FINITE CLUSTERS IN HIGH DENSITY BOOLEAN MODELS WITH BALLS OF VARYING SIZES

ANISH SARKAR,* *Indian Statistical Institute*

Abstract

In this paper we study finite clusters in a high density Boolean model with balls of two distinct sizes. Alexander (1993) studied the geometric structures of finite clusters in a high density Boolean model with balls of fixed size and showed that the only possible structure admitted by such events is that all Poisson points comprising the cluster are packed tightly inside a small sphere. When the balls are of varying sizes, the event that the cluster consists of k_1 big balls and k_2 small balls (both $k_1, k_2 \geq 1$) occurs only when the centres of all big balls are compressed in a small sphere and the centres of the small balls are distributed uniformly inside the region formed by the big balls in such a way that the small balls are totally contained inside the big balls. We also show that it is most likely that a finite cluster in a high density Boolean model with varying ball sizes is made up only of small balls.

Keywords: Poisson process; finite clusters

AMS 1991 Subject Classification: Primary 60K35
Secondary 60K10

1. Introduction

The continuum percolation model on a d-dimensional Euclidean space consists of overlapping balls of random radii centred at the points of a homogeneous Poisson point process. Formally, let $X' = \{x_1, x_2, \dots\}$ be a homogeneous Poisson point process of intensity λ on $\mathbb{R}^d$. Fix $x_0 = \mathbf{0}$, the origin, and let $X = \{x_0, x_1, \dots\}$. The point x_i, $i \geq 0$, is the centre of a ball $S_{r_i}(x_i)$ where each r_i is a positive random variable. The random variables $\{r_i : i \geq 0\}$ are independent and identically distributed according to the distribution of a positive random variable ρ, called the *radius random variable*. Further, the random variables $\{r_i : i \geq 0\}$ are independent of the process X. Let $C = \cup_{i=0}^{\infty} S_{r_i}(x_i)$ be the region covered by the balls and let $W(\mathbf{0})$, the cluster of the origin, be the connected component of C which contains the origin. Roy (1990), Meester and Roy (1994), Alexander (1993) and Penrose (1996) studied different aspects of the model. (See Meester and Roy (1996) for a more detailed account.)

Alexander (1993) studied the geometric structures of the event $E_k = \{\#(W(\mathbf{0})) = k\}$ in a high density Boolean model with balls of fixed size (i.e. when ρ is degenerate) where $\#(W(\mathbf{0}))$ denotes the number of Poisson points in the cluster $W(\mathbf{0})$ of the origin. Clearly, for any fixed $k \geq 1$, $\mathbb{P}_{(\lambda,\rho)}(\#(W(\mathbf{0})) = k)$ is very small for large λ and $\mathbb{P}_{(\lambda,\rho)}(\#(W(\mathbf{0})) = k) \to 0$ as $\lambda \to \infty$ where $\mathbb{P}_{(\lambda,\rho)}$ is the probability measure governing the model. Alexander showed that as $\lambda \to \infty$, such an event can occur only when all k points comprising the cluster $W(\mathbf{0})$ are packed tightly inside a small sphere of radius $O(k/\lambda)$ centred at the origin and there is an annular region surrounding the cluster which is free of any Poisson points. This gives rise

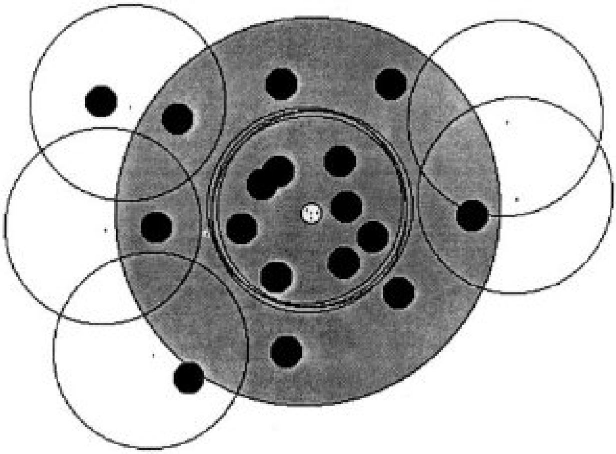

FIGURE 1: A possible realisation of $E(3, 8)$ with the centres of the big balls clustered in a small sphere near the origin and the centres of the small balls are distributed uniformly inside the region created by the big balls. The annular shaded region does not contatin any point that is the centre of a big ball; another annular region of width r_2 around the big balls clustered at the origin (not shown here) exists which will not contain any point that is the centre of a small ball.

to the phenomenon of compression as k, the number of Poisson points in this small sphere of radius $O(k/\lambda)$, is very large compared to the expected number of points $\lambda O((k/\lambda)^d)$ (as $\lambda \to \infty$) given by the ambient density λ of the underlying Poisson process.

Here we consider a continuum percolation model where ρ assumes two values r_1 and r_2 ($r_1 > r_2$) with probabilities p_1 and p_2 ($p_1 + p_2 = 1$) respectively, i.e.

$$\mathbb{P}_{(\lambda,\rho)}(\rho = r_1) = p_1 = 1 - \mathbb{P}_{(\lambda,\rho)}(\rho = r_2).$$

We refer to the balls of radius r_1 as *big balls* and the balls of radius r_2 as *small balls*. We consider the event that the cluster of the origin, $W(\mathbf{0})$, consists of k_1 big balls and k_2 small balls. Clearly, the probability of such an event goes to 0 as $\lambda \to \infty$. This paper is devoted to the study of the geometric structures admitted by such rare events when the intensity of the underlying process is very high.

When the origin is the centre of a big ball a possible structure of the event is that the centres of all big balls are compressed in a small sphere centred at the origin and the centres of the small balls are distributed uniformly inside the region formed by the big balls in such a way that the small balls are totally contained inside the big balls (see Figure 1). This requires that (a) an annular region of width r_1 surrounding the region created by the big balls be free of the Poisson points which are the centres of big balls, and (b) another annular region of width r_2 surrounding the region created by big balls be free of the Poisson points which are the centres of small balls. It is clear that the volumes of these two regions will determine the probability of the event we have considered. We show that the probability of the structure described above will be much higher than the probability of other possible structures as the given structure will minimise the volumes of the two annular regions just described (see Figures 1 and 2 for a comparison of the volumes) and thus it is most likely that the above event occurs with such a geometric structure.

When the origin is the centre of a small ball, the structure of the event is very similar. The possible structure here is that the centres of the big balls in $W(\mathbf{0})$ are clustered in a small sphere

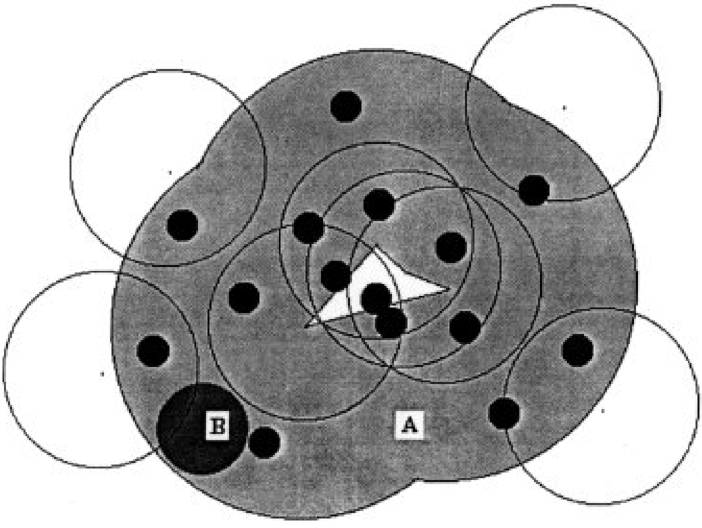

FIGURE 2: A possible realisation of $E(3, 8)$ when at least one centre of a big ball (here the one on the left-hand corner of the triangular region which contains all the centres of the big balls forming the cluster) is far away from the origin. The shaded region A does not contain any point that is the centre of a big ball; the shaded region B is the one we consider in Lemma 3.

(which is not necessarily centred at the origin; in fact the centre will be uniformly distributed inside a ball of radius $(r_1 - r_2)$ around the origin) and all the centres of the small balls are distributed uniformly inside the region formed by the big balls in such a way that the small balls are totally contained inside the big balls. As before, there are two annular regions: one of them contains no Poisson points which are the centres of big balls and the other is free of Poisson points which are the centres of small balls. Once again, it is this structure that has the largest probability and hence this is the structure we observe when the origin is the centre of a small ball. It is clear that this structure is obtained from the previous case (when the origin is the centre of a big ball) by just a change of the position of the origin to a random point which is uniformly distributed inside the sphere of radius $(r_1 - r_2)$.

If $W(\mathbf{0})$ consists only of big balls or only of small balls, the scenario observed is similar to the case when we have fixed sized balls. In these two cases, the centres of the balls are tightly packed in a small sphere near the origin and two regions are created, one of which does not contain any Poisson points which are the centres of big balls and the other is free of Poisson points which are the centres of small balls.

When the cluster $W(\mathbf{0})$ admits at least one Poisson point which is the centre of a big ball, the centres of the small balls are distributed uniformly over a sphere of radius $(r_1 - r_2)$. Typically such a region should contain $\lambda\pi_d(r_1 - r_2)^d$ Poisson points whereas the cluster $W(\mathbf{0})$ consists of only $(k_1 + k_2)$ Poisson points where π_d denotes the volume of the unit sphere in d dimensions. This gives rise to a different phenomenon, which we call the *rarefaction phenomenon* as the cluster contains fewer points than are allowed by the ambient density of the underlying process.

However, in the case when $W(\mathbf{0})$ comprises only small balls, the volume of the two regions described above is much smaller than the volume of the corresponding regions in the cases when $W(\mathbf{0})$ admits at least one big ball. Hence, the probability that $W(\mathbf{0})$ comprises only small balls dominates all other terms in $\mathbb{P}_{(\lambda,\rho)}(\#(W(\mathbf{0})) = k)$ and thus it is most likely that in a high density Boolean model a finite cluster comprises only small balls.

Our results hold for more general varying radius distribution; however, for the sake of simplicity we restrict ourselves to the case when there are only two distinct sizes of balls. Moreover, besides balls of varying radius, the results hold for more general convex shapes. We discuss this in the last section.

2. Statement of results

The independence of the radius random variable and the driving Poisson point process guarantees that the centres of the big balls, other than the point at the origin, form a homogeneous Poisson point process of intensity λp_1. We denote this process by Y and its points by $y_1, y_2, \dots$. Similarly, the point process consisting of the centres of the small balls, other than the point at the origin, form a homogeneous Poisson point process of intensity λp_2. This process is denoted by Z and its points by $z_1, z_2, \dots$. Moreover, Y and Z are independent point processes. Clearly, the union of the processes Y and Z comprises the original Poisson process of intensity λ without the point at the origin. Thus, to arrive at the continuum percolation model, we add one point at the origin to the union of the processes Y and Z and place either a big ball or a small ball at the origin, independently of the processes Y and Z, with probabilities p_1 and p_2 respectively. Hence, we view the model as the superposition of two independent Poisson processes Y and Z and the point at the origin, where all points of Y are the centres of a big ball and all points of Z are the centres of a small ball. The point at the origin is the centre of a big ball or a small ball with probabilities p_1 and p_2 respectively.

Now we encounter two possibilities: (a) the origin is the centre of a big ball and (b) the origin is the centre of a small ball. The conditional probability measure given that the origin is the centre of a big ball is denoted by $\mathbb{P}_{\mathrm{B}}$ while the conditional probability measure given that the origin is the centre of a small ball is denoted by $\mathbb{P}_{\mathrm{S}}$. The original probability measure $\mathbb{P}_{(\lambda,\rho)}$ can be recovered from these two measures by setting

$$\mathbb{P}_{(\lambda,\rho)}(\cdot) = p_1\mathbb{P}_{\mathrm{B}}(\cdot) + p_2\mathbb{P}_{\mathrm{S}}(\cdot). \tag{1}$$

We define two events $E(k_1, k_2)$ and $E'(k_1, k_2)$, as follows:
(i) given that the origin is the centre of a big ball, we define

$$E(k_1, k_2) = \{W(\mathbf{0}) \text{ consists of } (k_1 + 1) \text{ big balls (including one centred at the origin) and } k_2 \text{ small balls}\},$$

(ii) given that the origin is the centre of a small ball, we define

$$E'(k_1, k_2) = \{W(\mathbf{0}) \text{ consists of } k_1 \text{ big balls and } k_2 + 1 \text{ small balls (including one centred at the origin)}\}.$$

Using a simple marked point process argument, we can derive a relation between $\mathbb{P}_{\mathrm{B}}(E(k_1, k_2))$ and $\mathbb{P}_{\mathrm{S}}(E'(k_1, k_2))$. We say that a cluster is a *finite* (k_1, k_2)*-cluster* if it consists of only k_1 Poisson points which are the centres of big balls and k_2 Poisson points which are the centres of small balls.

Let us fix $\lambda > 0$ and $k_1 \geq 1$ and $k_2 \geq 1$. Let $B_n = [-n, n]^d$ and define $M_n(B)$ to be the number of Poisson points inside B_n, each of which is the centre of a big ball and is a constituent of a finite (k_1, k_2)-cluster.

We calculate the expectation of $M_n(B)$ using marked point process argument. Let $\mathcal{M}$ be the space of marks, which in our case is just the set $\{0, 1\}$ as we shall see shortly. Let M_i be the mark at the point x_i. Campbell's theorem for marked point processes (see Hall (1988), p. 200) guarantees that if the marked point process $\{x_i, M_i\}_{i\geq 1}$ is stationary then for any non-negative, measurable function f on $\mathbb{R}^d \times \mathcal{M}$ we have

$$\mathbb{E}\bigg(\sum_i f(x_i, M_i)\bigg) = \lambda\mathbb{E}\bigg(\int_{\mathbb{R}^d} f(x, M)\,\mathrm{d}x\bigg)$$
$$= \lambda\int_{\mathbb{R}^d} \mathbb{E} f(x, M)\,\mathrm{d}x, \tag{2}$$

where M is a random mark having the so-called 'mark distribution' and $\mathbb{E}$ is the expectation operator corresponding to the measure $\mathbb{P}_{(\lambda,\rho)}$. In our context, to apply Campbell's theorem we take the mark

$$M_i = \begin{cases} 1 & \text{if } x_i \text{ is a centre of a big ball and} \\ & x_i \text{ is a part of finite } (k_1, k_2)\text{-cluster,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f(x, M) = \begin{cases} M_i & \text{if } x = x_i \text{ for some } x_i \text{ in } B_n \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$M_n(B) = \sum_{i=1}^{\infty} f(x_i, M_i).$$

Hence, from (2) we obtain

$$\mathbb{E}(M_n(B)) = \mathbb{E}\bigg(\sum_{i=1}^{\infty} f(x_i, M_i)\bigg)$$
$$= \lambda\mathbb{E}\bigg(\int_{\mathbb{R}^d} f(x, M)\,\mathrm{d}x\bigg)$$
$$= \lambda(2n)^d p_1\mathbb{P}_{\mathrm{B}}(E(k_1 - 1, k_2)). \tag{3}$$

Now let $M_n(S)$ be the number of Poisson points inside B_n each of which is the centre of a small ball and is a constituent of a finite (k_1, k_2)-cluster. Using a similar marked point process argument we obtain

$$\mathbb{E}(M_n(S)) = \lambda(2n)^d p_2\mathbb{P}_{\mathrm{S}}(E'(k_1, k_2 - 1)). \tag{4}$$

Let R_n be the number of finite (k_1, k_2)-clusters inside B_n such that all the $(k_1 + k_2)$ points in each of these finite (k_1, k_2)-clusters are contained in B_n. In our definition of $M_n(B)$ and $M_n(S)$ the finite (k_1, k_2) cluster need not be completely contained in B_n, so it is clear that

$$k_1 R_n \leq M_n(B) \quad \text{and} \quad k_2 R_n \leq M_n(S). \tag{5}$$

Further, any finite (k_1, k_2)-clusters, at least one point of which is inside B_n, must be totally contained inside $B_{n+(k_1+k_2)2r_1}$. Hence, we also have

$$k_1 R_{n+(k_1+k_2)2r_1} \geq M_n(B) \text{ and } k_2 R_{n+(k_1+k_2)2r_1} \geq M_n(S). \tag{6}$$

Now combining Equations (3)–(6), we have

$$\begin{aligned}
\lambda p_1 \mathbb{P}_{\mathrm{B}}(E(k_1 - 1, k_2))/k_1 &\geq \limsup_{n\to\infty} \frac{\mathbb{E}(R_n)}{(2n)^d} \\
&= \limsup_{n\to\infty} \frac{\mathbb{E}(R_{n+(k_1+k_2)2r_1})}{(2(n+(k_1+k_2)2r_1))^d} \\
&\geq \limsup_{n\to\infty} \frac{\mathbb{E}(M_n(S))}{k_2(2n)^d} \times \frac{(2n)^d}{(2(n+(k_1+k_2)2r_1))^d} \\
&= \frac{1}{k_2} \lim_{n\to\infty} \frac{\mathbb{E}(M_n(S))}{(2n)^d} \times \lim_{n\to\infty} \frac{(2n)^d}{(2(n+(k_1+k_2)2r_1))^d} \\
&= \lambda p_2 \mathbb{P}_{\mathrm{S}}(E'(k_1, k_2 - 1))/k_2.
\end{aligned}$$

A similar calculation yields

$$\lambda p_2 \mathbb{P}_{\mathrm{S}}(E'(k_1, k_2 - 1))/k_2 \geq \lambda p_1 \mathbb{P}_{\mathrm{B}}(E(k_1 - 1, k_2))/k_1.$$

Combining the above two inequalities we obtain the following proposition.

Proposition 1. *For $\lambda > 0$ and $k_1, k_2 \geq 1$, we have*

$$p_1 \mathbb{P}_{\mathrm{B}}(E(k_1 - 1, k_2))/k_1 = p_2 \mathbb{P}_{\mathrm{S}}(E'(k_1, k_2 - 1))/k_2. \tag{7}$$

From this proposition, it follows that the results in the case when the origin is the centre of a small ball can be obtained from the results in the case when the origin is the centre of a big ball. So, unless specified, from now on we will assume that the origin is the centre of a big ball.

We define the measure of the size of the cluster $W(\mathbf{0})$ by

$$d(W(\mathbf{0})) := \max\{d(\mathbf{0}, x) : x \text{ is a Poisson point in } W(\mathbf{0})\}.$$

The relative density of the cluster $W(\mathbf{0})$ of the origin is defined by

$$\delta(\lambda) := \frac{\#(W(\mathbf{0}))}{\lambda \pi_d d(W(\mathbf{0}))^d}. \tag{8}$$

Alexander (1993) showed that when the balls are of fixed size, for k fixed or $k \to \infty$ but $k/\lambda \to 0$,

$$\mathbb{P}_{(\lambda,\rho)}(\delta(\lambda) \to \infty \mid \#(W(\mathbf{0})) = k) \to 1 \quad \text{as } \lambda \to \infty.$$

This phenomenon was termed *compression* by Alexander.

In the case when the balls are of varying sizes, the results are best understood when we divide them into different cases. We first consider the case when both k_1 and k_2 are fixed.

Theorem 1. *Suppose that both k_1 and k_2 are fixed. Then we have, as $\lambda \to \infty$,*

$$\mathbb{P}_{\mathrm{B}}(E(k_1, k_2)) = \exp(-\lambda \pi_d \mathbb{E}(\rho + r_1)^d + (k_2 - k_1(d-1)) \log \lambda + O(1))$$

and

(i) $\mathbb{P}_B(d(W(\mathbf{0})) > a_1(\lambda) \mid E(k_1, k_2)) \to 1$,
(ii) $\mathbb{P}_B(\delta(\lambda) \to 0 \mid E(k_1, k_2)) \to 1$
where $a_1(\lambda)$ is function of λ such that $a_1(\lambda) \to 0$ but $\lambda(a_1(\lambda))^d \to \infty$ as $\lambda \to \infty$ and $\mathbb{E}$ is the expectation operator according to the distribution of ρ.

Next we consider the case when k_2 is fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ as $\lambda \to \infty$.

Theorem 2. *Suppose that k_2 is fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ as $\lambda \to \infty$. Then, we have, as $\lambda \to \infty$,*

$$\mathbb{P}_B(E(k_1, k_2)) = \exp(-\lambda\pi_d\mathbb{E}(\rho + r_1)^d - (d-1)k_1 \log(\lambda/k_1) + k_2 \log\lambda + O(k_1))$$

and

(i) $\mathbb{P}_B(d(W(\mathbf{0})) > a_2(\lambda) \mid E(k_1, k_2)) \to 1$,
(ii) $\mathbb{P}_B(\delta(\lambda) \to 0 \mid E(k_1, k_2)) \to 1$
where $a_2(\lambda)$ is a function of λ such that $a_2(\lambda) \to 0$ and $\lambda(a_2(\lambda))^d/k_1 \to \infty$ as $\lambda \to \infty$.

Now we suppose that k_1 is fixed and $k_2 \to \infty$ but $k_2/\lambda \to 0$ as $\lambda \to \infty$.

Theorem 3. *Suppose that k_1 is fixed and $k_2 \to \infty$ but $k_2/\lambda \to 0$ as $\lambda \to \infty$. Then, we have, as $\lambda \to \infty$,*

$$\mathbb{P}_B(E(k_1, k_2)) = \exp(-\lambda\pi_d\mathbb{E}(\rho + r_1)^d - (d-1)k_1 \log\lambda + k_2 \log(\lambda p_2/k_2) + O(k_2))$$

and

(i) $\mathbb{P}_B(d(W(\mathbf{0})) > a \mid E(k_1, k_2)) \to 1$,
(ii) $\mathbb{P}_B(\delta(\lambda) \to 0 \mid E(k_1, k_2)) \to 1$
for every fixed $0 < a < r_1 - r_2$.

Now suppose that both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0$, $k_2/\lambda \to 0$ as $\lambda \to \infty$.

Theorem 4. *Suppose that both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0$, $k_2/\lambda \to 0$ as $\lambda \to \infty$. Then, we have, as $\lambda \to \infty$,*

$$\begin{aligned}\mathbb{P}_B(E(k_1, k_2)) = \exp(&-\lambda\pi_d\mathbb{E}(\rho + r_1)^d - (d-1)k_1 \log(\lambda/k_1)\\ &+ k_2 \log(\lambda p_2/k_2) + O(k_1) + O(k_2))\end{aligned}$$

and

(i) $\mathbb{P}_B(d(W(\mathbf{0})) > a \mid E(k_1, k_2)) \to 1$,
(ii) $\mathbb{P}_B(\delta(\lambda) \to 0 \mid E(k_1, k_2)) \to 1$
for every fixed $0 < a < r_1 - r_2$.

Next we consider the case when cluster of the origin $W(\mathbf{0})$ consists only of small balls or only of big balls. Let $E(k, 0)$ be the event that the cluster of the origin consists only of $k+1$ ($k \geq 0$) big balls. Similarly, let $E'(0, k)$ be the event that the cluster of the origin consists only of $k+1$ ($k \geq 0$) small balls.

Theorem 5. *Let k be fixed or $k \to \infty$ but $k/\lambda \to 0$ as $\lambda \to \infty$. Then, we have*
(i) $\mathbb{P}_B(E(k, 0)) = \exp(-\lambda\pi_d\mathbb{E}(\rho + r_1)^d - (d-1)k \log(\lambda/k) + O(k))$
(ii) $\mathbb{P}_S(E'(0, k)) = \exp(-\lambda\pi_d\mathbb{E}(\rho + r_2)^d - (d-1)k \log(\lambda/k) + O(k))$;
and finally

$$\mathbb{P}_{(\lambda,\rho)}(E'(0, k) \mid \#(W(\mathbf{0})) = k + 1) \to 1 \tag{9}$$

as $\lambda \to \infty$.

3. Lower bounds

In this section, we obtain the lower bounds of the probabilities of the events we have considered. Since we use Stirling's formula quite often we state below the results we need (Feller (1968), pp. 52–54).

Stirling's formula:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n}} = 1.$$

Also, for every $n \geq 1$,

$$\sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n} \leq n! \leq \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n} \exp(1/(12n)).$$

Let $N_Y(A)$ and $N_Z(A)$ be the number of Poisson points inside A of the point processes Y and Z respectively. Let rU be the ball of radius r centred at the origin and $x + rU$ be the ball of radius r centred at the point x.

Lemma 1. *Let (i)* k_1, k_2 *be both fixed or (ii)* k_1 *be fixed,* $k_2 \to \infty$ *but* $k_2/\lambda \to 0$ *or (iii)* k_2 *be fixed and* $k_1 \to \infty$ *but* $k_1/\lambda \to 0$ *or (iv) both* $k_1, k_2 \to \infty$ *but* $k_1/\lambda \to 0$ *and* $k_2/\lambda \to 0$ *(as* $\lambda \to \infty$*). Then, we have, for all large* λ

$$\begin{aligned}\mathbb{P}_{\mathrm{B}}(E(k_1, k_2)) \geq \exp(&-\lambda\pi_d \mathbb{E}(\rho + r_1)^d - (d-1)k_1 \log(\lambda/k_1) \\ &+ k_2 \log(\lambda p_2/k_2) + c_1 k_1 + c_2 k_2 + g(k_1, k_2, \lambda)), \end{aligned} \tag{10}$$

where c_1 *and* c_2 *are constants and* $g(k_1, k_2, \lambda)$ *is a function of* k_1, k_2 *and* λ.

Proof. Since the origin is the centre of a big ball, if k_1 points of Y in $W(\mathbf{0})$ are placed in a ball of radius $\alpha(k_1/\lambda)$, they will belong to the cluster $W(\mathbf{0})$ for all large λ where $\alpha = p_1(\pi_d(2r_1)^{d-1})^{-1}$ (see Figure 1). Note that if we assume that there are only big balls available for the cluster $W(\mathbf{0})$, then the optimal radius inside which all the Poisson points are packed is $\alpha(k_1/\lambda)$ (see also Alexander (1993)). Now the small balls can be placed inside a sphere of radius $(r_1 - r_2)$ centred at the origin without affecting the region covered by the balls. This creates two annular regions, one free of points of Y and the other free of points of Z. Thus we obtain

$$\begin{aligned}\mathbb{P}_{\mathrm{B}}(E(k_1, k_2)) &\geq \mathbb{P}_{\mathrm{B}}(N_Y((\alpha k_1/\lambda)U) = k_1, N_Z((r_1 - r_2)U) = k_2, \\ &\qquad N_Y((2r_1 + \alpha(k_1/\lambda))U \setminus (\alpha(k_1/\lambda))U) = 0, \\ &\qquad N_Z(((r_1 + r_2) + (\alpha k_1/\lambda))U \setminus (r_1 - r_2)U) = 0) \\ &= \exp(-\lambda\pi_d p_1(\alpha k_1/\lambda)^d) \frac{(\lambda\pi_d p_1(\alpha k_1/\lambda)^d)^{k_1}}{k_1!} \exp(-\lambda\pi_d p_2(r_1 - r_2)^d) \\ &\quad \times \frac{(\lambda\pi_d p_2(r_1 - r_2)^d)^{k_2}}{k_2!} \exp(-\lambda\pi_d p_1((2r_1 + (\alpha k_1/\lambda))^d - (\alpha k_1/\lambda)^d)) \\ &\quad \times \exp(-\lambda\pi_d p_2(((r_1 + r_2) + (\alpha k_1/\lambda))^d - (r_1 - r_2)^d)). \end{aligned}$$

We now use Stirling's approximation for $k_1!$ and $k_2!$ to obtain

$$\begin{aligned}\mathbb{P}_{\mathrm{B}}(E(k_1, k_2)) \geq \exp(&-\lambda\pi_d \mathbb{E}(\rho + r_1)^d - (d-1)k_1 \log(\lambda/k_1) + k_1 \log(e\pi_d p_1 \alpha^d) \\ &+ k_2 \log(\lambda p_2/k_2) + k_2 \log(e\pi_d(r_1 - r_2)^d) + g_1(k_1, k_2)) \\ &\times \exp\bigg(- \lambda\pi_d \sum_{j=1}^{d} \binom{d}{j} (\alpha k_1/\lambda)^j (p_1(2r_1)^{d-j} + p_2(r_1 + r_2)^{d-j})\bigg) \end{aligned}$$

where $g_1(k_1, k_2) = -(1/(12k_1) + 1/(12k_2)) - \log(k_1 k_2)/2 + \log(2\pi)$. Now, we choose λ so large that $k_1/\lambda < 1$. Then the last term in the exponential can be written as

$$\begin{aligned}
\lambda\pi_d \sum_{j=1}^{d} \binom{d}{j} (\alpha k_1/\lambda)^j (p_1(2r_1)^{d-j} + p_2(r_1+r_2)^{d-j}) \\
&= k_1 d\pi_d\alpha(p_1(2r_1)^{d-1} + p_2(r_1+r_2)^{d-1}) \\
&\quad + \lambda\pi_d \sum_{j=2}^{d} \binom{d}{j} (\alpha k_1/\lambda)^j (p_1(2r_1)^{d-j} + p_2(r_1+r_2)^{d-j}) \\
&\le k_1 d\pi_d\alpha(p_1(2r_1)^{d-1} + p_2(r_1+r_2)^{d-1}) \\
&\quad + \lambda\pi_d(k_1/\lambda)^2 \sum_{j=0}^{d} \binom{d}{j} (\alpha)^j (p_1(2r_1)^{d-j} + p_2(r_1+r_2)^{d-j}) \\
&= k_1 d\pi_d\alpha(p_1(2r_1)^{d-1} + p_2(r_1+r_2)^{d-1}) + C_1 k_1^2/\lambda,
\end{aligned}$$

where $C_1 = \pi_d[p_1(\alpha + (2r_1))^d + p_2(\alpha + (r_1+r_2))^d]$.

Now setting $c_1 = \log(e\pi_d p_1 \alpha^d) - d\pi_d\alpha(p_1(2r_1)^{d-1} + p_2(r_1+r_2)^{d-1})$ and $c_2 = \log(e\pi_d (r_1 - r_2)^d)$ and $g(k_1, k_2, \lambda) = g_1(k_1, k_2) - C_1 k_1^2/\lambda$, the lemma follows.

Next we consider the case when $W(\mathbf{0})$ comprises only big balls or only small balls. We prove the result in the case when $W(\mathbf{0})$ comprises only small balls, the other case being similar.

Lemma 2. *Let either (i) k be fixed or (ii) $k \to \infty$ but $k/\lambda \to 0$ as $\lambda \to \infty$. Then we have as $\lambda \to \infty$,*

$$\mathbb{P}_S(E'(0,k)) \ge \exp(-\lambda\pi_d \mathbb{E}(\rho + r_2)^d - (d-1)k\log(\lambda/k) + O(k)).$$

Proof. The proof in the case when $W(\mathbf{0})$ consists of only small balls follows a similar line to that of Lemma 1. A possible structure of the cluster $W(\mathbf{0})$ is that the centres of all small balls are packed tightly in a small sphere of radius $\alpha_2(k/\lambda)$ where $\alpha_2 = p_2(\pi_d(2r_2)^{d-1})^{-1}$ and there is a spherical region containing no points of Y and an annular region containing no points of Z. Thus, we have

$$\begin{aligned}
\mathbb{P}_S(E'(0,k)) &\ge \mathbb{P}_S(N_Z((\alpha_2(k/\lambda))U) = k,\ N_Y(((r_1+r_2) + (\alpha_2 k/\lambda))U) = 0, \\
&\qquad N_Z((2r_2 + \alpha_2(k/\lambda))U \setminus (\alpha_2(k/\lambda))U) = 0) \\
&= \exp(-\lambda\pi_d p_2((\alpha_2 k/\lambda) + 2r_2)^d)\frac{[\lambda\pi_d p_2(\alpha_2(k/\lambda)^d)]}{k!} \\
&\quad \times \exp(-\lambda\pi_d p_1((\alpha_2 k/\lambda) + (r_1+r_2))^d).
\end{aligned}$$

Calculations using Stirling's formula, similar to those used in the previous lemma, yield the result.

4. Upper bounds

In this section we obtain the upper bounds of the probability of the events we have defined. As discussed in the introduction, the occurrence of $E(k_1, k_2)$ creates two regions, one of which contains no points of Y (the centres of big balls) and the other contains no points of Z (the centres of small balls). To get an upper bound of $\mathbb{P}_B(E(k_1, k_2))$, we have to obtain estimates of the volumes of these two regions. To this end, we divide the event $E(k_1, k_2)$ into several smaller events depending on the size of the clusters and estimate the volumes, and thereby the probability, for each of these events separately.

The first two lemmas (Lemmas 3 and 4) deal with the big clusters where we show that the volumes of the regions described above are 'extremely large' (see Figure 2) and therefore the probability is negligible. In Lemma 5, we consider the case when the centres of the big balls are clustered in a small ball of 'optimal' size (see Figure 1)—this is the case which contributes significantly to $\mathbb{P}_B(E(k_1, k_2))$. Lemmas 6 and 7 take care of the moderate size clusters, and in Lemma 8 we consider the case when the cluster consists only of small balls.

In the next lemma, we consider the case when at least one point of Y in $W(\mathbf{0})$ is at a distance r_1 or more from the origin. To study this event, we define

$$d_Y(\mathbf{0}) := \max\{d(\mathbf{0}, y_i) : y_i \in W(\mathbf{0}) \cap Y\}.$$

Lemma 3. *Let (i) k_1, k_2 be both fixed or (ii) k_1 be fixed, $k_2 \to \infty$ but $k_2/\lambda \to 0$ or (iii) k_2 be fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ or (iv) both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0$ and $k_2/\lambda \to 0$ (as $\lambda \to \infty$). Then, for all large λ, we have*

$$\mathbb{P}_B(E(k_1, k_2), d_Y(\mathbf{0}) > r_1) \leq \exp(-\lambda\pi_d \mathbb{E}(\rho + r_1)^d - \lambda\pi_d p_1 (r_1/2)^d/2 + c_3 k_1 + c_4 k_2),$$

where c_3 and c_4 are positive constants not depending on k_1 and k_2.

Proof. Since the cluster has only $(k_1 + k_2)$ Poisson points (besides the origin), the Poisson point in $W(\mathbf{0})$ which is farthest from the origin is at most at a distance $2(k_1 + k_2)r_1$ from the origin. So we have

$$\mathbb{P}_B(E(k_1, k_2), d_Y(\mathbf{0}) > r_1) \leq \sum_{j=1}^{2(k_1+k_2)} \mathbb{P}_B(E(k_1, k_2),\ jr_1 < d_Y(\mathbf{0}) \leq (j+1)r_1).$$

Now we estimate the summands in the above inequality. Suppose that $jr_1 < d_Y(\mathbf{0}) \leq (j+1)r_1$. Then there is at least one Poisson point in $W(\mathbf{0}) \cap Y$ which lies outside the sphere $(jr_1)U$. Let $y_{\max}$ be the Poisson point in $W(\mathbf{0}) \cap Y$ which is farthest from the origin and hence $jr_1 < d(y_{\max}, \mathbf{0}) \leq (j+1)r_1$. Now, if we centre a ball of radius $2r_1$ at the point $y_{\max}$, the part of the ball which lies outside the sphere $((j+1)r_1)U$ contains no points of the process Y. A lower bound of the volume of the region of the ball of radius $2r_1$ which lies outside $((j+1)r_1)U$ can easily be obtained by observing that a ball of radius $(r_1/2)$ will always be contained inside such a region (see region B in Figure 2). To make this formal, we use a conditioning argument.

Let $C_\star$ be the positions of all Poisson points of Y and Z inside $((j+1)r_1)U$ and the origin, $\{\mathbf{0}\}$. Define, for $m, n \geq 0$,

$$A_m := \{N_Y((2r_1)U) = m\}$$
$$B_n := \{N_Z((r_1 + r_2)U) = n\}.$$

Since the event $E(k_1, k_2)$ occurs and the origin is the centre of a big ball, the ball $(2r_1)U$ may contain, besides the origin, at most k_1 Poisson points of Y and hence, the event $\cup_{m=0}^{k_1} A_m$ must occur. Similarly, $\cup_{n=0}^{k_2} B_n$ also occurs. Let $\mathbf{0}, y_{i_1}, \ldots, y_{i_{k_1}}$ be the Poisson points in $C_\star \cap W(\mathbf{0})$ which are also the centres of big balls. Thus we have

$$
\begin{aligned}
&\mathbb{P}_{\mathrm{B}}(E(k_1,k_2),\, jr_1 < d_Y(\mathbf{0}) \le (j+1)r_1) \\
&\quad = \mathbb{E}[\mathbb{P}_{\mathrm{B}}(E(k_1,k_2),\, jr_1 < d_Y(\mathbf{0}) \le (j+1)r_1 \mid C_\star)] \\
&\quad \le \mathbb{E}\Bigg[\mathbb{P}_{\mathrm{B}}(N_Y((\cup_{i=1}^{k_1}(y_{i_j} + 2r_1U)) \setminus (j+1)r_1U) = 0 \mid C_\star) \sum_{m=0}^{k_1}\sum_{n=0}^{k_2} 1_{A_m} 1_{B_n}\Bigg] \\
&\quad \le \sum_{m=0}^{k_1}\sum_{n=0}^{k_2} \mathbb{P}_{\mathrm{B}}(A_m)\mathbb{P}_{\mathrm{B}}(B_n)\exp(-\lambda\pi_d p_1(r_1/2)^d) \\
&\quad = \exp(-\lambda\pi_d \mathbb{P}(\rho + r_1)^d)\exp(-\lambda\pi_d p_1(r_1/2)^d) \\
&\qquad \times \sum_{m=0}^{k_1}\sum_{n=0}^{k_2} \frac{(\lambda\pi_d p_1(2r_1)^d)^m}{m!}\frac{(\lambda\pi_d p_2(r_1+r_2)^d)^n}{n!}.
\end{aligned}
$$

Choose constants $C_2, C_3 \ge 1$ such that

$$
(2r_1)^d < (r_1/2)^d C_2/4 \quad \text{and} \quad p_2(r_1+r_2)^d < p_1(r_1/2)^d C_3/4.
$$

Then, we have

$$
\begin{aligned}
&\sum_{m=0}^{k_1}\sum_{n=0}^{k_2} \frac{(\lambda\pi_d p_1(2r_1)^d)^m}{m!}\frac{(\lambda\pi_d p_2(r_1+r_2)^d)^n}{n!} \\
&\qquad \le \sum_{m=0}^{k_1}\sum_{n=0}^{k_2} \frac{(\lambda\pi_d p_1(r_1/2)^d C_2/4)^m}{m!}\frac{(\lambda\pi_d p_1(r_1/2)^d C_3/4)^n}{n!} \\
&\qquad \le C_2^{k_1} C_3^{k_2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\lambda\pi_d p_1(r_1/2)^d/4)^m}{m!}\frac{(\lambda\pi_d p_1(r_1/2)^d/4)^n}{n!} \\
&\qquad = C_2^{k_1} C_3^{k_2} \exp(\lambda\pi_d p_1(r_1/2)^d/2).
\end{aligned}
$$

Now, combining together, we have

$$
\begin{aligned}
\mathbb{P}_{\mathrm{B}}(E(k_1,k_2),\, d_Y(\mathbf{0}) > r_1) &\le \exp(-\lambda\pi_d \mathbb{P}(\rho + r_1)^d - \lambda\pi_d p_1(r_1/2)^d/2 \\
&\qquad + k_1 \log C_2 + k_2 \log C_3)(2(k_1+k_2)) \\
&\le \exp(-\lambda\pi_d \mathbb{P}(\rho + r_1)^d - \lambda\pi_d p_1(r_1/2)^d/2 + c_3k_1 + c_4k_2),
\end{aligned}
$$

where $c_3 = \log C_2 + 2$ and $c_4 = \log C_3 + 2$.

Lemma 3 shows that none of the centres of the big balls can be too far from the origin. Now we look at the case when at least one point of $W(\mathbf{0})$ which is the centre of a small ball is very far from the origin. For this, we define

$$
d_Z(\mathbf{0}) := \max\{d(\mathbf{0}, z_i) : z_i \in W(\mathbf{0}) \cap Z\}.
$$

Lemma 4. *Let (i) k_1, k_2 be both fixed or (ii) k_1 be fixed, $k_2 \to \infty$ but $k_2/\lambda \to 0$ or (iii) k_2 be fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ or (iv) both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0$ and $k_2/\lambda \to 0$ (as $\lambda \to \infty$). Then for all large λ, we have*

$$\mathbb{P}_{\mathrm{B}}(E(k_1, k_2), d_Y(\mathbf{0}) \le r_1, d_Z(\mathbf{0}) > r_1) \\ \le \exp(-\lambda\pi_d \mathbb{E}(\rho + r_1)^d - \lambda\pi_d p_2(r_2/2)^d/2 + c_5 k_1 + c_6 k_2)$$

where c_5 and c_6 are positive constants not depending on k_1 and k_2.

Proof. The proof of this lemma follows a similar line to that of Lemma 3.

$$\mathbb{P}_{\mathrm{B}}[E(k_1, k_2), d_Y(\mathbf{0}) \le r_1, d_Z(\mathbf{0}) > r_1] \\ \le \sum_{j=0}^{k_1+k_2} \mathbb{P}_{\mathrm{B}}[E(k_1, k_2), d_Y(\mathbf{0}) \le r_1, r_1 + jr_2 < d_Z(\mathbf{0}) \le r_1 + (j+1)r_2].$$

Now, we follow a similar method as in Lemma 3 to estimate the summands in the inequality. Suppose $r_1 + jr_2 < d_Z(\mathbf{0}) \le r_1 + (j+1)r_2$ and let $z_{\max}$ be the Poisson point farthest from the origin in $W(\mathbf{0})$ which is also the centre of a small ball. Clearly $r_1 + jr_2 < d(z_{\max}, \mathbf{0}) \le r_1 + (j+1)r_2$. Now a region lying outside $(r_1 + (j+1)r_2)U$ of volume at least $\pi_d(r_2/2)^d$ will contain no Poisson points which are the centres of small balls and hence, by a conditioning argument as in the previous lemma, we obtain

$$\begin{aligned} &\mathbb{P}_{\mathrm{B}}[E(k_1, k_2), d_Y(\mathbf{0}) \le r_1, r_1 + jr_2 < d_Z(\mathbf{0}) \le r_1 + (j+1)r_2] \\ &\quad \le \sum_{m=0}^{k_1}\sum_{n=0}^{k_2} \mathbb{P}_{\mathrm{B}}(A_m)\mathbb{P}_{\mathrm{B}}(B_n)\exp(-\lambda\pi_d p_2(r_2/2)^d) \\ &\quad = \exp(-\lambda\pi_d\mathbb{E}(\rho + r_1)^d)\exp(-\lambda\pi_d p_2(r_2/2)^d) \\ &\qquad \times \sum_{m=0}^{k_1}\sum_{n=0}^{k_2} \frac{[\lambda\pi_d p_1(2r_1)^d]^m}{m!}\frac{[\lambda\pi_d p_2(r_1 + r_2)^d]^n}{n!}. \end{aligned}$$

Now choosing c_5 and c_6 suitably the lemma follows.

Now we want to estimate the probability that all the centres of big balls comprising the cluster $W(\mathbf{0})$ are compressed in the optimal sphere about the origin (see Figure 1). Let $\alpha = p_1(\pi_d(2r_1)^{d-1})^{-1}$ and $h(n) = -\log(2\pi) - (\log n)/2$ where $n \ge 1$. We note here that if n is fixed then $h(n) = O(1)$ and if $n \to \infty$ then $h(n) = o(n)$.

Lemma 5. *Let (i) k_1, k_2 be both fixed or (ii) k_1 be fixed, $k_2 \to \infty$ but $k_2/\lambda \to 0$ or (iii) k_2 be fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ or (iv) both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0$ and $k_2/\lambda \to 0$ (as $\lambda \to \infty$). Then for all large λ and for some constants c_7 and c_8, we have*

$$\begin{aligned} &\mathbb{P}_{\mathrm{B}}(E(k_1, k_2), d_Y(\mathbf{0}) \le \alpha k_1/\lambda, d_Z(\mathbf{0}) \le r_1) \\ &\quad \le \exp(-\lambda\pi_d\mathbb{E}(\rho + r_1)^d - (d-1)k_1\log(\lambda/k_1) \\ &\qquad + k_2\log(\lambda p_2/k_2) + c_7 k_1 + c_8 k_2 + h(k_1)). \end{aligned}$$

Proof. We have

$$\begin{aligned}
&\mathbb{P}_{\mathrm{B}}(E(k_1,k_2), d_Y(\mathbf{0}) \le \alpha k_1/\lambda, d_Z(\mathbf{0}) \le r_1)\\
&\quad\le \mathbb{P}_{\mathrm{B}}(N_Y((\alpha k_1/\lambda)U) = k_1, N_Y((2r_1)U \setminus (\alpha k_1/\lambda)U) = 0,\\
&\qquad\quad N_Z(r_1U) = k_2, N_Z((r_1+r_2)U \setminus r_1U) = 0)\\
&\quad= \exp(-\lambda\pi_d p_1(\alpha k_1/\lambda)^d)\frac{(\lambda\pi_d p_1(\alpha k_1/\lambda)^d)^{k_1}}{k_1!}\exp(-\lambda\pi_d p_2 r_1^d)\\
&\qquad\times \frac{(\lambda\pi_d p_2 r_1^d)^{k_2}}{k_2!}\exp(-\lambda\pi_d p_1((2r_1)^d - (\alpha k_1/\lambda)^d))\\
&\qquad\times \exp(-\lambda\pi_d p_2((r_1+r_2)^d - r_1^d))\\
&\quad\le \exp(-\lambda\pi_d\mathbb{E}(\rho+r_1)^d - (d-1)k_1\log(\lambda/k_1)\\
&\qquad + k_2\log(\lambda p_2/k_2) + c_7k_1 + c_8k_2 + h(k_1)),
\end{aligned}$$

where $c_7 = \log(e\pi_d p_1\alpha^d)$ and $c_8 = \log(e\pi_d r_1^d) + 1$.

Next we look at the clusters which are of moderate size. For a fixed constant $\mu > 1$, we define,

$$\Psi_\mu(y) = p_1\pi_{d-1}(r_1)^{d-1}y - \log(ep_1\pi_d\mu^d y^d).$$

Note here that $\Psi_\mu(y) \to \infty$ as $y \to \infty$.

Lemma 6. *Let (i) k_1, k_2 be both fixed or (ii) k_1 be fixed, $k_2 \to \infty$ but $k_2/\lambda \to 0$ or (iii) k_2 be fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ or (iv) both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0$ and $k_2/\lambda \to 0$ (as $\lambda \to \infty$). Then for all large λ, we have*

$$\begin{aligned}
&\mathbb{P}_{\mathrm{B}}(E(k_1,k_2), d_Z(\mathbf{0}) \le r_1, yk_1/\lambda < d_Y(\mathbf{0}) \le \mu yk_1/\lambda)\\
&\quad\le \exp(-\lambda\pi_d\mathbb{E}(\rho+r_1)^d - (d-1)k_1\log(\lambda/k_1)\\
&\qquad - k_1\Psi_\mu(y)k_2\log(\lambda p_2/k_2) + c_9k_2 + h(k_1)).
\end{aligned}$$

where c_9 is a constant and $h(k_1)$ is as defined earlier.

Proof. We define

$$\begin{aligned}
G_\star &= \{\text{all points of } Y \text{ inside the ball } (\mu yk_1/\lambda)U\} \cup \{\mathbf{0}\},\\
H_\star &= \{\text{all points of } Z \text{ inside the ball } r_1U\};
\end{aligned}$$

and

$$\begin{aligned}
A &= \{N_Y((\mu yk_1/\lambda)U) = k_1\}\\
B &= \{N_Z(r_1U) = k_2\}.
\end{aligned}$$

The r-fattening of the set E is defined by

$$E^r = \{u \in \mathbb{R}^d : \text{ there exists } v \in E \text{ such that } u \in v + rU\}.$$

Now we have

$$\begin{aligned}
&\mathbb{P}_{\mathrm{B}}(E(k_1,k_2), yk_1/\lambda < d_Y(\mathbf{0}) \le \mu yk_1/\lambda, d_Z(\mathbf{0}) \le r_1)\\
&\quad= \mathbb{E}[\mathbb{P}_{\mathrm{B}}(E(k_1,k_2), yk_1/\lambda < d_Y(\mathbf{0}) \le \mu yk_1/\lambda, d_Z(\mathbf{0}) \le r_1 \mid (G_\star, H_\star))]\\
&\quad\le \mathbb{E}[1_A 1_B \mathbb{P}_{\mathrm{B}}(N_Y(G_\star^{2r_1} \cup H_\star^{r_1+r_2} \setminus (\mu yk_1/\lambda)U) = 0,\\
&\qquad N_Z(G_\star^{r_1+r_2} \cup H_\star^{2r_2} \setminus r_1U) = 0 \mid (G_\star, H_\star))].
\end{aligned}$$

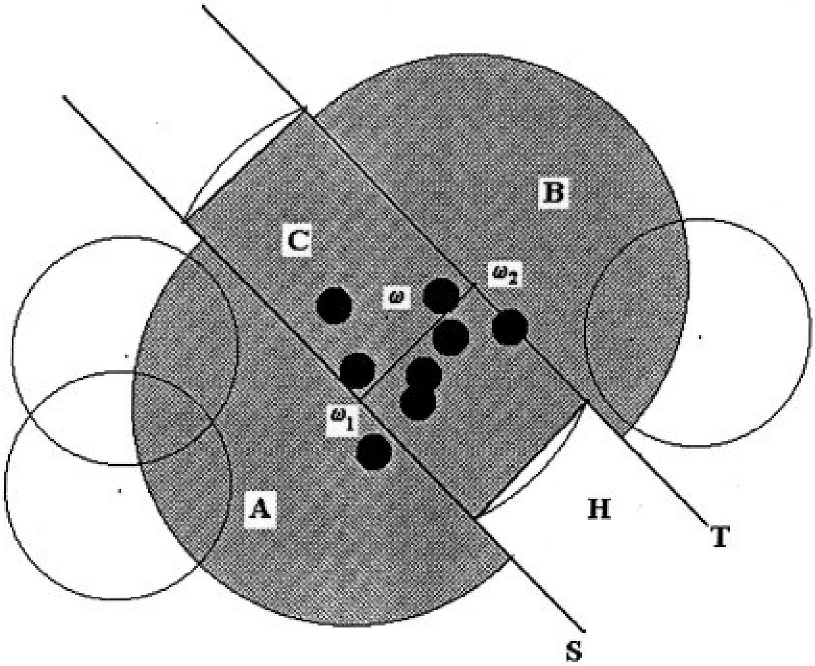

FIGURE 3: A realisation of the event $E(3, 8)$ when the cluster size is moderate and the regions A, B, C and H along with the hyperplanes S and T are considered in Lemma 6.

Since $\mathbf{0} \in G_\star$, we have $\ell(G_\star^{r_1+r_2} \cup H_\star^{2r_2} \setminus r_1U) \geq \ell(G_\star^{r_1+r_2} \setminus r_1U) \geq \ell(\{\mathbf{0}\}^{r_1+r_2} \setminus r_1U) = \pi_d((r_1 + r_2)^d - r_1^d)$, where $\ell(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^d$. Also we have $\ell(G_\star^{2r_1} \cup H_\star^{r_1+r_2} \setminus (\mu y k_1/\lambda)U) \geq \ell(G_\star^{2r_1} \setminus (\mu y k_1/\lambda)U) \geq \ell(G_\star^{2r_1}) - \pi_d(\mu y k_1/\lambda)^d$. Now to estimate $\ell(G_\star^{2r_1})$, we choose the pair of points (ω_1, ω_2) in $G_\star$ which has the maximum distance among all pairs of points in $G_\star$ and hence $d(\omega_1, \omega_2) > yk_1/\lambda$. Let S and T be the hyperplanes drawn at the points ω_1 and ω_2 respectively such that both of them are perpendicular to the line joining ω_1 and ω_2. Thus all the points of $G_\star$ will lie in the region H (see Figure 3) which lies in between the hyperplanes S and T. Hence there will be two half-spheres A and B (see Figure 3) of radius $2r_1$ centred at the points ω_1 and ω_2 respectively which are disjoint from H. For the part of $G_\star^{2r_1}$ which intersects H, we consider any point ω in $G_\star$, other than ω_1 and ω_2, and look at the region $H \cap (\omega + 2r_1U)$. A lower bound of the volume of this region can be obtained by noticing that a d-dimensional cylinder of height at least yk_1/λ and $(d-1)$-dimensional cross sectional area at least $\pi_{d-1}r_1^{d-1}$ will always be included in it (see region C in Figure 3). Thus $\ell(G_\star^{2r_1}) \geq \pi_d(2r_1)^d + \pi_{d-1}r_1^{d-1}yk_1/\lambda$. Thus, we have, for all large λ,

$$
\begin{aligned}
&\mathbb{P}_{\mathrm{B}}(E(k_1, k_2), yk_1/\lambda < d_Y(\mathbf{0}) \leq \mu y k_1/\lambda, d_Z(\mathbf{0}) \leq r_1) \\
&\quad \leq \exp(-\lambda\pi_d p_2 r_1^d)\frac{(\lambda\pi_d p_2 r_1^d)^{k_2}}{k_2!}\exp(-\lambda\pi_d p_1(\mu y k_1/\lambda)^d) \\
&\qquad \times \frac{(\lambda\pi_d p_1(\mu y k_1/\lambda)^d)^{k_1}}{k_1!}\exp(-\lambda\pi_d p_2((r_1 + r_2)^d - r_1^d)) \\
&\qquad \times \exp(-\lambda\pi_d p_1(((2r_1)^d - (\mu y k_1/\lambda)^d) - \pi_{d-1}(r_1)^{d-1}yk_1/\lambda))
\end{aligned}
$$

$$= \exp(-\lambda\pi_d \mathbb{E}(\rho + r_1)^d - (d-1)k_1 \log(\lambda/k_1) + k_2 \log(\lambda p_2/k_2) - k_1\Psi_\mu(y) + c_9 k_2 + h(k_1)),$$

where $c_9 = \log(e\pi_d r_1^d) + 1$.

We use this lemma repeatedly to obtain the upper bound of the probability when the cluster is of moderate size. The important thing to note in the above estimate is that the function $h(k_1)$ is independent of μ and y.

Lemma 7. *Let (i) k_1, k_2 be both fixed or (ii) k_1 be fixed, $k_2 \to \infty$ but $k_2/\lambda \to 0$ or (iii) k_2 be fixed and $k_1 \to \infty$ but $k_1/\lambda \to 0$ or (iv) both $k_1, k_2 \to \infty$ but $k_1/\lambda \to 0$ and $k_2/\lambda \to 0$ (as $\lambda \to \infty$). Then there exists $\beta > 0$ and constant c_{10} so that for all large λ, we have*

$$\mathbb{P}_{\mathrm{B}}(E(k_1,k_2), \beta k_1/\lambda < d_Y(\mathbf{0}) \le r_1, d_Z(\mathbf{0}) \le r_1)$$
$$\le \exp(-\lambda\pi_d\mathbb{E}(\rho + r_1)^d - (d-1)k_1\log(\lambda/k_1) + k_2\log(\lambda p_2/k_2) + c_{10}k_1 + c_9k_2 + h(k_1)),$$

where c_9 and $h(k_1)$ are as defined earlier.

Proof. We fix $\mu > 1$ and choose β large so that $\Psi_\mu(\mu^j\beta) \ge j$ for every $j \ge 1$. By the definition of $\Psi_\mu(\cdot)$ this is possible. For $M = \min\{j : \beta\mu^j k_1/\lambda > r_1\}$, we have

$$\begin{aligned}
&\mathbb{P}_{\mathrm{B}}(E(k_1,k_2), \beta k_1/\lambda < d_Y(\mathbf{0}) \le r_1, d_Z(\mathbf{0}) \le r_1)\\
&= \sum_{j=0}^{M-1} \mathbb{P}_{(\lambda,\rho)}(E(k_1,k_2), d_Z(\mathbf{0}) \le r_1, \mu^j\beta k_1/\lambda < d_Y(\mathbf{0}) \le \mu^{j+1}\beta k_1/\lambda)\\
&\le \sum_{j=0}^{M-1} \exp(-\lambda\pi_d\mathbb{E}(\rho + r_1)^d - (d-1)k_1\log(\lambda/k_1)\\
&\quad - k_1\Psi_\mu(\beta\mu^j) + k_2\log(\lambda p_2/k_2) + c_9k_2 + h(k_1))\\
&\le \sum_{j=0}^{\infty} \exp(-\lambda\pi_d\mathbb{E}(\rho + r_1)^d - (d-1)k_1\log(\lambda/k_1)\\
&\quad - k_1 j + k_2\log(\lambda p_2/k_2) + c_9k_2 + h(k_1))\\
&\le \exp(-\lambda\pi_d\mathbb{E}(\rho + r_1)^d - (d-1)k_1\log(\lambda/k_1)\\
&\quad + k_2\log(\lambda p_2/k_2) + c_{10}k_1 + c_9k_2 + h(k_1)),
\end{aligned}$$

where c_{10} is a suitably chosen constant.

Finally we look at the case when the origin is the centre of a small ball and $W(\mathbf{0})$ comprises only small balls. Calculations similar to that of the previous lemmas yield the next result whose proof we omit.

Lemma 8. *Let (i) k be fixed or (ii) $k \to \infty$ but $k/\lambda \to 0$ as $\lambda \to \infty$. Then, for some constants c_{11} and c_{12} and for all large λ, we have*

(i) $\mathbb{P}_{\mathrm{S}}(E'(0,k), d_Z(\mathbf{0}) > r_2) \le \exp(-\lambda\pi_d\mathbb{E}(\rho + r_2)^d - \lambda p_2(r_2/2)^d/2 + c_{11}k)$,

(ii) $\mathbb{P}_{\mathrm{S}}(E'(0,k), d_Z(\mathbf{0}) \le \alpha_1 k/\lambda) \le \exp(-\lambda\pi_d\mathbb{E}(\rho+r_2)^d - (d-1)k\log(\lambda/k) + c_{12}k + h(k))$.

(iii) *For $\Phi_\mu(y) = p_2\pi_{d-1}(r_2)^{d-1}y - \log(ep_2\pi_d\mu^d y^d)$,* *we have, for* $\mu > 1$ *and* $y > 0$ *and for all large* λ,

$$\mathbb{P}_{\mathrm{S}}(E'(0,k), yk/\lambda < d_Z(\mathbf{0}) \le y\mu k/\lambda) \le \exp(-\lambda\pi_d\mathbb{E}(\rho + r_2)^d - (d-1)k\log(\lambda/k) + k\Phi_\mu(y) + h(k)).$$

(iv) *There exists* $\beta > 0$ *such that for all large* λ *we have*

$$\mathbb{P}_{\mathrm{S}}(E'(0,k), \beta k/\lambda < d_Z(\mathbf{0}) \le r_1) \le \exp(-\lambda\pi_d\mathbb{E}(\rho + r_2)^d - (d-1)k\log(\lambda/k) + c_{13}k + h(k)),$$

where c_{13} *is a constant not depending on* k.

5. Proof of Theorems

The proofs of Theorems 1–4 are similar, so we prove only Theorem 1.

Proof of Theorem 1. For the first part, we note, for k_1 and k_2 fixed,

$$\begin{aligned}\mathbb{P}_{\mathrm{B}}(E(k_1,k_2)) &= \mathbb{P}_{\mathrm{B}}(E(k_1,k_2), d_Y(\mathbf{0}) > r_1)\\ &\quad + \mathbb{P}_{\mathrm{B}}(E(k_1,k_2), d_Y(\mathbf{0}) \le r_1, d_Z(\mathbf{0}) > r_1)\\ &\quad + \mathbb{P}_{\mathrm{B}}(E(k_1,k_2), d_Z(\mathbf{0}) \le r_1, d_Y(\mathbf{0}) \le \alpha k_1/\lambda)\\ &\quad + \mathbb{P}_{\mathrm{B}}(E(k_1,k_2), d_Z(\mathbf{0}) \le r_1, \alpha k_1/\lambda < d_Z(\mathbf{0}) \le r_1)\\ &\quad + \mathbb{P}_{\mathrm{B}}(E(k_1,k_2), d_Z(\mathbf{0}) \le r_1, \alpha k_1/\lambda < d_Y(\mathbf{0}) \le \beta k_1/\lambda).\end{aligned}$$

Now, using Lemmas 3–7 and Lemma 5 with $y = \alpha$ and $\mu = \beta/\alpha$ for the last term in the above equality, we obtain

$$\mathbb{P}_{\mathrm{B}}(E(k_1,k_2)) \le \exp(-\lambda\pi_d\mathbb{E}(\rho + r_1)^d + (k_2 - k_1(d-1))\log\lambda + O(1)).$$

This along with (10) proves the result.

To show the second part, we see that, for any $0 < a_1(\lambda) < (r_1 - r_2)$,

$$\mathbb{P}_{\mathrm{B}}(d(W(\mathbf{0}) > a_1(\lambda) \mid E(k_1,k_2)) \ge \mathbb{P}_{\mathrm{B}}(d_Z(\mathbf{0}) > a_1(\lambda) \mid E(k_1,k_2)). \tag{11}$$

If $d_Z(\mathbf{0}) \le r_1$, all points of the process Z are inside the sphere r_1U. As we have discussed earlier, we may place these points uniformly inside the sphere $(r_1 - r_2)$ without changing the region formed by the union of all balls (big and small). This is because any point inside this sphere will be totally contained inside the ball placed at the origin. Thus we have

$$\mathbb{P}_{\mathrm{B}}(d_Z(\mathbf{0}) > a_1(\lambda) \mid E(k_1,k_2), d_Z(\mathbf{0}) \le r_1) \ge 1 - \left(\frac{a_1(\lambda)}{r_1 - r_2}\right)^{dk_2}. \tag{12}$$

If we take $a_1(\lambda)$ such that $\lambda a_1(\lambda)^d \to \infty$ but $a_1(\lambda) \to 0$ as $\lambda \to \infty$ (one such choice is $a_1(\lambda) = \lambda^{-1/(2d)}$), we obtain from (12) and (11) as $\lambda \to \infty$,

$$\mathbb{P}_{\mathrm{B}}(d(W(\mathbf{0})) > a_1(\lambda) \mid E(k_1,k_2)) \to 1. \tag{13}$$

Now, we note that

$$\delta(\lambda) = \frac{k_1 + k_2 + 1}{\lambda\pi_d d(W(\mathbf{0}))^d} \le \frac{k_1 + k_2 + 1}{\lambda\pi_d a_1(\lambda)^d} \to 0 \qquad \text{as } \lambda \to \infty,$$

on the set $\{d(W(\mathbf{0})) > a_1(\lambda)\}$. So, by our choice of the $a_1(\lambda)$, we have

$$\mathbb{P}_{\mathrm{B}}(\delta(\lambda) \to 0 \mid E(k_1, k_2)) \to 1 \tag{14}$$

as $\lambda \to \infty$, proving the theorem.

Finally we are left with the proof of Theorem 5. The proofs of the first and the second parts follow a similar line to Theorem 1. To show the third part we need an upper bound of $\mathbb{P}_{\mathrm{S}}(E'(k_1, k_2))$ for $k_1 \geq 1$. For this we use Equation (7).

Proof of Theorem 5. We have

$$\begin{aligned}
&\mathbb{P}_{(\lambda,\rho)}(E'(0,k) \mid \#(W(\mathbf{0})) = k+1) \\
&\quad \geq 1 - \frac{\sum_{k_1=0}^{k} p_1 \mathbb{P}_{\mathrm{B}}(E(k_1, k-k_1)) + p_2 \sum_{k_1=1}^{k} \mathbb{P}_{\mathrm{S}}(E'(k_1, k-k_1))}{p_2 \mathbb{P}_{\mathrm{S}}(E'(0,k))} \\
&\quad \geq 1 - \frac{k+1}{p_2}\left[\frac{\max_{0 \leq k_1 \leq k} \mathbb{P}_{\mathrm{B}}(E(k_1, k-k_1))}{\mathbb{P}_{\mathrm{S}}(E'(0,k))} + \frac{\max_{1 \leq k_1 \leq k} \mathbb{P}_{\mathrm{S}}(E'(k_1, k-k_1))}{\mathbb{P}_{\mathrm{S}}(E'(0,k))}\right].
\end{aligned} \tag{15}$$

Let $\eta = \mathbb{E}(\rho + r_1)^d - \mathbb{E}(\rho + r_2)^d > 0$. For fixed k, we have

$$\max_{0 \leq k_1 \leq k} \mathbb{P}_{\mathrm{B}}(E(k_1, k-k_1)) \leq \exp(-\lambda \pi_d \mathbb{E}(\rho + r_1)^d + c_{14} \log \lambda + c_{15}),$$

where c_{14} and c_{15} are fixed positive constants not depending on λ. Further, from (7) and above, we have

$$\max_{1 \leq k_1 \leq k} \mathbb{P}_{\mathrm{S}}(E'(k_1, k-k_1)) \leq \exp(-\lambda \pi_d \mathbb{E}(\rho + r_1)^d + c_{16} \log \lambda + c_{17})$$

where c_{16}, c_{17} are positive constants not depending on λ. Thus from the lower bound of $\mathbb{P}_{\mathrm{S}}(E'(0,k))$ in Lemma 2, (15) and the above upper bounds, we have for some constants c_{18} and c_{19},

$$\begin{aligned}
&\mathbb{P}_{(\lambda,\rho)}(E'(0,k) \mid \#(W(\mathbf{0})) = k+1) \\
&\quad \geq 1 - \frac{2}{p_2} \exp(-\lambda \eta + (\max(c_{14}, c_{16}) + c_{18} \log \lambda + c_{19} k + \log(k+1))) \\
&\quad \to 1 \quad \text{as } \lambda \to \infty
\end{aligned}$$

proving (9).

When $k \to \infty$ but $k/\lambda \to 0$, we have to do a little more work. We first choose M so that $(\log x)/x \leq \eta/4$ if $x \geq M$. Now we choose λ_0 such that for all $\lambda > \lambda_0$ we have $\lambda p_1/k \geq M$ and $\lambda p_2/k \geq M$. From the upper bounds, whenever $\lambda \geq \lambda_0$, we have for some constant c_{20} (not depending on k and λ),

$$\begin{aligned}
\max_{0 \leq k_1 \leq k} \mathbb{P}_{\mathrm{B}}(E(k_1, k-k_1)) &\leq \max_{0 \leq k_2 \leq k} \exp(-\lambda \pi_d \mathbb{E}(\rho + r_1)^d + k_2 \log(\lambda p_2/k_2) + c_{19} k) \\
&\leq \exp(-\lambda \pi_d \mathbb{E}(\rho + r_1)^d + (\eta/4)\lambda + c_{20} k).
\end{aligned}$$

(writing $0 \cdot \infty = 0$). By using Equation (7) and the above inequality, we obtain

$$\begin{aligned}
&\max_{1 \le k_1 \le k} \mathbb{P}_S(E'(k_1, k - k_1)) \\
&\quad \le \max_{0 \le k_2 \le k} \exp(-\lambda \pi_d \mathbb{E}(\rho + r_1)^d + k_2 \log(\lambda p_2/k_2) + c_{20}k + \log(p_1/p_2) + \log k) \\
&\quad \le \exp(-\lambda \pi_d \mathbb{E}(\rho + r_1)^d + (\eta/2)\lambda + c_{21}k),
\end{aligned}$$

where c_{21} is a suitable constant not depending on λ. From the lower bound of $\mathbb{P}_S(E'(0, k))$ in Lemma 2 and the above upper bounds and (15), we conclude (9).

6. General radius and convex shapes

Our arguments in the preceding sections go through when instead of just 'big' and 'small' balls we have balls of more than two different sizes. Thus the results remain valid when the Boolean model is equipped with a radius random variable taking positive values $r_1, r_2, \dots, r_l$. Here the smallest balls constitute the finite cluster and the presence of any ball other than that of the smallest size in the finite cluster would result in rarefaction. Consider now a Boolean model with radius random variable ρ taking values $r_1 > r_2 > \cdots > r_n > \cdots$ with probabilities $p_1, p_2, \dots, p_n \dots$ where $r_n \downarrow 0$ as $n \to \infty$ and $p_n > 0$ for all $n \ge 1$. In this case, given that the origin has a ball of radius r_n, any finite cluster will consist of balls whose radii are smaller than r_n and are distributed uniformly inside the ball of radius r_n at the origin; therefore, a rarefaction phenomenon will be observed. This can be intuitively seen by considering another random variable ρ' taking values $r_1 > r_2 > \cdots > r_n > r_{n+1}$ with probabilities $p_1, p_2, \dots, p_n, \sum_{i=n+1}^{\infty} p_i$ and considering that the ball at the origin has radius r_n.

When the radius ρ of the underlying Boolean model has a distribution with support in $(0, R]$ for some $R > 0$ with $\mathbb{P}(\rho \le \epsilon) > 0$ for every $\epsilon > 0$, we expect that there will always be rarefaction because for any ball centred at the origin we could get balls of smaller size. This can be seen intuitively by considering a radius random variable ρ' taking values $r_1, r_2, \dots$ with $r_1 > r_2 > \cdots > 0$ and $r_n \to 0$ as $n \to \infty$.

Also, the arguments in the previous sections did not need the shapes to be spheres. Indeed, the results should go through for any convex shape. However, the calculations of the volume of the annular regions which we needed would be quite forbidding.

Acknowledgements

This article constitutes a chapter of my thesis (Sarkar (1994)) submitted to the Indian Statistical Institute. I am grateful to Rahul Roy for many helpful ideas and suggestions. I also thank the referee for his invaluable comments and helpful suggestions towards improving this paper.

References

ALEXANDER, K. (1993). Finite clusters in high-density continuous percolation: compression and sphericality. *Prob. Theory Rel. Fields* **97,** 35–63.

FELLER, W. (1968). *An Introduction to Probability Theory and its Applications*, Vol. I. Wiley, New York.

HALL, P. (1988). *Introduction to the Theory of Coverage Processes*. Wiley, New York.

MEESTER, R. AND ROY, R. (1994). Uniqueness of the unbounded occupied and vacant components in Boolean models. *Ann. of Appl. Prob.* **4,** 933–951.

MEESTER, R. AND ROY, R. (1996). *Continuum Percolation*. Cambridge University Press, New York.

PENROSE, M. D. (1996). Continuum percolation and Euclidean minimal spanning trees in high dimensions. *Ann. of Appl. Prob.* **6**, 528–544.

ROY, R. (1990). The RSW theorem and the equality of critical densities and the 'dual' critical densities for continuum percolation on $\mathbb{R}^2$. *Ann. Prob.* **18,** 1563–1575.

SARKAR, A. (1994). Some problems in continuum percolation. Ph.D. thesis, Indian Statistical Institute, New Delhi.