PRODUCT SYSTEMS OF ONE-DIMENSIONAL EVANS-HUDSON FLOWS

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Abstract: Every Evans-Hudson flow on the algebra of all bounded operators on a Hilbert space leads to a semigroup of $*$-endomorphisms, and then to a continuous tensor product system of Hilbert spaces. Here we have a new representation for exponential product systems. This helps us to show that only such product systems arise from one-dimensional Evans-Hudson flows.

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1 Introduction

Let $\mathcal{H}_0$ be a complex separable Hilbert space. An one-dimensional Evans-Hudson (EH) flow (with bounded structure maps) on $\mathcal{B}(\mathcal{H}_0)$ is a family of $J = \{J_t\}$ of unital $*$-homomorphisms mapping $\mathcal{B}(\mathcal{H}_0)$ into $\mathcal{B}(\mathcal{H}_0 \otimes \Gamma(L^2(I\!\!R_+)))$, satisfying the quantum stochastic stochastic differential equation [Hu]

$$\begin{aligned} dJ_t(X) &= J_t(LX - \sigma(X)L)dA^\dagger + J_t(\sigma(X) - X)d\Lambda \\ &\quad + J_t(XL^* - L^*\sigma(X))dA + J_t(\mathcal{L}(X))dt, \\ J_0(X) &= X \otimes I, \end{aligned} \tag{1.1}$$

where $L \in \mathcal{B}(\mathcal{H}_0)$, σ is a $*$-endomorphism of $\mathcal{B}(\mathcal{H}_0)$,

$$\mathcal{L}(X) = i[H, X] - \frac{1}{2}(L^*LX - 2L^*\sigma(X)L + XL^*L), \quad H = H^* \in \mathcal{B}(\mathcal{H}_0),$$

and $A^\dagger, \Lambda, A$ are creation, conservation, and annihilation processes [Pa] on symmetric Fock space $\Gamma(L^2(I\!\!R_+))$. The EH flow J is to be thought of as a dilation of quantum dynamical semigroup T with generator $\mathcal{L}$, as

$$\langle a, T_t(X)b\rangle = \langle ae(0), J_t(X)be(0)\rangle, \quad a, b \in \mathcal{H}_0$$

for all $X \in \mathcal{B}(\mathcal{H}_0), t \geq 0$. Here $e(0)$ is the vacuum in $\Gamma(L^2(I\!\!R_+))$ and $ae(0)$ means $a \otimes e(0)$.

In this article we assume that σ is a normal endomorphism. Then a fine observation of Bradshaw [Br] allows us to obtain an E_0-semigroup, that is, a strongly continuous semigroup of unital normal $*$-endomorphisms of $\mathcal{B}(\mathcal{H}_0 \otimes \Gamma(L^2(I\!\!R_+)))$, naturally associated with the flow J. Due to works of Arveson [Ar], Powers [Pr], and others now we have a fairly well-developed classification theory for E_0-semigroups. So perhaps it is worth-while to attempt a classification of Evans-Hudson flows through their E_0-semigroups. This was a suggestion of L. Accardi and here this program has been carried out completely for one-dimensional EH flows.

Arveson's approach is to associate a continuous tensor product system of Hilbert spaces with every E_0-semigroup. Here product system means a pair $(\mathcal{E}, U)$ where

$\mathcal{E} = \{\mathcal{E}_t, t > 0\}, U = \{U_{s,t} : s, t > 0\}$ is a family of separable Hilbert spaces along with unitary isomorphisms $U_{s,t} : \mathcal{E}_s \otimes \mathcal{E}_t \to \mathcal{E}_{s+t}$, having the associative property:

$$U_{s_1,s_2+s_3}(I_{s_1} \otimes U_{s_2,s_3}) = U_{s_1+s_2,s_3}(U_{s_1,s_2} \otimes I_{s_3}),$$

for $s_1, s_2, s_3 > 0$, as maps from $\mathcal{E}_{s_1} \otimes \mathcal{E}_{s_2} \otimes \mathcal{E}_{s_3}$ to $\mathcal{E}_{s_1+s_2+s_3}$. Strictly speaking there are some additional measurability conditions [Ar]. We do not stress them here as they are automatically satisfied in our context.

There is a product system naturally associated with the Fock space $\Gamma(L^2(\mathbb{R}_+, \mathcal{K}))$, for arbitrary separable Hilbert space $\mathcal{K}$. This is known as *exponential product system* with base space $\mathcal{K}$. In Section 2 we explain this concept and then with the help of an additional Hilbert space $\mathcal{K}_0$ construct a new product system which at the first look appears to be quite different from exponential product systems. But a closer analysis shows that we have just another representation of exponential product system with base $\mathcal{K}_0 \otimes \mathcal{K}$.

In the final section we determine product systems of all one-dimensional EH flows (that is, of their associated E_0-semigroups). It turns out that they are all exponential product systems. They have a base space of dimension higher than one if and only if the flow is not implemented by a Hudson-Parthasarathy type unitary cocycle. The results of Section 2 are quite handy in this analysis. It is to be mentioned that if $\mathcal{H}_0$ is infinite dimensional then generator of every uniformly continuous, unital, normal quantum dynamical semigroup can be written as above, with suitable choice of L, H, and σ [HS]. So one-dimensional quantum stochastic calculus is good enough to dilate all these semigroups. However as this paper shows intrinsically the EH flow may make use of a higher dimensional exponential product system.

2 A new representation of exponential product systems

First we explain the notion of exponential product systems in a way suitable for our further constructions. Let $\mathcal{K}$ be a complex separable Hilbert space. Then the symmetric Fock space $\mathcal{R}_t = \Gamma(L^2([0,t))$, is given by

$$\mathcal{R}_t = \oplus_{n \geq 0} h_t^{\textcircled{s}^n}, h_t = L^2([0,t)).$$

For $u_1, u_2, \cdots, u_n \in h_t$ the n-fold symmetric tensor product

$$u_1 \vee u_2 \cdots \vee u_n = \frac{1}{\sqrt{n!}} \sum_\sigma u_{\sigma(1)} \vee u_{\sigma(2)} \cdots u_{\sigma(n)}$$

where σ runs over all permutations, is an element of $h_t^{\circledS^n}$. Let $S_t : h_s \to h_{s+t}$ be the right shift

$$S_t f(x) = \begin{cases} f(x-t) & t \leq x \\ 0 & 0 \leq x < t \end{cases} \tag{2.1}$$

Then $U_{s,t} : \mathcal{R}_s \otimes \mathcal{R}_t \to \mathcal{R}_{s+t}$ defined by

$$U_{s,t}((u_1 \vee u_2 \cdots \vee u_m) \otimes (v_1 \vee v_2 \cdots \vee v_n)) = u_1 \vee u_2 \cdots \vee u_m \vee (S_s v_1) \vee (S_s v_2) \cdots \vee (S_s v_n)$$

extends to an associative family of unitaries. (Here $u_i \in h_{s+t}$ as $h_s \subset h_{s+t}$ in the natural way.) The resultant product system $\mathcal{R} = \{\mathcal{R}_t; t > 0\}$ is called as the exponential product system with base space $\mathcal{K}$. The dimension of the base space is a complete invariant for exponential product systems [Ar]. Observe that we actually obtain

$$h_{s+t}^{\circledS^p} \cong \sum_{m+n=p} h_s^{\circledS^m} \otimes h_t^{\circledS^n}. \tag{2.2}$$

Let $\mathcal{K}_0$ be another Hilbert space. We build a product a system $\mathcal{S} = \{\mathcal{S}_t; t > 0\}$ which depends upon both $\mathcal{K}$ and $\mathcal{K}_0$. In fact the construction is pretty simple. Take

$$\mathcal{S}_t = \oplus_{n \geq 0} (\mathcal{K}_0^{\otimes^n} \otimes h_t^{\circledS^n})$$

for $t > 0$. Define a map $V_{s,t} : \mathcal{S}_s \otimes \mathcal{S}_t \to \mathcal{S}_{s+t}$, by

$$V_{s,t}(\{a_1 \otimes a_2 \cdots \otimes a_m \otimes u_1 \vee u_2 \cdots \vee u_m\} \otimes \{b_1 \otimes b_2 \cdots \otimes b_n \otimes v_1 \vee v_2 \cdots \vee v_n\})$$

$$= a_1 \otimes a_2 \cdots \otimes a_m \otimes b_1 \otimes b_2 \cdots \otimes b_n \otimes (u_1 \vee u_2 \cdots \vee u_m \vee S_s v_1 \vee S_s v_2 \cdots \vee S_s v_n).$$

Using (2.2) it is not difficult to verify that $V_{s,t}$ extends to a unitary operator and the pair $(\mathcal{S}, V)$ is indeed a product system.

Theorem 2.1: The product system $(\mathcal{S}, V)$ is isomorphic to the exponential product system with base space $\mathcal{K}_0 \otimes \mathcal{K}$.

As a preparation to prove this result we introduce some notation and prove two simple lemmas. For any two subsets E, F of $\mathbb{R}_+$ we write $E < F$ to mean $x < y$ for all $x \in E$, and $y \in F$. Fix $t > 0$, and take $\tilde{\mathcal{R}}_t = \Gamma((L^2([0,t), \mathcal{K}_0 \otimes \mathcal{K}))$. Let $\mathcal{D}_t \subset \tilde{\mathcal{R}}_t$ be the set

$$\begin{aligned} \mathcal{D}_t &= \{e(0)\} \cup \{a_1 x_1 \chi_{E_1} \vee a_2 x_2 \chi_{E_2} \cdots \vee a_n x_n \chi_{E_n}; E_1 < E_2 \cdots < E_n, E_i \subset [0,t), \\ &\qquad a_i x_i = a_i \otimes x_i \in \mathcal{K}_0 \otimes \mathcal{K}, n \geq 1\} \end{aligned}$$

Lemma 2.2: The set $\mathcal{D}_t$ is total in $\tilde{\mathcal{R}}_t$.

Proof: Clearly it is enough to approximate vectors of the form $g_1 \vee g_2 \cdots \vee g_n$, where all $g_i \in L^2([0,t), \mathcal{K}_0 \otimes \mathcal{K})$ are simple. Hence it is enough to approximate vectors of the form

$$\xi = \vee_{i=1}^{p} \vee_{k=1}^{r_i} a_{ik} x_{ik} \chi_{E_i}, \quad E_1 < E_2 < \cdots < E_n$$

where $a_{ik} x_{ik} = a_{ik} \otimes x_{ik} \in \mathcal{K}_0 \otimes \mathcal{K}, r_i \geq 1$, with $\sum_i r_i = n$. Fix $M \geq 1$, and partition each E_i in to M parts as

$$E_i = \bigcup_{j=1}^{M} F_{ij}, \quad F_{i1} < F_{i2} < \cdots < F_{iM},$$

with $\mu(F_{ij}) = \frac{1}{M}\mu(E_i)$. ($\mu$ = Lebesgue measure.) Then

$$\xi = \vee_{i=1}^{p} \vee_{k=1}^{r_i} (\sum_{j=1}^{m} a_{ik} x_{ik} \chi_{F_{ij}}).$$

Expand the right hand side using multilinearity of symmetric tensor product to have M^n terms. With out loss of generality we can assume

$$\sup_{i,k} ||a_{ik}|| \leq 1, \quad \sup_i \mu(E_i) \leq 1.$$

Then norm of every term in the expansion is bounded by $(\frac{1}{M})^n$. The terms with subscripts of χ all distinct are in $\mathcal{D}_t$. Elementary combinatorics shows that there are precisely $\prod_{i=1}^{p}(M(M-1)\cdots(M-r_i+1))$ terms of this kind. We estimate sum of the norms of the rest by

$$(M^n - \prod_{i=1}^{p}(M(M-1)\cdots(M-r_i+1))\frac{1}{M^n} = (1 - \prod_{i=1}^{p} 1(1-\frac{1}{M})\cdots(1-\frac{r_i-1}{M}))$$

which clearly converges to zero as $M \to \infty$. ∎

Lemma 2.3: On $\mathcal{D}_t \times \mathcal{D}_t$

$$\langle a_1 x_1 \chi_{E_1} \vee a_2 x_2 \chi_{E_2} \cdots \vee a_n x_n \chi_{E_n}, b_1 y_1 \chi_{F_1} \vee b_2 y_2 \chi_{F_2} \cdots \vee b_m y_m \chi_{F_m} \rangle$$

$$= \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{\sqrt{n!}} \prod_{i=1} n \langle a_i, b_i \rangle \langle x_i, y_i \rangle \mu(E_i \cap F_i) & \text{if } n = m. \end{cases}$$

Proof: As an n-particle is orthogonal to any m-particle with $m \neq n$ one part is easy. To see the other consider the inner-product

$$\langle a_1 x_1 \chi_{E_1} \otimes \cdots \otimes a_n x_n \chi_{E_n}, b_{\sigma(1)} y_{\sigma(1)} \chi_{F_{\sigma(1)}} \otimes \cdots \otimes b_{\sigma(n)} y_{\sigma(n)} \chi_{F_{\sigma(n)}} \rangle$$

$$= \prod_{i=1}^{n} \langle a_i, b_{\sigma(i)} \rangle \langle x_i, y_{\sigma(i)} \rangle \mu(E_i \cap F_{\sigma(i)})$$

for some permutation σ. Suppose there exist p, q such that $p < q$ with $\sigma(p) > \sigma(q)$. Then if $z \in E_p \cup F_{\sigma(p)}$, we have $\{z\} < E_q$, as well as $F_{\sigma(q)} < \{z\}$, and hence $E_q \cup F_{\sigma(q)}$ is empty. We conclude that the inner-product under consideration is zero unless σ is the identity permutation. ∎

Proof of Theorem 2.1: Define $W_t : \mathcal{D}_t \to \mathcal{S}_t$ by $W_t(e(0)) = 1 \otimes e(0)$,

$$\begin{aligned} & W_t(a_1 x_1 \chi_{E_1} \vee a_2 x_2 \chi_{E_2} \cdots \vee a_n x_n \chi_{E_n}) \\ = \; & (a_1 \otimes a_2 \cdots \otimes a_n) \otimes (x_1 \chi_{E_1} \vee x_2 \chi_{E_{11}} \cdots \vee x_n \chi_{E_n}). \end{aligned}$$

From the lemmas it follows that W_t is isometric, and its domain, range are total in $\tilde{\mathcal{R}}_t$, $\mathcal{S}_t$ respectively. So W_t extends to a unitary isomorphism. Denote the extension also by W_t. We need to show that it respects the product system structure. Recall the definition of right shift S_s (2.1) and note that $S_s\chi_F = \chi_{F+s}$ for all F. Now for $a_1x_1\chi_{E_1} \vee a_2x_2\chi_{E_2} \cdots \vee a_mx_m\chi_{E_m} \in \mathcal{D}_s$, $b_1y_1\chi_{F_1} \vee b_2y_2\chi_{F_2} \cdots \vee b_ny_n\chi_{F_n} \in \mathcal{D}_t$

$$\begin{aligned}
& W_{s+t}(U_{s,t}(a_1x_1\chi_{E_1} \vee \cdots \vee a_mx_m\chi_{E_m}) \otimes (b_1y_1\chi_{F_1} \vee \cdots \vee b_ny_n\chi_{F_n})) \\
= \; & W_{s+t}(a_1x_1\chi_{E_1} \vee \cdots \vee a_mx_m\chi_{E_m} \vee (b_1y_1\chi_{F_1+s} \vee \cdots \vee b_ny_n\chi_{F_n+s})) \\
= \; & a_1 \otimes a_2 \cdots \otimes a_m \otimes b_1 \otimes b_2 \cdots b_n \\
& \quad \otimes(x_1\chi_{E_1} \vee \cdots \vee x_m\chi_{E_m} \vee y_1\chi_{F_1+s} \vee \cdots \vee y_n\chi_{F_n+s}) \\
= \; & V_{s,t}((a_1 \otimes a_2 \cdots \otimes a_m \otimes x_1\chi_{E_1} \otimes \cdots \otimes x_m\chi_{E_m}) \\
& \quad \otimes(b_1 \otimes b_2 \cdots b_n \otimes y_1\chi_{F_1} \otimes \cdots \otimes y_n\chi_{F_n})) \\
= \; & V_{s,t}(W_s(a_1x_1\chi_{E_1} \vee \cdots \vee a_mx_m\chi_{E_m}) \otimes W_t(b_1y_1\chi_{F_1} \vee \cdots \vee b_ny_n\chi_{F_n})).
\end{aligned}$$

Using the totality of $\mathcal{D}_s$, $\mathcal{D}_t$ the proof is complete. ■

3 The main result

Let α be a strongly continuous (strong operator topology) semigroup of unital, normal $*$-endomorphisms of $\mathcal{B}(\mathcal{H})$. We associate a product system $\mathcal{E} = \mathcal{E}^{(\alpha)}$, with α as follows. Fix a unit vector $a \in \mathcal{H}$. Take $\mathcal{E}_t =$ range $(\alpha_t(|a\rangle\langle a|))$. Define $V_{s,t} : \mathcal{E}_s \otimes \mathcal{E}_t \to \mathcal{E}_{s+t}$ by

$$V_{s,t}(\alpha_s(|a\rangle\langle a|)u \otimes \alpha_t(|a\rangle\langle a|)v) = \alpha_{s+t}(|a\rangle\langle a|)\alpha_s(|v\rangle\langle a|)u \tag{3.1}$$

for $u, v \in \mathcal{H}$. Then it is not difficult to see that $V_{s,t}$ is a unitary operator (normality of α_t used here). The pair $(\mathcal{E}, V_{s,t})$ becomes a product system and it is isomorphic to the product system obtained by Arveson [Ar] through more algebraic methods. This construction has been taken up from [Bh] with a minor modification so that $(\mathcal{E}, V)$ becomes anti-isomorphic to the product system $(\mathcal{P}, U)$ obtained there. (Compare with (4.14) of [Bh].)

Now let us recall the construction of Bradshaw [Br], the basic idea of which perhaps traces back to [Ac], and [Me]. The Hilbert space $\tilde{\mathcal{H}} = \mathcal{H}_0 \otimes \Gamma(L^2(\mathbb{R}_+))$ has the familiar decomposition as $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_{t]} \otimes \tilde{\mathcal{H}}_{[t}$, where

$$\tilde{\mathcal{H}}_{t]} = \mathcal{H}_0 \otimes \Gamma(L^2([0,t))), \quad \tilde{\mathcal{H}}_{[t} = \Gamma(L^2([t,\infty))),$$

for $t \geq 0$. Let $J = \{J_t, t \geq 0\}$ be an one-dimensional Evans-Hudson flow on $\mathcal{B}(\mathcal{H}_0)$. Then the homomorphisms $J_t : \mathcal{A}_0 \to \mathcal{B}(\tilde{\mathcal{H}})$ are of the form

$$J_t(X) = \bar{J}_t(X) \otimes I_{[t},$$

where $\bar{J}_t(X) \in \mathcal{B}(\tilde{\mathcal{H}}_{t]})$ and $I_{[t}$ is the identity operator on $\tilde{\mathcal{H}}_{[t}$. Observe that $L^2([0,\infty))$ is isomorphic to $L^2([t,\infty))$ through the shift operator S_t. This gives rise to a natural unitary isomorphism between $\tilde{\mathcal{H}}_{[0}$ and $\tilde{\mathcal{H}}_{[t}$, which maps exponential vector $e(f)$ to $e(S_t f)$. Now define $\alpha_t : \mathcal{B}(\tilde{\mathcal{H}}) \to \mathcal{B}(\tilde{\mathcal{H}})$ by

$$\alpha_t(X \otimes |e(f)\rangle\langle e(g)|) = \bar{J}_t(X) \otimes |e(S_t f)\rangle\langle e(S_t g)|, \quad X \in \mathcal{B}(\mathcal{H}_0), f, g \in L^2(I\!R_+).$$

Then it follows that α_t extends to a normal $*$-endomorphism of $\mathcal{B}(\tilde{\mathcal{H}})$, and $\alpha = \{\alpha_t : t \geq 0\}$ is strongly continuous. We shall call the product system associated with α as the product system of Evans-Hudson flow J.

At this moment it maybe observed that if J is minimal [BF] then α coincides with the minimal dilation E_0-semigroup associated with the the quantum dynamical semigroup of J. This is clear from Theorem 4.5 of [Bh] once we note that α_t maps $\tilde{J}_s(X) = \bar{J}_s(X) \otimes |e(0)\rangle\langle e(0)|$ to $\tilde{J}_{s+t}(X)$.

Now we determine the product system of special Evans-Hudson flow J^σ which satisfies the quantum stochastic differential equation

$$dJ_t^\sigma(X) = J_t^\sigma(\sigma(X) - X)d\Lambda, \quad J_0(X) = X \otimes I$$

where σ is a unital, normal $*$-endomorphism of $\mathcal{B}(\mathcal{H}_0)$. This EH flow is particularly simple to deal with as the homomorphisms J_t can be written down explicitly [Hu]. In fact

$$J_t^\sigma(X) = \sum_{n \geq 0} \sigma^n(X) \otimes Q_t^{(n)} \otimes I_{[t}$$

where $Q_t^{(n)}$ is the projection onto the n-particle space in $\Gamma(L^2([0,t)))$. With notation as in Section 2 (the special case, $\mathcal{K} \cong \mathbb{C}$) the range of $Q_t^{(n)}$ is $h_t^{\circledS^n}$, for $h_t = L^2([0,t))$.

Let β be the E_0-semigroup associated with J^σ. Now by the very definition of β we have

$$\beta_t(|ce(f)\rangle\langle de(g)|) = \sum_{n \geq 0} \sigma^n(|c\rangle\langle d|) \otimes Q_t^{(n)} \otimes |e(S_t f)\rangle\langle e(S_t g)|,$$

for $c, d \in \mathcal{H}_0,\ f, g \in L^2(I\!R_+)$. Fix a unit vector $a \in \mathcal{H}_0$, and take $\mathcal{E}_t$ as range of $(\beta_t(|ae(0)\rangle\langle ae(0)|))$. We try to visualize the product system formed by looking at some convenient vectors. Consider

$$\xi = c \otimes (u_1 \vee u_2 \cdots \vee u_m) \otimes e(f),$$

$$\eta = d \otimes (v_1 \vee v_2 \cdots \vee v_n) \otimes e(g)$$

where $c, d \in \mathcal{H}_0,\ u_i \in L^2([0, s)),\ v_i \in L^2([0, t)),\ f \in L^2([s, \infty))$, and $g \in L^2([t, \infty))$. Then clearly

$$\begin{aligned} \beta_s(|ae(0)\rangle\langle ae(0)|)\xi &= \sigma^m(|a\rangle\langle a|) \otimes (u_1 \vee u_2 \cdots \vee u_m) \otimes e(0), \\ \beta_s(|\eta\rangle\langle ae(0)|)\xi &= \sigma^m(|d\rangle\langle a|) \otimes (u_1 \vee u_2 \cdots \vee u_m) \\ &\quad \otimes (S_s v_1 \vee S_s v_2 \cdots \vee S_s v_n) \otimes e(S_s g). \end{aligned} \tag{3.2}$$

Now $e(S_s g) \in (\tilde{\mathcal{H}}_{s+t})$, and hence

$$\begin{aligned} & V_{s,t}(\beta_s(|ae(0)\rangle\langle ae(0)|)\xi \otimes \beta_t(|ae(0)\rangle\langle ae(0)|)\eta) \\ = \ & \beta_{s+t}(|ae(0)\rangle\langle ae(0)|)\beta_s(|\eta\rangle\langle ae(0)|)\xi \\ = \ & \sigma^{m+n}(|a\rangle\langle a|)\sigma^m(|d\rangle\langle a|)c \otimes (u_1 \vee u_2 \cdots \vee u_m \vee S_s v_1 \vee S_s v_2 \cdots \vee S_s v_n) \otimes e(0). \end{aligned} \tag{3.3}$$

Finally observe that $\{\sigma^n, n \geq 0\}$ is a discrete semigroup of endomorphisms of $\mathcal{B}(\mathcal{H}_0)$ and so there is an associated discrete product system. That is, on taking $\mathcal{K}_0 =$ range $\sigma(|a\rangle\langle a|)$, there exist unitary maps $W_n : [$ range $(\sigma^n(|a\rangle\langle a|))] \rightarrow \mathcal{K}_0^{\otimes^n}$, such that $W_1 = I$,

$$W_{m+n}(\sigma^{m+n}(|a\rangle\langle a|)\sigma^m(|d\rangle\langle a|)c) = W_m(\sigma^m(|a\rangle\langle a|c)) \otimes W_n(\sigma^n(|a\rangle\langle a|d)) \tag{3.4}$$

for $m, n \geq 1$. Combining (3.2), (3.3) and (3.4) as vectors of the form ξ, η along with $e(0)$ form total sets it should be now clear that Z_s defined by $Z_s e(0) = 1 \otimes e(0)$,

$Z_s(\beta_s(|ae(0)\rangle\langle ae(0)|)c \otimes (u_1 \vee u_2 \cdots \vee u_m) \otimes e(f)) = W_m(\sigma^m(|a\rangle\langle a|)c) \otimes u_1 \vee u_2 \cdots \vee u_m$

extends to a product system isomorphism between $\mathcal{E}$ and $\mathcal{S}$ (of Section 2 with $\mathcal{K} \cong \mathbb{C}$). Hence from Theorem 2.1 we can conclude that the product system of Evans-Hudson flow J^σ is the exponential product system with base space $\mathcal{K}_0$, where $\mathcal{K}_0^{\otimes^n}$ forms the discrete product system of σ.

Theorem 3.1: The product system of every one-dimensional Evans-Hudson flow is exponential.

Proof: Let J be an EH flow on $\mathcal{B}(\mathcal{H}_0)$ satisfying (1.1). Let α, β be product systems of J, J^σ respectively. Now from [Hu] we have

$$J_t(X) = U(t) J_t^\sigma(X) U(t)^*$$

where $\{U(t), t \geq 0\}$ is a strongly continuous family of unitaries satisfying the quantum stochastic differential equation

$$dU(t) = U(t)(L(t)dA^\dagger - L(t)^* dA + (iH(t) - \frac{1}{2}L(t)^* L(t))dt)$$

with $U(0) = I$, where $L(t) = J_t^\sigma(L)$, $H(t) = J_t^\sigma(H)$. Clearly $U(t)$ is of the form $\bar{U}(t) \otimes I_{[t}$, $\bar{U}(t) \in \mathcal{B}(\tilde{\mathcal{H}}_{t]})$ (adaptedness). Hence α, β are related by the relation

$$\alpha_t(W) = U(t)\beta_t(W)U(t)^*, \quad W \in \mathcal{B}(\tilde{\mathcal{H}}).$$

In other words α, β are exterior equivalent and from Theorem 3.18 of [Ar] they have isomorphic product systems. ∎

We conclude with a remark that perhaps some quantum stochastic differential equations with higher (maybe even infinite) degrees of freedom can be rephrased using the special representation of exponential product systems in Section 2. Then just one dimensional quantum stochastic calculus could be sufficient to handle them.

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