Weighted Moore-Penrose Inverse of a Boolean Matrix

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ABSTRACT

If A is a boolean matrix, then the weighted Moore-Penrose inverse of A (with respect to the given matrices M, N) is a matrix G which satisfies $AGA = A$, $GAG = G$, and that MAG and GAN are symmetric. Under certain conditions on M, N it is shown that the weighted Moore-Penrose inverse exists if and only if $ANA^TMA = A$, in which case the inverse is $N^TA^TM^T$. When M, N are identity matrices, this reduces to the well-known result that the Moore-Penrose inverse of a boolean matrix, when it exists, must be A^T.

1. INTRODUCTION

The binary boolean algebra $\mathscr{B}$ consists of the set $\{0, 1\}$ equipped with the operations of addition and multiplication defined as follows:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

By a boolean matrix we mean a matrix over $\mathscr{B}$. We confine our attention to boolean matrices. The operations of matrix addition, scalar multiplication, and matrix multiplication are defined in the usual way. For example, if

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

then

$$A + B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The transpose of the matrix A is denoted by A^T. The identity matrix of the appropriate order is denoted by I. For matrices A, B of the same order, $A \geqslant B$ means $a_{ij} \geqslant b_{ij}$ for all i, j (with the natural convention that $1 \geqslant 0$). For basic properties of boolean matrices we refer to [2].

Definition 1. Let A, M, N be matrices of order $m \times n$, $m \times m$, and $n \times n$ respectively. The *weighted Moore-Penrose inverse* of A (with respect to M, N, denoted by $A^{\dagger}_{M,N}$) is defined to be an $n \times m$ matrix G satisfying

(i) $AGA = A$,
(ii) $GAG = G$,
(iii) $(MAG)^T = MAG$,
(iv) $(GAN)^T = GAN$.

In case M, N are identity matrices, then the matrix G satisfying (i)–(iv) is simply the *Moore-Penrose inverse* (denoted by $A^{\dagger}$) of A.

It is well known that if A is a boolean matrix then A admits a Moore-Penrose inverse if and only if $AA^TA = A$, in which case A^T is the Moore-Penrose inverse; see for example, [5]. In the next result we present several characterizations of matrices admitting the Moore-Penrose inverse. Many of these characterizations are known, but the formulation of the result perhaps has some novelty. We will indicate a proof of Theorem 1.1 in Section 2, where we discuss the more general case of a weighted Moore-Penrose inverse.

THEOREM 1.1. *Let A be an $m \times n$ matrix. Then the following assertions are equivalent*:

(i) *The Moore-Penrose inverse of A exists.*

(ii) *The Moore-Penrose inverse of A exists and equals A^T.*

(iii) $AA^TA = A$.

(iv) $AA^TA \leqslant A$.

(v) *Any two rows of A are either identical or disjoint (i.e., there is no column with a 1 in both the rows).*

(vi) *Any two columns of A are either identical or disjoint.*

(vii) *The number of ones in any 2×2 submatrix of A is not 3.*

(viii) *Any 2×2 submatrix of A admits a Moore-Penrose inverse.*

(ix) *There exist permutation matrices, P, Q such that*

$$PAQ = \begin{bmatrix} J_1 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_t & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $J_1, \dots, J_t$ are matrices (not necessarily square) of all ones.

(x) *There exist permutation matrices P, Q such that*

$$PAQ = \begin{bmatrix} I & C \\ D & DC \end{bmatrix},$$

where C, D satisfy $CC^T \leqslant I$, $D^TD \leqslant I$.

(xi) *There exists a matrix G such that $GAA^T = A^T$ and $A^TAG = A^T$.*

The main purpose of the present paper is to generalize some aspects of Theorem 1.1 to the weighted case. The proof technique is new and may be used to obtain results for matrices over more general structures. Thus most of our statements are valid for matrices over a distributive lattice, whereas some require the structure of a completely ordered set. Such generalizations will be clear from the proofs. However, we have chosen to present the results only in the setting of binary boolean matrices. In the next section we consider the question of the existence of a weighted Moore-Penrose inverse and give a formula for it when it exists.

2. THE MAIN RESULT

We begin by showing that under some conditions on M, N, the inverse $A^-_{M,N}$, when it exists, is unique. We denote the row space of the matrix A by $\mathscr{R}(A)$, and the column space by $\mathscr{C}(A)$.

THEOREM 2.1. *Let A, M, N be matrices of order $m \times n$, $m \times m$, and $n \times n$ respectively, and suppose $A^-_{M,N}$ exists. Further suppose*

$$\mathscr{R}(A) = \mathscr{R}(MA), \qquad \mathscr{C}(A) = \mathscr{C}(AN),$$

i.e., there exist matrices X, Y such that

$$XMA = A, \qquad ANY = A.$$

Then

(a) $AN^TA^T = ANA^T$, $A^TM^TA = A^TMA$;
(b) $A^-_{M,N}$ *is unique.*

Proof. (a): Let $G = A^-_{M,N}$ exist. Then

$$\begin{aligned} AN^TA^T &= AN^TA^TG^TA^T \qquad (\text{since } AGA = A)\\ &= AGANA^T \qquad (\text{since } GAN \text{ is symmetric})\\ &= ANA^T \qquad (\text{since } AGA = A). \end{aligned}$$

The proof of the remaining part of (a) is similar to the above.

(b): Let, if possible, G_1, G_2 be two candidates for $A^-_{M,N}$. Then

$$\begin{aligned} G_1AN &= G_1AG_2AN \qquad (\text{since } A = AG_2A)\\ &= G_1AN^TA^TG_2^T \qquad (\text{since } G_2AN \text{ is symmetric})\\ &= G_1ANA^TG_2^T \qquad [\text{using (a)}]\\ &= N^TA^TG_1^TA^TG_2^T \qquad (\text{since } G_1AN \text{ is symmetric})\\ &= N^TA^TG_2^T \qquad (\text{since } AG_1A = A)\\ &= G_2AN \qquad (\text{since } G_2AN \text{ is symmetric}). \end{aligned}$$

Thus $G_1 ANY = G_2 ANY$, and hence $G_1 A = G_2 A$ (since $ANY = A$). It follows that $G_1 AG_1 = G_2 AG_1$ and therefore

$$G_1 = G_2 AG_1. \tag{1}$$

Now

$$\begin{aligned}
MAG_1 &= MAG_2 AG_1 \qquad (\text{since } A = AG_2 A)\\
&= G_2^T A^T M^T AG_1 \qquad (\text{since } MAG_2 \text{ is symmetric})\\
&= G_2^T A^T MAG_1 \qquad [\text{using (a)}]\\
&= G_2^T A^T G_1^T A^T M^T \qquad (\text{since } MAG_1 \text{ is symmetric})\\
&= G_2^T A^T M^T \qquad (\text{since } AG_1 A = A)\\
&= MAG_2 \qquad (\text{since } MAG_2 \text{ is symmetric}).
\end{aligned}$$

It follows that $XMAG_1 = XMAG_2$ and hence $AG_1 = AG_2$ (since $XMA = A$). Therefore $G_2 AG_1 = G_2 AG_2$, and thus

$$G_2 AG_1 = G_2. \tag{2}$$

It follows from (1), (2) that $G_1 = G_2$, and the proof is complete. ■

EXAMPLE. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad G_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Take $M = I$, and N to be the 2×2 zero matrix. Then it can be verified that both G_1, G_2 satisfy all conditions in Definition 1, and therefore the weighted Moore-Penrose inverse is not unique in this example. Observe that here the condition of Theorem 2.1 is not satisfied.

The next result will be used in the sequel.

LEMMA 1. *Let A be an $m \times n$ matrix. Then $A \leqslant AA^TA$.*

Proof. Let $B = AA^TA$. We must show that $a_{ij} \leqslant b_{ij}$ for all i, j. This is obvious if $a_{ij} = 0$. Now assume that $a_{ij} = 1$. We have

$$b_{ij} = \sum_{k=1}^{n} \sum_{l=1}^{m} a_{ik} a_{lk} a_{lj}. \tag{3}$$

If we set $k = j$, $l = i$, then $a_{ik} a_{lk} a_{lj} = a_{ij}^3 = a_{ij}$. It follows from (3) that $b_{ij} = 1$, and the proof is complete. ■

The following is the main result of this section.

THEOREM 2.2. *Let A, M, N be matrices of order $m \times n$, $m \times m$, and $n \times n$ respectively, and suppose*

(a) $\mathscr{R}(A) = \mathscr{R}(MA)$, $\mathscr{C}(A) = \mathscr{C}(AN)$,
(b) $M \geqslant I$, $N \geqslant I$.

Then the following assertions are equivalent:

(i) $A^{\dagger}_{M,N}$ *exists.*

(ii) *Any one of the following holds*[1]:

(1) $ANA^TMA = A$,
(2) $AN^TA^TMA = A$,
(3) $ANA^TM^TA = A$,
(4) $AN^TA^TM^TA = A$,

and thus $A^{\dagger}_{M,N} = N^TA^TM^T$.

(iii) *Any two rows of A are either identical or disjoint, and* $ANA^T = AA^T$, $A^TMA = A^TA$.

(iv) *Any two columns of A are either identical or disjoint, and* $ANA^T = AA^T$, $A^TMA = A^TA$.

Proof. (i) ⇒ (ii): Suppose $G = A^{+}_{M,N}$ exists. Since the number of boolean matrices of a given order is finite, there exist integers $k \geqslant 1$, $s \geqslant 1$ such that

$$(ANA^TM^T)^k = (ANA^TM^T)^{k+s}. \tag{4}$$

[1] The equalities in (1), (2), (3), (4) can be replaced by $\leqslant$.

Without loss we can assume $s > 1$, for if $s = 1$, then (4) clearly holds for $s = 2$ as well. Now, left multiplying Equation (4) by G and then using the fact that GAN is symmetric, we get

$$N^TA^TG^TA^TM^T(ANA^TM^T)^{k-1} = N^TA^TG^TA^TM^T(ANA^TM^T)^{k-1+s}.$$

Left multiply the above equation by Y^T and then use $AGA = A$, $ANY = A$ to get

$$A^TM^T(ANA^TM^T)^{k-1} = A^TM^T(ANA^TM^T)^{k-1+s}.$$

Left multiply the above equation by G^T and then use the facts that MAG is symmetric and that $AGA = A$ to get

$$M(ANA^TM^T)^{k-1} = M(ANA^TM^T)^{k-1+s}.$$

Finally, left multiply the above equation by X and use $XMA = A$ to get

$$(ANA^TM^T)^{k-1} = (ANA^TM^T)^{k-1+s}.$$

Continuing this way, we may assume, $k = 1$, without loss of generality, and therefore,

$$ANA^TM^T = (ANA^TM^T)^{s+1}.$$

Starting with the above equation, we get the following chain of implications:

$$\rightarrow \quad \underline{GAN}\,A^TM^T = \underline{GAN}\,A^TM^T(ANA^TM^T)^s$$

$$\Rightarrow \quad N^T\underline{A^TG^TA^T}M^T = N^T\underline{A^TG^TA^T}M^T(ANA^TM^T)^s$$

$$\Rightarrow \quad N^TA^TM^T = N^TA^TM^T(ANA^TM^T)^s$$

$$\Rightarrow \quad \underline{Y^TN^TA^T}M^T = \underline{Y^TN^TA^T}M^T(ANA^TM^T)^s$$

$$\Rightarrow \quad A^TM^T = A^TM^T(ANA^TM^T)^s$$

$$\Rightarrow \quad A^TM^TX^T = A^TM^T(ANA^TM^T)^sX^T$$

$$\Rightarrow \quad A^T = A^TM^T(ANA^TM^T)^{s-1}ANA^T.$$

and therefore

$$A = (AN^TA^TM)^k A. \tag{5}$$

By Lemma 1, $A \leqslant AA^TA$, and so, since $M \geqslant I$, $N \geqslant I$, we have

$$A \leqslant AN^TA^TMA, \tag{6}$$

and hence, postmultiplying by N^TA^TMA, we get

$$AN^TA^TMA \leqslant (AN^TA^TM)^2 A. \tag{7}$$

Repeated postmultiplication of (7) by N^TA^TMA gives

$$\begin{aligned} A &\leqslant AN^TA^TMA \leqslant (AN^TA^TM)^2 A \\ &\leqslant (AN^TA^TM)^3 A \leqslant \cdots \leqslant (AN^TA^TM)^k A = A, \end{aligned}$$

where the last equality follows in view of (5). Therefore

$$A = AN^TA^TMA = ANA^TMA = ANA^TM^TA = AN^TA^TM^TA, \tag{8}$$

where the last three equalities follow in view of Theorem 2.1 (a).

Now we show $G = A^{+}_{M,N} = N^TA^TM^T$. We have shown $AGA = A$. But $GAG = N^T\underline{A^TM^TAN^T}A^TM^T = N^TA^TM^T$ [in view of (8)], and

$$\begin{aligned} MAG &= MAN^TA^TM^T \\ &= MANA^TM^T \qquad \text{[using Theorem 2.1(a)]} \\ &= (MAN^TA^TM^T)^T = (MAG)^T. \end{aligned}$$

So MAG is symmetric. Showing GAN symmetric is similar. So $A^{-}_{M,N} = N^TA^TM^T$.

(ii) ⇒ (i): Let $ANA^TMA = A$. By Lemma 1, $A \leqslant AA^TA$. As $M \geqslant I$, $N \geqslant I$, we have $A \leqslant AA^TA \leqslant ANA^TA \leqslant ANA^TMA = A$. Thus

$$AA^TA = A = ANA^TA. \tag{9}$$

The second part of the above equation gives

$$\begin{aligned} AA^T &= ANA^TAA^T \\ &= ANA^T \qquad \text{[using the first part of (9)]} \\ &= AN^TA^T \qquad (\text{since } AA^T \text{ is symmetric}). \end{aligned}$$

Similarly it can be shown that $A^TMA = A^TM^TA = A^TA$. Now using these facts, Equation (9), and the assumption, one can easily see that $A^+_{M,N} = N^TA^TM^T$.

(ii) ⇒ (iii): Without loss we take $ANA^TMA = A$. Suppose two rows of A, say, the ith and the jth are not disjoint. Then there exists k such that $a_{ik} = a_{jk} = 1$. Now if $a_{ir} = 1$ for some r, then we have

$$a_{jr} \geqslant a_{jk} n_{kk} a_{ik} m_{ii} a_{ir} = 1,$$

and hence $a_{jr} = 1$. Thus the ith row of A is entrywise dominated by the jth row. Similarly we can show that the jth row of A is entrywise dominated by the ith row, and hence the two rows must be identical.

The proof of the remaining part is essentially contained in the proof of (ii) ⇒ (i).

(iii) ⇒ (ii): Let $B = AA^TA$, and suppose $b_{ij} = 1$. So there exist l_1, l_2 such that

$$a_{il_1} = a_{l_2 l_1} = a_{l_2 j} = 1.$$

Now observe that the l_1th column of A is nonzero. So by hypothesis we have that the ith row of A is equal to the l_2th row of A. But we also have $a_{l_2 j} = 1$. So $a_{ij} = 1$ and therefore $AA^TA \leqslant A$. It follows by Lemma 1 that $A = AA^TA$. Since $ANA^T = AA^T$, $A^TMA = A^TA$, then

$$ANA^TMA = AA^TMA = AA^TA = A,$$

and (ii) is proved. The equivalence of (iv) and (ii) is proved similarly. That completes the proof of the theorem. ■

An examination of the proof of Theorem 2.2 reveals that condition (b) may be replaced by the weaker condition $ANA^TMA \geqslant A$.

We now provide a proof of Theorem 1.1.

Proof of Theorem 1.1. The equivalence of (i)–(vi) of the theorem essentially follows from Theorem 2.2 by setting $M = N = I$. The equivalence of (v) and (vii) is easy to prove, and so is the equivalence of (vii) and (viii). The implications (v) $\Rightarrow$ (ix) and (ix) $\Rightarrow$ (iii) are easy to prove. Thus we have shown that assertions (i)–(ix) are equivalent.

It is easy to see that (ix) $\Rightarrow$ (x). If (x) holds, then it can be verified that A^T is the Moore-Penrose inverse of A and thus (i) holds.

We finally show the equivalence of (xi) with the remaining conditions. If (i) holds, then (iii) holds, and setting $G = A^T$, we see that (xi) holds as well. Conversely, suppose (xi) is true. Then $GAA^TG^T = A^TG^T = (GA)^T$, and thus GA is symmetric. Now from $GAA^T = A^T$ is follows that $AGA = A$. Similarly, using $A^TAG = A^T$, we conclude that AG is symmetric. Now it can be verified that GAG must be the Moore-Penrose inverse of A and thus (i) holds. That completes the proof. ■

We remark that all the assertions in Theorem 1.1 except (vii), (viii) are essentially contained in the literature; see [2, 4, 5]. However, we have given proofs for completeness.

As shown in Theorem 1.1, if A admits a Moore-Penrose inverse, then it must be A^T. Sometimes it happens that the weighted Moore-Penrose inverse $A^{+}_{M,N} = A^T$, the trivial case being $M = N = I$. So the obvious question is whether we can precisely point out the cases when $A^{+}_{M,N} = A^T$. To answer this question we need the following result.

THEOREM 2.3. *Let A, M, N, be as in Theorem 2.2. Then the following are equivalent:*

(i) *$A^{+}_{M,N}$ exists.*
(ii) *There exist permutation matrices P and Q such that*

$$\bar{A} = PAQ = \begin{bmatrix} J_1 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_k & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $J_1, \ldots, J_k$ are matrices (not necessarily square) of all ones,

$$\overline{M} = PMP^T = \begin{bmatrix} M_{11} & 0 & \cdots & 0 & M_{1,k+1} \\ 0 & M_{22} & \cdots & 0 & M_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{k,k} & M_{k,k+1} \\ M_{k+1,1} & M_{k+1,2} & \cdots & M_{k+1,k} & M_{k+1,k+1} \end{bmatrix}$$

where the rows and columns of $\overline{M}$ are partitioned according to the partitioning of the rows of $\overline{A}$, and

$$\overline{N} = Q^TNQ = \begin{bmatrix} N_{11} & 0 & \cdots & 0 & N_{1,k+1} \\ 0 & N_{22} & \cdots & 0 & N_{2,k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & N_{k,k} & N_{k,k+1} \\ N_{k+1,1} & N_{k+1,2} & \cdots & N_{k+1,k} & N_{k+1,k+1} \end{bmatrix}$$

where the rows and columns of $\overline{N}$ are partitioned according to the partitioning of the columns of $\overline{A}$.

Further, in the case that (i) *and* (ii) *hold, $\overline{A}^-_{\overline{M},\overline{N}}$ is given by*

$$\overline{G} = \begin{bmatrix} J_1^T & 0 & \cdots & 0 & J_1^TM_{k+1,1}^T \\ 0 & J_2^T & \cdots & 0 & J_2^TM_{k+1,2}^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_k^T & J_k^TM_{k+1,k}^T \\ N_{1,k+1}^TJ_1^T & N_{2,k+1}^TJ_2^T & \cdots & N_{k,k+1}^TJ_k^T & \sum_{i=1}^k N_{i,k+1}^TJ_i^TM_{k+1,i}^T \end{bmatrix}.$$

Proof. (i) $\Rightarrow$ (ii): First suppose that $A^-_{M,N}$ exists. In view of Theorem 2.2 and Theorem 1.1(ix), one can see that there exist permutation matrices P, Q such that $\overline{A} = PAQ$ is of the form given in (ii). Let $\overline{M} = PMP^T$ and $\overline{N} = Q^TNQ$.

Note that by using (9), we have

$$\overline{A}\overline{N}\overline{A}^T\overline{A} = \overline{A} \quad \text{and} \quad \overline{A}\overline{A}^T\overline{M}\overline{A} = \overline{A}. \tag{10}$$

Setting

$$\overline{N} = \begin{bmatrix} N_{11} & \cdots & N_{1,k+1} \\ \vdots & \ddots & \vdots \\ N_{k+1,1} & \cdots & N_{k+1,k+1} \end{bmatrix},$$

a conformal partition, and then using (10), we see that all the blocks $N_{ij} = 0$, $1 \leqslant i, j \leqslant k$, $i \neq j$, and that $N_{ii} \neq 0$, $1 \leqslant i \leqslant k$. We have a similar conclusion regarding $\overline{M}$.

It is easy to see that $\overline{G} = \overline{N}^T\overline{A}^T\overline{M}^T$ is the weighted Moore-Penrose inverse of $\overline{A}$ with respect to $\overline{M}, \overline{N}$. But

$$\overline{G} = (Q^TN^TQ)(Q^TA^TP^T)(PM^TP^T) = Q^T(N^TA^TM^T)P^T = Q^TGP^T. \quad (11)$$

Now carrying out the block multiplication in the equation $\overline{G} = \overline{N}^T\overline{A}^T\overline{M}^T$, we see that $\overline{G}$ is of the form given in the statement.

Now by (11), the proof of (i) ⇒ (ii) is complete.

Conversely, suppose (ii) holds. Defining $\overline{G}$ as in the statement of the theorem, it is easy to check that $\overline{A}\overline{G}\overline{A} = \overline{A}$. Since $\overline{G} = \overline{N}^T\overline{A}^T\overline{M}^T$, we have $\overline{A}\overline{N}^T\overline{A}^T\overline{M}^T\overline{A} = \overline{A}$. This implies $PAN^TA^TM^TAQ = PAQ$. Therefore $AN^TA^TM^TA = A$, and thus $A^+_{M,N}$ exists, by Theorem 2.2. ■

As a simple corollary we state the following result without proof.

COROLLARY 1. *Let A, M, N be as in Theorem 2.2. Then $A^+_{M,N} = A^T$ if and only if condition (ii) of Theorem 2.3 is satisfied with the additional proviso that $\overline{M}$ and $\overline{N}$ are block diagonal.*

We also have the following.

COROLLARY 2. *Let A, M, N be as in Theorem 2.2, and further suppose A has no zero row or zero column. Then, if $A^+_{M,N}$ exists, it equals A^T.*

Proof. Observe that $\overline{A}$ has no diagonal zero block. Hence by Theorem 2.3 $\overline{M}, \overline{N}$ are block diagonal. Furthermore, that $A^-_{M,N}$ exists implies that $\overline{G}$ exists. The result now follows by Corollary 1. ■

We conclude with an example which shows that the condition that A has no zero row or zero column is necessary in Corollary 2.

EXAMPLE. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad M = I.$$

Then $ANA^TMA = A$, but $NA^TM = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \neq A^T$.

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REFERENCES

1 D. A. Gregory, S. Kirkland, and N. J. Pullman, Power convergent boolean matrices, *Linear Algebra Appl.* 179:105–117 (1993).
2 K. H. Kim, *Boolean Matrix Theory and Applications*, Marcel Dekker, New York, 1982.
3 S. Kirkland and N. J. Pullman, Boolean spectral theory, *Linear Algebra Appl.* 175:177–190 (1992).
4 R. J. Plemmons, Generalized inverses of boolean matrices, *SIAM J. Appl. Math.* 20:426–433 (1971).
5 P. S. S. N. V. Prasada Rao, and K. P. S. Bhaskara Rao, On generalized inverses of boolean matrices, *Linear Algebra Appl.* 11:135–153 (1975).

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