Moore-Penrose Inverse of the Incidence Matrix of a Tree

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Let T be a tree with n vertices, where each edge is given an orientation, and let Q be its vertex-edge incidence matrix. It is shown that the Moore-Penrose inverse of Q is the $(n-1)\times n$ matrix M obtained as follows. The rows and the columns of M are indexed by the edges and the vertices of T respectively. If e,v are an edge and a vertex of T respectively, then the (e,v)-entry of M is, upto a sign, the number of vertices in the connected component of $T\setminus e$ which does not contain v. Furthermore, the sign of the entry is positive or negative, depending on whether e is oriented away from or towards v. This result is then used to obtain an expression for the Moore-Penrose inverse of the incidence matrix of an arbitrary directed graph. A recent result due to Moon is also derived as a consequence.

Keywords: Moore-Penrose inverse; incidence matrix; tree; distance matrix

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1. INTRODUCTION

We consider graphs which have no loops but which possibly have multiple edges. If G is a graph then $V(G), E(G)$ will denote the vertex set and the edge set of G respectively. Let G be a graph with $V(G)=\{1,\dots,n\}$, $E(G)=\{e_1,\dots,e_m\}$ and suppose each edge of G is assigned an orientation. The vertex-edge *incidence matrix* of G, denoted by $Q(G)$, (or simply by Q if there is no possibility of a confusion) is the $n\times m$ matrix defined as follows. The rows and the columns of Q are indexed by $V(G)$, $E(G)$ respectively. The (i,j) - entry of Q is 0 if vertex i and edge e_j are not incident and otherwise it is 1 or -1 according as e_j originates or terminates at i respectively.

If A is an $n\times n$ matrix, then an $m\times n$ matrix G is called a generalized inverse of A if $AGA=A$. The *Moore-Penrose inverse* of A, denoted by A^+, is an $m\times n$ matrix satisfying the equations $AGA=A$, $GAG=G$, $(AG)^T=AG$ and $(GA)^T=GA$. It is well-known that any complex matrix admits a unique Moore-Penrose inverse. We refer to [3, 6] for basic properties of the Moore-Penrose inverse.

The main purpose of this paper is to obtain a graph-theoretic description of the Moore-Penrose inverse of the incidence matrix of a tree. This is then used to describe the Moore-Penrose inverse of the incidence matrix of an arbitrary directed graph.

If G is a graph where each edge has an orientation, then $L(G)=Q(G)Q(G)^T$ is known as the *Laplacian matrix* of G and is important in many different areas (see [9] for a survey). The matrix $K(G)=Q(G)^TQ(G)$ has been called the *edge-version* of the Laplacian [7, 8]. Using our main result we also obtain expressions for the Moore-Penrose inverse of the Laplacian matrix of a tree. We also derive an expression for the inverse of $K(G)$ when G is a tree, which has recently been obtained by Moon [11].

2. INCIDENCE MATRIX OF A TREE

Let T be a tree with vertex set $V(T)=\{1,\dots,n\}$ and edge set $\{e_1,\dots,e_{n-1}\}$. Suppose each edge of T is assigned an orientation and let Q be the corresponding vertex-edge incidence matrix.

We now introduce some notation. If $e\in E(T)$, then the graph $T\backslash e$, obtained by removing e from T, has two connected components. Denote by $G_h(e,T)$, the component which contains the head (the terminating vertex) of e and by $G_t(e,T)$, the component which contains the tail (the initial vertex) of e. Also, let

$$\alpha_h(e,T)=|V(G_h(e,T))|,\quad \alpha_t(e,T)=|V(G_t(e,T))|,$$

where $|\cdot|$ denotes cardinality. Then clearly,

$$\alpha_h(e,T)+\alpha_t(e,T)=|V(T)|=n.$$

Let $i \in \{1, \ldots, n-1\}$, $j \in \{1, \ldots, n\}$. We define $\phi_T(e_i, j)$ to be the number of vertices in the component of $T \backslash e_i$ which does not contain j. Thus

$$\phi_T(e_i, j) = \begin{cases} \alpha_k(e_i, T) & \text{if} \quad j \in V(G_t(e_i, T)) \\ \alpha_t(e_i, T) & \text{if} \quad j \in V(G_k(e_i, T)). \end{cases}$$

Further, we define $\psi_T(e_i, j)$ to be 1 or 0 according as $j \in V(G_h(e_i, T))$ or otherwise.

The following is the main result.

THEOREM 1 *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $\{e_1, \ldots, e_{n-1}\}$. Suppose each edge of T is assigned an orientation and let Q be the corresponding vertex-edge incidence matrix. Let M be the $(n-1) \times n$ matrix defined as*

$$m_{ij} = \frac{1}{n}\{\alpha_h(e_i, T) - n\psi_T(e_i, j)\}, \quad i = 1, \ldots, n-1; \; j = 1, \ldots, n.$$

Then $M = Q^+$.

Proof We remark that $nm_{ij} = \pm\phi_T(e_i, j)$. Furthermore, the sign is negative if e_i is oriented towards j (*i.e.*, $j \in V(G_h(e_i, T))$) and positive otherwise.

We first claim that $MQ = I_{n-1}$, the identity matrix of order $n-1$. Let $MQ = W = [w_{ij}]$. Let $i, j \in \{1, \ldots, n-1\}$ and suppose e_j is oriented from vertex u to vertex v.

Then

$$w_{ij} = \sum_{k=1}^{n} m_{ik} q_{kj} = m_{iu} - m_{iv}. \tag{1}$$

Consider the case $i \neq j$ and first assume that $u \in V(G_k(e_i, T))$. Then clearly, $v \in V(G_k(e_i, T))$. Thus

$$m_{iu} = m_{iv} = -\frac{1}{n}\alpha_t(e_i, T)$$

and hence $w_{ij} = 0$.

Similarly, if $u \in V(G_t(e_i, T))$ then $v \in V(G_t(e_i, T))$ and

$$m_{iu} - m_{iv} = \frac{1}{n}\alpha_h(e_i, T)$$

and again $w_{ij} = 0$.

Finally, if $i = j$, then

$$m_{iu} = \frac{1}{n}\phi_T(e_i, u), m_{iv} = -\frac{1}{n}\phi_T(e_i, v).$$

Thus

$$w_{ij} = m_{iu} - m_{iv} = \frac{1}{n}(\phi_T(e_i, u) + \phi_T(e_i, v)) = \frac{1}{n}\cdot n = 1.$$

This shows that $W = MQ = I_{n-1}$ and the claim is proved. It also follows that $QMQ = Q, MQM = M$ and it remains to show that QM is symmetric.

We claim that $QM = I_n - \frac{1}{n}J_n$, where J_n is the $n \times n$ matrix of all ones.

Let $QM = Z = [z_{ij}]$ and let $i, j \in \{1, \ldots, n\}$. Let E_1, E_2 be the set of edges emanating from and terminating at i respectively. Then

$$z_{ij} = \sum_{k=1}^{n-1} q_{ik} m_{kj} = \sum_{k: e_k \in E_1} m_{kj} - \sum_{k: e_k \in E_2} m_{kj} \tag{2}$$

First let $i \neq j$ and suppose the (i, j)-path in T uses the edge $e_\ell \in E_1$. Then

$$nm_{kj} = \begin{cases} -\alpha_t(e_k, T) & \text{if} \quad k = \ell \\ \alpha_h(e_k, T) & \text{if} \quad e_k \in E_1, k \neq \ell \\ -\alpha_t(e_k, T) & \text{if} \quad e_k \in E_2 \end{cases}$$

Substituting in (2) we see that

$$nz_{ij} = -\alpha_t(e_\ell, T) + \sum_{e_k \in E_1, k \neq \ell} \alpha_h(e_k, T) + \sum_{e_k \in E_2} \alpha_t(e_k, T) = -1.$$

Now suppose the (i, j)-path in T uses the edge $e_r \in E_2$.

Then

$$nm_{kj} = \begin{cases} -\alpha_k(e_r, T) & \text{if } k = r \\ \alpha_r(e_k, T) & \text{if } e_k \in E_2, k \neq r \\ -\alpha_k(e_k, T) & \text{if } e_k \in E_1 \end{cases}$$

Again, substituting in (2) we see that

$$nz_{ij} = -\alpha_k(e_r, T) + \sum_{e_k \in E_2, k \neq r} \alpha_r(e_k, T) + \sum_{e_k \in E_1} \alpha_k(e_k, T) = -1$$

Finally, if $i = j$, then

$$\begin{aligned} nz_{ii} &= n \sum_{k=1}^{n-1} q_{ik} m_{ki} \\ &= n \sum_{e_k \in E_1} m_{ki} - n \sum_{e_k \in E_2} m_{ki} \\ &= \sum_{e_k \in E_1} \alpha_k(e_k, T) + \sum_{e_k \in E_2} \alpha_r(e_k, T) \\ &= n - 1. \end{aligned}$$

Therefore $z_{ii} = 1 - \frac{1}{n}, i = 1, \ldots, n$, and the proof is complete. ■

COROLLARY 2 *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $\{e_1, \ldots, e_{n-1}\}$. Suppose each edge of T is assigned an orientation and let Q be the corresponding vertex-edge incidence matrix. Then the class of generalized inverses of Q is given by*

$$\{Q^+ + w\mathbf{1}^T\}$$

where w is an arbitrary vector of order $(n-1) \times 1$ and $\mathbf{1}$ is a column vector of all ones.

Proof The class of all generalized inverses of Q is given by (see [12], p.25) matrices of the form

$$Q^+ + X - Q^+QXQQ^+, \tag{3}$$

where X is arbitrary. As seen in the proof of Theorem 1, $Q^+Q = I_{n-1}$ and $QQ^+ = I_n - \frac{1}{n}J_n$, where J_n is the $n \times n$ matrix of all ones.

Substituting in (3) we see that the class of generalized inverses of Q is given by matrices of the form.

$$Q^+ - \frac{1}{n} X J_n.$$

Setting $w = \frac{1}{n} X\mathbf{1}$ in the above expression, we get the result. ∎

As a consequence of Theorem 1 and Corollary 2 we see that the matrix $[\psi(e_i, j)]$, which is a 0–1 matrix, is a generalized inverse of Q.

We now derive an expression for the Moore-Penrose inverse of the incidence matrix of an arbitrary directed graph, using Theorem 1.

First we introduce more notation. For an $m \times n$ matrix B of rank r, the volume of B, denoted by $\mathrm{vol}(B)$, is defined to be the positive square root of the sum of squares of all $r \times r$ minors of B. This definition is due to Ben-Israel [2, 10].

Let B be an $n \times m$ matrix. For $1 \le i_1 < i_2 < \ldots < i_r \le m$, let $B(i_1, \ldots, i_r)$ denote the matrix obtained from B by retaining columns $i_1, \ldots, i_r$ and replacing all other columns by zero vectors. Then it can be seen using the determinantal formula for the Moore-Penrose inverse (see [1, 4, 10] that

$$B^+ = \sum_{(i_1, \ldots, i_r) \in \mathcal{M}} \frac{\mathrm{vol}^2 B(i_1, \ldots, i_r)}{\mathrm{vol}^2(B)} B(i_1, \ldots, i_r)^-, \tag{4}$$

where $\mathcal{M}$ is the set of all r-tuples $(i_1, \ldots, i_r)$ such that $1 \le i_1 < i_2 < \ldots < i_r \le m$, and the corresponding columns of B are linearly independent.

Let G be a connected, directed graph (possibly with parallel edges but with no loops) with $V(G) = \{1, \ldots, n\}$, $E(G) = \{e_1, \ldots, e_m\}$ and let Q be the corresponding incidence matrix. It is well-known, (see, for example, Bondy and Murty [5]), that columns $i_1, \ldots, i_{n-1}$ of Q are linearly independent if and only if the corresponding edges form a spanning tree of G, and in that case any submatrix of order $(n-1) \times (n-1)$ formed using these columns has determinant ± 1. Thus $\mathrm{vol}^2 Q(i_1, \ldots, i_r)$ equals n if the edges $e_{i_1}, \ldots, e_{i_r}$ form a spanning tree and equals zero otherwise. It follows that $\mathrm{vol}^2(Q) = n\chi(G)$, where $\chi(G)$ is the number of spanning trees in G. Substituting in (4) we have the following expression for the Moore-Penrose inverse of Q. If $M = Q^+$, then

$$m_{ij} = \frac{1}{n\chi(G)} \sum \{\alpha_n(e_i, T) - n\psi_T(e_i, j)\},$$

where the summation is over all spanning trees T containing e_i.

3. THE LAPLACIAN AND ITS EDGE VERSION

Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $\{e_1, \ldots, e_{n-1}\}$. Suppose each edge of T is assigned an orientation. Let Q be the corresponding vertex-edge incidence matrix. Let $L = QQ^T$ be the Laplacian matrix, and let $K = Q^TQ$ be the edge-version of the Laplacian. Since Q has rank $n-1$, K must be nonsingular. In this section we obtain expressions for the Moore-Penrose inverse of L and for K^{-1}. The expression for K^{-1} has been recently given by Moon [11].

By Campbell and Meyer [6, p.25], $L^+ = (Q^+)^TQ^+$. The following result now follows immediately from Theorem 1.

THEOREM 3 *Let T be a tree with vertex set $V(T) = \{1, \ldots, n\}$ and edge set $\{e_1, \ldots, e_{n-1}\}$. Suppose each edge of T is assigned an orientation. Let Q be the corresponding vertex-edge incidence matrix and let $L = QQ^T$ be the Laplacian. Then the (i,j)-entry of L^+ is given by*

$$\frac{1}{n^2}\sum_{k=1}^{n-1} \delta_T(i,j,e_k)\phi_T(e_k,i)\phi_T(e_k,j), \quad i = 1, \ldots, n-1, \; j = 1, \ldots, n;$$

where $\delta_T(i,j,e_k)$ is -1 or 1 according as e_k is on the (i,j)-path or otherwise.

We remark that using Theorem 3 and some simple manipulation we get the following interesting formula,

$$\ell^+_{ii} + \ell^+_{jj} - 2\ell^+_{ij} = d(i,j), \tag{5}$$

where $d(i,j)$ is the distance (i.e. the number of edges in the (i,j)-path) between vertices i, j.

We now obtain an expression for K^{-1}. As before, we may write $K^{-1} = K^{\dagger} = Q^{\dagger}(Q^{\dagger})^T = MM^t$.

Suppose edges e_i and e_j of T join vertices r and s and vertices u and v respectively. We suppose that $r = u$ and $s = v$ if $i = j$. We also assume that vertices s and u are on the (r, v)-path in T.

Consider the case where e_i and e_j are similarly oriented with respect to the (r, v)-path. The graph $T \setminus \{e_i, e_j\}$ has three connected components. Let a and b denote the number of vertices in the components containing r and v respectively. Let $c = n - (a + b)$. We assume, without loss of generality, that e_i is oriented towards e_j.

We have the partition

$$\{1, \ldots, n\}$$
$$= V(G_t(e_i, T)) \cup V(G_h(e_j, T)) \cup \{V(G_h(e_i, T)) \cap V(G_t(e_j, T))\}.$$

By Theorem 1,

$$n^2 m_{i\ell} m_{j\ell} = \begin{cases} b(b+c) & \text{if} \quad \ell \in V(G_t(e_i, T)) \\ a(a+c) & \text{if} \quad \ell \in V(G_h(e_j, T)) \\ -ab & \text{if} \quad \ell \in V(G_h(e_i, T)) \cap V(G_t(e_j, T)). \end{cases}$$

Thus the (i, j) − entry of $K^{-1} = MM^T$ is,

$$\sum_{\ell=1}^{n} m_{i\ell}\, m_{j\ell} = \frac{1}{n^2}(ab(b+c) + ab(a+c) - abc)$$
$$= \frac{1}{n^2} ab(a+b+c) = \frac{1}{n} ab.$$

The case when e_i and e_j are oriented oppositely with respect to the (r, v)-path can be handled similarly; the (i, j)-entry of K^{-1} turns out to be $-\frac{1}{n}\, ab$ in that case.

Let $\chi(i, j)$ be equal to 1 or −1 according as e_i and e_j are oriented similarly or oppositely with respect to the (r, v)-path respectively. We have thus proved the following result, which appears in a slightly different form and with different notation, in Moon [11].

THEOREM 4 *Using the notation introduced above, the (i, j)-entry of K^{-1} equals $\frac{1}{n}\chi(i, j)ab$.*

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