyā, Series A, 24, 281-286.

[CONTINUED]

A list of author's papers (continued)

4] 1962 : Recovery of inter-block information, Sankhyā, Series A, 24, 269-280 (jointly with J. Roy).

5] Use of inter-block information to obtain uniformly better estimators (awaiting publication).

6] On a result in the recovery of inter-block information (awaiting publication).

7] Analysis of two-period experiments (awaiting publication).