ON THE ESTIMATION OF PARAMETERS IN A RECURSIVE SYSTEM

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Introduction

The problem was suggested by Prof. H. Wold when he came to the Indian Statistical Institute as a visiting professor. This paper is exclusively devoted to model sampling experiments. Here, we may consider a process of the following type:

$$x_t^{(j)} = F_j\,(x_t^{(1)}, x_{t-1}^{(1)}, \ldots, x_t^{(2)}, x_{t-1}^{(2)}, \ldots, x_t^{(n)}, x_{t-1}^{(n)}, \ldots) + \xi_t^{(j)} \qquad \ldots \quad (1)$$

$$j = 1, 2, \ldots, n; \; t = 1, 2, \ldots, T$$

where $x_t^{(j)}$ denotes evariable $x^{(j)}$ as observed during the time-period t, $\xi_t^{(j)}$ are random disturbances, and the system is recursive of the type considered by Wold in the opening paper of this volume. As for example, the following is a recursive system of the type described above:

$$x_t^{(1)} = \frac{k}{x_t^{(2)}} + \xi_t^{(1)},$$

$$x_t^{(2)} = (a+bt)\{x_{t-1}^{(3)}\}^2 + \xi_t^{(2)},$$

$$x_{t+1}^{(3)} = x_t^{(3)} + c(x_t^{(1)} - x_t^{(2)}) + \xi_t^{(3)}$$

where k, a, b and c are constants. This type of process may occur in economic problems (Bentzel & Wold, 1946). Doubts have been expressed whether in this situation estimates of parameters of one or more of the relations can be obtained by the method of least squares. This problem was to some extent considered by Bentzel and Wold (1946) who thought that the least square estimation is permissible when the system of equations is recursive. To examine how far the least square method is successful, samples were constructed with known models and then the constants were derived for comparison with known true values.

Models and the sample series

The models considered here are of the type :

$$d_t = d(p_t) + \xi_t$$
$$s_t = s(p_{t-1}) + \eta_t$$
$$p_t = p(p_{t-1}, d_{t-1}, s_{t-1}) + \zeta_t\,;$$

where d_t, s_t and p_t denote demand, supply and price respectively at a particular time-point t. Then if the price-series be evolutive, but the demand and supply series be stationary, we may estimate the parameters of the functions $d(p_t)$ and $s(p_t)$ by the least-square method. But, if one of the latter two also be evolutive, we may estimate the parameters of the stationary one :

Model I : We consider :

$$d_t = \frac{1000}{p_t} + \xi_t,$$

$$s_t = \frac{1}{3}(1+0.04t)p_{t-1}^2 + \eta_t,$$

$$p_{t+1} = p_t + 0.05(d_t - s_t) + \zeta_t ;$$

where ξ_t, η_t, ζ_t are normal variates with zero mean and unit variance. The following sample series of size 50 with random variates taken from Mahalanobis (1984) has been obtained :

t	p_t	d_t	s_t	t	p_t	d_t	s_t
0	6.74	149.14	—	25	9.91	100.92	107.84
1	8.40	117.87	16.16	26	9.85	101.28	68.00
2	14.14	70.84	20.43	27	11.54	86.32	67.76
3	17.74	57.51	74.46	28	11.52	86.44	94.92
4	15.90	62.44	118.43	20	11.07	90.35	97.06
5	13.98	70.60	100.05	30	9.56	104.26	90.60
6	13.45	74.32	80.81	31	10.43	95.80	69.47
7	14.17	71.05	76.35	32	11.59	86.00	84.43
8	13.59	74.93	87.65	33	10.22	96.79	103.94
9	11.89	82.57	83.16	34	10.44	97.22	60.45
10	12.32	81.71	66.18	35	10.59	95.42	85.25
11	12.07	84.65	74.50	36	10.74	91.20	91.62
12	11.60	86.46	71.30	37	11.61	86.31	96.30
13	11.39	87.65	67.72	38	10.69	93.92	113.70
14	11.54	88.02	66.01	39	8.13	122.94	96.00
15	12.05	79.23	70.62	40	9.23	108.23	59.07
16	13.22	74.20	87.59	41	10.97	90.91	75.06
17	14.05	69.34	98.38	42	12.56	79.68	108.01
18	12.45	80.89	123.74	43	12.98	77.58	141.41
10	8.93	112.19	91.18	44	9.09	111.45	153.35
20	9.35	105.98	47.64	45	7.5·	132.44	76.74
21	11.45	80.10	52.65	46	9.30	106.01	54.30
22	12.02	77.80	80.88	47	10.27	97.70	83.77
23	13.46	76.08	99.88	48	11.95	83.82	103.23
24	12.72	78.03	117.76	49	10.93	91.42	140.11

The demand-price relation as a result of least square estimation of the parameters from the logarithms is,

$$d_t = \frac{972.30}{(p_t)^{0.988}}$$

which agrees quite well with the assumed model.

Model 2 :

$$d_t = 100-3\,p_t+\xi_t,$$

$$s_t = 50+p_{t-1}+\frac{t}{25}+\eta_t,$$

$$p_t = p_{t-1}+\frac{1}{20}(d_{t-1}-s_{t-1})+\frac{2t}{25}+\zeta_t\,;$$

and $V(\eta_t) = 1$, for $t \leqslant 25$, $V(\zeta_t) = 1$, for $t \leqslant 25$

$= 4$, for $t \geqslant 26$; $= 9$, for $t \geqslant 26$; $V(\xi_t) = 1$;

ξ_t, η_t, ζ_t are independent normal variates with zero mean. The following model-sample of size 50 has been considered for the estimation of parameters :

t	p_t	d_t	s_t	t	p_t	d_t	s_t
1	7.52	78.52	87.32	26	19.60	40.39	67.73
2	6.19	82.40	55.75	27	18.85	42.44	69.64
3	7.55	76.25	57.86	28	22.88	35.10	69.93
4	10.80	68.01	58.50	29	21.57	34.44	72.76
5	10.37	69.92	50.87	30	27.24	19.76	75.25
6	10.00	69.28	59.07	31	23.23	29.81	70.90
7	12.38	61.66	58.45	32	26.05	19.39	73.91
8	11.00	66.71	62.28	33	25.87	23.37	79.93
9	12.13	64.27	60.57	34	24.77	22.22	75.79
10	12.61	62.65	63.20	35	21.20	38.02	76.63
11	13.03	61.00	63.40	36	26.17	20.77	72.90
12	13.93	57.05	63.48	37	27.86	16.67	79.77
13	15.79	52.40	64.20	38	27.53	17.29	81.12
14	16.49	50.59	66.80	39	23.68	28.79	80.79
15	15.00	53.03	67.46	40	20.56	37.97	75.68
16	15.32	52.79	65.95	41	19.70	40.75	71.00
17	15.33	56.00	69.71	42	18.25	44.87	74.46
18	15.91	52.22	68.33	43	19.01	42.43	67.81
19	18.04	44.73	65.77	44	16.28	52.00	69.57
20	19.48	41.60	68.04	45	17.35	49.03	68.86
21	19.65	41.80	68.09	46	20.19	40.75	69.11
22	19.79	42.04	70.67	47	25.63	52.97	70.97
23	20.67	37.71	69.13	48	24.27	20.00	70.29
24	21.24	36.30	70.56	49	30.63	7.94	70.23
25	19.71	42.86	71.35	50	33.10	0.58	83.57

For this sample we get : $d_t = 99.661-2.968\,p_t,$

Splitting it up into five random samples, each of size ten, the following estimates for α_1 and β_1 in $d_t = \alpha_1+\beta_1\,p_t$ were obtained :

	1	2	3	4	5
α_1	100.927	99.641	97.209	96.879	100.944
β_1	−3.080	−2.906	−2.835	−2.844	−3.041

Model 3 :

$$d_t = 100 - p_t + \xi_t,$$
$$s_t = 50 + 0.04\, p_{t-1} - t + \eta_t,$$
$$p_t = p_{t-1} + 0.05(d_{t-1} - s_{t-1}) + \zeta_t\,;$$

ξ_t, η_t, ζ_t are normal variates with zero mean and unit variance. We take a random sample of size 50.

t	p_t	d_t	s_t	t	p_t	d_t	s_t
1	7.52	93.50	—	26	45.71	54.26	24.26
2	6.19	94.78	46.00	27	46.61	52.38	24.13
3	8.39	90.61	48.80	28	48.97	53.87	23.84
4	12.55	88.04	47.22	29	49.94	49.21	22.47
5	13.23	87.80	44.31	30	53.01	48.47	23.24
6	14.05	85.23	43.89	31	53.03	46.47	20.33
7	17.47	81.50	41.73	32	55.51	44.72	19.85
8	17.29	82.42	42.28	33	56.42	44.56	20.05
9	19.49	81.17	40.90	34	57.32	42.84	17.64
10	21.00	79.48	41.54	35	57.39	44.50	17.52
11	22.47	77.71	40.10	36	60.05	38.23	16.43
12	24.41	74.43	39.87	37	61.42	38.83	16.41
13	27.28	72.49	37.73	38	62.47	37.39	15.33
14	29.19	70.87	37.54	39	62.33	37.50	14.35
15	28.98	69.65	36.54	40	62.25	37.40	12.69
16	30.37	68.38	35.47	41	62.74	37.11	10.60
17	31.33	70.66	35.02	42	62.95	36.67	12.05
18	32.79	67.16	35.52	43	63.78	35.68	8.44
19	35.70	63.06	31.41	44	63.48	37.42	7.95
20	38.21	61.83	30.63	45	64.40	36.68	7.03
21	39.78	61.66	29.20	46	65.89	35.43	6.54
22	40.94	60.47	29.53	47	65.03	34.83	5.09
23	42.06	56.76	27.06	48	68.42	30.39	5.97
24	44.67	55.35	26.65	49	71.19	28.64	3.74
25	44.20	57.70	25.90	50	74.22	25.66	3.32

From this sample we get the following estimated equation

$$d_t = 100.110 - 0.909\, p_t$$

Model 4:

$$d_t = 100 - 3\, p_t + \xi_t,$$
$$s_t = 50 + p_{t-1} + \eta_t,$$
$$p_t = p_{t-1} + 0.05(d_{t-1} - s_{t-1}) + \zeta_t$$

The following are the samples :

Sample 1:

p_t	d_t	s_t
7.52	78.52	83.92
6.10	82.40	55.67
7.27	77.09	57.74
10.31	69.06	58.15
9.53	72.44	50.12
8.85	72.73	58.89
10.89	66.03	57.02
9.17	72.20	60.47
9.95	70.81	58.38
10.07	70.27	60.71

Sample 2:

p_t	d_t	s_t
10.12	69.82	60.42
10.65	66.89	60.09
12.13	63.38	60.40
12.45	62.71	62.58
10.58	66.80	62.82
10.51	67.22	60.89
10.14	71.57	61.32
10.33	68.00	62.42
12.08	62.01	59.43
13.08	60.80	61.28

Sample 3:

p_t	d_t	e_t
13.07	62.23	61.75
12.63	63.62	63.01
13.13	60.33	61.05
13.31	60.09	62.06
11.34	67.07	62.42
11.45	65.62	59.83
11.14	65.57	60.75
12.33	65.85	61.12
12.04	63.03	61.84
13.83	59.09	63.28

Sample 4:

p_t	d_t	e_t
12.43	62.21	63.04
13.56	59.55	62.16
13.10	61.68	64.30
12.63	62.27	62.38
11.42	67.63	62.80
12.97	59.37	61.55
13.14	60.83	63.98
12.91	61.12	64.01
11.53	65.24	63.76
10.36	68.57	61.73

Sample 5:

p_t	d_t	e_t	p_t	d_t	e_t
9.95	70.00	60.06	9.55	72.67	58.65
9.30	71.72	61.40	7.95	76.01	59.00
9.41	71.23	58.22	10.70	66.71	59.32
8.40	75.70	58.81	12.62	61.97	60.70
8.69	75.01	58.79	14.46	56.50	63.09

Assuming $V(\xi_t) = V(\eta_t) = V(\zeta_t) = 1$ and $e_t = 50+\beta_2 \; p_{t-1}+\eta_t$, the following are estimates of β_2: (sample size 10)

Sample no.	1	2	3	4	5
least square estimate	0.96	1.02	0.94	1.03	1.03

It appears that although some of the relationships are evolutive in character it is possible to estimate some of the parameters quite accurately by the method of least squares. A more exact method would be to consider the probability density of all the observations and derive the estimates by the method of maximum likelihood. This naturally leads to complicated computations. Some investigation is needed before a simple technique could be offered.

References

Bentzel, R. and Wold, H. (1946): On statistical demand analysis from the viewpoint of simultaneous equations. *Skandinavisk Aktuarietidskrift*, 29.

Mahalanobis, P. C. (1934): Tables of random samples from a normal population. *Sankhyā*, 1, 289-328.

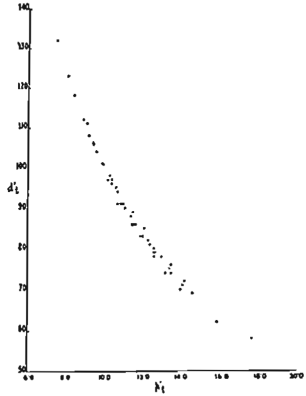

Chart I.1. Scatter diagram of model 1, where $d_t = \frac{1000}{p_t} + \xi_t$

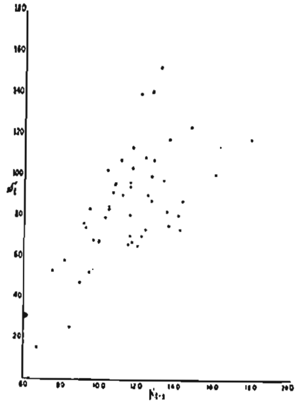

Chart I.2. Scatter diagram of model 1, where $s_t = \frac{1}{2}(1 + 0.04t)\, p_{t-1}^2 + \eta_t$

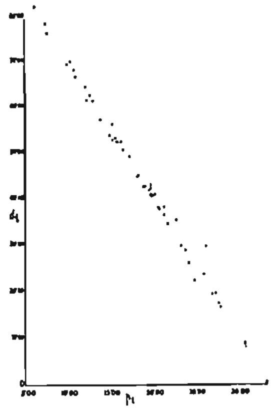

Chart 2.1. Scatter diagram of model 2, where $d_t = 100 - 3p_t + \epsilon_t$

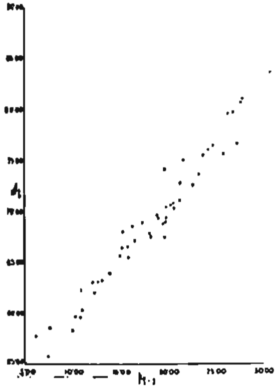

Chart 2.2. Scatter diagram of model 2, where $s_t = 50 + p_{t-1} + \frac{t}{25} + v_t$

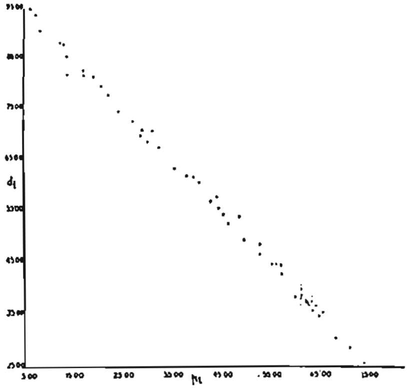

Chart 3.1. Scatter diagram of model 3, where $d_t = 100 - p_t + f_t$

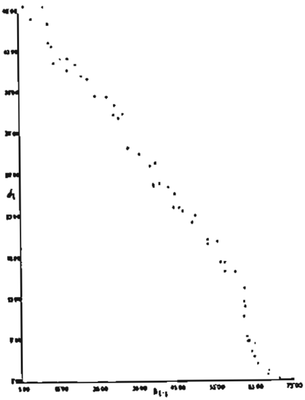

Chart 3.2. Scatter diagram of model 3, where $s = 50 + 0.04\, p_{t-1} - t + v_t$