Discussion of "The Contributions of Robert A. Wijsman to Sequential Analysis" by Adam T. Martinsek

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Adam has written a masterly review of all of Bob's work on sequential analysis. One can only add a few technical or personal details that may be of interest.

I read the three papers of Bob on sequential probability ratio tests (SPRTs) for simple hypotheses about i.i.d. random variables when I had just begun my doctoral research and was trying to solve a problem given to me by my adviser, Harikinkar Nandi. While trying to solve Nandi's problem, I had come across an article by Lindley (1953) and its review in the Mathematical Reviews by Wolfowitz (1954). Probably both had been pointed out to me by Nandi. Wolfowitz had a pretty argument that reduced Lindley's long and a bit incomplete proof to a page. I couldn't solve Nandi's problem till much later, in 1964, but I wrote a decision theoretic proof of some of Bob's results in *CSA Bulletin* (1961). I never found out if Bob had ever read my proof (Ghosh, 1961) of his results on the existence of SPRTs for given α, β $(\alpha + \beta < 1)$, "monotonicity," and optimality when the restriction to finite ASN is dropped.

Our friendship began a little later when he, I, and Jack Hall submitted independently somewhat different versions of our 1965 paper (Hall et al., 1965) to the *Annals* and Kruskal, as editor, suggested we write a joint paper. It took us several years to do that but it did seem worth the effort. How did we know about Stein's unpublished work? I can't recall anymore. I may have heard from the editor or from Bob, who may have heard from Lehmann. The paper was also closely related to a paper of Cox (1952), which was probably in our bibliography, and Nandi (1948). Cox had the same result without the invariance conditions and Nandi had an argument based on a heuristic approximation, also without invariance.

For me the main interest was to develop a sequential test for composite nulls against composite alternatives. The question of their termination arose later and led to the beautiful results that Adam describes in his paper.

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We also had a proof of transitivity of the sequence of an invariantly sufficient sequence. This is a property introduced by Bahadur to show that decision functions based on a transitive sufficient sequence form an essentially complete class.

Many years later my friends Rabi Bhattacharya and Ed Waymire independently rediscovered these results and applied them for an elegant construction of a Brownian motion with reflecting boundaries (Bhattacharya and Waymire, 1990). The proof and a reference to our paper appear in their very well-known book on stochastic processes.

Bob's results on stopping times are truly "masterly," as Adam calls them. They are based on very hard calculations and provide almost definitive answers in many of the examples. A key tool was Bob's representation of the likelihood ratio of a maximal invariant as integral over the group of transformations. This is a famous result by itself and has its origin in one of Stein's results for a compact group.

Another key result explained very well in the 1979 paper is an approximation to the integral over the group. This is a kind of Laplace approximation that I have not seen anywhere else. Its applications should go well beyond sequential analysis. In the mid-nineties, as part of our work on probability matching priors, Gauri Sankar Datta and I had found a new way of characterizing the right invariant Haar measure through a set of differential equations. Our proof was rather tricky and based on integration. I wanted to find out if this result was known and if one could have a more direct proof. The best person who could tell me was Bob, so I wrote to him. He wrote back that the result seemed to be new and, within a month, had constructed the kind of proof I had been looking for in vain.

In one of the last letters we exchanged, possibly four years ago, I mentioned I was getting old but still able to do research. He wrote back that I must go on doing research as long as I could. His 2004 paper shows how active he was till the end.

It was at Bob's initiative that in 1964 I was invited to spend a couple of years at the University of Illinois at Urbana-Champaign. That visit had a profound effect on my intellectual life and professional career. I remain profoundly in Bob's debt.

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