On Best Equivariance and Admissibility Of Simultaneous MLE For Mean Direction Vectors Of Several Langevin Distributions.[1]

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Abstract.
The circular normal distribution, $CN(\mu, \kappa)$, plays a role for angular data comparable to that of a normal distribution for linear data. We establish that for the curved and for the regular exponential family situations arising when κ is known, and unknown respectively, the MLE $\widehat{\mu}$ of the mean direction μ is the best equivariant estimator. These results are generalised for the MLE $\underset{\sim}{\widehat{\mu}}$ of the mean direction vector $\underset{\sim}{\mu} = (\mu_1, \cdots, \mu_p)'$ in the simultaneous estimation problem with independent $CN(\mu_i, \kappa)$, $i = 1, \cdots, p$, populations. We further observe that $\underset{\sim}{\widehat{\mu}}$ is admissible both when κ is known or unknown. Thus unlike the normal theory, Stein effect does not hold for the circular normal case. This result is generalised for the simultaneous estimation problem with directional data in q-dimensional hyperspheres following independent Langevin distributions, $L(\underset{\sim}{\mu}_i, \kappa), i = 1, \cdots, p$.

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1 Introduction and Summary :

Directional data arise in several situations, notably astrophysics, atmospheric sciences, geology, meteorology, oceanography etc. The von Mises or circular normal distribution, $CN(\mu, \kappa)$ with mean direction parameter $\mu, 0 \leq \mu < 2\pi$, and concentration parameter $\kappa, \kappa > 0$, plays the role in circular data parallel to that of the normal distribution in linear data. A natural extension of the CN distribution to the distribution on a ℓ-dimensional hypersphere leads to the Fisher-von Mises or the Langevin distribution $L(\mu, \kappa)$. For ℓ=3, i.e. for spherical data, this distribution was studied by Fisher and is often termed as the Fisher distribution. For further discussions see Mardia (1972). Here we study the exact properties of best equivariance and admissibility of the maximum likelihood estimator (MLE) of the mean directions and the mean direction vectors in simultaneous estimation with several independent circular normal and Langevin distributions, all having the same concentration parameter.

Conventional linear results if and when applied for the analyses of angular data are to be viewed with caution and often lead to paradoxes. However, we observe that for κ known, $CN(\mu, \kappa)$ reduces to a member of a (1,2) curved exponential family (Amari, 1985). Then, exploiting the associated results of Kariya (1989), we establish that the MLE for μ is the Best Equivariant Estimator (BEE) under a natural (angular) loss. This result is extended (Result 1) to the case when κ is unknown. These results are finally generalised to establish (Theorem 1) the Best Equivariant nature of the MLE $\hat{\underset{\sim}{\mu}}$ of $\underset{\sim}{\mu} = (\mu_1, \cdots, \mu_p)'$ in simultaneous estimation with several independent $CN(\mu_i, \kappa), i = 1, ..., p$ populations for both κ known and unknown cases. Next, we consider the property of exact admissibility. We note, either directly from Bagchi and Guttman (1988), Zhong (1992) or as a special case of the results in Watson (1986), that as in the normal case, the MLE for μ in $CN(\mu, \kappa)$ for a single population is admissible. This follows, for example, from the Bayes character of the MLE for an uniform prior. Admissibility of the simultaneous MLE $\hat{\underset{\sim}{\mu}}$ for $\underset{\sim}{\mu} = (\mu_1, \cdots, \mu_p)'$ in p independent $CN(\mu_i, \kappa), i = 1, \cdots, p, \kappa$ known or unknown, then becomes a natural question. It is known that as $\kappa \longrightarrow \infty$, $\sqrt{\kappa}(\theta - \mu) \xrightarrow{\mathcal{L}} N(0, 1)$ for $\theta \sim CN(\mu, \kappa)$. So, e.g., by Brown's results one would then expect the simultaneous MLE to be inadmissible for $p \geq 3$, at least for large κ. However, we show that the MLE is in fact admissible for all p, all κ and all sample sizes. This is a marked departure from the usual normal theory: i.e., we establish here that unlike in the normal theory, the Stein effect does not hold here. This result (Corollary 2.1) follows from the general theorem (Theorem 2) where we establish the admissibility of the MLE of the mean direction vectors in simultaneous estimation with p independent Langevin distributions, $L(\underset{\sim i}{\mu}, \kappa), i = 1, \cdots, p$.

For the sake of completeness it may be worthwhile to make a few observations for the case when κ is the parameter of interest. Note that for μ known, say $\mu = 0$, $CN(0, \kappa)$ is a member of the one parameter regular exponential family (REF) with κ as its canonical parameter. Then taking the convex support of the uniform measure on the perimeter of the unit circle as compact (say $\kappa \leq K < \infty$) yields the canonical statistic, $\cos\theta$, as an admissible estimator for κ. But this is not at all a sensible estimator of κ, since $0 \leq \kappa < \infty$) - also see, e.g., the caution in Exercise 4.17.4, p.p. 136-137 of Brown (1986). Also $\underset{\sim}{\mu} = (0, \cdots, 0, 1, 0, \cdots, 0)'$ in $L(\underset{\sim}{\mu}, \kappa)$ results in a one-dimensional REF with κ as its canonical parameter and hence the preceding comments for the CN case thus hold here too. In general, the ℓ-dimensional $L(\mu, \kappa)$ is a member of the ℓ-dimensional REF and hence standard results for estimation,e.g., existence and uniqueness of MLE (Jupp and Mardia, 1979) etc., hold with respect to its canonical parameters. However for studying the MLE of its usual parameters μ and κ, it is necessary to consider its not so convenient (Brown, p.p. 76-78) mean value parametrisation. Further difficulties for such study may also be encountered – see e.g., p.150 and also Exercise 6 21.3, p.p. 204-205 of Brown. Thus the admissibility for and any possible improvement on the MLE in simultaneous estimation of p concentration parameters κ_i of $L(\underset{\sim i}{\mu}, \kappa_i), i = 1, \cdots, p$ seem to be interesting problems for future research.

2 Best Equivariant Property of the Simultaneous MLE $\underset{\sim}{\widehat{\mu}}$ in Several Independent CN(μ_i, κ)

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Let $\theta_1, \theta_2, \ldots, \theta_p$ be independent with $\theta_i \sim \; CN(\mu_i, \kappa), \kappa$ known.
Then the p.d.f. of $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)'$ is given by :

$$(2.1) \qquad f(\underset{\sim}{\theta} \mid \underset{\sim}{\mu}, \kappa) = [2\pi I_o(\kappa)]^{-p} \exp\{\kappa \sum_{i=1}^{p} \cos(\theta_i - \mu_i)\}; \quad \underset{\sim}{0} \le \underset{\sim}{\theta}, \underset{\sim}{\mu} < 2\pi \underset{\sim}{1}.$$

Let $\underset{\sim}{\theta}^{(1)}, \underset{\sim}{\theta}^{(2)}, \ldots, \underset{\sim}{\theta}^{(n)}$ be a random sample from $f(\underset{\sim}{\theta} \mid \underset{\sim}{\mu}, \kappa)$.
Then the joint density of $\underset{\sim}{\theta}^{(1)}, \underset{\sim}{\theta}^{(2)}, \ldots, \underset{\sim}{\theta}^{(n)}$ is given by

$$\begin{aligned} f(\underset{\sim}{\theta}^{(1)}, \underset{\sim}{\theta}^{(2)}, \ldots, \underset{\sim}{\theta}^{(n)} \mid \underset{\sim}{\mu}, \kappa) &= [2\pi I_o(\kappa)]^{-np} \exp\{\kappa \sum_{j=1}^{n} \sum_{i=1}^{p} \cos(\theta_i^{(j)} - \mu_i)\} \\ &= [2\pi I_o(\kappa)]^{-np} \exp\{n\kappa \sum_{i=1}^{p} (cos\mu_i \bar{C}_i + \sin\mu_i \bar{S}_i)\}, \end{aligned}$$

$$\text{where } \bar{C}_i = \frac{1}{n} \sum_{j=1}^{n} \cos\theta_i^{(j)}, \qquad \bar{S}_i = \frac{1}{n} \sum_{j=1}^{n} \sin\theta_i^{(j)}, \qquad 1 \le i \le p.$$

The MLE of $\underset{\sim}{\mu}$ is given by $\underset{\sim}{\widehat{\mu}}$, the solution to :

$$(2.2) \qquad \cos\mu_i = \frac{\bar{C}_i}{(\bar{C}_i^2 + \bar{S}_i^2)^{1/2}}, \quad \sin\mu_i = \frac{\bar{S}_i}{(\bar{C}_i^2 + \bar{S}_i^2)^{1/2}}, \qquad 1 \le i \le p.$$

2.1 Case 1: κ known.

We consider the problem of estimating $\underset{\sim}{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)'$ with a natural loss in the circular context. Let $C_i = \cos\theta_i$, and $S_i = \sin\theta_i$. Then, consider the group $\mathcal{G}$ acting on $\mathcal{Z} = \prod_{i=1}^{p} \{\underset{\sim}{y_i}; \;\; \| \underset{\sim}{y} \le \underset{\sim}{y_i} \| = 1\}$ given by
$\mathcal{G} = \{g_{A_i} : \underset{\sim}{X_i} \to A_i \underset{\sim}{X_i} \mid A_i : 2 \times 2 \text{ orthogonal}; \; i = 1, 2, \ldots, p\}$

$$= \{g_\tau : \underset{\sim}{X_i} \to A_i \underset{\sim}{X_i}, \quad A_i = \begin{pmatrix} \cos\tau_i & -\sin\tau_i \\ \sin\tau_i & \cos\tau_i \end{pmatrix}, \quad \tau_i \in [0, 2\pi); 1 \le i < p\}.$$

Let us first consider the case when $p = 1$, i.e. a single CN population. Suppose that $\theta_1, \theta_2, \cdots, \theta_n$ are a random sample from $CN(\mu, \kappa)$.

Then, $f(\theta_1, \theta_2, \cdots, \theta_n) = [2\pi I_0(\kappa)]^{-n} \exp\{n\kappa\bar{C}\cos\mu + n\kappa\bar{S}\sin\mu\}$
where $\bar{C} = \frac{1}{n}\sum_{i=1}^{n} \cos\theta_i, \qquad \bar{S} = \frac{1}{n}\sum_{i=1}^{n} \sin\theta_i$
.

Clearly, $(\bar{C}, \bar{S})$ is sufficient for μ. Let, $\eta_1 = \cos\mu$ and $\eta_2 = \sin\mu$. Then, from Mardia (1972), the joint distribution of $(\bar{C}, \bar{S})$ is given by

$$(2.3) \quad g(\bar{C}, \bar{S}) = [2\pi I_0^n(\kappa)]^{-1} n^2 \exp\{n\kappa(\bar{C}\eta_1 + \bar{S}\eta_2)\}\phi_n(n^2(\bar{C}^2 + \bar{S}^2)), \quad (\bar{C}, \bar{S}) \in \{\underset{\sim}{y} \mid \underset{\sim}{y} \in IR^2, 0 < \| \underset{\sim}{y} \| \le 1\}.$$

Then the distribution of $(\bar{C},\bar{S})$ belongs to the curved exponential family with $\Theta = \{\underset{\sim}{\theta} \in \tilde{\Theta} : \underset{\sim}{\theta} = \psi(\mu), \mu \in \Upsilon\}$ where $\tilde{\Theta} = IR^2$, $\Upsilon = [0, 2\pi]$ and $\psi : \Upsilon \longrightarrow \tilde{\Theta}$ defined by $\psi(\mu) = (\cos\mu, \sin\mu)$ is clearly a bimeasurable bijection onto $\psi(\gamma) = \Theta \subset \tilde{\Theta}$.

Consider the group $\mathcal{G}$ acting on $\mathcal{Z} = \{\underset{\sim}{y} \in IR^2, 0 \leq \| \underset{\sim}{y} \| \leq 1\}$ given by

$$\mathcal{G} = \{g_A : \underset{\sim}{X} \to A\underset{\sim}{X} \mid A : n \times n \text{ orthogonal with } | A |= 1\}$$

$$= \{g_\tau : \underset{\sim}{X} \to A\underset{\sim}{X},\ A = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}, \quad 0 \leq \tau < 2\pi\}$$

Then, $\mathcal{G}$ is a topological group and the group action on $\mathcal{Z}$ is measurable (being continuous). Further, the joint distribution of $(y_1, y_2) \equiv g_\tau(\bar{C},\bar{S})$ is given by

$$\begin{aligned} f(y_1,y_2) &= K(\kappa,n)\exp[n\kappa\{y_1(\eta_1\cos\tau - \eta_2\sin\tau) \\ &\quad + y_2(\eta_1\sin\tau + \eta_2\cos\tau)\}]\phi_n(n^2(y_1^2+y_2^2)), \end{aligned}$$

with $K(\cdot)$ being a constant.

So, $g\mathcal{P}(\tilde{\Theta}) = \mathcal{P}(\tilde{\Theta})$ with $gP_\theta = P_\theta.g^{-1}\ \forall g \in \mathcal{G}$ i.e. $\mathcal{P}(\tilde{\Theta})$ is invariant under $\mathcal{G}$. Further,

$\bar{g}_\tau = g_\tau$ (i.e., $\bar{g}_A = g_A$) so that $\bar{\mathcal{G}} = \mathcal{G}$. Also, $\mathcal{G}$ acts homeomorphically on $\mathcal{Z}$ by $\begin{pmatrix} \bar{C} \\ \bar{S} \end{pmatrix} \to g_\tau \begin{pmatrix} \bar{C} \\ \bar{S} \end{pmatrix}$.

Defining $\tilde{g}_\tau = \psi^{-1}\ \bar{g}_\tau\ \psi$, we have, $\tilde{g}_\tau(\mu) = \psi^{-1}\bar{g}_\tau\ \psi(\mu) = \psi^{-1}\bar{g}_\tau \begin{pmatrix} \cos\mu \\ \sin\mu \end{pmatrix}$

$$= \psi^{-1}\begin{bmatrix} \cos\mu\ \cos\tau - \sin\mu\ \sin\tau \\ \cos\mu\ \sin\tau + \sin\mu\ \cos\tau \end{bmatrix} = (\mu + \tau) \bmod 2\pi.$$

Defining $\tilde{\mathcal{G}}$ acting on Υ by $\tilde{\mathcal{G}} = \{\tilde{g}_\tau(\theta) = (\theta+\tau) \bmod 2\pi, 0 \leq \tau < 2\pi\}$, we have $\tilde{\mathcal{G}}$ is a homeomorphic image of $\bar{\mathcal{G}}$ (and hence of $\mathcal{G}$) and the subfamily $\mathcal{P}(\Theta) = \{P_{\psi(\eta)} \mid \eta \in \Upsilon\}$ is $\tilde{\mathcal{G}}-$invariant.

It is clear that the orbit of $\bar{\mathcal{G}}$ is Θ so that the action of $\bar{\mathcal{G}}$ on Θ is transitive. It follows easily then that the action of $\tilde{\mathcal{G}}$ on Υ is transitive.

<u>Lemma 2.1</u> $R = u(\bar{C},\bar{S}) = (\bar{C}^2+\bar{S}^2)^{1/2}$ is a maximal invariant statistic under $\mathcal{G}$.

<u>Proof.</u> We first show that $u(\bar{C},\bar{S})$ is $\mathcal{G}$-invariant.

$$\begin{aligned} u(g_\tau(\bar{C},\bar{S})) &= [(g_\tau(\bar{C},\bar{S}))'(g_\tau(\bar{C},\bar{S}))]^{1/2} \\ &= [(\bar{C},\bar{S})'A'A(\bar{C},\bar{S})]^{1/2} = [(\bar{C},\bar{S})'(\bar{C},\bar{S})]^{1/2} \\ &= u(\bar{C},\bar{S}). \end{aligned}$$

Next, suppose that $u(\bar{C}_1,\bar{S}_1) = u(\bar{C}_2,\bar{S}_2)$. Then $R_1^2 = \bar{C}_1^2 + \bar{S}_1^2 = \bar{C}_2^2 + \bar{S}_2^2 = R_2^2$.

If $R_1 = R_2 = 0$, then $\bar{C}_1 = \bar{C}_2 = \bar{S}_1 = \bar{S}_2 = 0$ so that $g_o(\bar{C}_1,\bar{S}_1) = (\bar{C}_2,\bar{S}_2)$.

So, now suppose that $R_1 = R_2 > 0$. Let $P_i = \frac{1}{R_i}\begin{bmatrix} \bar{C}_i & \bar{S}_i \\ -\bar{S}_i & \bar{C}_i \end{bmatrix}, i = 1,2.$

Then P_1, P_2 are orthogonal matrices with determinant 1 and

$$P_1\begin{pmatrix} \bar{C}_1 \\ \bar{S}_1 \end{pmatrix} = (u(\bar{C}_1,\bar{S}_1)) = (u(\bar{C}_2,\bar{S}_2)) = P_2\begin{pmatrix} \bar{C}_2 \\ \bar{S}_2 \end{pmatrix}.$$

So, $\begin{pmatrix} \bar{C}_1 \\ \bar{S}_1 \end{pmatrix} = P_1'P_2\begin{pmatrix} \bar{C}_2 \\ \bar{S}_2 \end{pmatrix}$; $P_1'P_2 = \frac{1}{R_1R_2}\begin{bmatrix} \bar{C}_1\bar{C}_2+\bar{S}_2\bar{S}_1 & \bar{C}_1\bar{S}_2-\bar{S}_1\bar{C}_2 \\ \bar{S}_1\bar{C}_2-\bar{C}_1\bar{S}_2 & \bar{C}_1\bar{C}_2+\bar{S}_2\bar{S}_1 \end{bmatrix}$

Get τ such that $\cos\tau = \frac{\bar{C}_1\bar{C}_2+\bar{S}_2\bar{S}_1}{R_1R_2}$, $\sin\tau = \frac{-\bar{C}_1\bar{S}_2+\bar{S}_1\bar{C}_2}{R_1R_2}$.

Then $g_\tau \begin{pmatrix} \bar{C}_1 \\ \bar{S}_1 \end{pmatrix} = \begin{pmatrix} \bar{C}_2 \\ \bar{S}_2 \end{pmatrix}$, so that $u(\bar{C}, \bar{S}) = R$ is a maximal invariant under $\mathcal{G}$.

Lemma 2.2. Assumptions 2.1 and 2.2 of Kariya (1989) hold in our above set-up with the $CN(\mu, \kappa), \kappa$ known, model.

Proof : Note that $\lambda(\eta) = \| \eta \|$ is a maximal invariant parameter under $\mathcal{G}$. So Θ in (2.3) may be expressed as $\Theta = \{\theta \in \tilde{\Theta} \mid \lambda(\theta) = 1\}$. Further, the map $g_\tau \to \bar{g}_\tau \equiv g_\tau$ is measurable. So, Assumption (2.1) of Kariya is satisfied.

The MLE of μ is given by $\widehat{\mu}(\bar{C}, \bar{S})$, where $\widehat{\mu}(\bar{C}, \bar{S})$ is the solution to

$$\cos \mu = \frac{\bar{C}}{\| (\bar{C}, \bar{S}) \|}, \quad \sin \mu = \frac{\bar{S}}{\| (\bar{C}, \bar{S}) \|}. \tag{2.4}$$

It may be noted that excluding the set $\{\{0\} \times [-1, 1]\} \bigcup \{[-1, 1] \times \{0\}\}$ of measure zero on $\mathcal{Z}$, $\widehat{\mu}(\bar{C}, \bar{S})$ defines a bijection from $\mathcal{Z}$ onto $[0, 2\pi)$.

Define, $h(\bar{C}, \bar{S}) = \frac{1}{\|(\bar{C}, \bar{S})\|} \begin{bmatrix} \bar{C} & -\bar{S} \\ \bar{S} & \bar{C} \end{bmatrix}$. Then,

$$h(g_\tau(\bar{C}, \bar{S})) = \frac{1}{\|(\bar{C}, \bar{S})\|} \begin{bmatrix} \bar{C} \cos \tau - \bar{S} \sin \tau & -\bar{C} \cos \tau - \bar{C} \sin \tau \\ \bar{C} \sin \tau + \bar{S} \cos \tau & \bar{C} \cos \tau - \bar{S} \sin \tau \end{bmatrix} = g_\tau(h(\bar{C}, \bar{S})).$$

Define, $\pi(\bar{C}, \bar{S}) = (h(\bar{C}, \bar{S}), u(\bar{C}, \bar{S}))$. Then π is a continuous map defined from $\mathcal{Z}$ onto $\mathcal{G} \times \mathcal{U}$ where $\mathcal{U} = [0, 1]$ is a measurable space. Let $\pi(\bar{C}_1, \bar{S}_1) = \pi(\bar{C}_2, \bar{S}_2)$. Then,

$\frac{1}{\|(\bar{C}_1, \bar{S}_1)\|} \begin{bmatrix} \bar{C}_1 & -\bar{S}_1 \\ \bar{S}_1 & \bar{C}_1 \end{bmatrix} = \frac{1}{\|(\bar{C}_2, \bar{S}_2)\|} \begin{bmatrix} \bar{C}_2 & -\bar{S}_2 \\ \bar{S}_2 & \bar{C}_2 \end{bmatrix}$ and $\| (\bar{C}_1, \bar{S}_1) \| = \| (\bar{C}_2, \bar{S}_2) \|$.

Then $\bar{C}_1 = \bar{C}_2$ and $\bar{S}_1 = \bar{S}_2$, so that π is injective.

Next, we show that π is onto. Let $a \in \mathcal{G} \times \mathcal{U}$.

Then write $a = \left(\begin{bmatrix} x & \pm\sqrt{1 - x^2} \\ \mp\sqrt{1 - x^2} & x \end{bmatrix}, u \right)$. So, $\pi(ux, \mp u\sqrt{1 - x^2}) = a$ so that π is surjective.

Writing a as above, define $\pi^{-1} : \mathcal{G} \times \mathcal{U} \longrightarrow \mathcal{Z}$ as: $\pi^{-1}(a) = (ux, \mp u\sqrt{1 - x^2})$.

Then it follows, imitating the steps for the proof of the injectivity of π, that π^{-1} is well-defined.

We next show that π is continuous.

Let the metric d on $\mathcal{Z}$ be the usual one, i.e., for $\underset{\sim}{x}, \underset{\sim}{y} \in \mathcal{Z}$, $d(\underset{\sim}{x}, \underset{\sim}{y}) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2} = \| \underset{\sim}{x} - \underset{\sim}{y} \|$.

Viewing $\mathcal{G}$ as a subset of $I\!R^4$ we define the metric in the natural way on $\mathcal{G} \times \mathcal{U}$ by ρ, where

$$\rho(((\underset{\sim 1}{a}, \underset{\sim 1}{b}), u_1), (\underset{\sim 2}{a}, \underset{\sim 2}{b}), u_2)) = \{\| \underset{\sim 1}{a} - \underset{\sim 2}{a} \|^2 + \| \underset{\sim 1}{b} - \underset{\sim 2}{b} \|^2 + (u_1 - u_2)^2\}^{1/2}$$

$$= \{2 \| \underset{\sim 1}{a} - \underset{\sim 2}{a} \|^2 + (u_1 - u_2)^2\}^{1/2}, \text{ since } (\underset{\sim 1}{a}, \underset{\sim 1}{b}) \in \mathcal{G} \Rightarrow \underset{\sim}{b} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \underset{\sim}{a}.$$

Fix $\epsilon > 0$. For $(\bar{C}_1, \bar{S}_1) \in \mathcal{Z}$, by the continuity of $\frac{1}{\sqrt{1 + \left(\frac{\bar{S}}{\bar{C}}\right)^2}}, \frac{1}{\sqrt{1 + \left(\frac{\bar{C}}{\bar{S}}\right)^2}}$ and $(\bar{C}^2 + \bar{S}^2)^{1/2}, \exists \delta > 0$ with $(\bar{C}_1 - \bar{C}_2)^2 + (\bar{S}_1 - \bar{S}_2)^2 < \delta, \ni \left(\frac{\bar{C}_1}{R_1} - \frac{\bar{C}_2}{R_2}\right)^2 < \frac{\epsilon^2}{8}, \left(\frac{\bar{S}_1}{R_1} - \frac{\bar{S}_2}{R_2}\right)^2 < \frac{\epsilon^2}{8}$, and $(R_1 - R_2)^2 < \frac{\epsilon^2}{2}$. So, $\rho(\pi(\bar{C}_1, \bar{S}_1), \pi(\bar{C}_2, \bar{S}_2)) < \left(\frac{\epsilon^2}{8} \times 4 + \frac{\epsilon^2}{2}\right)^{1/2} = \epsilon$ for $d((\bar{C}_1, \bar{S}_1), (\bar{C}_2, \bar{S}_2)) < \delta$.

Hence, $\pi : \mathcal{Z} \longrightarrow \mathcal{G} \times \mathcal{U}$ is a continuous function. It follows that π^{-1} is also continuous.

So, there exists a bijective, bimeasureable map from $\mathcal{Z}$ onto $\mathcal{G} \times \mathcal{U}$ such that if $\pi(z) = (h(z), u(z))$, then $\pi(gz) = (gh(z), u(z))$, where $z = (\bar{C}, \bar{S}) \in \mathcal{Z}$ and $\mathcal{U}$ is a measurable space.

Hence, Assumption (2.2) of Kariya is satisfied. □

Now, let the loss function be given by

$$L(a,\mu) = 1 - \cos(a-\mu), \qquad 0 \le a, \mu < 2\pi.$$

Then, from Theorem 2.1 of Kariya it follows that a best equivariant estimator, when it exists is,

$$\widehat{\mu}(h(\bar{C},\bar{S}), u(\bar{C},\bar{S})) = \tilde{h}(\bar{C},\bar{S})\widehat{\mu}_1(e, u(\bar{C},\bar{S})) = \widehat{\mu}(\bar{C},\bar{S}) + \mu_1^*(u(\bar{C},\bar{S}))$$

where $\widehat{\mu}(\bar{C},\bar{S})$ is as defined in (2.4) and μ_1^* minimises the conditional expectation,

$$E_\mu[1-\cos(\widehat{\mu}+\mu_1(u(\bar{C},\bar{S}))-\mu) \mid u(\bar{C},\bar{S})] = E_0[1-\cos(\widehat{\mu}+\mu_1) \mid u(\bar{C},\bar{S})], \tag{2.5}$$

by the transitivity of $\mathcal{G}$.

To minimise the above w.r.t. μ_1 observe that μ_1^* satisfies the equation,

$$E_0[\sin(\widehat{\mu}+\mu_1^*) \mid u(\bar{C},\bar{S})] = 0.$$

Thus, $\sin\mu_1^* E_0[\cos\widehat{\mu} \mid u(\bar{C},\bar{S})] = 0$, since from Mardia (1972), $\widehat{\mu} \mid (\bar{C}^2+\bar{S}^2)^{1/2} \sim CN(0, \kappa(\bar{C}^2+\bar{S}^2)^{1/2})$, and for $\theta \sim CN(0,\kappa), E(\sin\theta) = 0$.
So, $\sin\mu_1^* A(\kappa(\bar{C}^2+\bar{S}^2)^{1/2}) = 0$, yielding $\mu_1^* = 0$ or π.

Now, $E_0[\cos(\widehat{\mu}+\pi) \mid u(\bar{C},\bar{S})] < 0$, while $E_0[\cos\widehat{\mu} \mid u(\bar{C},\bar{S})] > 0$, so that (2.5) is minimised for $\mu_1^* = 0$.

Consequently, the MLE $\widehat{\mu}$ in (2.4) is the best equivariant estimator.

Further, $\mathcal{G}$ being compact, the MLE $\widehat{\mu}$ is minimax in the class $\mathcal{D}$ of all estimators, (Ferguson, 1967). Further, it is also admissible in the class $\mathcal{D}$ of all estimators for μ.

2.2 Case 2: κ unknown

It may be noted that the distribution of $(\bar{C},\bar{S})$ belongs to a regular exponential family (REF) with $\Theta = \{(\mu,\kappa), \mu \in [0,2\pi), \kappa \ge 0\}$. Then, as before, consider the group $\mathcal{G}$ acting on $\mathcal{Z} = \{\underset{\sim}{y} \in I\!R^2, 0 \le \| \underset{\sim}{y} \| \le 1\}$ given by

$$\begin{aligned} \mathcal{G} &= \{g_A : \underset{\sim}{X} \to A\underset{\sim}{X} \mid A : n\times n \text{ orthogonal with } \mid A \mid = 1\} \\ &= \{g_\tau : \underset{\sim}{X} \to A\underset{\sim}{X}, A = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}, \ 0 \le \tau < 2\pi\}. \end{aligned}$$

Proceeding as before, we obtain the joint distribution of $(y_1,y_2) \equiv g_\tau(\bar{C},\bar{S})$ as

$$f(y_1,y_2) \;=\; \text{const. } (\kappa,n)\exp[n\kappa\{y_1(\eta_1\cos\tau - \eta_2\sin\tau) + y_2(\eta_1\sin\tau+\eta_2\cos\tau)\}]\phi_n(n^2(y_1^2+y_2^2)).$$

So, $g\mathcal{P}(\Theta) = \mathcal{P}(\Theta)$ with $gP_\theta = P_\theta.g^{-1}\forall g \in \mathcal{G}$ i.e. $\mathcal{P}(\Theta)$ is invariant under $\mathcal{G}$. Also, the induced group action on the parameter space Θ is given by $g_\tau(\mu,\kappa) = ((\mu+\tau) \mod 2\pi, \kappa)$. This shows that the induced group of transformations $\overline{\mathcal{G}}$ acting on the parameter space is not transitive. The same arguments as before provide us with $u(\overline{C},\overline{S}) = (\overline{C}^2+\overline{S}^2)^{1/2}$ as the maximal invariant statistic. Also, it can be shown, following the arguments similar to the ones for the case κ known, that the MLE of μ, there also given by $\widehat{\mu}(\overline{C},\overline{S})$ (in eqn. (2.4)), is equivariant under the group $\mathcal{G}$. To find the best equivariant estimator under the natural loss, we are to find μ_1, as a measurable function of the maximal invariant, $u(\overline{C},\overline{S})$ such that the risk, $E_{\mu,\kappa}[1-\cos(\widehat{\mu}+\mu_1(u(\overline{C},\overline{S}))-\mu) \mid u(\overline{C},\overline{S})]$ is minimized uniformly for all $(\mu,\kappa) \in \Theta$. To minimize the above risk w.r.t. μ_1, observe that μ_1^* satisfies the equation

$$E_{\mu,\kappa}[\sin(\widehat{\mu}+\mu_1^*-\mu) \mid u(\overline{C},\overline{S})] = 0.$$

Thus, $\sin\mu_1^* E_{\mu,\kappa}[\sin(\widehat{\mu}-\mu) \mid u(\overline{C},\overline{S})] = 0$, since from Mardia (1972), $\widehat{\mu} \mid (\overline{C}^2+\overline{S}^2)^{1/2} \sim CN(\mu, \kappa(\overline{C}^2+\overline{S}^2)^{1/2})$, and for $\theta \sim CN(\mu,\kappa), E(\sin(\theta-\mu)) = 0$. Hence, $\sin\mu_1^* A(\kappa(\overline{C}^2+\overline{S}^2)^{1/2}) = 0$, yielding $\mu_1^* = 0$ or π. Now, $E_{\mu,\kappa}[\cos(\widehat{\mu}-\mu+\pi) \mid u(\overline{C},\overline{S})] < 0$, while $E_{\mu,\kappa}[cos(\widehat{\mu}-\mu) \mid u(\overline{C},\overline{S})] > 0$, so that the risk is uniformly minimized for $\mu_1^* = 0$. This leads us to conclude that the MLE is the Best Equivariant Estimator of the mean direction, under the given natural loss, even when the concentration parameter is unknown.

Result 1 The MLE $\widehat{\mu}$ in (2.4) for μ in $CN(\mu,\kappa)$ population, κ known or unknown, is the Best Equivariant, Admissible and Minimax estimator in the class of all estimators for μ.

Let us now consider the simultaneous estimation problem. Following the same arguments as for the case $p=1$ above, the induced action on the parameter space is given by $\bar{g}\,\underset{\sim}{\mu} = \{(\mu_i+\tau_i) \bmod 2\pi,\ i=1,2,\ldots,p\}$. Further, the action is transitive with the parameter space as the orbit. It is easy to see that the MLE for $\underset{\sim}{\mu}$ is given by $\underset{\sim}{\widehat{\mu}}$ and is an equivariant estimator for $\underset{\sim}{\mu}$. Imitating the steps in Section 2, we get the constant vector to be a maximal invariant statistic. Then, the only equivariant estimators for $\underset{\sim}{\mu} = (\mu_1,\mu_2,\ldots,\mu_p)'$ are of the form $\delta(\underset{\sim}{\theta}) = ((\theta_1+c_1) \bmod 2\pi, (\theta_2+c_2) \bmod 2\pi, \ldots, (\theta_p+c_p) \bmod 2\pi)'$. To find the best equivariant estimator under the natural loss function,

$$L(\underset{\sim}{\mu},\underset{\sim}{a}) = p - \sum_{i=1}^{p}\cos(\mu_i - a_i)$$

we look for the equivariant estimator having minimum risk. To minimize the risk, we are to find the values of $c_1, c_2, \ldots, c_p$ such that the risk,

$E[p - \sum_{i=1}^{p}\cos(\mu_i - (\widehat{\mu}_i + c_i) \bmod 2\pi)] = E[p - \sum_{i=1}^{p}\cos(\mu_i - \widehat{\mu}_i - c_i)]$ is minimized. This gives us, following steps similar to those used above for $p=1$, that the minimum risk is achieved when $c_i = 0$; $i = 1,2,\ldots,p$. Thus, combining cases 1 and 2 as in result 1 above, we get,

Theorem 1. The MLE $\underset{\sim}{\widehat{\mu}}$ of $\underset{\sim}{\mu}$ in the simultaneous estimation problem with p independent CN $(\mu_i,\kappa), i = 1,\cdots,p$, κ known or unknown, is the Best Equivariant Estimator.

Remarks 2.1 Generalization of theorem 1 above to several independent $L(\mu_i,\kappa)$ populations seems to be non-trivial. For example, even with a single Langevin population and even with κ known, the approach of Kariya seems to fail when one tries to find the BEE of the mean direction vector. We are able to claim only that provided a BEE exists, the MLE, being both admissible and equivariant, is the BEE. However, it has not been possible to confirm or deny the existence of the BEE.

3 Admissibility of the MLE in Several Independent $L(\mu_i,\kappa)$.

3.1 Case 1: κ known.

Let $\Theta^{(1)}, \Theta^{(2)}, \ldots, \Theta^{(n)}$ be a random sample of size n from p independent populations of l-dimensional Langevin (or von-Mises-Fisher) distributions $L(\underset{\sim i}{\mu},\kappa), i = 1,\ldots,p$. Then, the random variables are matrix-valued and are from,

$$f(\Theta \mid \mathcal{M},\kappa) = a_\ell{}^{-p}(\kappa)\exp\{\kappa\sum_{i=1}^{p}\underset{\sim}{u}'(\underset{\sim i}{\theta})\underset{\sim}{u}(\underset{\sim}{\mu})\}\prod_{i=1}^{p}\prod_{j=1}^{\ell-1}\sin^{\ell-j}\theta_{i(j-1)},$$

$$\Theta = (\underset{\sim 1}{\theta},\ldots,\underset{\sim p}{\theta}); \mathcal{M} = (\underset{\sim 1}{\mu},\ldots,\underset{\sim p}{\mu}),$$

$$0 < \theta_{ij}, \mu_{ij} < \pi, \quad j = 1, 2, \ldots, , \ell - 2; i = 1, 2, \ldots, p,$$

$0 < \theta_{i(\ell-1)}, \mu_{i(\ell-1)} < 2\pi, \quad i = 1, 2, \ldots, p$, and $\underset{\sim}{u}(\underset{\sim}{x})$ is such that

$$\begin{aligned}
u_1(\underset{\sim}{x}) &= \cos x_1 \\
u_j(\underset{\sim}{x}) &= \cos x_j \prod_{m=1}^{j-1} \sin x_m, \quad 2 \leq j \leq \ell - 1 \\
u_\ell(\underset{\sim}{x}) &= \prod_{m=1}^{\ell-1} \sin x_m, \qquad 0 < x_i < \pi, i = 1, 2, \ldots, \ell - 2; \quad 0 < x_{\ell-1} < 2\pi.
\end{aligned}$$

Then, as before, the likelihood function of $\mathcal{M}$, given the observations is

$$L(\mathcal{M} \mid \Theta^{(1)}, \Theta^{(2)}, \ldots, \Theta^{(n)}) = a_\ell{}^{-np}(\kappa)[\exp\{n\,\kappa \sum_{i=1}^{p} \underset{\sim}{\bar{u}}'(\underset{\sim i}{\theta})\, \underset{\sim}{u}(\underset{\sim i}{\mu})\} \prod_{i=1}^{p} \prod_{k=1}^{n} \prod_{j=1}^{\ell-1} \sin^{\ell-j} \theta_{i(j-1)}^{(k)}],$$

where

$$\underset{\sim}{\bar{u}}(\underset{\sim i}{\theta}) = \frac{1}{n} \sum_{k=1}^{n} \underset{\sim}{u}(\underset{\sim}{\theta}{}_i^{(k)}), \quad i = 1, 2, \ldots, p.$$

The MLE of $\mathcal{M}$ is given by $\widehat{\mathcal{M}}$, the solution to the system of equations

$$\frac{\partial}{\partial \mu_{ij}} \underset{\sim}{u}'(\underset{\sim i}{\mu}) \underset{\sim}{\bar{u}}(\underset{\sim i}{\theta}) = 0 \text{ and } \| \underset{\sim}{u}(\underset{\sim i}{\widehat{\mu}}) \| = 1, \qquad 1 \leq i \leq p. \tag{3.1}$$

Let $\mathcal{M}$ have the prior density

$$\pi(\mathcal{M}) = K \prod_{i=1}^{p} \prod_{j=2}^{\ell-1} \sin^{\ell-j} \mu_{i(j-1)}, \quad 0 < \mu_{ij} < \pi, 1 \leq j \leq \ell - 2, 0 < \mu_{i(\ell-1)} < 2\pi, 1 \leq i \leq p,$$

where the constant K is such that

$$\int \pi(\mathcal{M}) d\mu_1 \ldots d\mu_{p(\ell-1)} = 1. \tag{3.2}$$

Then, the posterior distribution of $\mathcal{M}$ is given by,

$$\pi(\mathcal{M} \mid \Theta^{(1)}, \Theta^{(2)}, \ldots, \Theta^{(n)}; \kappa) = \prod_{i=1}^{P} \left[a_\ell^{-1}(n^\kappa \| \underset{\sim}{\bar{u}}(\underset{\sim i}{\theta}) \|) \right] \left[\exp\{n\,\kappa \underset{\sim}{u}'(\underset{\sim i}{\mu})\, \underset{\sim}{\bar{u}}(\underset{\sim i}{\theta})\} \prod_{j=2}^{\ell-1} \sin^{\ell-j} \mu_{i(j-1)} \right].$$

Then, given the observations and κ, $\underset{\sim 1}{\mu}, \underset{\sim 2}{\mu}, \cdots, \underset{\sim p}{\mu}$ are independent with the posterior distribution of $\underset{\sim i}{\mu}$ as,

$$\pi(\underset{\sim i}{\mu} \mid \Theta^{(1)}, \Theta^{(2)}, \ldots, \Theta^{(n)}, \kappa) \sim \mathcal{L}_\ell \left(\frac{\underset{\sim}{\bar{u}}(\underset{\sim i}{\theta})}{\| \underset{\sim}{\bar{u}}(\underset{\sim i}{\theta}) \|}, n\kappa \| \underset{\sim}{\bar{u}}(\underset{\sim i}{\theta}) \| \right), i = 1, \ldots, p.$$

Under the loss function,

$$L(\mathcal{M}, A) = p - \sum_{i=1}^{p} \underset{\sim}{u}'(\underset{\sim i}{\mu})\, \underset{\sim}{u}(\underset{\sim i}{a}), \qquad A = (\underset{\sim 1}{a}, \ldots, \underset{\sim p}{a}), \tag{3.3}$$

the posterior Bayes risk for a decision rule $\triangle$ is given by

$$r(\pi,\triangle) = EE(p - \sum_{i=1}^{p} \underset{\sim}{u}'(\underset{\sim i}{\mu})\, \underset{\sim}{u}(\underset{\sim i}{\delta})|\Theta^{(1)},\ldots,\Theta^{(n)};\kappa), \qquad \triangle = (\underset{\sim 1}{\delta},\ldots,\underset{\sim p}{\delta}).$$

To minimize $r(\pi,\triangle)$ it is enough to minimize

$$E(p - \sum_{i=1}^{p} \underset{\sim}{u}'(\underset{\sim i}{\delta})\, \underset{\sim}{u}(\underset{\sim i}{\mu}) \mid \Theta^{(1)},\ldots,\Theta^{(n)};\kappa) \qquad = p - \sum_{i=1}^{p} \underset{\sim}{u}'(\underset{\sim i}{\delta}) E^{\pi}[\underset{\sim}{u}(\underset{\sim i}{\mu})|\Theta^{(1)},\ldots,\Theta^{(n)};\kappa]$$

$$= p - \sum_{i=1}^{p} \underset{\sim}{u}'(\underset{\sim i}{\delta}) A_{\ell}(n\kappa \parallel \bar{u}(\underset{\sim i}{\theta}) \parallel) \underset{\sim}{u}'(\delta_i) \frac{\underset{\sim}{\bar{u}}(\underset{\sim i}{\theta})}{\parallel \underset{\sim}{\bar{u}}(\underset{\sim i}{\theta}) \parallel}.$$

To obtain the Bayes estimator, it is enough to obtain the rule minimizing the above. We thus have the first order conditions

$$\frac{\partial}{\partial \delta_{ij}} \underset{\sim}{u}'(\delta_i) \underset{\sim}{\bar{u}}(\underset{\sim i}{\theta}) = 0 \qquad \text{and} \qquad \parallel \underset{\sim}{u}(\underset{\sim i}{\delta}) \parallel = 1.$$

So the Bayes estimator is given by,

$$\triangle^B = (\underset{\sim 1}{\delta}^B,\ldots,\underset{\sim p}{\delta}^B) \qquad \text{where } \underset{\sim i}{\delta}^B \text{ satisfies } \underset{\sim}{u}(\delta_i) = \frac{\underset{\sim}{\bar{u}}(\underset{\sim i}{\theta})}{\parallel \underset{\sim}{\bar{u}}(\underset{\sim i}{\theta}) \parallel}.$$

Thus $\triangle^B$ satisfies (3.1) and hence, the MLE of $\mathcal{M}$ is the unique Bayes estimator w.r.t. the prior (3.2). So the MLE is admissible.

3.2 Case 2: κ unknown.

As before, here also the MLE for $\mathcal{M}$ is given by $\widehat{\mathcal{M}}$ which satisfies the equations (3.1). Let
(3.4)

$$\pi(\mathcal{M},\kappa) \propto \prod_{i=1}^{p}\prod_{j=2}^{\ell-1} sin^{\ell-j}\, \mu_{i(j-1)}\, I(\kappa=\kappa_o), \qquad 0<\mu_{ij}<\pi, 1\le j\le \ell-2,\ 0<\mu_{i(\ell-1)} < 2\pi, 1\le i\le p.$$

Then, the posterior density of $\mathcal{M}$ and κ is given by

$$\pi(\mathcal{M},\kappa \mid \Theta^{(i)}, 1\le i\le n) \propto \prod_{i=1}^{p}\prod_{j=2}^{\ell-1} \sin^{\ell-j} \mu_{i(j-1)} \exp\{n\kappa \underset{\sim}{\bar{u}}'(\underset{\sim i}{\theta}) \underset{\sim}{u}(\mu_i)\} a_{\ell}^{-1}(n\kappa \parallel \underset{\sim}{\bar{u}}(\theta_i) \parallel).I(\kappa=\kappa_o),$$

$$0<\mu_{ij}<\pi, 1\le j\le \ell-2, 0<\mu_{i(\ell-1)}<2\pi; 1\le i\le p.$$

Then, under the loss (3.3), the posterior Bayes risk is given by,

$$r(\pi,\triangle) = E[p - \sum_{i=1}^{p} \underset{\sim}{u}'(\underset{\sim i}{\delta}) E\{\underset{\sim}{u}(\underset{\sim i}{\mu})|\Theta^{(1)},\cdots,\Theta^{(n)}\}]$$

To minimize $r(\pi,\triangle)$, it is enough to minimize,

$$E_1^{\kappa}[p - \sum_{i=1}^{p} \underset{\sim}{u}'(\underset{\sim i}{\delta}) E\{\underset{\sim}{u}(\underset{\sim i}{\mu})|\Theta^{(1)},\Theta^{(2)},\cdots,\Theta^{(n)},\kappa_0\}] = p - \sum_{i=1}^{p} \underset{\sim}{u}'(\underset{\sim i}{\delta}) E[\underset{\sim}{u}(\underset{\sim i}{\mu})|\Theta^{(1)},\cdots,\Theta^{(n)},\kappa_0].$$

This is the same as minimizing the above for κ known and equal to κ_0. Then, the Bayes estimator for $\mathcal{M}$ w.r.t. the prior (3.4) is given by,

$$\triangle(\Theta^{(1)}, \cdots, \Theta^{(n)}) = \widehat{\mathcal{M}}, \text{ the MLE .}$$

Thus, we get

Theorem 2. For $p \geq 1, \kappa$ known or unknown, the MLE of $\mathcal{M} = (\underset{\sim i}{\mu}, \cdots, \underset{\sim p}{\mu})$ in p independent, ℓ-dimensional, $\ell \geq 2$, Langevin populations, $L(\underset{\sim i}{\mu}, \kappa), i = 1, \cdots, p$, is admissible.

Corollary 2.1. For $p \geq 1, \kappa$ known or unknown, the MLE of $\underset{\sim}{\mu} = (\mu_1, \ldots, \mu_p)'$ in p independent Circular Normal populations, $CN(\mu_i, \kappa), i = 1, \ldots, p$, is admissible.

Remarks 3. (i) The above corollary is a special case of Theorem 2 for $\ell = 2$. This also follows directly (SenGupta and Maitra, 1994) by taking an uniform prior on $\underset{\sim}{\mu}$.

(ii) In particular, for the case $n = 1$, we have $\underset{\sim}{\theta}$ is admissible for $\underset{\sim}{\mu}$ where $\underset{\sim}{\theta}$ is an observation from (2.1). But, from Mardia we have,

$$\sqrt{\kappa}(\underset{\sim}{\theta} - \underset{\sim}{\mu}) \xrightarrow{\mathcal{L}} N(\underset{\sim}{0}, \mathrm{I}) \text{ as } \kappa \to \infty.$$

By Brown's results or otherwise, one would then expect, intuitively, the MLE to be inadmissible, at least for large κ. However, our result holds for all κ, in particular for large κ also; something that runs counter to our intuition. The authors are thankful to Prof. S.R. Jammalamadaka (formerly, J.S. Rao) for drawing their attention to this interesting point.

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