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SOLUTIONS OF SOME BALANCED DOUBLY BALANCED AND PARTIALLY BALANCED STATISTICAL DESIGNS

by

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INTRODUCTION

The theory and methods of construction of statistical designs have connections with Modern algebraic systems, theory of numbers, arithmetic theory of quadratic forms, information theory and construction of codes. The properties of finite linear spaces have been used for the construction of (i) complete set of mutually orthogonal latin squares (Bose and Nair, 1941) (ii) balanced incomplete block designs (Bose 1939) (iii) partially balanced incomplete block designs (Bose and Nair 1939) (iv) designs where some effects of treatments are confounded (Bose, 1947). Primrose (1951) studied quadric surfaces and used them in the construction of balanced incomplete block designs. Roychoudhuri (1962) has generalised his study on quadrics and obtained several series of partially balanced incomplete block designs through linear spaces contained in a quadric. Shrikhande and Singh (1962) have observed some relations between association schemes and balanced incomplete block designs and using them they have given solutions to some practical designs.

In this thesis solutions to balanced, doubly balanced and partially balanced incomplete block designs some of which are not listed in the known tables have been obtained. A detailed summary of work done in each chapter is given at the beginning of that chapter. Below is given a brief summary of results in various chapters.

Chapter I deals with methods of constructing balanced incomplete block designs from association schemes. The incidence matrix of the design is determined as the partitioned matrix $(B^1 \vdots B^2 \vdots \ldots\ldots \vdots B^t)$ where $B^i = (b^i_{jk})$ is the i-th association matrix and $b^i_{jk} = 1$ if the objects j and k are i-th associates or zero otherwise. In the same chapter a new general series of balanced incomplete block designs is obtained through difference sets which contains a series of balanced incomplete block designs given by Gassner (1965). This series is obtained by taking a special set of elements of cartesian product of Galois fields in the initial blocks.

In chapter II geometries imbedded in finite projective geometry PG(n, s) of n dimensions based on a Galois field of order s are investigated. The concept of generating Restricted Linear Analytic Independent set of points is introduced. It is shown that such a set generates a geometry isomorphic to a PG(r, s_1) imbedded in

PG(n, s). The properties of PG(1, s_1) - Line segments imbedded in PG(1, s) are studied to a greater extent than higher dimensional spaces. Singular and nonsingular imbedded planes are defined and non-singular planes imbedded in a plane are used to construct a series of Regular Group Divisible designs. This series contains new designs not listed previously as shown in the appendix B. In the general case of imbedded geometries PG(r, s_1) in PG(n, s) the truncated configuration of lines is used to construct Pairwise balanced designs (Bose and Shrikhande, 1960) which are useful in the study of orthogonal latin squares. The properties of line segments contained in a line are used to construct a series of doubly balanced incomplete block designs such that every triplet of treatments appears exactly once in the design.

In chapter III non-degenerate quadrics in PG(2t-1, s) are studied. The form A of a non-degenerate quadric is classified as hyperbolic or elliptic according as $(-1)^t$ det A is a square or a non-square where the characteristic of the field is different from 2. Non linear configurations like cones and vertex-less cones contained in a nondegenerate quadric in finite projective geometry are studied and the explicit member of such configurations obtained. Their properties are used to construct several series of symmetric balanced and partially balanced incomplete block designs. A non-isomorphic solution to the wellknown hyperplanes solution of the symmetric balanced incomplete block design is obtained through tangent cones of a nondegenerate quadric Q_{2t} in PG(2t, s). This series includes a non-isomorphic solution to the symmetric design with parameters $v = 15$, $k = 7$, $\lambda = 3$ for which Fisher and Yates (1963) show only one solution- (a, b, c, e, f, i, k) - the cyclic one.

N O T A T I O N

Symbol used	Meaning of the symbol
$\cap$	Intersection (of sets)
$\subseteq$	is contained in
$\in$	belongs to
$\notin$	does not belong to
$\Rightarrow$	implies
[P]	The set containing the element P
PG(n,s)	Finite Projective Geometry based on a Galois Field G.F. (s)
$\mid B \mid$	Number of elements in the set B
det A	Determinent of the matrix A
Δ_r	Imbedded Projective Geometry of dimensions r and of order s_1

BALANCED INCOMPLETE BLOCK DESIGNS FROM ASSOCIATION SCHEMES AND DIFFERENCE SETS

0. SUMMARY

In this chapter three theorems on the existence of balanced incomplete block designs are proved using the existence of association schemes and difference sets. The main results on association schemes are that if an association scheme exists with v objects and n_i i^{th} associates, P_i the matrix of p^i_{jk}'s, $i = 1, 2, \ldots, m$; such that

$$n_1 = n_2 = \ldots = n_t = n \quad \text{for } 1 \leq t \leq m$$

and

$$\sum_{j=1}^{t} P^i_{jj} = \lambda \quad \text{for } i = 1, 2, \ldots, m$$

there exists a balanced incomplete block design* with parameters:

$$(v, vt, tn, n, \lambda),$$

and that if $t = m$ then under the same conditions of the above theorem there exists anothdr series of balanced incomplete block designs with parameters:

$$\begin{matrix} v & b & r & k & \lambda \\ (mk+1, & m(mk+1), & m(k+1), & k+1, & k+1) \end{matrix}$$

This series is new for $m > 2$. A new design of this series has parameters

$$(31, 93, 33, 11, 11)$$

obtained from a three associate scheme of Bose and Nair (1939).

In section 4 the following observations are made on difference sets: Let v be any integer and $v = s_1 . s_2 \ldots s_m$ be its prime power

*As far as the author is aware there is only one result of S.S. Shrikhande and N. K. Singh (1962) known in this direction connecting the existence of association schemes and Balanced Incomplete Block designs decomposition. Consider the cartesian product of the m Galois Fields G.F.(s_i), $i = 1, 2, \ldots, m$. Let x_i denotes the primitive root of G.F.(s_i), $i = 1, 2, \ldots, m$. Then label the points of the product space with

$$\alpha_{j+1} = (x_1^j, x_2^j, \ldots, x_m^j), \quad j = 0, 1, \ldots, s_1 - 2;$$

where s_1 is the least prime power factor of v, label arbitrarily the remaining points. Addition and multiplication of these points is defined in co-ordinate-wise manner. Let β_j $j = 1, 2, \ldots, \frac{v-1}{2}$ be a set of points such that no two β_j's add to the null vector. Then the $\frac{v-1}{2}$ initial blocks:

$$(\beta_1\alpha_o, \beta_1\alpha_1, \ldots, \beta_1\alpha_{k-1}), \ldots \quad \ldots, (\beta_{\frac{v-1}{2}}\alpha_o, \beta_{\frac{v-1}{2}}\alpha_1, \ldots, \beta_{\frac{v-1}{2}}\alpha_{k-1}),$$

where $k \leq s_1$ if $m > 1$ and $k < s_1$ if $m = 1$, and $\alpha_o = (0, 0, \ldots, 0)$. generate the following new series of balanced incomplete block designs*: $(v, \frac{v.v-1}{2}, \frac{k-v-1}{2}, k, \frac{k.k-1}{2})$ by adding the v points of the product space to each of these $\frac{v-1}{2}$ initial blocks.

1. B.I.B.D.'s FROM ASSOCIATION SCHEMES

Theorem 1.1. If an m- associate scheme on v objects exists where

$$n_1 = n_2 = \ldots = n_t = n \quad \text{for } 1 \leq t \leq m$$

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* This generalises the series given by B.J. Gassner ('Equal difference BIB designs', Proc. Amer. Math. Soc. Vol. 16, 3, 1965, 378-380).

and

$$p^i_{11} + p^i_{22} + \dots + p^i_{tt} = \lambda$$

for all $i = 1, 2, \dots, m$;

There exists a balanced incomplete block design with the following parameters

$$(v, vt, tn, n, \lambda).$$

A constructive proof is given here. Identify the objects with the treatment of a design. Also each object i, $i = 1, 2, \dots, v$, construct t blocks of size n each

$$B_{iL} \quad L = 1, 2, \dots, t;$$

where the Lth block contains as n treatments the n, Lth associates of the object i. This is an arrangement of v treatments in vt blocks, each block of size n. Let us note that in all there are $tn = r$ objects to be denoted by $i_1, i_2, \dots\, i_r$; which are either first associates or second associates or .... or t-th associates of an object j. Therefore treatment j will appear in all r times in the design, once in the set of t blocks

$$B_{i_k, L} \quad , L = 1, 2, \dots t$$

for each $k = 1, 2, \dots, r$.

Consider a pair of objects j and j' which are i-th associates $i = 1, 2, \dots, m$. Treatment j and treatment j' appear in a block

$B_{i,L}$ if there exists an object i to which both j and j' are L-th associates. Hence the number of times such a pair of treatments j and j' appears together in a block of the design is

$$p^i_{11} + p^i_{22} + \dots + p^i_{tt} = \lambda$$

Since is independent of i the design obtained is a balanced incomplete block design with parameters

$$(v,\ vt,\ tn,\ n,\ \lambda)$$

Corollary 1.1. If there exists a two classes association scheme with parameters n_1, n_2 and matrices P_1, and P_2 such that either

( ) $p^1_{11} = p^2_{11} = \lambda$ or (ii) $p^1_{22} = p^2_{22} = \lambda$

then we can construct a symmetrical BIBD with parameters $v,\ r = n_1,\ \lambda = \lambda$ or $v,\ r = n_2,\ \lambda = \lambda$ according as condition (i) or (ii) is satisfied. (Theorem 2 of Shrikhande, S.S. and N.K. Singh (1962)).

By taking t = 1 in the above theorem this result follows easily.

Corollary 1.2. If a two associate scheme with parameters

$$v = 4t + 1$$

$$n_1 = n_2 = 2t$$

$$p^1_{11} = t-1$$

$$p^2_{11} = t$$

exists then a balanced incomplete block design with parameters

$$v = 3t + 1$$
$$b = 8t + 2$$
$$r = 4t$$
$$k = 2t$$
$$\lambda = 2t - 1$$

exists.

This corollary follows by solving for p^i_{jk}'s under the additional condition that

$$p^1_{11} + p^1_{22} = p^2_{11} + p^2_{22} = 2t-1.$$

Theorem 1.2. If an association scheme with $v = mk + 1$ objects exists such that

$$n_1 = n_2 = \ldots. = n_m = (v - 1) / m = k$$

and

$$p^i_{11} + p^i_{22} + \ldots. + p^i_{mm} = \lambda = k - 1$$

then there exists the following series of balanced incomplete block designs with parameters

I. $(mk + 1,\ m(mk + 1),\ mk,\ k,\ k - 1)$

II. $(mk + 1,\ m(mk + 1),\ m(k + 1),\ k + 1,\ k + 1)$.

The existence of series I is obvious by theorem 2.1. In fact series I is known to exist for all prime powers $v = mk + 1$ by a theorem of Sprout (1954) which; uses difference sets. Theorem 1.1 enables us to construct designs of this series even for non-prime power values of v.

Series II is obtained from series I by adding the ith treatment to each of the m blocks

$$B_{iL}, \quad L = 1, 2, \ldots., \quad m.$$

Note: If m = 2 the two series are complementary. But if m > 2 it is no more the case and series II is a new one.

1.1. Association Schemes Satisfying the conditions of the theorem.

1.1.1. Latin Square Association Scheme:

L_i denotes the latin square association scheme defined by Bose and Shimamoto (1952) on the $v = n^2$ objects. Here the objects are identified with the n^2 cells of a square with n rows and n columns. Two objects are first associates if the corresponding cells are either in the same row or in the same column of the square or they contain the same letter of any one latin square in a chosen set of (i-2) mutually orthogonal latin squares of order n. We know its parameters are:

$n_1 = i(n-1)$; $\quad n_2 = (n+1-i)(n-1)$;

$p^1_{11} = (n-2) + (i-1)(i-2)$, $\quad p^1_{12} = (i-1)(n+1-i)$,

$p^1_{22} = (n-i)(n+1-i)$; $\quad p^2_{11} = i\,(i-1)$,

$p^2_{12} = i\,(n-i)$, $\quad p^2_{22} = (n-2)+(n-i)(n-i-1)$.

For a given n if there exist (n-3)/2 mutually orthogonal latin squares of order n, then the latin square association scheme $L_{(n+1)/2}$ satisfies the conditions of theorem 2.1. If n is a prime power, a complete set of mutually orthogonal latin squares exists and $L_{(n+1)/2}$ could be constructed leading to designs with parameters

$$v = n^2, \qquad b = 2n^2$$

$$r = n^2 - 1 \qquad k = \frac{n^2 - 1}{2}$$

$$\lambda = \frac{n^2 - 3}{2}$$

The complementary design has the following parameters:

$$v = n^2 \qquad b = 2n^2$$

$$r = n^2 + 1 \qquad k = \frac{n^2 + 1}{2}$$

$$\lambda = \frac{n^2 + 1}{2}$$

1.1.2. Some Cyclic Association Schemes.

Now let us consider the extended Partial Youden squares listed in Shrikhande (1951). The designs with rdference numbers 1,4,6 & 8 in this list satisfy the conditions of theorem 1.1. The cyclic designs of B.C.S. Catalogue (1954) with reference numbers C_1, C_5, C_8 and C_{10} also satisfy the conditions of theorem 1.1. From these designs balanced incomplete block designs that can be obtained or listed in the following page. (It may be noted that these designs can be obtained through difference sets of Sproutt 1954).

v	b	r	k	λ	Ref.no. in Shrikhande	Ref. no. in B.C.S.
13	26	12	6	5	1	C_1
17	34	16	8	7	4	C_5
25	50	24	12	11	6	-
29	58	28	14	13	8	C_8
37	74	36	18	17	-	C_{10}

1.1.3. A Three Associate Scheme From Difference Sets*.

Bose and Nair (1939) have obtained a three associate design through the difference set

(1, 2, 4, 8, 15, 16, 23, 27, 29, 30)

in the module of residue classes modulo 31. It may be verified that the association scheme of this design satisfies the condition of theorem 1.1. Hence we have the following two designs by theorem 1.1:

sl.	v	b	r	k	λ
1	31	93	30	10	9
2	31	93	33	11	11

The design sl. 2 is new balanced incomplete block design. Its lay-out is indicated in the appendix and listed as B.3.

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* Let G be an abelian group of order v. A set D of k distinct elements from G is called a difference set, if the k(k-1) differences of the elements of D contain every non-zero element of G exactly times. These definitions are generalised and in place of a single set D, one can take t initial sets of k elements each and demand that t.k(k-1) intra set differences from the t sets contain every non-zero element of G the same number of times.

2. BIBD'S FROM DIFFERENCE SETS

Let G.F.(s_i) denote a Galois Field of order $s_i = p_i^{e_i}$ and x_i be a primitive root in the field G.F.(s_i), $i = 1, 2, \ldots, m$. Let v be an odd integer with the following prime power decomposition:

$$v = p_i^{e_1} \ldots \ldots p_m^{e_m} = s_i \, s_2 \ldots s_m .$$

Let s_1 be the least prime power factor of the integer v and β denote a point of the cartesian product G of the m finite fields

$$\text{G.F.}(s_i), \; i = 1, 2, \ldots, m$$

$$G = \text{G.F.}(s_1) * \text{G.F.}(s_2) * \ldots * \text{G.F.}(s_m) ;$$

where the operations of addition and multiplication of two points are defined coordinatewise in their respective fields. Let us label some of the points β's by α's as follows:

$$\alpha_{j+1} = (x_1^j, \; x_2^j, \ldots \ldots, x_m^j)$$

$$j = 0, 1, \ldots \ldots, s_1 - 2$$

$$\alpha_o = (0, 0, \ldots \ldots, 0).$$

Let B denote the set of points

$$B : (\alpha_o, \alpha_1, \ldots \ldots, \alpha_{k-1})$$

where $k \leq s_1$ if $m > 1$ and $k < s_1$ if $m = 1$.

2.1. Some Lemmas on the set B.

Lemma 2.1.1. If α_c and α_d are any two distinct elements of the set B, then $(\alpha_c - \alpha_d)$ has a multiplicative inverse.

Proof follows easily since no coordinate of the point $(\alpha_c - \alpha_d)$ is zero and hence a multiplicative inverse exists for each coordinate in their respective fields.

Lemma 2.1.2. If $\alpha_c \in B$ and $c \neq 0$ or 1 then $\alpha_c^{-1} \notin B$.

For, otherwise if a suffix d exists such that

$$\alpha_c \alpha_d = \alpha_{c+d-1} = \alpha_1$$

then

$$c + d - 2 = 0 \mod [(s_1 - 1), (s_2 - 1), \ldots, (s_m - 1)] \ldots 2.1.2.$$

Since c and d are both not greater than (k-1) which is $\leq (s_1 - 1)$, the fields being of odd order and $c \neq d$ the maximum value which (c+d-2) can take is $2(s_1 - 1) - 3$. Hence (c+d-2) can take the value $(s_1 - 1)$ in which case (c+d-2) can not be equal to

$$0 \mod (s_i - 1);\ i = 2, 3, \ldots \ldots m.$$

Hence in no case can 2.1.2 be satisfied.

PROPOSITION 2.1.3. A set T of (v-1)/2 points $[\beta_j]$ $j = 1, 2, \ldots, (v-1)/2$; can be selected from the product space G such that if $\beta_j \in T$ then its additive inverse $-\beta_j \notin T$.

A constructive proof is give below:

The set T does not contain α_o since it is its own additive inverse. From the remaining (v-1) elements of the product space G, form (v-1)/2 distinct pairs of distinct points, each pair containing a point of G and its additive inverse. The set T can be obtained now by selecting one point from each of such pairs. The point selected for the set T from the jth pair is denoted by β_j and the other member of the pair by $\beta_{j+(v-1)/2}$. We have clearly

$$\beta_{j+(v-1)/2} = -\beta_j \; ; \quad j = 1, 2, \ldots \ldots, (v-1)/2.$$

For convenience the point α_0 is also denoted by β_0.

Theorem 2.1. From the initial blocks

$$B_1 : (\beta_1\alpha_0, \beta_1\alpha_1, \ldots \ldots \ldots, \beta_1\alpha_{k-1})$$
$$B_2 : (\beta_2\alpha_0, \beta_2\alpha_1, \ldots \ldots \ldots, \beta_2\alpha_{k-1})$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$B_{(v-1)/2} : (\beta_{(v-1)/2}\,\alpha_0, \ldots, \beta_{(v-1)/2}\alpha_{k-1})$$

on adding the v points of the product space G to each element of each block a balanced incomplete block design with the following parameters results in:

$$v = v$$
$$b = v.(v-1)/2$$
$$r = k.(v-1)/2$$
$$k = k$$
$$\lambda = k.(k-1)/2.$$

Proof: The v points $\{\beta_j\}$, $j=0, 1, 2, \ldots \ldots, (v-1)$; of the product space G are taken as the v treatments of the design. First we establish that each initial block contains distinct treatments, then that every treatment appears in r blocks and finally that every pair of treatments appears in λ blocks.

If the initial block B_j had contained two identical treatments then we should have:

$$\beta_j \alpha_c = \beta_j \alpha_d \quad , \quad c \neq d$$

i.e.

$$\beta_j (\alpha_c - \alpha_d) = \alpha_o$$

which would imply that

$$\beta_j = \alpha_o (\alpha_c - \alpha_d)^{-1} = \alpha_o$$

and since α_o is not in the set T which contains β_j the initial block B_j could not have had two identical points. Hence every block of the design has distinct treatments.

Now we will show that each treatment appears exactly λ times. Let β be any point in the product space G. Consider the v blocks generated by the initial block B_j for some fixed j, j=1, 2, ...(v-1)/2

Let

$$\beta - \beta_j \alpha_c = \beta_c \quad ,$$

then the point β appears in the k blocks

$$B_j + \beta_c \quad , \quad c = 0, 1, \ldots \ldots, (k-1)$$

Hence as j = 1, 2, ..., (v-1)/2; we observe that every treatment appears in r = k. (v-1)/2 blocks.

To show that every pair of treatments appears λ times in the design, consider the differences

$$\beta_j (\alpha_c - \alpha_d) \text{ and } -\beta_j (\alpha_c - \alpha_d) = \beta_{j+(v-1)/2} (\alpha_c - \alpha_d)$$

arising from the pair of points

$$(\beta_j \alpha_c , \beta_j \alpha_d)$$

of the initial block B_j and the (v-1) such differences arising from the (v-1)/2 initial blocks B_j, j=1, 2, ... (v-1)/2. We shall show that among these (v-1) differences all the non-null elements of G appear for some

$$j \neq j' \quad (j, j' = 1, 2, \ldots \ldots (v-1)/2)$$

we must have

$$\beta_j (\alpha_c - \alpha_d) = \beta_{j'} (\alpha_c - \alpha_d)$$

By multiplying both sides by the multiplication inverse of $(\alpha_c - \alpha_d)$ we have

$$\beta_j = \beta_{j'}$$

which is impossible.

Among the k.(k-1)(v-1)/2 differences from the initial blocks every point appears λ = k.(k-1)/2 times and hence every pair of treatments appears λ times.

2.2. A property of the design: The b blocks, of design derived in theorem 2.1, are all distinct.

Consider two initial blocks:

$$B_j = (\beta_j \alpha_o, \beta_j \alpha_1, \ldots \ldots, \beta_j \alpha_{k-1})$$

and

$$B_i = (\beta_i \alpha_o, \beta_i \alpha_1, \ldots \ldots, \beta_i \alpha_{k-1}).$$

If

$$\beta_j \alpha_1 \not\in B_i$$

then B_j and B_i are distinct blocks.

If

$$\beta_j \alpha_1 \in B_i$$

then we have that

$$\beta_i \alpha_1 \not\in B_j ,$$

for, otherwise for some c, we should have

$$\beta_j \alpha_1' = \beta_i \alpha_c , \quad 1 \leq c \leq k - 1$$

and hence

$$\beta_i \alpha_1 = \beta_j \alpha_c^{-1} \alpha_1^2 = \beta_j \alpha_c^{-1}$$

since $\alpha_1^2 = \alpha_1$ as $\alpha_1 = (1, 1, \ldots, 1)$, showing that $\alpha_c^{-1} \in B$ which is not true by proposition 2.1.3.

DOUBLY BALANCED AND PARTIALLY BALANCED DESIGNS FROM IMBEDDED GEOMETRIES

0. SUMMARY

Let $PG(n, s)$ denote a finite projective geometry of n dimensions based on a Galois Field G.F.(s) of order s. Let $G.F.(s_1) = G_1$ be a sub-field in G.F.(s). Every geometric point has s-1 analytic representations. A set P_r of r+1 geometric points is said to be Restricted Linear Analytic (RLA) independent with respect to (S_r, G_1) where S_r is a set of fixed analytic points one corresponding to each geometric point of P_r, if no linear combination of the analytic points of S_r with coefficients chosen from the subfield G_1 vanishes unless all the coefficients are zero. Taking a set P_r which is RLA independent with respect to (S_r, G_1) consider all linear combinations of analytic points S_r with coefficients restricted to the sub-field. A geometric point, for which one of its associated analytic points lies in this set of Restricted linear combinations will have (s_1-1) analytic representations in this set. An RLA independent set P_r is said to be Generating if a geometric point has either zero or (s_1-1) analytic points in the set of all Restricted Linear (RL) combinations of points of S_r.

It is proved that the space Δ_r of geometric points associated with the analytic points of the set of RL combinations of points of S_r is

isomorphic to a PG(r, s_1) if P_r is a Generating RLA independent set with respect to (S_r, G_1). Δ_0 are points, Δ_1 are called Line segments. These are also defined to be nonsingular imbeeeded geoemtries of dimentions 0 and 1 respectively. The imbedded geometry Δ_r obtained by the generating set P_r of RLA independent set (S_r, G_1) is said to be non-singular if Δ_{r-1} obtained by the generating set P_{r-1} of RLA independent set (S_{r-1}, G_1) is non-singular and the geometric point of P_r which is not in P_{r-1} is not incident to any line generated by points of Δ_{r-1}. Necessary conditions for the existence of non-singular geometry Δ_r are obtained.

A number of combinatorial properties of Line segments are obtained in detail and they are used in the construction of Doubly Balanced Incomplete Block designs (Calvin, 1954) where every triplet of objects occurs the same number of times. This contains new designs in the practical range.

Let Δ_2 be a non-singular imbedded finite projective plane of order s_1 (which could be generated as indicated above) in a PG(2, s). A line of PG(2, s) is classified as an outsideline (it does not have any points in common with the imbedded plane) or a tangent (it has exactly one point in common with the imbedded plane) or a secant (it has a line segment of the imbedded plane in it). If $s=s_1^2$ every line is shown to be either a tangent or a secant. In this case by cutting off the imbedded plane and taking the tangents to the imbedded plane without the cut-off points as blocks and with the remaining points as treatments one obtains a Regular Group Divisible Partially Balanced Incomplete block designs.

1. INTRODUCTION

Let GF(s) = G denote a Galois field of order s and P.G.(n, s) a finite projective geometry of n dimensions based on the field GF(s). A point χ of the geometry has (s-1) analytic points associated with it represented by

$$\chi : (\rho\alpha_0, \rho\alpha_1, \ldots, \rho\alpha_n) = \rho\alpha$$

where ρ is any nonzero element in G and $\alpha_i \in G$, i=0, 1, ..., n; not all zero.

Let us note here that a finite projective geometry P.G.(n, s) of n dimensions based on GF(s) contains subsets of points χ of the geometry, called its subspaces or flats satisfying the following properties:

(i) there exist subspaces Δ_h of dimension h in P.G.(n, s), for h = -1, 0, 1, ..., n where $\Delta_{-1} = \emptyset$: the empty set Δ_0 are points Δ_1 are lines, Δ_2 are planes etc. and P.G.(n, s) = Δ_n itself.

(ii) $\Delta_h \subseteq \Delta_k$ (h, k = -1, 0, 1, ..., n) $\Longrightarrow h \leq k$ and h = k if and only if $\Delta_h = \Delta_k$.

(iii) The points common to Δ_h and Δ_k constitute a subspace Δ_r called its intersection. The space Δ_s of minimum dimensions containing Δ_h and Δ_k (necessarily unique) is called the join of Δ_h and Δ_k which is the intersection of all subspaces that contain both Δ_h and Δ_k.

(iv) $h + k = r + s$ where h, k, r, s are as obtained above.

A Δ_h is defined in F.G.(n, s) as the set of points χ such that $\alpha A = 0$, where A is an $n + 1$ by m matrix of rank $n - h$ and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ denotes the point χ.

The geometric points are denoted with upper suffices on Greek letters and elements of GF(s) by $\rho_i, \alpha_j^i, \mu_i$.

Let $G_1 = GF(s_1) \subseteq GF(s)$. The elements of $GF(s_1)$ are denoted by λ's and are called Restricted elements. A Restricted linear combination means linear combinations with coefficients being chosen from G_1 only.

2. RESTRICTED LINEAR ANALYTIC (RLA) INDEPENDENCY

Let

$$P_r = [\chi^0, \chi^1, \ldots, \chi^r]$$

denote a set of $r + 1$ geometric points and

$$S_r = [\alpha^0, \alpha^1, \ldots, \alpha^r]$$

a set of analytic points where α^i as a fixed analytic representation of χ^i, $i = 0, 1, \ldots, r$.

2.1. The set P_r is said to be Restricted Linear Analytic Independent with redpect to (S_r, G_1) or briefly RLA independent (S_r, G_1) if

$$\left.\begin{array}{c} \lambda_0, \lambda_1, \ldots, \lambda_r \in G_1 \\ \lambda_0 \alpha^0 + \lambda_1 \alpha^1 + \ldots, + \lambda_r \alpha^r = 0 \end{array}\right\} \Longrightarrow \lambda_0 = \lambda_1 = \ldots = \lambda_r = 0.$$

2.2. A set P_r, which is RLA independent (S_r, G_1) for every analytic set S_r of P_r, is called Restricted Linear (RL) independent set with respect to G_1.

The concept of RL independency may appear to be more general since when $GF(s_1) = GF(s)$ RL independency is same as Linear independency. We shall however prove:

Theorem 2.3. A necessary and sufficient condition that P_r is an RL independent set is that P_r is a linearly independent set.

Proof: Sufficiency of the condition is obvious. To prove necessity, let P_r be RL independent but not linearly independent, i.e., there exist elements

$$\varrho_o, \varrho_1, \dots \dots, \varrho_r$$

not all zero in $GF(s)$ such that

$$\varrho_o \alpha^o + \varrho_1 \alpha^1 + \dots + \varrho_r \alpha^r = 0 \qquad \text{——I.}$$

Consider the set

$$S'_r = [\, \beta_o, \beta_1, \dots, \beta_r \,]$$

where

$$\beta^i = \begin{cases} \varrho_i \alpha^i & \text{if } \varrho_i \neq 0 \\ \alpha^i & \text{if } \varrho_i = 0 \qquad i = 0, 1, \dots, r. \end{cases}$$

P_r is supposed to be RLA independent with respect to every one of its S_r sets and hence with respect to S'_r in particular. Consider

$$\lambda_o \alpha^o + \lambda_1 \alpha^1 + \dots + \lambda_r \alpha^r = \alpha$$

where

$$\lambda_i = \begin{cases} 1 & \text{if } \varrho_i \neq 0 \\ 0 & \text{if } \varrho_i = 0 \end{cases} \qquad i = 0, 1, \ldots, r.$$

By I, α is reduced to the null vector though not all λ_i's are zero. Thus RL independency is equivalent to linear independency but RLA independency is not. RLA independency is a more general concept than linear independency.

2.4. The set

$$P_1 : [\, \chi^0, \chi^1 \,]$$

is always linearly independent, hence RL independent (G_1) and hence RLA independent (S_1, G_1) with reference to any specific analytic representation S_1 and subfield G_1 if only the two geometric points χ^0 and χ^1 are distinct.

2.4.1. But P_1 can be RLA independent with respect to some S_1 and G_1 though χ^0 and χ^1 are not distinct, as for example the set

$$P_1 = [\, \chi^0, \chi^1 \,]$$

is RLA independent with respect to (S_1, G_1) if

$$S_1 = [\, \alpha^0, \varrho\, \alpha^0 \,]$$

and $\varrho \notin G_1$.

3. GENERATING RLA INDEPENDENT SET:

3.1. Let the set P_r be RLA independent set (S_r, G_1). The set P_r is said to be of the generating type and referred to as generating RLA independent (S_r, G_1) set if in addition*

$$\left.\begin{array}{ll}\text{(i)} & \lambda_o, \lambda_1, \ldots, \lambda_r; \ \lambda'_o, \lambda'_1, \ldots, \lambda'_r \ \varepsilon\ G_1 \\ \text{(ii)} & (\lambda_o - \lambda'_o \varrho)\, \alpha^o + \ldots + (\lambda_r - \lambda'_r \varrho)\, \alpha^r = 0\end{array}\right\} \Rightarrow \varrho\ \varepsilon\ G_1$$

Note that the set in 2.4.1. though RLA independent set (S_1, G_1) is not of generating type.

Example 3.1.a: Consider $GF(2^4)$ associated with the minimum function:

$$x^4 + x^3 + 1 = 0.$$

In $P.G.(2, 2^4)$ let us consider the set

$$P_2 = [\chi^o, \chi^1, \chi^2]$$

of points with respect to

$$S_2 = [\alpha^o, \alpha^1, \alpha^2]$$

where

$$\alpha^o = (0,0,x), \quad \alpha^1 = (0,x,0), \quad \alpha^2 = (0, x^2, x^2 + 1)$$

observe that P_2, though not linearly independent, is a generating RLA independent set $(S_2, GF(2))$.

Example 3.1.b: In $P.G.(2, 2^4)$ as above the set P_2 with S_2 where

$$\alpha^o = (0,0,x), \quad \alpha^1 = (0,x,0), \quad \alpha^2 = (1,0,0)$$

is also a generating RLA independent set.

---

* This additional condition guarantees that in the set of analytic points generated by taking all non-null linear combinations of the points α^i, $i=0, 1, \ldots, r$ with coefficients restricted to the subfield G_1, a geometric point χ would have exactly $s_1 - 1$ analytic representations, if it has one in this set.

Example 3.1.c: Again in P.G.$(2, 2^4)$ with the same minimum function the set P_2 with respect to

$$S_2 = [\ \beta^o, \beta^1, \beta^2\ ] \qquad \text{and } GF(2)$$

where

$$\beta^o = (0, 0, x), \quad \beta^1 = (0, x, 0) \quad \beta^2 = (0, x^2, x^2 + 1)$$

is generating RLA independent set but with respect to

$$S_2' = [\ \alpha^o, \alpha^1, \alpha^2\ ] \text{ and } GF(2)$$

where

$$\alpha^o = (0, 0, x) = \beta^o, \quad \alpha^1 = (0, x, 0) = \beta^1, \quad \alpha^2 = (0, x, x^3 + x^2 + x) = (x^3 + x^2).\beta^2$$

is not a generating RLA independent set.

3.2. We may note here that linear independency of the set P_r is a sufficient condition for P_r to be a generating RLA independent set with respect some S_r and G_1 but the condition is not necessary as shown in example 3.1.a.

4. IMBEDDED GEOMETRIES

Henceforth

$$P_r = [\ \chi^o, \chi^1, \ldots, \chi^r\ ]$$

is supposed to be a generating RLA independent (S_r, G_1) set.

4.1. Restricted Subset and Restricted Subspace

The set C_r:

$$C_r = [\ \alpha \,/\, \alpha = \lambda_o \alpha^o + \lambda_1 \alpha^1 + \ldots + \lambda_r \alpha^r, \text{ not all } \lambda_i = 0, \lambda_i \in G_1]$$

is defined as the Restricted subset of analytic points generated by the generating RLA indepdndent set S_r. The set C_r contains $(s_1^{r+1} - 1)$ analytic points, with each of which a geometric point χ of P.G.(n, s) is associated.

The set Δ_r of all geometric points which are associated with the analytic points of C_r is defined as the Restricted subspace Δ_r generated by P_r through S_r or simply Restricted subspace Δ_r. It is also denoted by the following symbols:

$$\Delta_{\alpha^o, \alpha^1, \ldots, \alpha^r}^{\chi^o, \chi^1, \ldots, \chi^r}, \quad \Delta^{\chi^o, \chi^1, \ldots, \chi^r}, \quad \Delta_{\alpha^o, \alpha^1, \ldots, \alpha^r}.$$

4.2. The Restricted subspace Δ_r generated by P_r a generating RLA independent set with respect to (S_r, G_1) is isomorphic to a projective geometry P.G.(r, s_1) of r dimensions based on GF(s_1).

Proof: The Restricted subset C_r contains $(s_1^{r+1} - 1)$ analytic points. No two of these points are identical, for let

$$\alpha = \lambda_o \alpha^o + \lambda_1 \alpha^1 + \ldots + \lambda_r \alpha^r$$

and

$$\alpha' = \lambda'_o \alpha^o + \lambda'_1 \alpha^1 + \ldots + \lambda'_r \alpha^r$$

then

$$\alpha - \alpha' = (\lambda_o - \lambda'_o) \alpha^o + \ldots + (\lambda_r - \lambda'_r) \alpha^r$$

does not vanish, since $[\alpha^o, \alpha^1, \ldots, \alpha^r]$ is an RLA independent set of P_r.

To every geometric point χ of Δ_r, (s_1-1) analytic points of C_r are associated and no more; since if we have

$$\alpha = \lambda_0 \alpha^0 + \lambda_1 \alpha^1 + \dots + \lambda_r \alpha^r$$

and

$$\beta = \lambda'_0 \alpha^0 + \lambda'_1 \alpha^1 + \dots + \lambda'_r \alpha^r$$

associated with the same geometric point, then $\alpha = \varrho \beta$. As ϱ ranges over the nonzero values of G_1, we get the (s_1-1) analytic representations of χ in C_r and if $\varrho \notin G_1$ we have:

$$\alpha - \varrho \beta = (\lambda_0 - \varrho \lambda'_0) \alpha^0 + \dots + (\lambda_r - \varrho \lambda'_r) \alpha^r \neq 0$$

(since P_r is a generating RLA independent set) and hence

$$\alpha \neq \varrho \beta \quad \varrho \notin G_1.$$

Thus the restricted subspace Δ_r contains $\dfrac{s_1^{r+1}-1}{s_1-1}$ geometric points χ. The isomorphism between Δ_r and P.G.(r, s_1) will be clear with the following correspondence between their points: for a point χ in Δ_r consider an analytic representation which can always be written in the form $\lambda_0 \alpha^0 + \lambda_1 \alpha^1 + \dots + \lambda_r \alpha^r$ with $\lambda_i \in G_1 = GF(s_1), (i=0, 1, \dots, r)$. Consider also a point ξ in P.G.(r, s_1) determined by the analytic point $(\lambda_0, \lambda_1, \dots, \lambda_r)$ then the correspondence $\chi \longleftrightarrow \xi$ is an isomorphism preserving incidences. Thus Δ_r is an imbedded geometry of dimensions r and order s_1 in P.G.(n, s).

5. LINE SEGMENTS

The Restricted subspace $\Delta^{\chi^o, \chi^1}_{\alpha^o, \alpha^1}$ is also called a line segment and denoted by $L^{\chi^o, \chi^1}_{\alpha^o, \alpha^1}$ or L^{χ^o, χ^1}. We shall study here the properties of line segments and finally obtain the number of line segments generated by the two distinct geometric points χ^o and χ^1 as we take different analytic representations for them.

It may be recalled that in case of P_1, Ω is a generating RLA independent (S_1, G_1) set iff the two points χ^o and χ^1 are distinct.

5.1. $L^{\chi^o, \chi^1}_{\alpha^o, \alpha^1} \equiv L^{\chi^1, \chi^o}_{\alpha^1, \alpha^o}$

5.2. We have $L^{\chi^o, \chi^1}_{\alpha^o, \alpha^1} \equiv L^{\sigma^o, \sigma^1}_{\partial^o, \partial^1}$ where σ^o and σ^1 are any two distinct geometric points of $L^{\chi^o, \chi^1}_{\alpha^o, \alpha^1}$ and ∂^o and ∂^1 respectively their analytic representations in C_1. (Reproductive property).

This follows from the fact that C_1 is a vector-space of which (α^o, α^1) and (∂^o, ∂^1) are just two different bases.

Thus the same line segment is reproduced by any two of its analytic points provided they correspond two distinct geometric points.

5.3. $L^{\chi^o, \chi^1}_{\alpha^o, \alpha^1} \equiv L^{\chi^o, \chi^1}_{\rho_o \alpha^o, \rho_1 \alpha^1}$ if and only if $\rho_o \rho_1^{-1} \in G_1$.

To establish the necessity part consider a third point χ other than χ^o and χ^1 which belongs to both the sides of the identity. Consider also the two possibly different restricted analytic representations of χ one $\lambda_o \alpha^o + \lambda_1 \alpha^1$ generated by (α^o , α^1) and the other $\lambda'_o \rho_o \alpha^o + \lambda'_1 \rho_1 \alpha^1$ generated by ($\rho_o \alpha^o$, $\rho_1 \alpha^1$). It is seen that clearly

$$\lambda_o \alpha^o + \lambda_1 \alpha^1 = \rho (\lambda'_o \rho_o \alpha^o + \lambda'_1 \rho_1 \alpha^1) .$$

Since α^o and α^1 are also linearly independent this equality implies

$$\frac{\lambda'_o \rho_o}{\lambda_o} = \frac{\lambda'_1 \rho_1}{\lambda_1}$$

the common value being equal to ρ^{-1} , from which one obtains

$$\rho_o \rho_1^{-1} = (\lambda'_1 \lambda_o) (\lambda_1 \lambda'_o)^{-1} \varepsilon G_1$$

It may be noted that since the selected point χ is different from both χ^o and χ^1 none of $\lambda_o, \lambda'_o, \lambda_1$ and λ'_1 could be zero.

The sufficiency part is obvious.

This property shows that two different analytic representations of the same two geometric points could give rise to two different line segments (which are distinct except of course for the two defining geometric points. Two line segments generated by two distinct geometric points χ^o and χ^1 are then either identical or have only χ^o and χ^1 in common.

5.4. The line segment $L^{\chi^o, \chi^1}_{\alpha^o, \alpha^1}$ has $s_1 + 1$ points.

5.**5.** The number of distinct line segments L^{χ^o, χ^1} generated by two distinct geometric points χ^o and χ^1 is $(s - 1) / (s_1 - 1)$.

Since the analytic pairs ($\rho_o \alpha^o$, $\rho_1 \alpha^1$) and ($\rho\, \alpha^o$, α^1) with generate the same line segment, in counting the number of distinct line segments one may without any loss of generality restrict one's attention only to the $(s - 1)$ generators of the type $(\rho\alpha^0, \alpha^1)$, determined by the $s - 1$ nonnull elements ρ in $GF(s)$. Result 5.5 follows once it is noticed that for a fixed value of ρ the $(s_1 - 1)$ generators $(\rho\lambda\alpha^0, \alpha^1)$ with λ ranging over the s_1 nonnull elements of the subfield G_1 generate identical line-segments. (Result 5-3). The number of distinct line segments is thus equal to the number of cosets of multiplicative group of G_1 with respect to the multiplicative subgroup of G_1 i.e. $\frac{s - 1}{s_1 - 1}$.

5.6. A distinct triplet of points appears in exactly one line segment.

This property follows from 5.4.

5.7. The total number of distinct line segments generated by pairs of distinct points of a line P.G.(r, s) is

$$\frac{s}{s_1} \cdot \frac{(s - 1)}{(s_1^2 - 1)}.$$

A pair of distinct geometric points generate $\frac{s - 1}{s_1 - 1}$ distinct line segments. Hence the $\binom{s+1}{2}$ pairs of points generate $\binom{s+1}{2}\frac{(s - 1)}{(s_1 - 1)}$ line segments; but each line segment could be generated by any two of its geometric points. Hence the result.

It could also be obtained by noting that a line segment is uniquely determined by any three of its points and hence the number of distinct line segments is $\binom{s+1}{3} / \binom{s_1+1}{3}$.

5.8. A point χ of (P.G.(1, s) appears in $\frac{s(s-1)}{s_1(s_1-1)}$ line segments.

Choose any other point χ' of the line which can be done in s ways. For each choice of χ' the number of line segments generated by χ and χ' is $\frac{s-1}{s_1-1}$. Hence χ appears in $s.\frac{s-1}{s_1-1}$ line segments; but a line segment is counted s_1 times in this process as many times as a point of the line segment (other than χ) a... chosen as χ'. Note that the given line segment is one of the $\frac{s-1}{s_1-1}$ line segments associated with any one of the s_1 pairs of geometric points so constituted. Hence the preposition.

5.9. A pair of distinct points χ and χ' appears in $\frac{s-1}{s_1-1}$ line segments.

That a pair of distinct points χ and χ' appears in at least $\frac{s-1}{s_1-1}$ line segments is obvious. If the pair had appeared in a one more line segment generated by two points σ^0 and σ^1 then the same line segment can be generated by χ and χ' and hence is already counted in $\frac{s-1}{s_1-1}$ line segments. Thus every pair of distinct points appears in $\frac{s-1}{s_1-1}$ line segments.

5.10. Examples illustrating the properties of line segments.

Consider P G (2, 2^4) based on GF(2^4) whose minimum function is $x^4 + x^3 + 1 = 0$. Let the subfield be GF(2^2) whose elements are:

$$\{0,\ 1,\ x^3+x,\ x^3+x+1\} \subseteq GF(2^4)$$

two geometric points χ^{0} and χ^{1} be taken with the analytic points $\alpha^{0} = (0,\ x,\ x+1)$ and $\alpha^{1} = (0,\ 0,\ 1)$ respectively. The other 15 points of the entire line generated by χ^{0} and χ^{1} in P.G.$(2,\ 2^{4})$ are given below:

$\chi^{2} : (0,\ x,\ x)$ $\quad$ $\chi^{3} : (0,\ x,\ x^{3}+1)$ $\quad$ $\chi^{4} : (0,\ x,\ x^{3})$

$\chi^{5} : (0,\ x,\ 1)$ $\quad$ $\chi^{6} : (0,\ x,\ x^{3}+x^{2}+x)$ $\quad$ $\chi^{7} : (0,\ x,\ x^{3}+x^{2})$

$\chi^{8} : (0,\ x,\ x^{2}+x+1)$ $\quad$ $\chi^{9} : (0,\ x,\ 0)$ $\quad$ $\chi^{10} : (0,\ x,\ x^{2})$

$\chi^{11} : (0,\ x,\ x^{3}+x+1)$ $\quad$ $\chi^{12} : (0,\ x,\ x^{2}+1)$ $\quad$ $\chi^{13} : (0,\ x,\ x^{3}+x^{2}+1)$

$\chi^{14} : (0,\ x,\ x^{2}+x)$ $\quad$ $\chi^{15} : (0,\ x,\ x^{3}+x)$ $\quad$ $\chi^{16} : (0,\ x,\ x^{3}+x^{2}+x+1)$

We may verify that $L^{\chi^{0},\chi^{1}}_{\alpha^{0},\alpha^{1}}$ contains the points χ^{2}, χ^{3} and χ^{4} in addition to the two defining points χ^{0} and χ^{1} and

$$L^{\chi^{2},\chi^{4}}_{\alpha^{2},\alpha^{4}} \equiv L^{\chi^{0},\chi^{1}}_{\alpha^{0},\alpha^{1}} \equiv L^{\chi^{0},\chi^{1}}_{\varrho\alpha^{0},\mu\alpha^{1}}$$

where $\varrho = x^{2}+1$ and $\mu = x^{3}+x^{2}$ whence $\varrho\,\mu^{-1} = x^{3}+1+x \ \varepsilon\ GF(s_{1})$. Taking $\{(\varrho,\ \mu)\} = \{(1,\ 1),\ (1,\ x),\ (1,\ x^{2}),\ (1,\ x^{3}),\ (1,\ x^{2}+1)\}$ we generate 5 lines segments which are all disjoint, the four other line segments than the one obtained above being as shown below respectively:

$\{\chi^{0},\ \chi^{1},\ \chi^{5},\ \chi^{6},\ \chi^{7}\}$ $\qquad$ $\{\chi^{0},\ \chi^{1},\ \chi^{8},\ \chi^{9},\ \chi^{10}\}$

$\{\chi^{0},\ \chi^{1},\ \chi^{11},\ \chi^{12},\ \chi^{13}\}$ $\qquad$ $\{\chi^{0},\ \chi^{1},\ \chi^{14},\ \chi^{15},\ \chi^{16}\}$

It is obvious that the above line in PG$(2,\ 2^{4})$ with these 17 points has the equation $x_{o} = 0$. It is covered by these 5 line segments.

6. CLASSIFICATION OF IMBEDDED GEOMETRIES

In this section we classify the imbedded geometries and later obtain the conditions of their existence after studying some preliminary cases:

6.1. Let us consider Δ_2 generated by $P_2 = [\chi^o, \chi^1, \chi^2]$ of generating RLA independent set. The Restricted subspace Δ_2 contains the Line segment $L^{\chi^o, \chi^1}_{\alpha^o, \alpha^1}$ say. Then the point χ^2 may or may not belong to the entire line generated by the points χ^o and χ^1 in the geometry P G.(n, s). If it belongs to the line we call P_2 a singular generating RLA independent set and Δ_2 a singular Restricted subplane, otherwise we call Δ_2 a nonsingular Restricted subplane and P_2 a nonsingular generating RLA independent set.

6.1.a. Example: A singular Restricted subplane Δ_2 in a plane.

Let us consider P G $(2, 2^4)$ and G_1 be the field of $\{0, 1\}$ elements. and the set

$$P_2 = [\chi^o, \chi^1, \chi^2]$$

where $\chi^o = (0, 0, x)$, $\chi^1 = (0, x, 0)$ and $\chi^2 = (0, x^2, x^2 + 1)$. It is clear that χ^2 is coincident to the line through the points χ^o and χ^1. The line segment

$$L^{\chi^o, \chi^1} = [\chi^o, \chi^1, \chi^3]$$

where the point χ^3 is given by $(0, x, x)$.

The Restricted subplane $\Delta_2 = [\chi^0, \chi^1, \chi^3, \chi^2, \chi^4, \chi^5, \chi^6]$ where

$\chi^4 = (0,\ x^2,\ x^2 + x + 1)$, $\chi^5 = (0,\ x^2+x,\ x^2+1)$ $\chi^6 = (0,\ x^2+x, x^2+x+1)$

and its 7 line segments are as given below:

1: (χ^0,χ^1,χ^3), 2: (χ^0,χ^2,χ^4), 3: (χ^0,χ^5,χ^6), 4: (χ^1,χ^2,χ^5),

5: (χ^1,χ^4,χ^6), 6: (χ^2,χ^3,χ^6), 7: (χ^3,χ^4,χ^5).

These seven points and seven line segments are isomorphic to P.G (2, 2) contained in a line of P.G.(2, 2^4).

6.1.b. Example: A nonsingular Restricted subplane Δ_2 in a plane.

Let us consider in the above example

$$P_2 = [\ \chi^0,\ \chi^1,\ \chi^2\ ]$$

where $\chi^0 = (0,\ 0,\ x)$, $\chi^1 = (0,\ x,\ 0)$ and $\chi^2 = (1,\ 0,\ 0)$. It is obvious that χ^2 does not lie on the line generated by χ^0 and χ^1 in the geometry F G (2, 2^4). The remaining four points of the nonsingular Restricted subplane Δ_2 are:

$\chi^3 = (0,\ x,\ x)$, $\chi^4 = (1,\ 0,\ x)$, $\chi^5 = (1,\ x,\ 0)$, $\chi^6 = (1,\ x,\ x)$,

with the seven line segments.

1: (χ^0,χ^1,χ^3), 2: (χ^0,χ^2,χ^4), 3: (χ^0,χ^5,χ^6), 4:($\chi^1,\chi^2,\chi^5)$,

5: (χ^1,χ^4,χ^6), 6: $(\chi^2,\chi^3\chi^6)$, 7: (χ^3,χ^4,χ^5).

It is obvious that these seven points and seven line segments are also isomorphic to a P G (2, 2).

Thus in both the cases the imbedded geometries are isomorphic to P G (2,2). But the important difference is that in the singular case all the points lie on a line of the geometry in which they are imbedded while the nonsingular case has a structurally different imbedding in the geometry P.G.(2, 2^4).

6.2. In general let P_r be a set of generating RLA independent points that generates the Restricted subspace Δ_r. Then Δ_r is said to be a nonsingular Restricted subspace if the Restricted subspace Δ_{r-1} generated by P_{r-1} is a nonsingular Restricted subspace and the point χ^r of the set P_r is such that it does not lie on any of the lines in P.G.(n, s) which are generated by any pair of points of the Restricted subspace Δ_{r-1}. The point Δ_o is nonsingular Restricted subspace by definition.

In all other cases the imbedded geometry is said to be singular.

6.2.a. Example: A nonsingular Restricted subplane Δ_2 of order two imbedded in a finite projective plane P.G.(2, 2^2).

The minimum function is $x^2 + x + 1 = 0$, the elements of the field are $0, 1, x, x^2$ and the subfield is of the elements $\{0, 1\}$. The points

$$\chi^o = (0, 0, 1) \qquad \chi^1 = (0, 1, 0), \qquad \chi^2 = (1, 0, 0)$$

generate a nonsingular Restricted subplane Δ_2 in P.G.(2, 2^2).

6.2.b. Example: A nonsingular Restricted 3-dimensional subspace Δ_3 of order two imbedded in a finite projective plane P.G.(2, 2^4).

The minimum function is $x^4 + x^3 + 1 = 0$ and the subfield is of elements $\{0, 1\}$. Let the points χ^o and χ^1 be taken with the analytic points $\alpha^{o} = (0, 0, x)$ and $\alpha^{1} = (0, x^3+1, 1)$ respectively. The Restricted subset C_1 generated by them has the only third point χ^2 given by

$$\alpha^2 = \alpha^o + \alpha^1 = (0, x^3+1, x+1)$$

and the line segment $\Delta_1 = [\chi^o, \chi^1, \chi^2]$ where χ^2 is given by α^2. It may be noted that the point χ^3 given by $(1, 0, 0)$ does not lie on the line generated by the points χ^o and χ^1 which infact consists of 17 points. Hence the set

$$P_2 = [\chi^o, \chi^1, \chi^3]$$

is a nonsingular generating RLA independent set with respect to the analytic set

$$S_2 = [\alpha^o, \alpha^1, \alpha^3]$$

and the nonsingular Restricted subplane Δ_2 consists of the following additional points:

$\chi^4 : (1, 0, x)$, $\chi^5 : (1, x^3+1, 1)$ $\chi^6 : (1, x^3+1, x+1)$

and the line segments:

$$L_1 : (\chi^o, \chi^1, \chi^2),\ L_2 : (\chi^o, \chi^3, \chi^4),\ L_3 : (\chi^o, \chi^5, \chi^6),$$
$$L_4 : (\chi^1, \chi^3, \chi^5),\ L_5 : (\chi^1, \chi^4, \chi^6),\ L_6 : (\chi^2, \chi^3, \chi^6),\ L_7 : (\chi^2, \chi^4, \chi^5).$$

The fourth point to be included in P_3 so that it may constitute a nonsingular generating RLA independent set should not be incident to any of these 7 (entire) lines in P.G.(2, 2^4) whose equations are given below:

L_1: $x_o = 0$

L_2: $x_1 = 0$

L_3: $(x^3+1)\ x_o + x_1 = 0$

L_4: $x_1 + (x^3+1)\ x_2 = 0$

L_5: $(x^3+x+1)\ x_o + x_1 + (x^3+1)\ x_2 = 0$

L_6: $x_1 + (x^2+x+1)\ x_2 = 0$

L_7: $(x^3+x^2+x)\ x_o + x_1 + (x^2+x+1)\ x_2 = 0$

It is easily verified that the point χ^7 with the analytic representation $\alpha^7 = (1, 1, 1)$ has the required property. Hence the set

$$P_3 = [\ \chi^o, \chi^1, \chi^3, \chi^7\ ]$$

is a nonsingular generating RLA independent set with respect to the analytic set

$$S_3 = [\ \alpha^o, \alpha^1, \alpha^3, \alpha^7\ ].$$

The restricted subspace Δ_3 generated by P_3 has a total of 15 points of which 8 points χ^i (i = 1, 2, ..., 8) have already been enumerated earlier. The remaining 7 points are as follows:

χ^8 : (1, 1, x + 1), χ^9: (1, x^3, 0), χ^{10}: (1, x^3, x), χ^{11}: (0, 1, 1)

χ^{12}: (0, 1, x+1), χ^{13}: (0, x^3, 0), χ^{14}: (0, x^3, x).

It was observed in example 3.1.a that a generating RLA independent set P_r need not necessarily be a linearly independent set. The example here shows that even a nonsingular generating RLA independent set P_r is not necessarily a linearly independent set of points.

7. EXISTENCE CONDITIONS FOR NONSINGULAR RESTRICTED SUBSPACE

Theorem 7.1. A necessary condition that a nonsingular 3-dimensional Restricted subspace of order s_1 exist in P.G.(n, s) is that

$$(s^{n+1}-1)/(s-1) - [(s_1^2+s_1+1)(s-s_1)] > 0.$$

Proof. If a nonsingular Restricted subspace Δ_2 exists and its $(s_1^2+s_1+1)$ line segments all intersect in Δ_2 itself, every line which has a line segment in Δ_2 contains $(s-s_1)$ distinct points of P.G.(n, s) which are not in the imbedded plane. Hence the fourth point to generate a nonsingular Restricted 3-space must be outside these lines and the choices for that point are:

$$(s^{n+1}-1)/(s-1) - (s_1^3-1)(s-s_1)/(s_1-1)$$

which must be strictly positive.

7.2. Theorem: A necessary condition for the existence of nonsingular Restricted subspace Δ_r in P.G.(n, s) is that

$$(s^{n+1}-1)/(s-1) \geq (s_1^{r+1}-1)/(s_1-1).$$

This condition follows if we note that the number of geometric points in the imbedded space Δ_r are at most as many as the total number of points in the geometry P C.(n, s,).

8. STRUCTURAL RELATIONS OF NONSINGULAR RESTRICTED SUBPLANES OF A PLANE

Let $P_2 = [\chi^o, \chi^1, \chi^2]$ be a generating RLA independent set with respect to a (S_2, G_1) in P C.(2, s). The Restricted subplane Δ_2 generated by P_2 has been shown to be isomorphic to a P.G.(2, s_1) in section 4. Hence if we consider any two geometric points of Δ_2, there is a unique line segment, generated by these two geometric points with analytic representations taken from C_2, which belongs to the subplane Δ_2. This property holds because of generating RLA independent property of the set P_2 with respect to (S_2, G_1).

If we consider any line L of the geometry P.G.(2, s), the line may cut the imbedded geometry Δ_2 generated by P_2 in either no points or in one point or in more than one point. If the line L cuts the imbedded plane in more than one point say two points then from the isomorphism of Δ_2 to a P.G.(2, s_1) the line L cuts the imbedded plane in a line segment uniquely determined by the two geometric points with analytic representations from the Restricted subset C_2. The line L may have some more points in common with the plane Δ_2 than the above line segment. But if the imbedded plane is a nonsingular Restricted subplane then we show that it will have no more points in common.

Henceforth we shall consider a nonsingular Restricted subplane Δ_2 generated by nonsingular generating RLA independent set P_2 with respect (S_2, G_1).

8.1. If a line of P.G.(2, s) cuts a nonsingular Restricted subplane Δ_2 in a line segment, then the line has no more points in common with the nonsingular Restricted subplane Δ_2.

If a line L cuts the nonsingular Restricted subplane Δ_2 in a line segment $L^{\sigma^0, \sigma^1}_{\alpha^0, \alpha^1}$ and also has one more point σ^2 of the line L in common with the subplane Δ_2, then consider the set

$$S_2' = [\alpha^0, \alpha^1, \alpha^2]$$

of analytic points of

$$P_2' = [\sigma^0, \sigma^1, \sigma^2]$$

of the line L.

It is clear that P_2' is RLA independent and S_2' generates the Restricted subset C_2 since C_2 is a vector space and S_2' is another basis for C_2 and thus we obtain the Restricted subplane Δ_2 from P_2' which shows that the Restricted subplane Δ_2 is generated by a set of three collinear geometric points σ^0, σ^1 and σ^2 and that Δ_2 is contained in the line L. It implies that the three points χ^0, χ^1 and χ^2 constituting P_2(which are in Δ_2) are also collinear, violating the fact that the Restricted subplane Δ_2 is nonsingular.

We may now classify the lines of a plane I . 7 . 2, s) into three classes, in an exclusive and exhaustive manner. A line L is called an OUTSIDE LINE, or a TANGENT or a SECANT with respect to a nonsingular Restricted subplane Δ_2 according as the line L cuts the imbedded plane in no points or in one point or in a line segment. Similarly the points of the plane not in the imbedded plane are classified as follows: a point P belongs to class I if it has the property that every line through P intersects the imbedded plane in atmost one point, and the rest of points which are not in the imbedded plane and class I belong to class II. It may be noted that all lines through a point Q of class II cannot be outside lines or tangents and hence there must be at least one line L among the lines which is a secant to the subplane Δ_2 and by property 8.1 that for a line L through Q there cannot be two line segments in common with the nonsingular Restricted subplane Δ_2.

8.2. Through a point Q of class II there cannot be two lines which are secants with respect to the Restricted subplane Δ_2.

Since the Restricted subplane Δ_2 is isomorphic to a PG(2, s_1) every two line segments intersect in a point of the Restricted subplane Δ_2 and hence the point Q has to belong to the subplane contradictorily.

Thus through a point Q of class II there is a unique line by 8.2 which has a unique line segment by 8.1 in common with the non-singular Restricted subplane Δ_2. In other words the points of class II are nothing but the points on the extensions of the line segments of the nonsingular imbedded subplane.

8.3. The number of points in class II is

$$(s_1^2 + s_1 + 1)(s - s_1)$$

As has been noted in the preceeding paragraph of 8.2 every point Q of class II lies on a unique secant and every secant has $(s - s_1)$ points on it which do not belong to the subplane. The number of secants is as many as the number of line segments in the nonsingular Restricted subplane which is $(s_1^2 + s_1 + 1)$. No two secants intersect outside the subplane as shown in 8.2. Hence the number of points of class II is

$$(s_1^2 + s_1 + 1)(s - s_1).$$

8.4. The number of points in class I is

$$(s^2 + s + 1) - (s_1^2 + s_1 + 1)(s - s_1 + 1)$$

as can be obtained easily by subtraction.

8.5. Through every point of class I the number of lines that do not cut the subplane is $(s - s_1^2 - s_1)$.

For, through every point P class I, $s + 1$ lines pass through and of them $s_1^2 + s_1 + 1$ lines cut the subplane Δ_2 in one and only one point each. It may be noted that each line joining the point P with a point of the subplane Δ_2 is a tangent since P is a point of class I.

8.6. The number of tangents from a point Q of class II to the subplane Δ_2 is s_1^2.

8.7. The number of outside lines through each point class II is $s - s_1^2$.

Through each point Q of class II there is a secant to the subplane Δ_2 containing $s_1 + 1$ points of the subplane Δ_2 and hence $(s_1^2 + s_1 + 1 - \overline{s_1 + 1})$ points remain in the subplane through each of which a tangent passes originating at Q. Thus the lines which do not cut at all the subplane Δ_2 from Q are in number $s + 1 - s_1^2 - 1 = s - s_1^2$.

8.8. Class I is empty if and only if s is either s_1 or s_1^2.

The number of points in class I is equal to

$$(s_1^2 + s + 1) - (s_1^2 + s_1 + 1)(s - s_1 + 1)$$

as shown in 8.4, Note that for a given s_1, this expression a quadratic in s, reduces to zero if an only if either $s = s_1$ or $s = s_1^2$.

9. STRUCTURAL RELATIONS OF NONSINGULAR IMBEDDED GEOMETRIES IN F.G (n, s)

We shall consider a nonsingular Restricted subspace Δ_r of order s_1 imbedded in a P.G.(n, s). In 8.1 it has been proved that if a line cuts an imbedded plane Δ_2 in a line segment then it will have no more points in common with the subplane Δ_2. In general,

9.1. It may be verified in a similar fashion that if a line L cuts a nonsingular Restricted subspace Δ_r in P G.(n, s) in a line segment, then the line L will have no more points in common with Δ_r.

Again we can classify all the lines of P G.(n, s) into three mutually exclusive and exhaustive classes: the class of OUTSIDE LINES where every line of this class cuts Δ_r in no points; the class of TANGENTS where every line of this class cuts Δ_r in exactly one point and the class of SECANTS where every line of this class cuts Δ_r in a line segment. of Δ_r.

Similarly the classification for points is as follows: A point not in Δ_r belongs to class I if every line through it is either a tangent or an outside line; all other points not in Δ_r and class I beong to class II.

9.2. The number of tangents through each point of class I is $(s_1^{r+1} - 1) / (s_1 - 1)$.

9.3. The total number of tangent to Δ_r from all the points of P.G.(r, s^2) is

$$(s^{r-1}-1)(s^r-1)(s^{r+1}-1)/(s-1)(s^2-1)$$

where the subfield is of order s.

Every point P in the Restricted subspace Δ_r has $(s^r-1)/(s-1)$ line segments through it in Δ_r. Each of these $(s^r-1)/(s-1)$ lines through P has a line segment in common with Δ_r. Hence the number of lines through P which have no line segments in Δ_r is by subtraction from the total number of lines in P G (r, s^2) through the point P :

$$(s^{2r}-1)/(s^2-1)-(s^r-1)/(s-1).$$

No two such lines as above which originates from any two distinct points of the $(s^{r+1}-1)/(s-1)$ points of Δ_r being identical we get the required number of tangents to be

$$[(s^{r+1}-1)/(s-1)][(s^{2r}-1)/(s^2-1)-(s^r-1)/(s-1)].$$

10. APPLICATIONS OF IMBEDDED GEOMETRIES TO STATISTICAL DESIGNS

With the help of results of the previous sections we obtain now a series of doubly balanced incomplete block designs and a series of partially balanced incomplete block designs of two associate classes which include new designs. Some pairwise balanced designs are also obtained and are used to improve lower bounds on the number of mutually orthogonal latin squares.

10.1. Doubly Balanced Block designs through line segments:

In P.G (1, s) the line containing s + 1 points consider all line segments based on a subfield of order s_1. The points of P.G.(1, s) as treatments and the line segments as blocks gives the following balanced incomplete block design with parameters of (say) series LS

$$v = s + 1$$

$$b = s (s^2 - 1) / s_1 (s_1^2 - 1)$$

$$r = s (s - 1) / s_1 (s_1 - 1)$$

$$k = s_1 + 1$$

$$\lambda = (s - 1) / (s_1 - 1)$$

$$\delta = 1$$

The values of these parameters are obvious from the properties of line segments proved in sections 5.1 through 5.9.

An important property of these designs is that every triplet of treatments appears exactly once in the design. Very few such balanced incomplete block designs are known (Calvin, L.D; 1954)* where triplets are also known to occur a constant number of times. A list of designs with r and k $\leq$ 20 of this LS series is given in the appendix B.

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* "Doubly Balanced Incomplete Block Designs for experiments in which the treatment effects are correlated", Biometrics 10, 61-88. Here their use in organoleptic experiments is also discussed by him.

10.2. Regular Group Divisible designs through Nonsingular Restricted subplanes imbedded in a plane.

Let Δ_2 be a nonsingular Restricted subplane of order s imbedded in a P G (2, s^2).

Let U be the set of points which are not in Δ_2 and V be the set of all tangents to Δ_2 from points of U. The configuration (U, V) where U is the set of treatments and V is the set of blocks is a partially balanced incomplete block design which is a Regular group divisible design with the following parameters of series (say) IM:

$$V = s^4 - s = b$$

$$r = s^2 = k$$

$$\lambda_1 = 0$$

$$\lambda_2 = 1$$

$$n_1 = s^2 - s - 1$$

$$p^1_{11} = n_1 - 1$$

$$p^2_{11} = n_2 - n_1 - 1 .$$

First let us notice that it is a group divisible design. The number of treatments is easily obtained to be $(s^4 - s)$. These treatments are partitioned into $(s^2 + s + 1)$ groups each containing $(s^2 - s)$ treatments. Each group corresponds to a secant i.e. a line segment of the subplane Δ_2. Two treatments which belong to the same group do not appear together in the design since no tangent passes through them both as already they are on a secant. Two treatments which do not belong to the same group appear exactly once together since the line in P G.$(2, s^2)$ which is generated by these two points has to be a tangent to Δ_2, for it cannot be a secant in which case the two treatments belong to the same group and it cannot be an outside line since the set of outside lines is empty in this case. Since each group contains s^2-s treatments it is clear that the number of first associate n_1 is $(s^2 - s - 1)$.

Two treatments that appear in a group have exactly $n_1 - 1$ treatments which also do not appear together with them. Hence

$$p^1_{11} = s^2 - s - 2 = n_1 - 2.$$

If two treatments A and B appear in a group the number of treatments which are 1st associates of A and second associates of B is zero, since the same line cannot be a secant and a tangent. Thus $p^1_{12} = 0$ and similarly $p^1_{21} = 0$. Let A and B be first associates, hence a tangent passes through them. The number of points C such that **CA** is a tangent as well as **CB** is also a tangent is the number of points not in the group to which A and B belong and that number is n_2. Hence $p^1_{22} = n_2$. Similarly the other association matrix can be verified.

The number of tangents is obtained to be $s^4 - s$ by taking $r = 2$ in section 9.3.

Every tangent cuts the subplane Δ_2 in a single point, hence $k = s^2 + 1 - 1 = s^2$ and through every point of U, there are exactly s^2 tangents which gives the number of replications.

The following designs are obtained for r and k less than 17 by taking $s = 2, 3, 2^2$.

Sl.no.	v	b	r	k	λ_1	λ_2	n_1	n_2	p^1_{11}	p^2_{11}
R.1.	14	14	4	4	0	1	1	12	0	0
R.2.	78	78	9	9	0	1	5	72	4	0
R.3.	252	252	16	16	0	1	11	240	10	0

The design no.2 is a new design whose construction is indicated below:

Let us consider the projective plane of order 9 with the minimum function $x^2 + 1 = 0$. Let the subfield be the set of elements $\{0, 1, -1\}$.

Consider the set

$$P_2 = [\, \bar{\chi}^0, \bar{\chi}^1, \bar{\chi}^2 \,]$$

of geometric points where the analytic set

$$S_2 = [\, \bar{\alpha}^0, \bar{\alpha}^1, \bar{\alpha}^2 \,]$$

with $\bar{\alpha}^0 = (0,0,1)$, $\bar{\alpha}^1$ $(0,1,0)$, $\bar{\alpha}^2 = (1, 0, 0)$.

The set P_2 generates a nonsingular Restricted subplane Δ_2 of 13 points and 13 lines. The tangents of the plane to the subplane now give us the design sl.no. R.2.

10.3. Let us consider a nonsingular subspace Δ_r or order s_1 imbedded in a P G.(n, s). Let U be the set of all points not in Δ_r and V be the set of all outside lines, tangents and secants to Δ_r from points of U. The configuration (U, V) is a Pairwise balanced design of index unity and type (v, k_1, k_2, k_3) where:

$$v = (s^{n+1}-1)/(s - 1) - (s_1^{r+1}-1)/(s_1 - 1)$$

$$k_1 = s$$

$$k_2 = s - s_1$$

$$k_3 = s + 1$$

For v is the number of points outside the imbedded geometry. Since we take all lines as blocks through every two points of U there is either a secant or a tangent or an outside line. Hence every pair of treatments appears once and only once in the design. It is obvious that we have the block sizes k_1, k_2, k_3 as above depending on whether the block is derived from a tangent, secant or an outside line.

As a corollary it follows that if the N(t) denotes the maximum number of mutually orthogonal latin squares of order t then the existence of the above pairwise balanced design implies the following inequality:

$$N\left[(s^{n+1}-1)/(s-1) - (s_1^{r+1}-1)/(s_1-1)\right] \geq \min\left\{N(s),\ N(s+1),\ N(s-s_1)\right\} - 1,$$

Note that the expression on the right hand side is independent of r and n. Taking $n = r$ and $s = s_1^2$ we have

$$N\left[(s^{2(r+1)}-1)/(s^2-1) - (s^{r+1}-1)/(s-1)\right] \geq \min\left\{N(s),\ N(s+1),\ N(s^2-s)\right\} - 1.$$

10.4. Let us consider a nonsingular Restricted subplane of order s imbedded in P.G $(2, s^2)$. In this case it is proved in section 8.3 that Class I of points is empty i.e. through every point there is a unique secant to the Restricted subplane Δ_2. Consider the points of P.G $(2, s^2)$ not belonging to the subplane Δ_2 as the set U of treatments and the set V of tangents and secants as blocks. This configuration is a Pairwise balanced design of index unity and type (v, k_1, k_2) where:

$$v = s^4 - s$$

$$k_1 = s^2$$

$$k_2 = s^2 - s.$$

Hence the last inequality in section 10.3 can be improved in the special case of $r = 2$, to

$$N(s^4 - s) \geq \min \{N(s^2), N(s^2 - s)\} - 1,$$

$$= \min \{s^2 - 1, N(s^2 - s)\} - 1.$$

Taking $s = 2^2 = 4$ we thus have

$$N(252) \geq \min \{15, N(12)\} - 1$$
$$= 4$$

which is an improvement on the known* lower bound 3 for N(252).

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* Mann, H.B; (1949): Design and Analysis of Experiments, Dover Publications.

BALANCED AND PARTIALLY BALANCED DESIGNS FROM NON-LINEAR CONFIGURATIONS IN FINITE PROJECTIVE GEOMETRY

0. SUMMARY

In PG(n, s) the set of all solutions of a second degree homogenous equation in n+1 variables which are general coordinates of a point in PG(n, s) is known as a Quadric Q_n in PG(n, s). It is said to be non-degenerate if there exists no nonsingular linear transformation of the geometry PG(n, s) by which the equation of Q_n can be transformed to an equation containing n+1-r ($r \geq 1$) variables, otherwise degenerate. A degenerate quadric Q_n in PG(n, s) which cannot be expressed in fewer variables than n is called a cone of order 1. The tangent space of a point of the quadric has cone of order 1, with that point as vertex, in common with the quadric.

In PG(2t-1, s) where s is odd, the form A of a nondegenerate quadric is classified as elliptic or hyperbolic according as $(-1)^t$ det A is a nonsquare or square.

Taking all tangent cones of order 1 as blocks and the points of a non-degenerate quadric Q_{2t} in PG(2t, s) we get the series of Symmetric Balanced Incomplete Block (BIB) designs

$$v = \frac{s^{2t}-1}{s-1} = b, \quad r = \frac{s^{2t-1}-1}{s-1} = k \quad \lambda = \frac{s^{2t-2}-1}{s-1}$$

This series is known, but the actual plans obtained here through quadrics are proved to be non-isomorphic to the known plans. Known plans are

based on hyperplanes of dimension (2t-2) in PG(2t - 1, s) whereas Q_{2t} does not contain such hyperplanes of dimension (2t - 2). For example this series gives the second solution to the design with parameters

$$v = b = 15, \; r = k = 7, \; \lambda = 3$$

for which only the cyclic solution (a b c e f i k) is known (Fisher and Yates Tables, 2nd Edition 1963).

Similarly taking tangent ... and vertex-less tangent cones in non-degenerate elliptic and hyperbolic quadrics in PG(2t-1, s) four new series of PBIB designs are obtained.

1. INTRODUCTION

Let Q_n be a quadric in PG(n, s) defined by the set of all points

$$x = (x_o, x_1, \ldots, x_n)$$

that satisfy the equation:

$$\sum_{j \geq i = 0}^{n} a_{ij} \; x_i \; x_j = 0$$

where all the elements a_{ij} and x_i belong to G.F.(s).

If by a nonsingular transformation of the geometry Q_n goes to Q'_{n-r} with the equation:

$$\sum_{j \geq i = 0}^{n-r} b_{ij} \; y_i y_j = 0, \qquad r \geqslant 1$$

then Q_n is said to be a degenerate quadric. Otherwise Q_n is a non-degenerate quadric in PG(n, s). If Q'_{n-r} is nondegenerate in PG(n-r, s), then Q_n is said to be a cone of order r with the vertex given by the equations:

$$y_o = y_1 = \ldots = y_{n-r} = 0,$$

and a base given by the equations:

$$y_{n-r+1} = y_{n-r+2} = \ldots\ldots = y_n = 0.$$

Two points $\alpha = (\alpha_o, \alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_o, \beta_1, \ldots, \beta_n)$ are said to be conjugate with respect to a nondegenerate Q_n if:

$$\sum_{j \geq i = 0}^{n} a_{ij} (\alpha_i \beta_j + \alpha_j \beta_i) = 0$$

The set of all points which are conjugate to a given point is called its polar space. The polar space T(P) of a point P of the quadric is called the tangent space of P. It is known that (Roychoudhuri 1962) $Q_n \cap T(P)$ is a cone of order 1 in the plane T(P) (theorem A2 of appendix A). In case of characteristic 2 Dickson (1958) has shown that a nondegenerate quadric Q_n in $PG(n, 2^m)$ can be reduced to one of the following forms:

(1) when n = 2t all quadrics reduce the form:

$$x_o^2 + x_1 x_2 + \ldots \ldots + x_{2t-1} x_{2t} = 0$$

(2) when n = 2t-1 a quadric reduces to one of the two forms below:

(a) $x_o x_1 + x_2 x_3 + \ldots \ldots + x_{2t-1} x_{2t} = 0$

(b) $\lambda (x_o^2 + x_1^2) + x_o x_1 + x_2 x_3 + \ldots + x_{2t-2} x_{2t-1} = 0$

where

$$\lambda \ (x_o^2 + x_o^1) + x_o \ x_1$$

is irreducible in G.F.(2^m). It is shown by Roychoudhuri (1962) that the quadrics (a) and (b) above represent hyperbolic and elliptic quadrics respectively.

2. CLASSIFICATION OF FORMS OF QUADRICS IN PG(2t-1, s)

Consider a nondegenerate quadric in PG(2t-1, s) . Its equation can always be written in the cannonical form

$$\alpha_o x_o^2 + \alpha_1 x_1^2 + \dots \dots + \alpha_{2t-1} x_{2t-1}^2 = 0$$

by a nonsingular transformation where no is zero, $i=0, 1, \dots, (2t-1)$. In this section these forms will be characterised as elliptic or hyperbolic depending on the nature of the determinant of the form .

Let N(0, n) denote the number of points of a non-degenerate quadric in PG(n, s). From the results of Primrose (1951) it is known that

$$N(0, 2t-1) = (s^t+1)(s^{t-1}-1)/(s-1) \text{ if } Q_{2t-1} \text{ is elliptic}$$

and

$$N(0, 2t-1) = (s^t-1)(s^{t-1}+1)/(s-1) \text{ if } Q_{2t-1} \text{ is hyperbolic.}$$

Theorem 2.1. The quadratic form

$$\alpha_o x_o^2 + \alpha_1 x_1^2 + \dots \dots + \alpha_{2t-1} x_{2t-1}^2$$

represents as nondegenerate hyperbolic or an elliptic quadric according as the product

$$(-1)^t \alpha_o \alpha_1 \cdots \cdots \alpha_{2t-1} \qquad \text{-----(A)}$$

is a square or a nonsquare.

Proof: Consider the following equation

$$\alpha_o x_o^2 + \alpha_1 x_1^2 + \cdots \cdots + \alpha_{2t-1} x_{2t-1}^2 = 0 \qquad \text{---(2.1)}$$

Case I: Let the product (A) be a square.

The number of solutions x where

$$x = (x_o, x_1, \ldots\ldots, x_{2t-1})$$

which satisfy the equation 2.1, as shown by Dickson (1958) is

$$(s^t - 1)(s^{t-1} + 1) + 1.$$

The number of nonzero solutions x of equation 2.1 is hence

$$(s^t - 1)(s^{t-1} - 1).$$

The number of geometric points in PG(n, s) which lie on the quadric of equation (2.1) is precisely

$$(s^t - 1)(s^{t-1} + 1)/(s-1).$$

It is clear that only a nondegenerate hyperbolic quadric Q_{2t-1} has these many solutions.

Case II: Let the product (A) be a nonsquare.

The number of nonzero solutions in this case is also known (Dickson, 1958) to be

$$(s^t + 1)(s^{t-1} - 1)$$

and hence the number of geometric points of equation 2.1 is now

$$(s^{2t} - 1)(s^{2t-1} + 1)/(s - 1)$$

showing that Q_{2t-1} is an elliptic quadric.

In both the cases since none of the

$$\alpha_i \ , \quad i = 0, 1, \ldots \ldots, (n-1)$$

could be zero, it is obvious that the quadric is nondegenerate.

It may be notedthat the product

$$(-1)^t \alpha_o \alpha_1 \ldots \ldots \alpha_{2t-1}$$

and the product

$$-(1)^t \det A$$

where A is the form of a Quadric not necessarily in cannonical form, are either both squares or both non-squares (being identical) and hence we can re-state theorem 2.1 in the following form:

Theorem 2.1': Let A be the form of a Quadric in PG(2t-1, s). The form represents a nondegenerate hyperbolic or elliptic quadric according as $(-1)^t \det A$ is a square or a non-square.

Examples: 2.a. In PG(5, 2) the equation of a nondegenerate elliptic quadric

$$(x_o^2 + x_1^2) + x_o x_1 + x_2 x_3 + x_4 x_5 = 0$$

2.b. In PG(5, 2) the equation of a hyperbolic quadric is

$$x_o x_1 + x_2 x_3 + x_4 x_5 = 0$$

2.c. In PG(5, 3) the equation of an elliptic quadric is

$$x_o^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0$$

2.d. In PG(5, 3) the equation of a hyperbolic quadric is

$$-x_o^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0$$

3. NON-LINEAR CONFIGURATIONS CONTAINED IN NONDEGENERATE QUADRICS

In this section we shall study the properties of cones and other non-linear configurations contained in nondegenerate quadrics. Let Q_n be a nondegenerate quadric in PG(n, s) and P be a point on it. Let, T(P) denote the tangent space. Then it is known (Roychoudhuri 1962, result quoted as Theorem A.2 of appendix A) that

$$Q_n \cap T(P) = Q_{n-1}$$

is a cone of order 1 with vertex at P and a base a nondegenerate quadric Q_{n-2} in PG(n-2, s).

Theorem 3.1. The number of tangent cones of order one contained in a non-degenerate Q_n is N(0, n).

Proof: First we shall prove the following lemma.

Lemma 3.1. Let P_1 and P_2 be any two distinct points of the quadric. The number of points of the quadric which are conjugate to P_1 and P_2 is non-zero whether P_1 and P_2 are conjugate or not.

The points of Q_n which are conjugate to P_1 are the points of

$$Q_n \cap T(P_1).$$

The points of Q_n which are conjugate to both P_1 and P_2 are the points of

$$Q_n \cap T(P) \cap T(P_2)$$

Hence all that is needed to be shown is that

$$|Q_n \cap T(P_1)| - |Q_n \cap T(P_1) \cap T(P_2)| > 0$$

where $|A|$ denote the number of points in the set A.

The set $Q_n \cap T(P_1) \cap T(P_2)$ contains $(s+1) + s^2 N(0, n-4)$ points if P_1 and P_2 are conjugate as obtained by Roychoudhuri (result quoted in the appendix A as Theorem A.5). If P_1 and P_2 are not conjugate then the intersection

$$Q_n \cap T(P_1) \cap T(P_2)$$

contains $N(0, n-2)$ points as obtained by Roychoudhuri (result quoted in the appendix A as Theorem A.6).

Proof of the theorem: If the points P_1 and P_2 are conjugate then the inequality to be shown is

$$1 + s\,N(0, n-2) - [s + 1 + s^2 N(0, n-4)] > 0 \qquad ---(I)$$

Case 1: Let Q_n be taken in PG(2t, s). Then the left hand side of the inequality is

$$1 + \frac{s.(s^{2t-2} - 1)}{s - 1} - \left[s + 1 + \frac{s^2 \cdot s^{2t-4} - 1}{s - 1}\right] = s^{2t-2} + 1,$$

which is > 0, infact > 1.

Case 2. Let Q_n be taken in PG(2t-1, s).

a) Let Q_{2t-1} be elliptic, then the left hand side of (I) is

$$1 + \frac{s.(s^{t-1}+1)(s^{t-2}-1)}{s-1} - \left[s+1 + \frac{s^2.(s^{t-2}+1)(s^{t-3}-1)}{s-1}\right]$$
$$= s^{2t-3}+1,$$

which is > 0 and which is $\geq s+1$ since $t \geq 2$.

b) Let Q_{2t-1} be hyperbolic, then we have

$$1 + s\,N(0, n-2) - [s + 1 + s^2\,N(0, n-4)]$$
$$= 1 + \frac{s.(s^{t-1}-1)(s^{t-2}+1)}{s-1} - \left[s + 1 + \frac{s^2.(s^{t-2}-1)(s^{t-3}+1)}{s-1}\right]$$
$$= s^{2t-3} + 1$$

which is $\geq s + 1$ since $t \geq 2$.

If P_1 and P_2 are not conjugate then the inequality to be shown becomes as follows:

$$1 + s\,N(0, n-2) - N(0, n-2) > 0$$

i.e. $$1 + (s - 1)\ N(0, n-2) > 0$$

Since $s \geq 2$ and $N(0, n-2) \geq 0$ we have that left hand side of the inequality to be always ≥ 1.

As for the theorem, corresponding to each point P we have

$$Q_n \cap T(P)$$

to be a cone of order one and these cones are all distinct by the above lemma. Hence we have N(0, n) distinct cones contained in the quadric. Let us call these cones as tangent cones.

Corollary 3.1. The number of tangent cones of order one without vertices contained in a nondegenerate quadric Q_n is N(0, n).

This directly follows from the above theorem.

Theorem 3.2. The number of cones that pass through a given point of the quadric and contained in the quadric is 1 + s N(0, n-2).

Proof. Let C be a point of the quadric. A cone of order 1 contained in the quadric passes through this point, if the vertex C is conjugate to P with respect to the quadric, or by symmetry of conjugacy relation if P is conjugate to C. The number of such cones is equal to the number of points conjugate to C and contained in the quadric which is exactly given by the number of points in the tangent cone with C as vertex since only these are the points conjugate to C in the quadric. By theorem 3.1, all the cones with these points is vertices including C are distinct. Hence the number of cones that pass through a point of the quadric and contained in it are 1 + s N(0, n-2).

Corollary 3.2. The number of tangent cones without their vertices that pass through a given point is s N(0, n-2).

It is obvious from the above theorem that only one tangent cone that has been given point itself as vertex does not pass through this point as it is suppressed in counting the tangent cones without vertices.

Theorem 3.3. The number of cones which are contained in the quadric and which pass through two distinct points of the quadric is either

$$(s+1) + s^2 \ N(0, n-1) \quad \text{or} \quad N(0, n-2).$$

Proof. Let P_1 and P_2 be two distinct points of the quadric. If there is a point P which is conjugate to both P_1 and P_2, then the tangent cone with P as vertex passes through both P_1 and P_2. Two tangent cones are distinct if their vertices are distinct by lemma 3.1. Hence the required number of points is given by the number of points which are conjugate to both P_1 and P_2

$$\text{i.e.} \quad Q_n \cap T(P_1) \cap T(P_2)$$

which is $s + 1 + s^2 N(0, n-4)$ if P_1 and P_2 are conjugate and $N(0, n-2)$ if P_1 and P_2 are not conjugate.

Corollary 3.3. The number of configurations which are vertex-less tangent cones that pass through a pair of distinct points of a nondegenerate quadric is

$$(s+1) + s^2 \ N(0, n-4) \quad \text{or} \quad N(0, n-2)$$

according as the two points are conjugate or not with respect to the quadric.

This follows from the above theorem since all cones contained in the quadric are distinct even after suppressing their vertices.

4. APPLICATION TO STATISTICAL DESIGNS

We shall consider a nondegenerate quadric Q_n in PG(n, s). An association scheme is defined on the N(0, n) points of the quadric by the conjugacy relation. Two points of the quadric are first associates if they are conjugate with respect to the quadric and second associates if they are not conjugate. Using theorems in the last section one can now construct partially balanced incomplete block designs as follows:

Let the v treatments be represented by the N(0, n) points of the quadric and let the blocks be represented by the tangent cones contained in the quadric. The parameters of this design are

$$v = N(0,\ n) = b$$

$$r = 1 + s\,N(0,\ n-2) = k$$

$$\lambda_1 = (s+1) + s^2\,N(0,\ n-4)$$

$$\lambda_2 = N(0,\ n-2)$$

$$n_1 = s\,N(0,\ n-2)$$

$$P^1_{11} = (s-1) + s^2\,N(0,\ n-4) = \lambda_1 - 2$$

$$P^2_{11} = N(0,\ n-2) = \lambda_2$$

4.1. Let Q_n be a nondegenerate quadric Q_{2t} in PG(2t, s). Taking all its tangent cones as blocks we get in fact balanced incomplete block designs which have the parameters (Series N_1)

$$v = (s^{2t} - 1)/(s-1) = b$$

$$r = (s^{2t-1} - 1)/(s - 1) = k$$

$$\lambda = (s^{2t-2} - 1)/(s - 1)$$

It may be noted that if we take hyperplanes $\sum_{2t-2}$ in PG(2t-1, s) as blocks and all points in PG(2t-1, s) as treatments we obtain a balanced incomplete block designs with the same parameters above.

But the designs obtained here through quadrics are non-isomorphic, i.e. a design with parameters (v, k, λ) of one series cannot be obtained by substitution on its letters or objects from a (v, k, λ) design of the other series. This fact is clear since if there were such isomorphism between the two designs then there has to exist a one to one isomorphism between the tangent cones of the nondegenerate quadric Q_{2t} in $PG(2t, s)$ and the $2t-2$ dimensional hyperplanes of $PG(2t-1, s)$ such that the incidence propetties are preserved. In other words the quadric has a tangent cone isomorphic to a hyperplane of dimension $2t-2$ but the non-degenerate quadric Q_{2t} in $PG(2t, s)$ is known to contain linear spaces of dimension $t-1$ and no higher dimensional linear spaces and certainly it does not contain linear spaces of dimension $2t-2$. Thus there does not exist a one to one correspondence between the designs; proving that the series N_1 is non-isomorphic to the known series.

4.2. Taking a nondegenerate elliptic quadric in $PG(2t-1, s)$ and all its tangent cones as blocks we have the following series N_2 :

$$v = (s^t + 1)(s^{t-1}-1) / (s - 1) = b$$

$$r = (s^{2t-2}+ s^{t-1} - s^t - 1)/(s - 1) = k$$

$$\lambda_1 = (s^{2t-3}+ s^{t-1} - s^t - 1)/(s - 1)$$

$$\lambda_2 = (s^{2t-3}- s^{t-1} + s^{t-2} - 1) / (s - 1)$$

$$n_1 = s \lambda_2$$

$$p_{11}^{1} = \lambda_1 - 2$$

$$p_{11}^{2} = \lambda_2$$

This series contains one practical design for s = 2, t = 3 whose parameters are

v	b	r	k	λ_1	λ_2	n_1	p_{11}^1	p_{11}^2
27	27	11	11	3	5	10	1	5

4.3. Taking a nondegenerate hyperbolic quadric in PG(2t-1, s) and with all its tangent cones as blocks we get the Series N_3:

$$v = (s^t-1)(s^{t-1}+1)/(s-1) = b$$

$$r = (s^{2t-2} - s^{t-1} + s^t - 1)/(s-1) = k$$

$$\lambda_1 = (s^{2t-2} - s^{t-1} + s^t - 1)/(s-1)$$

$$\lambda_2 = (s^{t-1} - 1)(s^{t-2} + 1)/(s-1)$$

$$n_1 = (s(s^{t-1} - 1)(s^{t-2} + 1)/(s-1)$$

$$p_{11}^1 = \lambda_1 - 2$$

$$p_{11}^1 = \lambda_2$$

This series contains only one design with r and k smaller than 15 obtained for s = 2 and t = 3. The parameters are :

v	b	r	k	λ_1	λ_2	n_1	p_{11}^1	p_{11}^2
35	35	11	11	11	9	18	9	9

Hence symmetric balanced inoomplete block design

$$v = b = 35$$

$$r = k = 18$$

$$\lambda = 9$$

All these designs are obtained by considering tangent cones as blocks and points of the quadric as treatments.

Now using the corollaries of the theorems of section 3 one can obtain further designs as follows. In a nondegenerate Q_n, but all tangent cones be taken as blocks from which the vertices are left over, then we have the following series of partially balanced incomplete block designs with parameters

$$v', b', r', k', n_1', \lambda_1', \lambda_2', p_{ij}^{k'}$$

where

$$v' = b' = v = b$$

$$r' = k' = k-1 = r-1$$

$$\lambda'_1 = \lambda_1 - 2$$

$$\lambda'_2 = \lambda_2$$

$$n_1 = n_1$$

$$p_{ij}^{k'} = p_{ij}^{k}$$

referring to v, b, r, k, $\lambda_1, \lambda_2, n_1, p_{ij}^k$ of section 4 above.

4.4. Taking a nondegenerate quadric Q_{2t} in PG(2t, s), its vertexless cones and points produce the following general Series N_1':

$$v = (s^{2t} - 1)/(s - 1) = b$$

$$r = s.(s^{2t-2} - 1) / (s - 1) = k$$

$$\lambda_1 = (s^{2t-2} - 1) / (s - 1) - 2$$

$$\lambda_2 = (s^{2t-2} - 1) / (s - 1)$$

$$p^1_{11} = \lambda_1$$

$$p^2_{11} = \lambda_2$$

This series produces the following designs with r and k smaller than 16. Taking s = 3 and t = 2 we have the design

v	b	r	k	λ_1	λ_2	n_1	p^1_{11}	p^2_{11}
15	15	6	6	1	3	6	1	3
40	40	12	12	2	4	12	2	4

4.5. Taking elliptic nondegenerate quadric in PG(2t-1, s) we get the Series N_2':

$$v = (s^t + 1)(s^{t-1} - 1) / (s - 1) = b$$

$$r = s.(s^{t-1} + 1)(s^{t-2} - 1) / (s - 1) = k$$

$$\lambda_1 = (s^{2t-3} - s^t + s^{t-1} - 1) / (s - 1) - 2 = p^1_{11}$$

$$\lambda_2 = (s^{2t-3} - s^{t-1} + s^{t-2} - 1) / (s - 1) = p^2_{11}$$

$$n_1 = s.(s^{t-1} + 1)(s^{t-2} - 1) / (s - 1)$$

This series gives one design with r and k smaller than 16 which is not included in B.C.S. catalogue (19 4) by taking s = 2, t = 3 with the parameters

v	b	r	k	λ_1	λ_2	n_1	p^1_{11}	p^2_{11}
27	27	10	10	1	5	10	1	5

Lay-out of **this design** is indicated in B.8 of appendix B.

4.6. Taking hyperbolic nondegenerate quadric in PG(2t-1, s) we get the following Series N_3' :

$$v = (s^t - 1)(s^{t-1} + 1) / (s - 1) = b$$

$$r = s.(s^{t-1} - 1)(s^{t-2} + 1) / (s - 1) = k$$

$$\lambda_1 = (s^{2t-3} + s^t - s^{t-1} - 1) / (s - 1) - 2 = p^1_{11}$$

$$\lambda_2 = (s^{2t-3} + s^{t-1} - s^{t-2} - 1)/(s - 1) = p^2_{11}$$

$$n_1 = s.(s^{t-1} - 1)(s^{t-2} + 1) / (s - 1)$$

This series contains the following design for s = 2, t = 3 with the parameters

v	b	r	k	λ_1	λ_2	n_1	p^1_{11}	p^2_{11}
35	35	18	18	11	9	18	9	9

APPENDIX A

RELATED THEOREMS OF RAYCHOUDHURI AND PRIMROSE

In chapter III (Non-linear configurations in finite projective geometry) the proofs of many theorems require a previous knowledge of certain results of Roychoudhuri. Those results are stated below without proof and for proofs reference may be made to Raychoudhuri's paper (1962) and his Thesis (1959) and Primrose (1955).

Let Q_n be a nondegenerate quadric in PG(n, s) and P a point of Q_n. Let T(P) be its tangent space and $_{n-1}$ an n-1 dimensional hyperplane in PG(n, s) which does not pass through P and Σ'_{n-1} be a plane which passes through P and not identical to the hyperplane T(P).

Theorem A.1. Let A_o, $A_1, \ldots,$ A_p be linearly independent points of a quadric Q_n in PG(n, s). The p-flat $_p$ determined by these points is completely contained in the quadric if and only if the (p+1) points are pairwise conjugate (Lemma 2.3 of Raychoudhuri, 1962).

Theorem A.2. $Q_n \cap T(P)$ is a cone of order 1 in the (n-1)-flat T(P). (Theorem 2.1. of Roychoudhuri 1962).

Theorem A.3. $Q_n \cap T(P) \cap \Sigma'_{n-1}$ is a nondegenerate quadric in PG(n-2, s) which is elliptic or hyperbolic according as Q_n is elliptic or hyperbolic. (Theorem 2.1 of Raychoudhuri 1962).

Theorem A. 4. Let $N(p,n)$ denote the number of p-flats or linear subspaces of dimensionality p, contained in a non-degenerate quadric Q_n in $PG(n, s)$. Primrose (1951) has shown by stereographic projection that

$$N(0,\ 2t) = (s^{2t} - 1)/(s - 1)$$

$$N(0,\ 2t-1) = (s^t + 1)(s^{t-1} - 1)/(s - 1) \quad \text{if } Q_{2t} \text{ is elliptic}$$

$$N(0,\ 2t-1) = (s^t - 1)(s^{t-1} + 1)/(s - 1) \quad \text{if } Q_{2t-1} \text{ is hyperbolic}$$

Raychoudhuri's results (1962) further show that

$$(N(p,n) = N(p-1,n-2)\,N(0,n)\,(s-1)/(s^{p+1} - 1).$$

Theorem A.5. Let P_1 and P_2 be two points of a nondegenerate quadric Q_n in $PG(n, s)$ such that the line $P_1\ P_2$ is a generator (i.e. P_1 and P_2 are conjugate). Then the number of points P other than P_1 and P_2 such that both PP_1 and PP_2 are generators of Q_n is

$$(s - 1) + s^2 .\ N(0,\ n-4)$$

(Lemma 3.1.2 of Raychoudhuri, 1959).

Theorem A.6. If P_1 and P_2 be two points of a nondegenerate quadric Q_n in $PG(n, s)$ such that the line $P_1\ P_2$ is not a generator. The number of points P such that both the lines PP_1 and PP_2 are generators of the quadric is $N(0,\ n-2)$

(Lemma 3.1.1 of Raychoudhuri 1959).

APPENDIX B

LIST OF DESIGNS AND SOME LAY-OUTS

This section contains a list of the different series of designs that have been obtained in this thesis and also lay-outs some of these designs.

The following PBIB series of designs is derived from non-singular imbedded planes:

B.1. Series IM: By taking tangents to a nonsingular imbedded finite projective plane PG(2, s) in a finite projective plane PG(2, s^2) after cutting off the points of the imbedded plane we have the Regular Group Divisible design:

$$v = b = s^2 - s,$$
$$r = k = s^2,$$
$$\lambda_1 = 0,$$
$$\lambda_2 = 1,$$
$$n_1 = s^2 - - 1,$$
$$p^1_{11} = n_1 - 1,$$
$$p^2_{11} = n_2 - n_1 - 1$$

with a solution for the parameters:

v	b	r	k	n_1	p^1_{11}	p^2_{11}
78	78	9	9	5	4	0

by taking s = 3.

The following two series of BIB designs are derived from association schemes:

B.2 Series A_1: The BIB design with parameters

$$(v, vt, tn, n, \lambda)$$

exists if an m-associate scheme with $n_i = n$ for i = 1, 2, ..., t m exists such that

$$p^i_{11} + \ldots + p^i_{tt} = \lambda \quad \text{for } i = 1, 2, \ldots \ldots, m.$$

B.3. Series A_2: If t = m above we have the series of BIB designs

$$(mn + 1, m \quad mn + 1, \quad m \quad n + 1, \quad n + 1, \quad n + 1)$$

The lay out of the design with parameters in the above series

(31, 93, 33, 11, 11)

is indicated below:

Let us consider the difference set:

(1, 2, 4, 8, 15, 23, 27, 29, 30)

in the module of residue classes modulo 31. Two treatments denoted by i and j are first associates if:

$(i-j) \bmod 31 \quad A_1 = (3, 6, 7, 12, 14, 17, 19, 24, 25, 28)$

second associates if

$(i-j) \bmod (31) \quad A_2 = (1, 2, 4, 8, 15, 16, 23, 27, 29, 30)$

and third associates if

(i-j) mod 31 $A_3 = (5, 9, 10, 11, 13, 18, 20, 21, 22, 26)$.

Corresponding to treatment i, the block

$$B_{iL}, \ (L = 1, 2, 3)$$

is obtained by putting i and i+t mod 31 where t ranges over the elements of Lth set A_L above for i = 0, 1, ..., 30.

The following series of BIB designs is obtained from Difference Sets:

B.4. Series D_1: $\left(v, \frac{v.v-1}{2}, \frac{k.v-1}{2} \ k, \frac{k.k-1}{2}\right)$

where k v if v = p^h and k the least prime power otherwise. The lay-out of the design with parameters:

(9, 36, 16, 4, 6)

is displayed here under. It is constructed using the initial blocks (0, 1, -1, x); (0, x, -x, -1); (0, x+1, -x-1, x-1); (0, x-1, -x+1. -x-1), in the field G.F.(3^2) with the irreducible function $x^2 + 1 = 0$:-

(1 2 3 4) (2 3 1 6) (3 1 2 8) (1 4 5 3) (2 6 9 1) (3 8 7 2)

(1 6 7 8) (2 8 4 5) (3 4 9 6) (1 8 9 7) (2 4 5 7) (3 6 5 9)

(4 6 8 5) (5 9 7 1) (6 8 4 9) (4 5 1 8) (5 1 4 7) (6 9 2 4)

(4 9 3 7) (5 2 8 3) (6 7 1 5) (4 7 2 3) (5 3 6 8) (6 5 3 1)

(7 5 9 3) (8 4 6 7) (9 7 5 2) (7 3 8 9) (8 7 3 6) (9 2 6 5)

(7 1 6 2) (8 5 2 9) (9 3 4 1) (7 2 4 6) (8 9 1 2) (9 1 8 4)

Below are series of designs from nonlinear configuration in quadrics:

B.5. Series N_1 : All the tangent cones of a nondegenerate quadric Q_{2t} in PG(2t, s) give the Symmetric Balanced Incomplete Block design with parameters

$$v = (s^{2t-1} - 1) / (s - 1) = b$$

$$r = (s^{2t-2} - 1) / (s - 1) = k$$

$$\lambda = (s^{2t-3} - 1) / (s - 1)$$

This series contains a design with parameters

$$v = 15 = b$$

$$r = 7 = k$$

$$\lambda = 3$$

for which Fisher and Yates Tables (1963) show only one solution the cyclic one: (a, b, c, e, f, i, k). The solution through quadrics is obtained by taking the quadric

$$Q_4 : \; x_o^2 + x_1 x_2 + x_3 x_4 = 0$$

in PG(4, 2)..It has 15 points which are our treatments and with respect to each point P:

$$P = (\alpha_o, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

the tangent space T(P) is given by:

$$\alpha_1 x_2 + \alpha_2 x_1 + \alpha_3 x_4 + \alpha_4 x_3 = c$$

The points common to T(P) and Q_4 give us the block corresponding to the point F. Thus we have the 15 blocks as follows:

(1, 4, 7, 5, 8, 6, 9) (6, 1, 9, 2, 12, 3, 15) (11, 2, 5, 7, 15, 9, 13)

(2, 4, 10, 5, 11, 6, 12) (7, 11, 15, 12, 14, 1, 4) (12, 2, 6, 7, 14, 8, 13)

(3, 4, 13, 5, 14, 6, 15) (8, 1, 5, 10, 15, 12, 13) (13, 3, 4, 8, 12, 9, 11)

(4, 1, 7, 2, 10, 3, 13) (9, 1, 6, 10, 14, 11, 13) (14, 3, 5, 7, 12, 9, 10)

(5, 1, 8, 2, 11, 3, 14) (10, 2, 4, 8, 15, 9, 14) (15, 3, 6, 7, 11, 8, 10)

B.6. Series N'_1: The vertex-less cones of a nondegenerate quadric Q_{2t} produce the designs with parameters:

$$v = (s^{2t} - 1) / (s - 1) = b$$

$$r = (s.(s^{2t-1} - 1) / (s - 1) = k$$

$$\lambda_1 = (s^{2t-2} - 1) / (s - 1) \ -2$$

$$\lambda_2 = (s^{2t-2} - 1) / (s - 1)$$

$$p^1_{11} = \lambda_1$$

$$p^1_{11} = \lambda_2$$

The series contains the design which does not appear in B.C.S. catalogue with parameters

v	b	r	k	λ_1	λ_2	n_1	p^1_{11}	p^2_{11}
15	15	6	6	1	3	6	1	3

The lay-out of this design can be obtained by deleting the first treatment in each of the 15 blocks of the design constructed above.

B.7. Series N_2: In a nondegenerate elliptic quadric PG(5, 2) taking its tangent cones we have the PBIB design with parameters:

v	b	r	k	λ_1	λ_2	n_1	p^1_{11}	p^2_{11}
27	27	11	11	3	5	10	1	5

This design is constructed using the following quadric Q_5 in PG(5, 2):

$$x_o^2 + x_1^2 + x_o x_1 + x_2 x_3 + x_4 x_5 = 0$$

The tangent space of any point

$$\alpha = (\alpha_o, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

is given by:

$$Q_5 \; ( ) \quad \alpha_1 x_o + \alpha_o x_1 + \alpha_3 x_2 + \alpha_2 x_3 + \alpha_5 x_4 + \alpha_4 x_5 = 0$$

The 27 points and their tangent cones are shown here under:

P_1: (000010)	P_2: (000001)	P_3: (000100)
P_4: (001000)	P_5: (001111)	P_6: (000101)
P_7: (000110)	P_8: (001010)	P_9: (001001)
P_{10}: (011100)	P_{11}:(010011)	P_{12} (011011)
P_{13} (010111)	P_{14}:(011101)	P_{15}: (011110)
P_{16} (101100)	P_{17}: (100011)	P_{18}: (101011)
P_{19}: (100111)	P_{20}: (101101)	P_{21}: (101110)
P_{22}: (111100)	P_{23}: (110011)	P_{24}: (111110)
P_{25}: (110111)	P_{26}: (111101)	P_{27}: (111110)

These are the twenty seven points of the elliptic quadric in PG(5, 2).

Below we show the blocks of the design, representing the tangent cones by the set of suffices of points:

1: (1, 3, 4, 7, 8, 10, 15, 16, 21, 22, 27)

2: (2, 3, 4, 6, 9, 10, 14, 16, 20, 22, 26)

3: (3, 1, 2, 6, 7, 11, 13, 17, 19, 23, 25)

4: (4, 1, 2, 8, 9, 11, 12, 17, 18, 23, 24)

5: (5, 6, 7, 8, 9, 10, 11, 16, 17, 22, 23)

6: (6, 2, 3, 5, 8, 12, 15, 18, 21, 24, 27)

7: (7. 1, 3, 5, 9, 12, 14, 18, 20, 24, 26)

8: (8, 1, 4, 5, 6, 13, 14, 19, 20, 25, 26)

9: (9, 2, 4, 5, 7, 13, 15, 19, 21, 25, 27)

10: (10, 2, 5, 11, 14, 15, 19, 24, 25, 1, 2)

11: (3, 4, 5, 10, 11, 12, 13, 20, 21, 26, 27)

12: (4, 6, 7, 11, 12, 14, 15, 16, 19, 22, 25)

13: (3, 8, 9, 11, 13, 14, 15, 16, 18, 22, 24)

14: (2, 7, 8, 10, 12, 13, 14, 17, 21, 23, 27)

15: (1, 6, 9, 10, 12, 13, 15, 17, 20, 23, 26)

16: (1, 2, 5, 12, 13, 16, 17, 20, 21, 24, 25)

17: (3, 4, 5, 14, 15, 16, 17, 18, 19, 26, 27)

18: (4, 6, 7, 10, 13, 17, 18, 20, 21, 22, 25)

19 : (3, 8, 9, 10, 12, 17, 19, 20, 21, 22, 24)

20 : (2, 7, 8, 11, 15, 16, 18, 19, 20, 23, 27)

21 : (1, 6, 9, 11, 14, 16, 18, 19, 21, 23, 26)

22 : (1, 2, 5, 12, 13, 18, 19, 22, 23, 26, 27)

23 : (3, 4, 5, 14, 15, 20, 21, 22, 23, 24, 25)

24 : (4, 6, 7, 10, 13, 16, 19, 23, 24, 26, 27)

25 : (3, 8, 9, 10, 12, 16, 18, 23, 25, 26, 27)

26 : (2, 7, 8, 11, 15, 17, 21, 22, 24, 25, 26)

27 : (1, 6, 9, 11, 14, 17, 20, 22, 24, 25, 27)

B.8. Series N'_2: In the nondegenerate elliptic quadric in PG(5, 2) taking all tangent oones and deleting their vertices we get the PBIB design with parameters:

v	b	r	k	λ_1	λ_2	n_1	p^1_{11}	p^2_{11}
27	27	10	10	1	5	10	1	5

where the 27 blocks can be written down explicitly from the above design by leaving off the treatment number corresponding to that block which appears in that block.

B.9. Series N_3: In a non-degenerate hyperbolic quadric in PG(2t-1, s) taking tangent cones as blocks and points of the quadric as treatment we get the designs of this series.

B.10. Series N'_3: Hyperbolic nondegenerate quadric with its vertex-less tangent cones gives the designs of this series.

The following is a series of Doubly Balanced Incomplete Block Designs from line segments of a line.

B.11. Series LS: These are constructed by taking all possible line segments of order s_1 in a projective line of order s.

The parameters are:

$$v = s + 1,$$

$$b = s.(s^2 - 1) / s_1.(s_1^2 - 1),$$

$$r = s.(s - 1) / s_1.(s_1 - 1)$$

$$k = s_1 + 1,$$

$$\lambda = (s - 1) / (s_1 - 1)$$

$$\delta = 1$$

Table : List of Doubly Balanced Incomplete block designs

with

with r and k smaller than 21.

Sl.no.	v	b	r	k	λ	δ
LS 1	5	10	6	3	3	1
LS 2	10	30	12	4	4	1
LS 3	17	68	20	5	5	1

The doubly balanced incomplete block design with parameters:

v	b	r	k	λ	δ
10	30	12	4	4	1

where is the number of times every triplet of treatments occurs is constructed using the line segment of order 3 imbedded in a finite projective line of order 9 based on G.F.(3^2) with the minimum function:

$$x^2 + 1 = 0$$

These are ten points namely:

1: (0, 1)	6: (1, -x)
2: (1, 0)	7: (1, x+1)
3: (1, 1)	8: (1, x-1)
4: (1, -1)	9: (1, -x+1)
5: (1, x)	10: (1, -x-1).

Taking every pair of points of the line and generating all the line segments using G.F.(3) $\subsetneq$ G.F.(3^2) the 30 distinct line segments are as follows:

1 2 3 4	1 2 8 9	1 3 5 10
1 2 5 6	1 3 7 9	1 4 8 10
1 2 7 10	1 3 8 6	1 4 5 9
1 4 6 7	1 5 7 8	1 6 9 10
2 3 8 10	2 3 7 5	2 3 6 9
2 4 9 7	2 4 10 6	2 4 5 8
2 5 10 9	2 6 7 8	3 4 6 5
3 4 9 10	3 4 8 7	3 5 8 9
3 6 7 10	4 5 10 7	4 6 9 8
5 6 10 8	5 6 7 9	7 8 9 10

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