Structure of a Nonnegative Regular Matrix and Its Generalized Inverses

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ABSTRACT

A nonnegative matrix is called regular if it admits a nonnegative generalized inverse. The structure of such matrices has been studied by several authors. If A is a nonnegative regular matrix, then we obtain a complete description of all nonnegative generalized inverses of A. In particular, it is shown that if A is a nonnegative regular matrix with no zero row or column, then the zero-nonzero pattern of any nonnegative generalized inverse of A is dominated by that of A^T, the transpose of A. We also obtain the structure of nonnegative matrices which admit nonnegative least-squares and minimum-norm generalized inverses.

1. PRELIMINARIES

A matrix A is *nonnegative* (*positive*) if each entry of the matrix is nonnegative (positive), in which case we write $A \geqslant 0$ ($A > 0$). For matrices A, B, the notation $A \geqslant B$ is used to denote the fact that $a_{ij} \geqslant b_{ij}$ for all i, j. The transpose of the matrix A will be denoted by A^T.

Let A be an $m \times n$ matrix, and consider the equations

$$AGA = A, \tag{1}$$

$$GAG = G, \tag{2}$$

$$(AG)^T = AG, \tag{3}$$

$$(GA)^T = GA. \tag{4}$$

A matrix G satisfying (1) is called a generalized inverse (or a g-inverse) of A. If G satisfies (1), (3), then it is a {1, 3}-inverse or a *least-squares g-inverse* of A, and if it satisfies (1), (4), then it is a {1, 4}-inverse or a *minimum-norm g-inverse* of A. Finally, if G satisfies (1)–(4), then it is the *Moore-Penrose inverse* of A. The Moore-Penrose inverse of a real matrix A, denoted by A^+, always exists and is unique.

A nonnegative matrix is *regular* if it admits a nonnegative g-inverse. In this paper, by a regular matrix we always mean a nonnegative regular matrix.

A square matrix is a *monomial matrix* if it has precisely one nonzero entry in each row and column.

Let P be an $m \times n$ $0-1$ matrix. A nonnegative matrix A is said to have *pattern* P if a_{ij} is nonzero if and only if p_{ij} is nonzero. If A, B are nonnegative $m \times n$ matrices, then A, B are said to have the same pattern if $a_{ij} = 0$ if and only if $b_{ij} = 0$, whereas B is said to be dominated by A *in pattern* if $b_{ij} = 0$ whenever $a_{ij} = 0$.

2. NONNEGATIVE G-INVERSES OF A REGULAR MATRIX

The structure of regular matrices has been investigated by several authors; see [1–4, 6]. In particular, it is known that a nonnegative matrix of rank r is regular if and only if it has a monomial submatrix of order r. More precisely, we have the following.

THEOREM 1. *Let A be an $m \times n$ regular matrix of rank r. Then there exist permutation matrices P, Q such that*

$$PAQ = \begin{bmatrix} M & MU \\ VM & VMU \end{bmatrix},$$

where M is an $r \times r$ diagonal matrix and U, V are nonnegative matrices of order $r \times (n-r)$, $(m-r) \times r$ respectively.

The main purpose of this section is to describe the structure of all nonnegative g-inverses of a regular matrix. For this purpose it will be sufficient to consider matrices with no zero row or column, in view of the following simple result. The proof is omitted.

LEMMA 2. *Let A be an $m \times n$ matrix partitioned as*

$$A = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the class of all g-inverses of A is given by matrices of the form

$$\begin{bmatrix} B^{-} & X \\ Y & Z \end{bmatrix},$$

where B^{-} is a g-inverse of B and X, Y, Z are arbitrary matrices of appropriate dimension.

We will need the following result; see, for example, [5, p. 25]. A proof is included for completeness.

LEMMA 3. *Let A be an $m \times n$ matrix. Then the class of all g-inverses of A is given by matrices of the form*

$$A \quad + X - A \quad AXAA \quad , \tag{5}$$

where A is a specific g-inverse of A and X is arbitrary.

Proof. Clearly, for any X, (5) is indeed a g-inverse of A. Conversely, if G is a g-inverse of A, then G can be put in the form (5) by setting $X = G - A^{-}$. ■

The following is the main result.

THEOREM 4. *Let A be an $m \times n$ regular matrix of rank r with no zero row or column, and let G be a nonnegative g-inverse of A. Then G is* *dominated by A^T in pattern. Furthermore, there exist permutation matrices P, Q such that*

$$PAQ = \left[\begin{array}{cccc|c} A_{11} & 0 & \cdots & 0 & \\ 0 & A_{22} & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & * \\ 0 & 0 & \cdots & A_{rr} & \\ \hline & & * & & * \end{array}\right],$$

$$Q^T G P^T = \left[\begin{array}{cccc|c} G_{11} & 0 & \cdots & 0 & \\ 0 & G_{22} & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & G_{rr} & \\ \hline & & 0 & & * \end{array}\right], \tag{6}$$

where A_{ii} is a positive, rank-one matrix and $A_{ii}G_{ii}A_{ii} = A_{ii}$, $i = 1, 2, \dots, r$. (Specifically, if $A_{ii} = x_i y_i^T$ for positive vectors x_i, y_i, then G_{ii} is a nonnegative matrix satisfying $y_i^T G_{ii} x_i = 1$, $i = 1, 2, \dots, r$.)

Proof. We remark that the $*$'s in the expression for PAQ in the theorem are not completely arbitrary, since, for example, there is a rank restriction on the matrix, so that the $*$ blocks cannot contribute to the rank of PAQ. By Theorem 1 we may assume, after permuting the rows and the columns if necessary, that

$$A = \begin{bmatrix} M & MU \\ VM & VMU \end{bmatrix},$$

where $M = \operatorname{diag}(m_{11}, \dots, m_{rr})$ is an $r \times r$ diagonal matrix and U, V are nonnegative matrices of order $r \times (n-r)$, $(m-r) \times r$ respectively. Then

$$A^- = \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

is a g-inverse of A. We have

$$A^-A = \begin{bmatrix} I & U \\ 0 & 0 \end{bmatrix}, \qquad AA^- = \begin{bmatrix} I & 0 \\ V & 0 \end{bmatrix},$$

If G is a g-inverse of A, then by Lemma 3,

$$G = \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} - \begin{bmatrix} I & U \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ V & 0 \end{bmatrix}$$

for some $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$. Thus

$$G = \begin{bmatrix} M^{-1} - (UX_{21} + X_{12}V + UX_{22}V) & X_{12} \\ X_{21} & X_{22} \end{bmatrix}.$$

If $G \geqslant 0$, then $X_{12} \geqslant 0$, $X_{21} \geqslant 0$, and $X_{22} \geqslant 0$. Furthermore,

$$M^{-1} \geqslant UX_{21} + X_{12}V + UX_{22}V \geqslant 0,$$

and therefore UX_{21}, $X_{12}V$ and $UX_{22}V$ must be diagonal matrices.

Note that $UX_{21} = \sum_j U_{.j}(X_{21})_{j.}$, where $U_{.j}$ is the jth column of U and $(X_{21})_{j.}$ is the jth row of X_{21}. Each $U_{.j}(X_{21})_{j.}$ is a diagonal matrix and no $U_{.j}$ is the zero vector, so we conclude that if $U_{.j}$ has more than one positive entry, then $(X_{21})_{j.}$ is zero, while if $U_{.j}$ has one positive entry in position i, then $(X_{21})_{j.}$ has at most one positive entry, necessarily in position i. In particular, X_{21}^T is dominated by U (and hence by MU) in pattern. A similar argument applies for X_{12}^T and VM.

We now claim that X_{22}^T is dominated by VU (and hence by VMU) in pattern. Suppose the (i, j) entry of VU is zero but the (i, j) entry of X_{22}^T is positive. Then there exist p, q such that the (i, p) entry of V and the (q, j) entry of U are both positive. Observe that, since the (i, j) entry of VU is zero, then $p \neq q$. We have $u_{qj}(X_{22})_{ji}v_{ip} > 0$ and hence the (q, p) entry of $UX_{22}V$ is positive. However, this is a contradiction, since $UX_{22}V$ is a diagonal matrix. Thus the claim is proved.

We have therefore shown that G is dominated by A^T in pattern.

We now take a closer look at the structure of A and G. Assume, without loss of generality, that

$$U = [U_1, \ldots, U_r, U_{r+1}],$$

where U_i has all positive entries in the ith row and zeros elsewhere, $i = 1, 2, \dots, r$, and in U_{r+1}, each column has at least two nonzero entries. (It is possible that some U_i is vacuous). Let

$$X_{21}^T = [W_1, \dots, W_r, W_{r+1}]$$

be a conformal partitioning of X_{21}^T.

Similarly, we assume, without loss of generality, that

$$V = \begin{bmatrix} V_1 \\ \vdots \\ V_r \\ V_{r+1} \end{bmatrix}, \qquad X_{12}^T = \begin{bmatrix} Z_1 \\ \vdots \\ Z_r \\ Z_{r+1} \end{bmatrix},$$

where V_i has all positive entries in the ith column and zeros elsewhere, $i = 1, 2, \dots, r$, and in V_{r+1}, each row has at least two nonzero entries. The matrix A thus has the form

$$A = \left[\begin{array}{ccccc|c} M & MU_1 & MU_2 & \cdots & MU_r & MU_{r+1} \\ V_1 M & V_1 MU_1 & 0 & \cdots & 0 & \vdots \\ V_2 M & 0 & V_2 MU_2 & \cdots & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ V_r M & 0 & 0 & \cdots & V_r MU_r & V_r MU_{r+1} \\ \hline V_{r+1} M & V_{r+1} MU_1 & \cdots & \cdots & V_{r+1} MU_r & V_{r+1} MU_{r+1} \end{array}\right],$$

in view of the fact that $V_i MU_j = 0$, $1 \leqslant i \neq j \leqslant r$.

Now partition G accordingly. As seen before, the blocks of G corresponding to the blocks $MU_{r+1}, \dots, V_r MU_{r+1}$ as well as to the blocks $V_{r+1} M, \dots, V_{r+1} MU_r$ must be zero.

Set

$$A_{ii} = \begin{bmatrix} m_{ii} & (MU_i)_{i\cdot} \\ (V_i M)_{\cdot i} & (V_i)_{\cdot i}(MU_i)_{i\cdot} \end{bmatrix}, \qquad i = 1, 2, \dots, r,$$

where the subscript $i\cdot$ denotes ith row and the subscript $\cdot i$ denotes ith column.

Now it can be seen that the form given in (6) is obtained by another permutation of the rows and the columns in A, and simultaneously in G^T. That completes the proof. ■

We now note some consequences of Theorem 4. Nonnegative matrices which admit nonnegative {1, 3}-inverse and nonnegative {1, 4}-inverse have been characterized in [2]. However, our characterization, given in the next result, describes the form of such matrices in a more explicit way.

COROLLARY 5. *Let A be $m \times n$ nonnegative matrix of rank r with no zero row or column. Then A admits a nonnegative {1,3}-inverse if and only if there exist permutation matrices P, Q such that*

$$PAQ = \begin{bmatrix} B & * \end{bmatrix},$$

where B is a direct sum of r positive, rank-one matrices. The matrix A admits a nonnegative {1, 4}-inverse if and only if there exist permutation matrices P, Q such that

$$PAQ = \begin{bmatrix} B \\ * \end{bmatrix},$$

where B is a direct sum of r positive, rank-one matrices.

Proof. First suppose that G is a nonnegative {1, 3}-inverse of A. Considering GAG if necessary, we may assume, without loss of generality, that G satisfies (1), (2), (3). By Theorem 4, there exist permutation matrices such that (6) holds. Since A has rank r, we have

$$PAQ = \begin{bmatrix} B & * \\ XB & * \end{bmatrix}$$

for some matrix X, where B denotes the direct sum of $A_{11}, \dots, A_{rr}$.

Since G is a reflexive g-inverse of A, it also has rank r, and therefore we have

$$Q^T G P^T = \begin{bmatrix} H & 0 \\ 0 & 0 \end{bmatrix}.$$

where H denotes the direct sum of $G_{11}, \dots, G_{rr}$. Since AG is symmetric, then

$$PAQQ^TGP^T = \begin{bmatrix} BH & 0 \\ XBH & 0 \end{bmatrix}$$

is also symmetric. Thus $XBH = 0$. However, $AGA = A$ implies that $BHB = B$ and hence $XBHB = XB = 0$. Now the $*$ in the (2, 2) block of PAQ must also be zero, since PAQ has rank r. Since A has no zero row, the (2, 1), (2, 2) blocks in PAQ must in fact be vacuous.

Conversely, suppose PAQ has the form asserted in the corollary where B is a direct sum of $A_{11}, \dots, A_{rr}$. Let $A_{ii} = x_i y_i^T$ for positive vectors x_i, y_i, $i = 1, \dots, r$. Set

$$G_{ii} = \frac{1}{(x_i^T x_i)(y_i^T y_i)} y_i x_i^T, \qquad i = 1, \dots, r.$$

Then

$$Q\begin{bmatrix} G_{11} & 0 & \cdots & 0 \\ 0 & G_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_{rr} \\ 0 & 0 & \cdots & 0 \end{bmatrix}P$$

is a nonnegative {1, 3}-inverse of A. The second part of the result is proved similarly. ■

The next result has been obtained in [4], and it follows immediately from Corollary 5.

COROLLARY 6. *Let A be an $m \times n$ nonnegative matrix of rank r. Then $A^+ \geqslant 0$ if and only if there exist permutation matrices P, Q such that*

$$PAQ = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix},$$

where B is a direct sum of r positive, rank-one matrices.