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UNIFORM APPROXIMATION FOR FAMILIES OF STOCHASTIC INTEGRALS

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SUMMARY. Uniform approximations for families of stochastic integrals of Rubin-Fisk-Stratonovich type and Ito type are studied. It is shown that the approximants of Rubin-Fisk-Stratonovich type obtained from partitions of equal size converge faster to the corresponding stochastic integral than the approximants of Ito type for such partitions converge to the corresponding stochastic integral. Methods used for the study of families of stochastic integrals are of independent interest.

1. **Introduction**

In view of the extensive use of stochastic integrals and stochastic differential equations in modeling of systems in engineering, and economic systems especially in mathematical finance and other applied problems, it is necessary to find whether there are *good* approximants to the stochastic integrals and the stochastic differential equations which can be used for simulation purposes. Some work in the area of approximations for the stochastic differential equations is in Rao *et al.* (1974) and Milshtein (1978). More recently, Kloeden and Platen (1992) gives a comprehensive discussion on the numerical solution of stochastic differential equations.

Our aim in this paper is to study the uniform approximations for families of stochastic integrals both of the Ito type and Rubin-Fisk-Stratonovich type. The problem is of major interest especially when modelling is done by a stochastic differential equation involving unknown parameters and the uniform approximation of the stochastic integrals involved becomes important for simulation

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purposes. The problem is of importance not only from the probabilistic point of view but also from the statistical modelling purpose due to its applications in statistical inference for stochastic processes (cf. Prakasa Rao (1997)). An earlier version of this paper appeared as Prakasa Rao and Rubin (1979). As far as the authors are aware, there are no articles dealing with the uniform aspect of the problem till now. We show that the standard approximants of Rubin-Fisk-Stratonovich integral converge under some conditions to the corresponding stochastic integral faster than the Ito approximants to the corresponding Ito-stochastic integral. Further we obtain uniform bounds in probability for the errors in approximating families of stochastic integrals. Our method is similar to the one used in Prakasa Rao and Rubin (1981) in the study of the large sample theory for estimation for parameters in non-linear stochastic differential equations.

2. Approximation of a Stochastic Integral

Consider the Ito stochastic differential equation

$$dX(t) = a(X(t))dt + dW(t),\ 0 \le t \le T,\ X(0) = X_0 \qquad \ldots (2.1)$$

where $\{W(t)\}$ is the standard Wiener process,

(A1) $a(\cdot)$ satisfies the Lipschitz and growth conditions i.e.,

$$|a(x) - a(y) \le L|x - y|,$$

$$|a(x)| \le L(l + |x|)$$

for some constant $L > 0$, and

(A2) $E[X_0^8] < \infty$.

In addition, suppose that

(A3) $f(\cdot)$ is a real valued function with bounded first and second derivatives and $E[f^2(X_0)] < \infty$.

It is well known that the equation has a unique solution $\{X(t)\}$ under the Condition (A1). Suppose that $\{(X(t)\}$ is a stationary process. Conditions for the existence of a stationary solution are given in Gikhman and Skorokhod (1972). Then

$$I \equiv \int_0^T f(X(t))dW(t)$$

exists as an Ito-integral almost surely under (A3) and

$$I = \lim_{n\to\infty} \sum_{i=1}^{n} f(X(t_i))[W(t_{i+1}) - W(t_i)],$$

where $\pi_n : 0 = t_1 < \ldots < t_{n+1} = T$ is a subdivision of $[0,T]$ such that $\Delta_n =$ norm of $\pi_n = \max\{|t_{i+1} - t_i| : 1 \leq i \leq n\}$ tends to zero as $n \to \infty$, and lim denotes limit in quadratic mean. If

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left\{ \frac{f(X(t_i)) + f(X(t_{i+1}))}{2} \right\} [W(t_{i+1}) - W(t_i)]$$

exists as $\Delta_n \to 0$, then one obtains the Rubin-Fisk-Stratonovich integral here after denoted by

$$S \equiv \int_0^T f(X(t))dW(t).$$

It can be checked that the integral S exists as defined above under the condition (A3) following Stratonovich (1966).

We now obtain the rate of approximation for stochastic integrals of the type S for integrands of the form $f(X(t))$ where $\{X(t),\ 0 \leq t \leq T\}$ is a stationary solution of a Ito stochastic differential equation.

Let

$$S_{\pi_n} = \sum_{i=1}^{n} \left\{ \frac{f(X(t_i)) + f(X(t_{i+1}))}{2} \right\} [W(t_{i+1}) - W(t_i)].$$

We shall now estimate $E|S_{\pi_n} - S|^2$ to obtain the rate of convergence. Let $t_i = (i-1)T/n,\ 1 \leq i \leq n+1$. Throughout this paper, C denotes a generic constant.

Let π'_n be a partition, finer than π_n, obtained by choosing the mid point $\tilde{t}_i$ from each of the intervals $t_i < \tilde{t}_i < t_{i+1},\ i = 1, \ldots, n$. Let $0 = t'_1 < t'_2 < \ldots < t'_{2n+1} = T$ be the points of subdivision of the refined partition π'_n. Define the approximating sum $S_{\pi'_n}$ as before. We shall first obtain bounds on $E|S_{\pi_n} - S_{\pi'_n}|^2$ to get bounds on $E|S_{\pi_n} - S|^2$.

Let $0 \leq t^*_0 < t^*_1 < t^*_2 \leq T$ be three points in $[0,T]$ and let us denote $X(t^*_i)$ by X_i and $W(t^*_i)$ by W_i. Define

$$\begin{aligned} Z \equiv\ & \left\{ \tfrac{f(X_2)+f(X_0)}{2} \right\} (W_2 - W_0) - \left\{ \left(\tfrac{f(X_2)+f(X_1)}{2} \right) (W_2 - W_1) \right. \\ & \left. + \left(\tfrac{f(X_1)+f(X_0)}{2} \right) (W_1 - W_0) \right\} \\ =\ & \left(\tfrac{W_1 - W_0}{2} \right) (f(X_2) - f(X_1)) + \left(\tfrac{W_2 - W_1}{2} \right) (f(X_0) - f(X_1)). \end{aligned} \qquad \ldots (2.2)$$

Clearly

$$\begin{aligned} f(X_2) - f(X_1) &= (X_2 - X_1)f'(X_1) + \tfrac{1}{2}(X_2 - X_1)^2 f''(\mu) \\ &= (W_2 - W_1 + I_2)f'(X_1) + \tfrac{1}{2}(X_2 - X_1)^2 f''(\mu) \end{aligned} \qquad \ldots (2.3)$$

and

$$\begin{aligned} f(X_0) - f(X_1) &= (X_0 - X_1)f'(X_1) + \tfrac{1}{2}(X_0 - X_1)^2 f''(\nu) \\ &= -(W_1 - W_0 + I_1)f'(X_1) + \tfrac{1}{2}(X_0 - X_1)^2 f''(\nu) \end{aligned} \qquad \ldots (2.4)$$

where $|X_1 - \mu| \leq |X_2 - X_1|$, $|X_1 - \nu| \leq |X_0 - X_1|$ and

$$I_1 = \int_{t_0^*}^{t_1^*} a(X(t))dt, \quad I_2 = \int_{t_1^*}^{t_2^*} a(X(t))dt. \qquad \ldots (2.5)$$

Equations (2.2) to (2.4) show that

$$\begin{aligned} Z &= \left(\tfrac{W_1 - W_0}{2}\right) I_2 f'(X_1) + \left(\tfrac{W_1 - W_0}{4}\right)(X_2 - X_1)^2 f''(\mu) \\ &\quad - \left(\tfrac{W_2 - W_1}{2}\right) I_1 f'(X_1) + \left(\tfrac{W_2 - W_1}{4}\right)(X_1 - X_0)^2 f''(\nu). \end{aligned} \qquad \ldots (2.6)$$

Let

$$J_1 = (W_1 - W_0)\left(\tfrac{I_2}{2} f'(X_1) + \tfrac{(X_2 - X_1)^2}{4} f''(\mu)\right) \qquad \ldots (2.7)$$

and

$$J_2 = (W_2 - W_1)\left(\tfrac{I_1 f'(X_1)}{2} + \tfrac{(X_1 - X_0)^2}{4} f''(\nu)\right). \qquad \ldots (2.8)$$

Clearly

$$E(Z^2) \leq 2(E(J_1^2) + E(J_2^2)). \qquad \ldots (2.9)$$

Furthermore, the J_2's, corresponding to different subintervals of $[0, T]$ generated by π_n, form a martingale difference sequence and the J_1's corresponding to different subintervals of $[0, T]$ generated by π_n form a reverse martingale difference sequence.

Observe that

$$\begin{aligned} E(J_2^2) &= E(W_2 - W_1)^2 E\left(\tfrac{I_1 f'(X_1)}{2} + \tfrac{(X_1 - X_0)^2}{4} f''(\nu)\right)^2 \\ &\leq C\, E(W_2 - W_1)^2 \{E(I_1^2) + E(X_1 - X_0)^4\} \end{aligned} \qquad \ldots (2.10)$$

for some constant $C > 0$ by the boundedness of derivatives of f and by the C_r-inequality. Note that there exists $C > 0$ such that

$$E(X_1 - X_0)^4 \leq C(E(X_0^4) + 1)(t_1^* - t_0^*)^2 \qquad \ldots (2.11)$$

by Theorem 4 of Gikhman and Skorokhod (1972) p.48 and

$$\begin{aligned} E(I_1^2) &= E\left(\int_{t_0^*}^{t_1^*} a(X(t))dt\right)^2 \\ &\leq (t_1^* - t_0^*) E\left(\int_{t_0^*}^{t_1^*} a^2(X(t))dt\right) \\ &\leq 2L(t_1^* - t_0^*)^2\, E(1 + |X_0|^2) \end{aligned} \qquad \ldots (2.12)$$

by stationarity of the process $\{X(t)\}$.

Relations (2.10) - (2.12) prove that

$$E(J_2^2) \leq C(t_2^* - t_1^*)(t_1^* - t_0^*)^2 \qquad \ldots(2.13)$$

for some constant $C > 0$ independent of t_0^*, t_1^* and t_2^*. Let us now estimate $E(J_1^2)$. Note that

$$\begin{aligned} E(J_1^2) &= E\left[(W_1 - W_0)\left\{\tfrac{I_2 f'(X_1)}{2} + \tfrac{(X_2-X_1)^2}{4} f''(\mu)\right\}\right]^2 \\ &= E\left[(W_1 - W_0)^2\left\{\tfrac{I_2 f'(X_1)}{2} + \tfrac{(X_2-X_1)^2}{4} f''(\mu)\right\}^2\right] \\ &\leq \left[E(W_1 - W_0)^4 E\left\{\tfrac{I_2 f'(X_1)}{2} + \tfrac{(X_2-X_1)^2}{4} f''(\mu)\right\}^4\right]^{1/2} \qquad \ldots(2.14) \\ &\qquad \text{(by Cauchy-Schwartz inequality)} \\ &\leq C(t_1^* - t_0^*)[E\{I_2^4 + (X_2 - X_1)^8\}]^{1/2} \end{aligned}$$

for some constant $C > 0$, by the boundedness of derivatives of f, C_r-inequality and the fact that $E(W_1 - W_0)^4 = 3(t_1^* - t_0^*)^2$. Note that there exists a constant $C > 0$ such that

$$E(X_2 - X_1)^8 \leq C(t_2^* - t_1^*)^4 \qquad \ldots(2.15)$$

by Theorem 4 of Gikhman and Skorokhod (1972), p. 48. Furthermore, it is easy to check that

$$\begin{aligned} E(I_2^4) &= E\left[\int_{t_1^*}^{t_2^*} a(X(t))dt\right]^4 \\ &\leq L^4 E\left[\int_{t_1^*}^{t_2^*} (1 + |X(t)|)dt\right]^4 \qquad \ldots(2.16) \\ &\leq 4L^4(t_2^* - t_1^*)^4 E(| + |X(0)|^4) \end{aligned}$$

by the stationarity of the process $\{X(t)\}$. Relations (2.15) and (2.16) prove that

$$E(J_1^2) \leq C(t_1^* - t_0^*)(t_2^* - t_1^*)^2 \qquad \ldots(2.17)$$

for some constant $C > 0$ independent of t_0^*, t_1^* and t_2^*. Inequalities (2.13) and (2.17) prove that there exists a constant $C > 0$ independent of t_0^*, t_1^* and t_2^* such that

$$E(J_i^2) \leq C(t_2^* - t_0^*)^3, \quad i = 1, 2. \qquad \ldots(2.18)$$

Using the property that J_2's corresponding to different subintervals form a martingale difference sequence and J_l's form a reverse martingale difference sequence, it follows that

$$E|S_{\pi_n} - S_{\pi'_n}|^2 \leq C\frac{T^3}{n^2} \qquad \ldots(2.19)$$

for some constant $C > 0$.

Let $\{\pi_n^{(p)},\ p \leq 0\}$ be the sequence of partitions such that $\pi_n^{(i+1)}$ is a refinement of $\pi_n^{(i)}$ by choosing the midpoints of subintervals generated by $\pi_n^{(i)}$. Note that $\pi_n^{(0)} = \pi_n$ and $\pi_n^{(1)} = \pi_n'$. The analysis given above proves that

$$E|S_{\pi_n}(p) - S_{\pi_n}(p+1)|^2 \leq C\tfrac{T^3}{2^p n^2},\ p \geq 0 \qquad \ldots(2.20)$$

where $S_{\pi_n}(p)$ is the approximant corresponding to $\pi_n^{(p)}$ and $S_{\pi_n}(0) = S_{\pi_n}$. Therefore

$$\begin{aligned} E|S_{\pi_n} - S_{\pi_n}(p+1)|^2 &\leq \{\sum_{k=0}^{p}(E|S_{\pi_n}(k) - S_{\pi_n}(k+1)|^2)^{1/2}\}^2 \\ &\leq \left\{\sum_{k=0}^{p}\left(\frac{CT^3}{2^k n^2}\right)^{1/2}\right\}^2 \leq C\tfrac{T^3}{n^2} \end{aligned} \qquad \ldots(2.21)$$

for all $p \geq 0$. Let $p \to \infty$. Since the integral S exists, $S_{\pi_n}(p+1)$ converges in quadratic mean as $p \to \infty$. Note that $\{\pi_n(p+1),\ p \geq 0\}$ is a sequence of partitions such that the norms of the partition tends to zero as $p \to \infty$ for any fixed n. Therefore

$$E|S_{\pi_n} - S|^2 = O(n^{-2}), \qquad \ldots(2.22)$$

where

$$S = \lim_{n\to\infty} S_{\pi_n} = \int_0^T f(X(t))dW(t).$$

We have the following result.

THEOREM 2.1. *Let $\{X(t),\ 0 \leq t \leq T\}$ be a stationary stochastic process satisfying the Ito stochastic differential equation (2.1). Suppose the conditions (A1), (A2) and (A3) hold. Define S_{π_n} as given above as an approximation for the Rubin-Fisk-Stratonovich integral S of $f(X(t))$ with respect to the Wiener process on $[0,T]$. Suppose π_n is a sequence of equidistant partitions. Then $E|S_{\pi_n} - S|^2 = O(n^{-2})$.*

On the other hand, let us consider

$$I_{\pi_n} = \sum_{i=1}^{n} f(X(t_i))[W(t_{i+1}) - W(t_i)]$$

as an approximating sum for the Ito integral

$$I = \int_0^T f(X(t))dW(t).$$

REMARKS. It can be easily shown that

$$E|I_{\pi_n} - I|^2 = O(n^{-1}) \quad \ldots (2.23)$$

by arguments analogous to those given above and by noting that $\{I_{\pi_n},\ n \geq 1\}$ is a martingale. It is sufficient to assume the existence and boundedness of first derivative of f in this case.

In other words, the sequence of Rubin-Fisk-Stratonovich approximating sums converge to the corresponding Rubin-Fisk-Stratonovich integral faster than the sequence of Ito approximating sums converge to the corresponding Ito integral. The assumption about the equidistant partition is not essential for the result. However, smoothness of f and stationarity of the process $X(\cdot)$ are crucial for the method adapted here for obtaining the rates.

3. Uniform Equi-continuity of Ito Stochastic Integrals Indexed by a Parameter

Let us now consider a family of stochastic integrals

$$I(\theta) = \int_0^T f(X(t), \theta) dW(t),\ \theta \in [-1, 1], \quad \ldots (3.1)$$

where $f(X, \theta)$ is differentiable with respect to θ and the partial derivative f_θ is Lipschitz in θ of order $\alpha > 0$ i.e.,

$$|f_\theta(x, \phi_1) - f_\theta(x, \phi_2)| \leq g(x)|\phi_1 - \phi_2|^\alpha \quad \ldots (3.2)$$

with

$$E[g^2(X(0))] < \infty, \quad \ldots (3.3)$$

and $\{X(t),\ 0 \leq t \leq T\}$ is the stationary process satisfying (2.1).

We shall suppose that $\{I(\theta),\ \theta \in [-1, 1]\}$ is separable. It is easy to see from (3.2) that

$$\begin{aligned} E|f(X(t), \theta) \quad &- \quad \tfrac{1}{2}\{f(X(t), \theta + \varepsilon) + f(X(t), \theta - \varepsilon)\}|^2 \\ &\leq \quad \tfrac{1}{4}\varepsilon^{2+2\alpha} E[g^2(X(t))] \end{aligned} \quad \ldots (3.4)$$

for every $\varepsilon > 0$. Let $\pi_n : 0 = t_1 < t_2 < \ldots < t_{n+1} = T$ be a partition of $[0, T]$. Denote $X(t_k)$ by X_k and $\Delta W_k = W(t_{k+1}) - W(t_k)$. Define

$$I_{\pi_n}(\theta) = \sum_{k=1}^{n} f(X_k, \theta) \Delta W_k. \quad \ldots (3.5)$$

Note that $I_{\pi_n}(\theta)$ is an approximating sum for the Ito stochastic integral $I(\theta)$ defined by (3.1). Let

$$Q_{\pi_n}(\theta,\varepsilon) = I_{\pi_n}(\theta) - \frac{1}{2}[I_{\pi_n}(\theta+\varepsilon) + I_{\pi_n}(\theta-\varepsilon)].$$

Relation (3.4) implies that

$$\begin{aligned} E[Q_{\pi_n}(\theta,\varepsilon)]^2 &\leq \tfrac{1}{4}\varepsilon^{2+2\alpha}\sum_{i=1}^{n} E[g^2(X_i)](t_{i+1}-t_i) \\ &\leq CT\varepsilon^{2+2\alpha} \end{aligned} \qquad \ldots (3.6)$$

for some $C > 0$ by the stationarity of $\{X(t)\}$. In view of the remarks made in the Appendix, $f(x,\theta)$ can be expanded in the form

$$f(x,\theta) = \sum_{i=0}^{\infty}\sum_{j=1}^{2^i} \lambda_{ij}(x) q_{ij}(\theta)$$

where

$$\lambda_{00} = 1$$

$$\lambda_{ij}(x) = f(x,-1+\frac{2j-1}{2^i}) - \frac{1}{2}\{f(x,-1+\frac{j-1}{2^{i-1}}) + f(x,-1+\frac{j}{2^{i-1}})\}$$

for $1 \leq j \leq 2^i$, $i \geq 1$ and q_{ij}'s are as defined in the Appendix. Furthermore, for any θ, at most only one of q_{ij}'s is non-zero. It is obvious from the fact that $\{X(t)\}$ is a stationary process that

$$E(\lambda_{ij}^2(X(0))) \leq C2^{-(2+2\alpha)i} \text{ for } 1 \leq j \leq 2^i \qquad \ldots (3.7)$$

and hence

$$P(\max_{1\leq j\leq 2^i} |\lambda_{ij}(X(0))| \geq \varepsilon_i) \leq C2^{-(1+2\alpha)i}\varepsilon_i^{-2}. \qquad \ldots (3.8)$$

Let $\varepsilon > 0$ and $\varepsilon_i = \varepsilon A 2^{-\tau i}$ where $A = (1-2^{-\tau})$ and $\alpha < \tau < \frac{1+2\alpha}{2}$. Then

$$\sum_{i=1}^{\infty} P(\max_{1\leq j\leq 2^i} |\lambda_{ij}(X(0))| \geq \varepsilon_i) \leq C\sum_{i=1}^{\infty} 2^{-(1+2\alpha-2\tau)i}\,\varepsilon^{-2}A^{-2} = C\varepsilon^{-2} \qquad \ldots (3.9)$$

for some constant $C > 0$. Therefore, by the Borel-Cantelli lemma, it follows that for any $\varepsilon > 0$,

$$\max_{1\leq j\leq 2^i} |\lambda_{ij}(X(0))| \leq \varepsilon A 2^{-\tau i}$$

for sufficiently large i with probability one and by the stationarity of the process $X(t)$,

$$\max_{1\leq j\leq 2^i} |\lambda_{ij}(X(t))| \leq \varepsilon A 2^{-\tau i},\ 0 \leq t \leq T \qquad \ldots (3.10)$$

for sufficiently large i with probability one.

Let $\{\pi_n\}$ be a sequence of partitions such that the norm of $\{\pi_n\}$ tends to zero as $n \to \infty$ and define $I_{\pi_n}(\theta)$ by (3.5). Note that

$$\begin{aligned} I_{\pi_n}(\theta) &= \sum_k \{\sum_i \sum_j \lambda_{ij}(X_k) q_{ij}(\theta)\} \Delta W_k \\ &= \sum_i \sum_j \{\sum_k \lambda_{ij}(X_k)\Delta W_k\} q_{ij}(\theta) \qquad \ldots (3.11) \\ &= \sum_i \sum_j R_{ij} q_{ij}(\theta), \end{aligned}$$

where

$$R_{ij} = \sum_k \lambda_{ij}(X_k)\Delta W_k. \qquad \ldots (3.12)$$

Now, for any $\varepsilon_i > 0$,

$$\begin{aligned} P(\max_j |R_{ij}| > \varepsilon_i) &\leq \frac{1}{\varepsilon_i^2} \sum_{j=1}^{2^i} E(R_{ij}^2) \\ &= \frac{1}{\varepsilon_i^2} \sum_{j=1}^{2^i} \{\sum_{k=1}^{n} E(\lambda_{ij}^2(X_k))(t_{k+1} - t_k)\} \\ &= \frac{T}{\varepsilon_i^2} \sum_{j=1}^{2^i} E(\lambda_{ij}^2(X_0)) \quad \text{(by stationarity)} \\ &\leq \frac{T}{\varepsilon_i^2} C 2^{-(2+2\alpha)i} 2^i \quad \text{(by (3.7))} \\ &= \frac{CT}{\varepsilon_i^2} 2^{-(1+2\alpha)i}. \end{aligned}$$

Let $\varepsilon_i = \varepsilon 2^{-\tau i}$ where $\alpha < \tau < \frac{1+2\alpha}{2}$. Then

$$\begin{aligned} \sum_{i=1}^{\infty} P(\max_j |R_{ij}| > \varepsilon_i) &\leq \frac{CT}{\varepsilon^2} \sum_{i=1}^{\infty} \frac{1}{2^{(1+2\alpha-2\tau)i}} \\ &\leq \frac{C}{\varepsilon^2} < \infty. \end{aligned}$$

Hence, by the Borel-Cantelli Lemma, it follows that

$$\max_{1 \leq j \leq 2^i} |R_{ij}| \leq \varepsilon 2^{-\tau i} \qquad \ldots (3.13)$$

for sufficiently large i with probability one. For any fixed i, at most one of the $q_{ij}(\theta)$ is non-zero and if θ and ϕ are such that $|\theta - \phi| < \delta$, then θ and ϕ are in adjacent intervals of size $\frac{1}{2^i}$, if i is sufficiently large. Hence it follows that

$$\sup_{|\theta-\phi|<\delta} |I_{\pi_n}(\theta) - I_{\pi_n}(\phi)| \leq \sum_i \{\max_{i \leq j \leq 2^i} |R_{ij}|\} \leq \sum_i \varepsilon 2^{-\tau i} \leq C\varepsilon$$

for some constant $C > 0$ with probability approaching one. Therefore, for every $\varepsilon > 0$,

$$\overline{\lim}_{\delta \to 0} \lim_{n\to\infty} P(\sup_{|\theta-\phi|\leq\delta} |I_{\pi_n}(\theta) - I_{\pi_n}(\phi)| \geq \varepsilon) = 0. \qquad \ldots (3.14)$$

Note that $I_{\pi_n}(\theta) \to I(\theta)$ in probability for every $\theta \in [-1,\ 1]$. Let F be any finite set of θ's in $[-1, 1]$. it is clear that, for any $\varepsilon > 0$, there exist N_1 and N_2 and $\delta > 0$ such that

$$P(\max_{\theta \in F} |I_{\pi_n}(\theta) - I(\theta)| \geq \frac{\varepsilon}{3}) < \frac{\varepsilon}{3}$$

for $n \geq N_1$ and

$$P\left(\sup_{|\theta-\phi|\leq\delta} |I_{\pi_n}(\theta) - I_{\pi_n}(\phi)| \geq \frac{\varepsilon}{3}\right) < \frac{\varepsilon}{3}$$

for $n \geq N_2$ (independent of F) by (3.14). Let $N = \max(N_1, N_2)$. It is now easily seen that

$$\begin{aligned} & P(\sup_{\substack{|\theta-\phi|\leq\delta \\ \theta,\phi\in F}} |I(\theta) - I(\phi)| \geq \varepsilon) \\ \leq \ & P\left(\sup_{\theta\in F} |I(\theta) - I_{\pi_n}(\theta)| \geq \frac{\varepsilon}{3}\right) + P\left(\sup_{\phi\in F} |I(\phi) - I_{\pi_N}(\phi)| \geq \frac{\varepsilon}{3}\right) \\ & + P\left(\sup_{\substack{|\theta-\phi|\leq\delta \\ \theta,\phi\in F}} |I_{\pi_N}(\theta) - I_{\pi_N}(\phi)| \geq \frac{\varepsilon}{3}\right) \\ < \ & \varepsilon, \end{aligned}$$

for any finite set F of θ's in $[-1, 1]$. As an immediate consequence, it follows that

$$P\left(\sup_{\substack{|\theta-\phi|\leq\delta \\ \theta,\phi\in K}} |I(\theta) - I(\phi)| \geq \varepsilon\right) < \varepsilon$$

for any countable set K of θ's in $[-1, 1]$. By the sep of the process, it follows that

$$P\left(\sup_{|\theta-\phi|\leq\delta} |I(\theta) - I(\phi)| \geq \varepsilon\right) < \varepsilon.$$

Hence the process $\{I(\theta),\ \theta \in [-1,1]\}$ is uniformly equicontinuous in probability and we have the following result.

THEOREM 3.1. *Suppose* $\{X(t),\ 0 \le t \le T\}$ *is a stationary stochastic process satisfying the Ito Stochastic differential equation (2.1). Define* $I(\theta)$ *by (3.1). Then the process* $\{I(\theta),\ \theta \in [-1,1]\}$ *is uniformly equicontinuous in probability in the sense that for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$P\left(\sup_{|\theta-\phi|\le\delta} |I(\theta) - I(\phi)| \ge \varepsilon\right) < \varepsilon.$$

4. Uniform Approximation of Families of Rubin-Fisk-Stratonovich Stochastic Integrals

We shall now obtain uniform bounds in probability for the family of Rubin-Fisk-Stratonovich stochastic integrals

$$S(\theta) \equiv \int_0^T f(X(t),\theta)dW(t),\ \theta \in [-1,1], \qquad \ldots(4.1)$$

where $\{X(t),\ 0 \le t \le T\}$ is the unique stationary solution of the Ito stochastic differential equation

$$dX(t) = a(X(t))dt + dW(t),\ 0 \le t \le T,\ X(0) = X_0.$$

Conditions for the existence of such a solution are given in Gikhman and Skorokhod (1972). We assume that the following additional conditions hold :

(B1) $f(x,\theta)$ is differentiable with respect to x and θ with partial derivatives $f_{x\theta}(x,\theta)$ and $f_{xx\theta}(x,\theta)$. Furthermore $f_{x\theta}(x,\theta)$ is Lipschitz in θ of order $\alpha > \frac{1}{2}$ and $f_{xx\theta}(x,\theta)$ is Lipschitz in θ of order α uniformly in x;

(B2) (i) $|a(x)| \le L(1+|x|),\ x \in R$, (ii) $|a(x) - a(y)| \le L|x-y|,\ x \in R$ for some constant $L > 0$; and

(B3) $E(X_0^8) < \infty$.

In view of the approximation discussed in the Appendix

$$f(x,\theta) = \sum_i \sum_j \lambda_{ij}(x) q_{ij}(\theta),$$

where $q_{ij}(\theta)$ is defined in the Appendix and

$$\lambda_{ij}(x) = f(x, -1 + \frac{2j-1}{2^i}) - \frac{1}{2}\{f(x,-1 + \frac{j-1}{2^{i-1}}) + f(x,-1 + \frac{j}{2^{i-1}})\}$$

for $1 \leq j \leq 2^i,\ i \geq 1$. In particular

$$f_x(x,\theta) = \sum_i \sum_j \lambda'_{ij}(x) q_{ij}(\theta) \qquad \ldots (4.2)$$

and

$$f_{xx}(x,\theta) = \sum_i \sum_j \lambda''_{ij}(x) q_{ij}(\theta). \qquad \ldots (4.3)$$

Let

$$\pi_n : 0 = t_{-1} < t_1 < t_3 < \ldots < t_{2n-1} < t_{2n+1} = T$$

be a partition of $[0,T]$ and

$$\pi_n(1) : 0 = t_{-1} < t_0 < t_1 < t_2 < t_3 < \ldots < t_{2n-1} < t_{2n} < t_{2n+1} = T$$

be obtained from π_n by taking the midpoints of the subintervals of π_n i.e. $t_{2k} = \frac{1}{2}(t_{2k-1} + t_{2k+1}),\ k = 0, 1, \ldots, n$. Let

$$S_{\pi_n}(\theta) = \sum_{k=1}^{n} \left\{ \frac{f(X(t_{2k+1}),\theta) + f(X(t_{2k-1}),\theta))}{2} \right\} [W(t_{2k+1}) - W(t_{2k-1})] \qquad \ldots (4.4)$$

and define $S_{\pi_n(1)}(\theta)$ in a similar way. For simplicity, write

$$X(t_s) = X_s \text{ and } \Delta W_s = W(t_s) - W(t_{s-1}) \text{ and } \Delta X_s = X(t_s) - X(t_{s-1}).$$

It is easy to see that

$$\begin{aligned} S_{\pi_n}(\theta) - S_{\pi_n(1)}(\theta \quad &= \quad \sum_k \frac{1}{2}\{f(X_{2k+1},\theta) - f(X_{2k},\theta)\}\Delta W_{2k} \\ &\quad + \sum_k \frac{1}{2}\{f(X_{2k-1},\theta) - f(X_{2k,}\theta)\}\Delta W_{2k+1} \qquad \ldots (4.5) \\ &= \quad \sum_k J_1(X_{2k},\theta) + \sum_k J_2(X_{2k},\theta) \end{aligned}$$

where

$$J_1(X_{2k},\theta) = \{\frac{1}{2} I_2(X_{2k}) f_x(X_{2k},\theta) + \frac{1}{4} X^2_{2k+1} f_{xx}(\mu_{2k},\theta)\}\Delta W_{2k}, \qquad \ldots (4.6)$$

$$J_2(X_{2k},\theta) = \{\frac{1}{2} I_1(X_{2k}) f_x(X_{2k},\theta) + \frac{1}{4} X^2_{2k} f_{xx}(\nu_{2k},\theta)\}\Delta W_{2k+1}, \qquad \ldots (4.7)$$

$$I_1(X_{2k}) = \int_{t_{2k-1}}^{t_{2k}} a(X(s))ds, \qquad \ldots (4.8)$$

and

$$I_2(X_{2k}) = \int_{t_{2k}}^{t_{2k+1}} a(X(s))ds, \qquad \ldots(4.9)$$

by arguments analogous to those in Section 2. Note that

$$\begin{aligned}\sum_k J_2(X_{2k},\theta) &= \tfrac{1}{2}\sum_k I_1(X_{2k})\{\sum_i\sum_j \lambda'_{ij}(X_{2k})q_{ij}(\theta)\}\Delta W_{2k+1} \\ &\quad + \tfrac{1}{4}\sum_k \Delta X_{2k}^2\{\sum_i\sum_j \lambda''_{ij}(\nu_{2k})q_{ij}(\theta\}\Delta W_{2k+1} \\ &= \sum_i\sum_j[\sum_k\{\frac{1}{2}I_1(X_{2k})\lambda'_{ij}(X_{2k})\Delta W_{2k+1}\}]q_{ij}(\theta) \\ &\quad + \sum_i\sum_j[\sum_k\{\frac{1}{4}\Delta X_{2k}^2\lambda''_{ij}(\nu_{2k})\Delta W_{2k+1}\}]q_{ij}(\theta).\end{aligned}$$

$$\ldots(4.10)$$

Let

$$R_{ij}^{(1)} = \sum_k I_1(X_{2k})\lambda'_{ij}(X_{2k})\Delta W_{2k+1} \qquad \ldots(4.11)$$

and

$$R_{ij}^{(2)} = \sum_k \Delta X_{2k}^2\lambda''_{ij}(\nu_{2k})\Delta W_{2k+1}. \qquad \ldots(4.12)$$

Suppose that i is sufficiently large so that there exists $C > 0$ and $\alpha < \tau < \frac{1+2\alpha}{2}$ such that $|\lambda'_{ij}(X(t))| \leq C2^{-\tau i}$ a.s. for all t. This is possible by arguments similar to those used to derive (3.10). It is easy to see that $|\lambda''_{ij}(x)| \leq C2^{-\alpha i}$ for all x by (B1). Hence

$$\begin{aligned}P(\max_j |R_{ij}^{(1)}| > \varepsilon_i) &\leq \tfrac{1}{\varepsilon_i^2}\sum_{j=1}^{2^i} E(R_{ij}^{(1)})^2 \\ &= \tfrac{1}{\varepsilon_i^2}\sum_{j=1}^{2^i}\{\sum_k E(I_1(X_{2k})\lambda'_{ij}(X_{2k}))^2(t_{2k+1}-t_{2k})\} \\ &\leq \tfrac{C}{\varepsilon_i^2}\sum_{j=1}^{2^i}\{\sum_k E(I_1(X_{2k})^2 2^{-2\tau i}(t_{2k+1}-t_{2k})\} \\ &= \tfrac{C}{\varepsilon_i^2}\sum_{j=1}^{2^i} 2^{-2\tau i}\{\sum_k (t_{2k}-t_{2k-1})^2(t_{2k+1}-t_{2k})\}\end{aligned}$$

$$\ldots(4.13)$$

by arguments similar to those used to derive (2.12). Therefore

$$P(\max_j |R_{ij}^{(1)}| > \varepsilon_i) \leq \frac{C}{\varepsilon_i^2} 2^{(1-2\tau)i} \sum_k (t_{2k+1} - t_{2k-1})^3$$

for some constant $C > 0$.

Similarly

$$\begin{aligned} P(\max_j |R_{ij}^{(2)}| > \varepsilon_i) &\leq \frac{1}{\varepsilon_i^2} \sum_{j=1}^{2^i} E(R_{ij}^{(2)})^2 \\ &= \frac{1}{\varepsilon_i^2} \sum_{j=1}^{2^i} \sum_k \{E(\Delta X_{2k}^2 \lambda_{ij}''(\nu_{2k}))^2 (t_{2k+1} - t_{2k})\} \\ &\leq \frac{C}{\varepsilon_i^2} \sum_{j=1}^{2^i} \{\sum_k E(\Delta X_{2k}^4) 2^{-2\alpha i} (t_{2k+1} - t_{2k})\} \\ &\leq \frac{C}{\varepsilon_i^2} 2^{(1-2\alpha)i} \sum_k (t_{2k+1} - t_{2k})(t_{2k} - t_{2k-1})^2 \end{aligned}$$

for some constant $C > 0$ by Theorem 4 of Gikhman and Skorokhod (1972), p.48. Hence

$$P(\max_j |R_{ij}^{(2)}| > \varepsilon_i) \leq \frac{C}{\varepsilon_i^2} 2^{(1-2\alpha)i} \sum_k (t_{2k+1} - t_{2k-1})^3. \qquad \ldots (4.14)$$

Combining (4.10)-(4.14), we get that

$$P(\sup_\theta |\sum_k J_2(X_{2k}, \theta)| > \sum_i \varepsilon_i) \leq C \sum_k (t_{2k+1} - t_{2k})^3 \sum_{i=1}^\infty \frac{2^{(1-2\alpha)i}}{\varepsilon_i^2}$$

for some constant $C > 0$. Similar estimate can be obtained for the term

$$\sum_k J_1(X_{2k}, \theta)$$

by using the reverse martingale property of J_1's and the stationarity of the process $\{X(t)\}$. Hence

$$P(\sup_\theta |S_{\pi_n}(\theta) - S_{\pi_n(1)}(\theta)| \leq \sum \varepsilon_i) \leq C((\sum_k (t_{2k+1} - t_{2k-1})^3 \{\sum_{i=1}^\infty \frac{2^{(1-2\alpha)i}}{\varepsilon_i^2}\}.$$

Let γ be such that $0 < 2\gamma < 2\alpha - 1$. Choosing $\varepsilon_i = \varepsilon 2^{-\gamma i}$, we have the inequality

$$P(\sup_\theta |S_{\pi_n}(\theta) - S_{\pi_n(1)}(\theta)| > \varepsilon A) \leq \frac{C}{\varepsilon^2} \sum_k (t_{2k+1} - t_{2k-1})^3$$

where $A = \sum_i 2^{-i\gamma}$. Hence, if $t_{2k+1} - t_{2k} = T/n$, then

$$P(\sup_\theta |S_{\pi_n}(\theta) - S_{\pi_n(1)}(\theta)| > \varepsilon A) \le \frac{C}{\varepsilon^2}\frac{T^3}{n^2}.$$

Let $\{\pi_n(p),\ p \ge 0\}$ be a sequence of partitions of $[0,T]$ as defined in Section 2. Then

$$P(\sup_\theta |S_{\pi_n}(\theta) - S_{\pi_n(p+1)}(\theta)| > \varepsilon) \le C \sum_{k=0}^{p} \frac{1}{\varepsilon_k^2 (2_n^k)^2},$$

where $\sum_{k=1}^{\infty} \varepsilon_k \le \varepsilon$. Choosing ε_k suitably, we obtain that

$$P(\sup_\theta |S_{\pi_n}(\theta) - S_{\pi_n(p+1)}(\theta)| > \varepsilon) \le C n^{-2}\varepsilon^{-2},\ p \ge 0.$$

Letting $p \to \infty$, we see that $S_{\pi_n}(\theta) \xrightarrow{p} S(\theta)$ uniformly in θ and

$$P(\sup_\theta |S_{\pi_n}(\theta) - S(\theta)| > \varepsilon) \le C n^{-2}\varepsilon^{-2}$$

for some positive constant C. In fact

$$E(\sup_\theta |S_{\pi_n}(\theta) - S(\theta)|^2) = O(\frac{1}{n^2})$$

by the arguments given earlier under the assumptions (B1), (B2) and (B3). We have the following result.

THEOREM 4.1. *Let $\{X(t),\ 0 \le t \le T\}$ be a stationary stochastic process satisfying the stochastic differential equation (2.1). Let $S(\theta)$ be the Rubin-Fisk-Stratonovich integral defined by (4.1) and suppose that conditions (B1)-(B3) hold. Let $S_{\pi_n}(\theta)$ be an approximating sum for $S(\theta)$ where $\{\pi_n\}$ is a sequence of equidistant partitions. Then*

$$E(\sup_\theta |S_{\pi_n}(\theta) - S(\theta)|^2) = O(n^{-2}).$$

REMARKS. If one considers the Ito integral $I(\theta)$ given by (3.1) and the Ito approximating sum defined by (3.5), then it can be shown that

$$E(\sup_\theta |I_{\pi_n}(\theta - I(\theta)|^2) = O(\frac{1}{n})$$

for the sequence of partitions defined above under the assumptions (B2), (B3) and (B1)* given below :
(B1)* $f(x,\theta)$ is differentiable with respect to x and θ with partial derivative $f_{x\theta}(x,\theta)$. Furthermore $f_{x\theta}(x,\theta)$ is Lipschitz in θ of order $\alpha > \frac{1}{2}$ uniformly in x.

Since the method of proof is similar to that given above, we omit the details.

REMARKS. The techniques used in Section 2 of this paper have recently been used in Mishra and Bishwal (1995) in their work on approximate maximum likelihood estimation for diffusion processes from discrete observations based on an unpublished earlier version of this paper (cf. Prakasa Rao and Rubin (1979)).

Appendix

Let $h(\theta)$ be continuous on $[-1,1]$. We construct a family of functions $q_{ij}(\theta),\ 1 \le j \le 2^i,\ i \ge 0$ as follows. Define

$$q_{00}(0) = h(-1),\ q_{oo}(1) = h(1)$$

and suppose that $q_{00}(\cdot)$ is linear in $[-1,1]$. For $i \ge 1$, at the ith stage, divide $[-1,1]$ into 2^i equal intervals and define, for $1 \le j \le 2^i$,

$$\begin{aligned} q_{ij}(\theta) &= 0 \text{ if } \theta \notin (-1+\frac{j-1}{2^{i-1}},\ -1+\frac{j}{2^{i-1}}) \\ &= 1 \text{ if } \theta = -1+\frac{2j-1}{2^i} \end{aligned}$$

and $q_{ij}(\theta$ linear in the intervals

$$[-1+\frac{j-1}{2^{i-1}}, -1+\frac{2j-1}{2^i}] \text{ and } [-1+\frac{2j-1}{2^i},\ -1+\frac{j}{2^{i-1}}].$$

Observe that at each stage exactly one of the q's is non-zero for any given $\theta \in [-1,1]$. Let

$$\lambda_{00} \equiv 1$$

and

$$\lambda_{ij} = h(-2+\frac{2j-1}{2^i}) - \frac{1}{2}\left\{h(-1+\frac{j-1}{2^{i-1}}) + h(-1+\frac{j}{2^{i-1}})\right\}$$

for $1 \le j \le 2^i$ and $i \ge 1$. Then

$$h(\theta) = \sum_{i=0}^{\infty}\sum_{j=1}^{2^i} \lambda_{ij} q_{ij}(\theta),\ \theta \in [-1,1].$$

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