On Interacting Systems of Hilbert-Space-Valued Diffusions*

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Abstract. A nonlinear Hilbert-space-valued stochastic differential equation where L^{-1} (L being the generator of the evolution semigroup) is not nuclear is investigated in this paper. Under the assumption of nuclearity of L^{-1}, the existence of a unique solution lying in the Hilbert space H has been shown by Dawson in an early paper. When L^{-1} is not nuclear, a solution in most cases lies not in H but in a larger Hilbert, Banach, or nuclear space. Part of the motivation of this paper is to prove under suitable conditions that a unique strong solution can still be found to lie in the space H itself. Uniqueness of the weak solution is proved without moment assumptions on the initial random variable.

A second problem considered is the asymptotic behavior of the sequence of empirical measures determined by the solutions of an interacting system of H-valued diffusions. It is shown that the sequence converges in probability to the unique solution Λ_0 of the martingale problem posed by the corresponding McKean–Vlasov equation.

Key Words. Martingale problem, Nuclear, Interacting Hilbert-space-valued diffusions, McKean–Vlasov equation, Propagation of chaos.

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1. Introduction

In his pioneering paper on stochastic evolution equations, Dawson [3] considered equations of the type

$$dX_t = -LX_t\,dt + B(t, X_t)\,dW_t + A(t, X_t)\,dt \tag{1.1}$$

for an H-valued process, H being a separable Hilbert space, where W_t is a cylindrical Brownian motion and L is an unbounded operator such that L^{-1} is a nuclear operator. The presence of L allowed him to consider coefficients B which take values in the space of bounded operators (as opposed to the Hilbert–Schmidt operator) and still have the solution as an H-valued process. The solution obtained in [3] was also shown to have continuous sample paths. Such equations have also been investigated by other authors [5], [8]. We are concerned with two problems in this paper. In Section 2 we study, in some detail, the regularity properties of the unique H-valued solution of (1.1) under stronger conditions on the coefficients A and B but without assuming that L^{-1} is nuclear. The principal results of this section, Theorem 2.6, as well as Theorems 2.9 and 2.10 in which the martingale problem corresponding to (2.2) is shown to be well posed, may be regarded as extensions or generalizations of previous results in [3] or [5]. That, however, is not our main reason for presenting them here.

In a large class of stochastic partial differential equations (SPDEs) or H-valued SDEs the equation can be written as a stochastic evolution equation in which L^{-p} is nuclear for some $p > 1$ but L^{-1} is not. The examples that come to mind are the SPDEs which can be formally written as H-valued SDEs where the order m of the elliptic operator L does not exceed $\frac{1}{2}d$, d being the number of spatial variables. (See [5].) In such cases, only a generalized solution can be shown to exist. By the latter we mean a stochastic process taking values in some larger Hilbert space, H_{-p}, say, where H_{-p} is a suitable space of distributions.

The role played by the results of Section 2 is to provide a method of obtaining H or $C([0, T], H)$-valued approximations to the distribution valued, i.e., $C([0, T], H_{-p})$-valued solutions of more general SDEs. This problem is studied in Section 6 in which a systematic procedure is given for approximating many types of SPDEs which only have distribution-valued solutions by "smoother" SPDEs whose solutions are ordinary random fields. An application to Walsh's SDE for a two-dimensional neuron model is given [11].

The second problem is addressed in Sections 3–5 which are devoted to interacting H-valued SDEs. The main aim in these sections is to study the asymptotic behavior of the sequence of empirical measures

$$\Gamma^N := \frac{1}{N}\sum_{j=1}^{N}\delta_{X^{N,j}},$$

where δ_x is the Dirac measure at x and $\{X^{N,j}\}$, $j = 1, 2, \ldots, N$, is the unique solution of the interacting system. As a first step, a law of large numbers (propagation of chaos) is derived (Theorems 5.3 and 5.4) by showing that the above sequence of random elements with values in the class $\mathcal{P}(\mathcal{C})$ of probability measures on $\mathcal{C} = C([0, T], H)$ converges in probability to a nonrandom limit $\Lambda_0 \in \mathcal{P}(\mathcal{C})$, which is the unique solution to the martingale problem corresponding to the McKean–Vlasov SDE (4.3) introduced in Section 4. The latter equation is central to the proof of the propagation of chaos result in the H-valued context as it is in the case of finite-dimensional interacting diffusions. Furthermore, in our discussion of the McKean–Vlasov equation the following interesting fact emerges which may be of independent interest: If Λ_0 is as described above and (η_t) is the coordinate process on $\mathcal{C}$, then $\mu_t := \Lambda_0 \circ \eta_t^{-1}$ satisfies the remarkable nonlinear (and nonrandom) measure-valued equation (4.11), and under additional assumptions on the coefficients, the nonlinear equation has a unique solution.

2. Infinite-Dimensional Diffusions

Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. $\mathcal{L}(H, H)$ denotes the class of all continuous linear operators on H and $\mathcal{L}_2(H, H)$ the class of all Hilbert–Schmidt operators. For an operator A, the Hilbert–Schmidt norm is denoted by $\|\cdot\|_{\text{HS}}$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with filtration $(\mathcal{F}_t)$ assumed to satisfy the usual conditions. Let (W_t) be an $(\mathcal{F}_t)$-cylindrical Brownian motion on H. Recall that, for a progressively measurable H-valued process f such that $\int_0^t \|f_s\|^2\,ds < \infty$ a.s. for all t, the stochastic integral $\int_0^t \langle f_s, dW_s\rangle$ is defined as follows. Let $\{\varphi_k\}$ be a CONS in H. Let $W_t^k := W_t(\varphi_k)$. Then W_t^k is a sequence of independent real-valued Brownian motions and

$$\int_0^t \langle f_s, dW_s\rangle := \sum_{k=1}^{\infty} \int_0^t \langle f_s, \varphi_k\rangle dW_s^k.$$

It can be proved that the series appearing above converges uniformly in $t \in [0, \infty)$ for all ω outside a null set.

The indefinite integral is a continuous local martingale with quadratic variation process $\int_0^t \|f_s\|^2\,ds$. For a progressively measurable $\mathcal{L}_2(H, H)$-valued process F, with $\int_0^t \|F_s\|_{\text{HS}}^2\,ds < \infty$ a.s. for all t, the stochastic integral $\int_0^t F_s\,dW_s$ is defined and satisfies, for $\varphi \in H$,

$$\left\langle \int_0^t F_s\,dW_s, \varphi \right\rangle = \int_0^t \langle F_s^* \varphi, dW_s\rangle. \tag{2.1}$$

Here F_s^* denotes the adjoint of the operator F_s. Indeed, (2.1) can be taken as the definition of the stochastic integral

$$\int_0^t F_s\,dW_s.$$

We need the following estimate, which is Burkholder's inequality in this context. It is stated in a weaker form (without sup over t), suitable for use later.

Lemma 2.1. *For $2 \le p < \infty$, there exist constants C_p depending only on p such that for a progressively measurable $\mathcal{L}_2(H, H)$-valued process F with*

$$E\left[\left\{\int_0^t \|F_s\|_{\text{HS}}^2\,ds\right\}^{p/2}\right] < \infty$$

one has

$$E\left[\left\|\int_0^t F_s\,dW_s\right\|^p\right] \le C_p E\left[\left\{\int_0^t \|F_s\|_{\mathrm{HS}}^2\,ds\right\}^{p/2}\right].$$

Outline of the Proof. Let $\{\varphi_k\}$ be a CONS in H. Using Burkholder's inequality (see, e.g., p. 117 of [9]), we get

$$E\left[\left\{\sum_{k=1}^d \left(\int_0^t \langle F_s^*\varphi_k, dW_s\rangle\right)^2\right\}^{p/2}\right] \le C_p E\left[\left\{\sum_{k=1}^d \int_0^t \|F_s^*\varphi_k\|^2\,ds\right\}^{p/2}\right].$$

The required inequality follows from this, using Fatou's lemma. □

We consider the following SDE:

$$dX_t = -LX_t\,dt + B(t, X_t)\,dW_t + A(t, X_t)\,dt, \tag{2.2}$$

where X_0 is independent of (W_t). Here the operator L is assumed to satisfy the following conditions:

$$T_t \equiv e^{-tL} \qquad \text{is a contraction semigroup on } H, \tag{2.3}$$

$$L^{-1} \text{ is a bounded self-adjoint operator with discrete spectrum.} \tag{2.4}$$

Let $\{\varphi_k\}$ be the eigenfunctions of L^{-1}, which constitutes a CONS in H and let $\{\lambda_k\}$ be the corresponding eigenvalues. We assume also that A: $[0, T] \times H \to H$ and B: $[0, T] \times H \to L(H, H)$ are continuous functions satisfying

$$|\langle A(t, h), \varphi_k\rangle| \le a_k(1 + \|h\|^2)^{1/2}, \tag{2.5}$$

$$\|B^*(t, h)\varphi_k\| \le b_k(1 + \|h\|^2)^{1/2}, \tag{2.6}$$

$$|\langle A(t, h_1) - A(t, h_2), \varphi_k\rangle| \le a_k\|h_1 - h_2\|, \tag{2.7}$$

$$\|(B^*(t, h_1) - B^*(t, h_2))\varphi_k\| \le b_k\|h_1 - h_2\| \tag{2.8}$$

for all $k \ge 1$, $t \in [0, T]$, $h, h_1, h_2 \in H$, where B^* is the adjoint of the operator B and $\{a_k\}$, $\{b_k\}$ satisfy

$$\sum_{k=1}^\infty a_k^2\lambda_k^{-1} \equiv C_{2,1} < \infty, \tag{2.9}$$

$$\sum_{k=1}^\infty b_k^2\lambda_k^{-1} \equiv C_{2,2} < \infty. \tag{2.10}$$

Under these conditions the stochastic integral $\int_0^t B(s, X_s)\,dW_s$ may not be defined. However, for any progressively measurable process (X_t),

$$\int_0^t \|T_{t-s}B(s, X_s)\|_{\mathrm{HS}}^2\,ds \le \int_0^t \sum_{k=1}^{\infty} e^{-2(t-s)\lambda_k} b_k^2(1 + \|X_s\|^2)\,ds$$
$$= \int_0^t f_B(t-s)(1 + \|X_s\|^2)\,ds, \tag{2.11}$$

where

$$f_B(t) = \sum_{k=1}^{\infty} e^{-2\lambda_k t} b_k^2. \tag{2.12}$$

Since $\int_0^t f_B(u)\,du \le \sum_{k=1}^{\infty} b_k^2 \lambda_k^{-1} = C_{2,2}$ it follows that the stochastic integral referred to above exists if

$$\int_0^T \|X_s\|^2\,ds < \infty \qquad \text{a.s.} \tag{2.13}$$

Similarly,

$$\left[\int_0^t \|T_{t-s}A(s, X_s)\|\,ds\right]^2 \le T \int_0^t \sum_{k=1}^{\infty} e^{-2(t-s)\lambda_k} a_k^2(1 + \|X_s\|^2)\,ds$$
$$= \int_0^t f_A(t-s)(1 + \|X_s\|^2)\,ds, \tag{2.14}$$

where

$$f_A(t) = T \sum_{k=1}^{\infty} e^{-2\lambda_k t} a_k^2 \tag{2.15}$$

and again we have that $\int_0^t f_A(u)\,du \le T C_{2,1}$. Thus for every ω such that (2.13) holds, we also have that the integral

$$\int_0^t T_{t-s}A(s, X_s)\,ds \tag{2.16}$$

is well defined.

Definition 2.1. A progressively measurable process (X_t) is said to be a *mild solution* or *evolution solution* to (2.2) if (2.13) holds and, for every t,

$$X_t = T_t X_0 + \int_0^t T_{t-s}B(s, X_s)\,dW_s + \int_0^t T_{t-s}A(s, X_s)\,ds \qquad \text{a.s.} \tag{2.17}$$

Note that the progressive measurability of (X_t) implies that X_0 is independent of (W_t). It is easy to see that if (X_t) is a solution and (X_t') is a progressively measurable modification of (X_t), i.e., $P(X_t = X_t') = 1$ for all t, then (X_t') is also a solution to (2.2).

It is convenient to define a new probability measure $\tilde{P}$ on $\mathcal{F}$,

$$\tilde{P}(C) = \int_C \exp\{-\|X_0\|\}\,dP \Big/ \int \exp\{-\|X_0\|\}\,dP. \tag{2.18}$$

Clearly, $\tilde{P}$ and P are mutually absolutely continuous and the Radon–Nikodym derivative $d\tilde{P}/dP$ is $\mathcal{F}_0$ measurable. Hence (W_t) is again a cylindrical Brownian motion on $(\Omega, \mathcal{F}, \tilde{P})$. If $M_t = \int_0^t F_s\,dW_s$ on $(\Omega, \mathcal{F}, P)$ and $\tilde{M} = \int_0^t F_s\,dW_s$ on $(\Omega, \mathcal{F}, \tilde{P})$ where $\int_0^T \|F_s\|_{\mathrm{HS}}^2\,ds < \infty$ a.s. (P or $\tilde{P}$), then

$$P(M_t = \tilde{M} \text{ for all } t) = \tilde{P}(M_t = \tilde{M} \text{ for all } t) = 1.$$

Thus (X_t) is a solution to (2.2) on $(\Omega, \mathcal{F}, P)$ if and only if (X_t) is a solution to (2.2) on $(\Omega, \mathcal{F}, \tilde{P})$. Further, we have, for all $p < \infty$,

$$E^{\tilde{P}}\|X_0\|^p < \infty.$$

Here is a version of Gronwall's lemma which will be used in proving existence and uniqueness results for the solution.

Lemma 2.2. (i) *Let f, g, and δ be nonnegative functions on $[0, T]$. Let $\alpha \in [0, \infty)$ such that $\int_0^T e^{-\alpha t} f(t)\,dt \le \frac{1}{2}$. Suppose that either g is bounded or g is integrable and δ is bounded. If, for all $t \le T$,*

$$g(t) \le c + \int_0^t f(s)\{g(t-s) + \delta(t-s)\}\,ds, \tag{2.19}$$

then there exists a nonnegative Borel measure μ on $[0, T]$ such that $\mu[0, t] \le e^{\alpha t}$ and

$$g(t) \le c(1 + e^{\alpha t}) + \int_0^t \delta(t-s)\mu(ds). \tag{2.20}$$

(ii) *Let f, g be nonnegative functions on $\{0, 1, \ldots, n\}$. Let $\alpha \in [0, \infty)$ such that $\sum_{i=1}^n e^{-\alpha i} f(i) \le \frac{1}{2}$. If, for all $0 \le i \le n$,*

$$g(i) \le c + \sum_{j=1}^{i} f(j) g(i-j), \tag{2.21}$$

then

$$g(i) \le c(1 + e^{\alpha i}). \tag{2.22}$$

Proof. Iterating inequality (2.19), we get

$$\begin{aligned} g(t) \le{}& c + \int_0^t f(s_1)\delta(t-s_1)\,ds_1 \\ &+ \int_0^t f(s_1)\left[c + \int_0^{t-s_1} f(s_2)\{g(t-s_1-s_2) + \delta(t-s_1-s_2)\}\,ds_2\right]ds_1 \end{aligned} \tag{2.23}$$

$$
\begin{aligned}
&= c + \int_0^t \{c + \delta(t - s_1)\} f(s_1)\, ds_1 \\
&\quad + \int_0^t \int_0^t \{g(t - s_1 - s_2) + \delta(t - s_1 - s_2)\} f(s_1) f(s_2) 1_{s_1 + s_2 \le t}\, ds_1\, ds_2 \\
&\le \cdots \\
&\le c + \sum_{j=1}^{k} \int_0^t \{c + \delta(t - s)\} \mu_j(ds) - c\mu_k([0, t]) + \int_0^t g(t - s) \mu_k(ds),
\end{aligned}
$$

where

$$
\mu_j([0, t]) = \int_0^t \cdots \int_0^t f(s_1) \cdots f(s_j) 1_{s_1 + \cdots + s_j \le t}\, ds_1 \cdots ds_j .
$$

As

$$
\begin{aligned}
\mu_j([0, t]) &\le e^{\alpha t} \int_0^t \cdots \int_0^t e^{-\alpha(s_1 + \cdots + s_j)} f(s_1) \cdots f(s_j)\, ds_1 \cdots ds_j \\
&\le e^{\alpha t} (\tfrac{1}{2})^j ,
\end{aligned}
$$

$\mu(C) \equiv \sum_{j=1}^{\infty} \mu_j(C)$, $C \in \mathcal{B}([0, T])$ is a well-defined nonnegative Borel measure on $[0, T]$ such that $\mu[0, t] \le e^{\alpha t}$. Letting $k \to \infty$ on the right-hand side of (2.23), we have

$$
\begin{aligned}
g(t) &\le c + \int_0^t \{c + \delta(t - s)\} \mu(ds) + \liminf_{k \to \infty} \int_0^t g(t - s) \mu_k(ds) \\
&\le c(1 + e^{\alpha t}) + \int_0^t \delta(t - s) \mu(ds) + \liminf_{k \to \infty} \int_0^t g(t - s) \mu_k(ds). \qquad (2.24)
\end{aligned}
$$

If g is bounded, then $\liminf_{k \to \infty} \int_0^t g(t - s) \mu_k(ds) = 0$ and hence (2.20) holds. If g is integrable and δ is bounded, then

$$
\begin{aligned}
\int_0^T \liminf_{k \to \infty} \int_0^t g(t - s) \mu_k(ds)\, dt &\le \liminf_{k \to \infty} \int_0^T \int_0^t g(t - s) \mu_k(ds)\, dt \\
&\le \int_0^T g(t)\, dt \liminf_{k \to \infty} \mu_k([0, T]) = 0,
\end{aligned}
$$

i.e., $\liminf_{k \to \infty} \int_0^t g(t - s) \mu_k(ds) = 0$ for a.e. $t \in [0, T]$, and hence, for a.e. $t \in [0, T]$,

$$
g(t) \le c(1 + e^{\alpha t}) + \|\delta\|_\infty e^{\alpha t}.
$$

By (2.19), $\forall t \in [0, T]$,

$$
g(t) \le c + \int_0^T f(s)\, ds\, (c + \|\delta\|_\infty) \left(1 + e^{\alpha T}\right),
$$

i.e., g is bounded and hence (2.20) holds. Inequality (2.22) can be proved similarly. □

We now obtain an estimate on the second moment of a solution.

Theorem 2.3. *If* (X_t) *is a solution to* (2.2) *satisfying* $E\|X_0\|^2 < \infty$, *then*

$$\sup_{t\le T} E\|X_t\|^2 \le C_{2,3}[1 + E\|X_0\|^2], \tag{2.25}$$

where $C_{2,3}$ *is a constant,*

Proof. Let (X_t) be a solution to (2.2) satisfying (2.13). Then it follows that

$$\langle X_t, \varphi_k\rangle = e^{-\lambda_k t}\langle X_0, \varphi_k\rangle + \int_0^t \langle e^{-\lambda_k(t-s)} B^*(s, X_s)\varphi_k, dW_s\rangle + \int_0^t e^{-\lambda_k(t-s)}\langle A(s, X_s), \varphi_k\rangle\, ds \tag{2.26}$$

and hence that

$$d\langle X_t, \varphi_k\rangle = \langle B^*(t, X_t)\varphi_k, dW_t\rangle + \langle A(t, X_t) - \lambda_k X_t, \varphi_k\rangle dt. \tag{2.27}$$

Fix n and define a stopping time τ_n by

$$\tau_n = \inf\left\{t \ge 0\colon \int_0^t \|X_s\|^2\, ds \ge n\right\} \wedge T \tag{2.28}$$

and let

$$\xi_t^k \equiv e^{\lambda_k(t\wedge\tau_n)}\langle X_{t\wedge\tau_n}, \varphi_k\rangle.$$

Note that $\tau_n \to T$ since (X_t) is assumed to satisfy (2.13). It is easy to see that

$$\xi_t^k = \xi_0^k + \int_0^{t\wedge\tau_n} e^{\lambda_k s}\langle B^*(s, X_s)\varphi_k, dW_s\rangle + \int_0^{t\wedge\tau_n} e^{\lambda_k s}\langle A(s, X_s), \varphi_k\rangle ds$$

and hence that

$$\begin{aligned} E|\xi_t^k|^2 &\le 3E\left[|\xi_0^k|^2 + \int_0^{t\wedge\tau_n} e^{2\lambda_k s}\|B^*(s, X_s)\varphi_k\|^2\, ds + t\int_0^{t\wedge\tau_n} e^{2\lambda_k s}\langle A(s, X_s), \varphi_k\rangle^2\, ds\right] \\ &\le 3\left[E|\xi_0^k|^2 + \int_0^t e^{2\lambda_k s}(b_k^2 + Ta_k^2)E\{(1 + \|X_s\|^2)1_{s<\tau_n}\}\, ds\right]. \end{aligned}$$

Using that $E[\|X_t\|^2 1_{t<\tau_n}] \le \sum_k e^{-2\lambda_k t}E|\xi_t^k|^2$ we get

$$\begin{aligned} E[\|X_t\|^2 1_{t<\tau_n}] &\le 3\left[E\|X_0\|^2 + \int_0^t \sum_k e^{-2\lambda_k(t-s)}(b_k^2 + Ta_k^2)E\{(1 + \|X_s\|^2)1_{s<\tau_n}\}\, ds\right] \\ &\le 3\left[E\|X_0\|^2 + TC_{2,1} + C_{2,2} + \int_0^t f_0(t-s)E\{\|X_s\|^2 1_{s<\tau_n}\}\, ds\right], \end{aligned}$$

where $f_0(u) = f_B(u) + f_A(u)$ is an integrable function (see (2.12) and (2.15)).

Let $g(t) = E[\|X_t\|^2 1_{t<\tau_n}]$, $f(t) = 3f_0(t)$, $c = 3\left[E\|X_0\|^2 + TC_{2,1} + C_{2,2}\right]$, and $\delta(t) = 0$, $\forall t \in [0, T]$. Then (2.19) holds. As $\int_0^T g(t)\,dt = \int_0^T E[\|X_s\|^2 1_{s<\tau_n}]\,ds \le n$ by the choice of τ_n, g is integrable. It is clear that δ is bounded. Since

$$\int_0^T e^{-\alpha t} f(t)\,dt = 3\int_0^T e^{-\alpha t} f_0(t)\,dt \le 3\sum_{k=1}^{\infty} \frac{Ta_k^2 + b_k^2}{\alpha + 2\lambda_k} \to 0$$

as $\alpha \to \infty$, there exists α such that

$$3\sum_{k=1}^{\infty} \frac{Ta_k^2 + b_k^2}{\alpha + 2\lambda_k} \le \frac{1}{2}. \tag{2.29}$$

Therefore, all conditions in Lemma 2.2(i) are satisfied and, hence,

$$E[\|X_t\|^2 1_{t<\tau_n}] = g(t) \le 3[E\|X_0\|^2 + TC_{2,1} + C_{2,2}](1 + e^{\alpha T}).$$

Now the result follows from Fatou's lemma by letting $n \to \infty$. □

The next result proves the existence and uniqueness of the solution to (2.2).

Theorem 2.4. *Suppose that L, A, and B satisfy (2.3)–(2.10). Let X_0 be an $\mathcal{F}_0$-measurable H-valued random variable and let (W_t) be an $(\mathcal{F}_t)$-cylindrical Brownian motion. Then:*

(i) *There exists a solution $(\hat{X}_t)$ of (2.2) satisfying (2.13) with $\hat{X}_0 = X_0$.*
(ii) *If $\{X_t\}$ and $\{U_t\}$ are solutions to (2.2) satisfying (2.13) such that $X_0 = U_0$, then*

$$P(X_t = U_t) = 1 \qquad \text{for all } t. \tag{2.30}$$

Proof. (i) Let $\tilde{P}$ be defined by (2.18). It suffices to construct a solution on $(\Omega, \mathcal{F}, \tilde{P})$. For $n \ge 1$, let $t_i^n = (i/n)T$, $0 \le i \le n$. Let $X_0^n = X_0$ and define $\{X_t^n, t_i^n < t \le t_{i+1}^n\}$, $i \ge 0$, inductively as follows. For $t_i^n < t \le t_{i+1}^n$, let

$$X_t^n = T_{t-t_i^n} X_{t_i^n}^n + \int_{t_i^n}^t T_{t-u} B(u, X_{t_i^n}^n)\,dW_u + \int_{t_i^n}^t T_{t-u} A(u, X_{t_i^n}^n)\,du. \tag{2.31}$$

Similar to (2.11) and (2.14), $\forall t_i^n < t \le t_{i+1}^n$, we have

$$\begin{aligned}\tilde{E}\|X_t^n\|^2 &\le 3\left[\tilde{E}\|X_{t_i^n}^n\|^2 + \int_{t_i^n}^t \sum_k e^{-2\lambda_k(t-s)}(b_k^2 + Ta_k^2)\tilde{E}(1 + \|X_{t_i^n}^n\|^2)\,ds\right] \\ &\le 3(1 + TC_{2,1} + C_{2,2})(1 + \tilde{E}\|X_{t_i^n}^n\|^2).\end{aligned} \tag{2.32}$$

Let $Y_t^n = X_{t_i^n}^n$ for $t_i^n < t \le t_{i+1}^n$. Then

$$X_t^n = T_t X_0 + \int_0^t T_{t-u} B(u, Y_u^n)\,dW_u + \int_0^t T_{t-u} A(u, Y_u^n)\,du. \tag{2.33}$$

Proceeding as in (2.11) and (2.14), it follows that

$$\tilde{E}\|X_t^n\|^2 \leq 3\left[\tilde{E}\|X_0\|^2 + TC_{2,1} + C_{2,2} + \int_0^t f_0(t-s)\,\tilde{E}\|Y_s^n\|^2\,ds\right], \tag{2.34}$$

where $f_0 = f_A + f_B$. Let $g_n(i) = \tilde{E}\|X_{t_i^n}^n\|^2$, $0 \leq i \leq n$. By (2.32) and induction in i, it is easy to show that $g_n(i)$ is a finite-valued function on $i \in \{0, 1, \ldots, n\}$. It follows from (2.34) that

$$\begin{aligned} g_n(i) &\leq 3\left[\tilde{E}\|X_0\|^2 + TC_{2,1} + C_{2,2} + \sum_{j=0}^{i-1}\int_{t_j^n}^{t_{j+1}^n}\sum_k (b_k^2 + Ta_k^2)e^{-2\lambda_k(t_i^n - s)}\,ds\,g_n(j)\right] \\ &\leq 3(\tilde{E}\|X_0\|^2 + TC_{2,1} + C_{2,2}) + \sum_{j=1}^{i} f_n(j)g_n(i-j), \end{aligned} \tag{2.35}$$

where

$$f_n(i) = 3\sum_k \frac{b_k^2 + Ta_k^2}{2\lambda_k}e^{-2\lambda_k t_i^n}(e^{(2\lambda_k/n)T} - 1).$$

Let α be given by (2.29). Then

$$\begin{aligned} \sum_{i=1}^n f_n(i)e^{-(\alpha T/n)i} &= \sum_{i=1}^n 3\sum_k \frac{b_k^2 + Ta_k^2}{2\lambda_k}e^{-((2\lambda_k T + \alpha T)/n)i}(e^{(2\lambda_k/n)T} - 1) \\ &\leq 3\sum_{k=1}^\infty \frac{Ta_k^2 + b_k^2}{2\lambda_k}\,\frac{e^{(2\lambda_k T/n)} - 1}{e^{((2\lambda_k + \alpha)/n)T} - 1} \\ &\leq 3\sum_{k=1}^\infty \frac{Ta_k^2 + b_k^2}{\alpha + 2\lambda_k} \leq \frac{1}{2} \end{aligned}$$

and hence, by Lemma 2.2,

$$\tilde{E}\|X_{t_i^n}^n\|^2 = g_n(i) \leq 3(\tilde{E}\|X_0\|^2 + TC_{2,1} + C_{2,2})(1 + e^{(\alpha T/n)i}).$$

It then follows from (2.32) again that

$$\sup_{n\geq 1}\sup_{0\leq t\leq T}\tilde{E}\|X_t^n\|^2 \leq C'[1 + \tilde{E}\|X_0\|^2] = C''. \tag{2.36}$$

Using (2.33) for n, m and using the Lipschitz conditions on A, B we get (the calculations are similar to those in (2.11) and (2.14))

$$\begin{aligned} \tilde{E}\|X_t^n - X_t^m\|^2 &\leq 2\tilde{E}\left\{\int_0^t \|T_{t-u}(B(u, Y_u^n) - B(u, Y_u^m))\|_{\mathrm{HS}}^2\,du \right. \\ &\qquad \left. + T\int_0^t \|T_{t-u}(A(u, Y_u^n) - A(u, Y_u^m))\|^2\,du\right\} \\ &\leq 2\int_0^t f_0(t-u)\tilde{E}\|Y_u^n - Y_u^m\|^2\,du. \end{aligned}$$

Let $g_{n,m}(t) = \tilde{E}\|X_t^n - X_t^m\|^2$ and $\delta_{n,m}(t) = \tilde{E}\|X_t^n - Y_t^n\|^2 + \tilde{E}\|X_t^m - Y_t^m\|^2$. Then $g_{n,m}$, $\delta_{n,m}$ are uniformly bounded (by (2.36)) and

$$g_{n,m}(t) \le 6\int_0^t f_0(t-u)\{g_{n,m}(u) + \delta_{n,m}(u)\}\,du.$$

Similar to (2.32), it follows from (2.31) and (2.36) that, for $t_i^n < t \le t_{i+1}^n$,

$$\begin{aligned}\tilde{E}\|X_t^n - Y_t^n\|^2 &= \tilde{E}\|X_t^n - X_{t_i^n}^n\|^2 \\ &\le 3\sum_{k=1}^{\infty}(e^{-\lambda_k T/n} - 1)^2\tilde{E}\langle X_{t_i^n}^n, \varphi_k\rangle^2 \\ &\quad +3(1+C'')\sum_{k=1}^{\infty}\frac{Ta_k^2 + b_k^2}{2\lambda_k}(1 - e^{-2(\lambda_k T/n)}).\end{aligned} \tag{2.37}$$

It follows from (2.33) that

$$\begin{aligned}\langle X_t^n, \varphi_k\rangle &= e^{-\lambda_k t}\langle X_0, \varphi_k\rangle + \int_0^t e^{-\lambda_k(t-u)}\langle B(u, Y_u^n)^*\varphi_k, dW_u\rangle \\ &\quad + \int_0^t e^{-\lambda_k(t-u)}\langle A(u, Y_u^n), \varphi_k\rangle\,du\end{aligned}$$

and then

$$\begin{aligned}&\tilde{E}\langle X_t^n, \varphi_k\rangle^2 \\ &\quad\le 3\tilde{E}\langle X_0, \varphi_k\rangle^2 + 3\int_0^t (b_k^2 + Ta_k^2)e^{-2\lambda_k(t-u)}(1 + \tilde{E}\|Y_u^n\|^2)\,du \\ &\quad\le 3\tilde{E}\langle X_0, \varphi_k\rangle^2 + 3(1+C'')\frac{b_k^2 + Ta_k^2}{2\lambda_k}.\end{aligned}$$

Hence, by the dominated convergence theorem, it follows from (2.37) that $\delta_{n,m}(t) \to 0$. By Lemma 2.2 and the dominated convergence theorem again,

$$g_{n,m}(t) \le \int_0^T \delta_{n,m}(t)\mu(dt) \to 0.$$

Therefore

$$\sup_{t\le T}\tilde{E}\|X_t^n - X_t^m\|^2 \to 0, \qquad \sup_{t\le T}\tilde{E}\|Y_t^n - Y_t^m\|^2 \to 0. \tag{2.38}$$

Note that since Y^n is a piecewise constant, left-continuous, adapted process it is progressively measurable. In view of (2.38) we can choose a subsequence $\{n_k\}$ such that $Z_s^k \equiv Y_s^{n_k}$ satisfies

$$\sup_{s\le T}\tilde{E}\|Z_s^k - Z_s^{k+1}\|^2 \le 2^{-k}.$$

Then it follows that $\sum_k \|Z_s^k - Z_s^{k+1}\| < \infty$ a.s. for all s. Thus Z_s^k converges a.s. for each s. Define

$$\hat{X}_s(\omega) = \begin{cases} \lim\limits_{k\to\infty} Z_s^k(\omega) & \text{if it exists in } H, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\hat{X}_s$ is a progressively measurable process. Further, it follows that

$$\sup_{s\le T} \tilde{E}\|Y_s^n - \hat{X}_s\|^2 \to 0, \qquad \sup_{s\le T} \tilde{E}\|X_s^n - \hat{X}_s\|^2 \to 0.$$

From this, it can be verified that $\hat{X}$ is a solution to (2.2) (on $(\Omega, \mathcal{F}, \tilde{P})$) with $\hat{X}_0 = X_0$ and that (2.13) holds. This completes the proof of (i).

For (ii), again let $\tilde{P}$ be given by (2.18). Then $\{X_t\}$ and $\{U_t\}$ are solutions to (2.2) on $(\Omega, \mathcal{F}, \tilde{P})$ and, in view of Theorem 2.3, $\int_0^T \tilde{E}\|X_s - U_s\|^2\,ds < \infty$. Using the Lipschitz conditions on A, B, we can deduce

$$\tilde{E}\|X_t - U_t\|^2 \le 2\left[\int_0^t f_0(t-s)\tilde{E}\|X_s - U_s\|^2\,ds\right].$$

An application of Lemma 2.2, with $c = 0$ and $\delta = 0$, yields

$$\tilde{E}\|X_t - U_t\|^2 = 0$$

for all t. Thus $\tilde{P}(X_t = U_t) = 1$ and hence (2.30) follows. □

We are now in a position to obtain an estimate on the growth of the pth moment of the solution.

Theorem 2.5. *Let $\{X_t\}$ be a solution to (2.2) satisfying (2.13). Then, for $p \ge 2$, there exists a constant C_p' depending only on the constant C_p in Lemma 2.1 and on $C_{2,1}, C_{2,2}$ such that if $E\|X_0\|^p < \infty$, then*

$$\sup_{s\le T} E\|X_s\|^p \le C_p'[1 + E\|X_0\|^p]. \tag{2.39}$$

Proof. Let X_t^n be the approximation constructed in the proof of the previous theorem. Using Lemma 2.1, it follows from (2.31) that, for $t_i^n < t \le t_{i+1}^n$,

$$\begin{aligned} E\|X_t^n\|^p &\le 3^{p-1}\Bigg[E\|X_{t_i^n}^n\|^p + C_pE\left(\int_{t_i^n}^t f_B(t-s)\,ds(1 + \|X_{t_i^n}^n\|^2)\right)^{p/2} \\ &\qquad + E\left(\int_{t_i^n}^t f_A(t-s)\,ds(1 + \|X_{t_i^n}^n\|^2)\right)^{p/2}\Bigg] \\ &\le 3^{p-1}[E\|X_{t_i^n}^n\|^p + (C_pC_{2,2}^{p/2} + (TC_{2,1})^{p/2})E(1 + \|X_{t_i^n}^n\|^2)^{p/2}]. \end{aligned} \tag{2.40}$$

Let $h_n(i) = \tilde{E}\|X^n_{t^n_i}\|^p, 0 \le i \le n$. By (2.40) and by induction in i, we see that $h_n(\cdot)$ is a finite-valued function. By (2.33), proceeding as in (2.40), we have

$$E\|X^n_t\|^p \le 3^{p-1}\left[E\|X_0\|^p + C_p E\left(\int_0^t f_B(t-s)(1+\|Y^n_s\|^2)\,ds\right)^{p/2} + E\left(\int_0^t f_A(t-s)(1+\|Y^n_s\|^2)\,ds\right)^{p/2}\right]. \tag{2.41}$$

Using Hölder's inequality for the ds integrals, we get

$$\begin{aligned} E\|X^n_t\|^p \le 3^{p-1}\Bigg[&E\|X_0\|^p + C_p\left(\int_0^t f_B(t-s)\,ds\right)^{p/2-1} \\ &\times E\left(\int_0^t f_B(t-s)(1+\|Y^n_s\|^2)^{p/2}\,ds\right) \\ &+\left(\int_0^t f_A(t-s)\,ds\right)^{p/2-1} \\ &\times E\left(\int_0^t f_A(t-s)(1+\|Y^n_s\|^2)^{p/2}\,ds\right)\Bigg]. \end{aligned} \tag{2.42}$$

It then follows from similar arguments as in (2.34)–(2.36) that there exists a constant C'_p depending only on p and on $C_{2,1}$, $C_{2,2}$ such that

$$\sup_{n\ge 1}\sup_{0\le t\le T} E\|X^n_t\|^p \le C'_p[1+E\|X_0\|^p]. \tag{2.43}$$

As noted in the previous result, a subsequence of X^n_t converges to $\hat{X}_s$, where $\hat{X}$ is a solution to (2.2). Hence, using Fatou's lemma, it follows that the required moment estimate holds for $\hat{X}$. The result follows from this as $\hat{X}$, X have the same finite dimensional distributions by the uniqueness part of the previous theorem. □

We now look at the regularity of paths of the solution to (2.2).

In order to prove sample continuity of the solution, we impose a stronger condition than (2.10):

$$\sum_{k=1}^{\infty} b_k^2 \lambda_k^{-\theta} \equiv C_{2,4} < \infty \tag{2.44}$$

for some θ, $0 < \theta < 1$.

Theorem 2.6. *Let (X_t) be a solution to* (2.2). *Then (X_t) admits a continuous modification, which is of course a solution to* (2.2).

Proof. Let $\tilde{P}$ be defined by (2.18). It suffices to prove that X has a continuous modification on $(\Omega, \mathcal{F}, \tilde{P})$. We write

$$X_t = T_t X_0 + Y_t + Z_t$$

where $Y_t = \int_0^t T_{t-u} B(u, X_u)\, dW_u$ and $Z_t = \int_0^t T_{t-u} A(u, X_u)\, du$. Clearly, $T_t X_0(\omega)$ is continuous for all ω. For $0 \le s \le t \le T$,

$$\begin{aligned}
\|Z_t - Z_s\|^2 &= \left\| \int_0^s (T_{t-u} - T_{s-u}) A(u, X_u)\, du + \int_s^t T_{t-u} A(u, X_u)\, du \right\|^2 \\
&\le 2 \left[\int_0^s \|(T_{t-u} - T_{s-u}) A(u, X_u)\|\, du \right]^2 \\
&\quad + 2 \left[\int_s^t \|T_{t-u} A(u, X_u)\|\, du \right]^2 \\
&\le 2 \left[\int_0^s \left\{ \sum_k (e^{-\lambda_k (t-u)} - e^{-\lambda_k (s-u)})^2 a_k^2 (1 + \|X_u\|^2) \right\}^{1/2} du \right]^2 \\
&\quad + 2 \left[\int_s^t \left\{ \sum_k e^{-2\lambda_k (t-u)} a_k^2 (1 + \|X_u\|^2) \right\}^{1/2} du \right]^2 \\
&\le \left[\int_0^T (1 + \|X_u\|^2)\, du \right] \alpha(s, t) \qquad (2.45)
\end{aligned}$$

by Hölder's inequality where

$$\alpha(s, t) = \int_0^s \sum_k (e^{-\lambda_k (t-u)} - e^{-\lambda_k (s-u)})^2 a_k^2\, du + \int_s^t \sum_k e^{-2\lambda_k (t-u)} a_k^2\, du.$$

α can be computed and we can verify that $\alpha(s, t) \le \beta(t - s)$ with

$$\beta(\delta) \equiv \sum_{k=1}^{\infty} \frac{a_k^2}{2\lambda_k} [(1 - e^{-\delta\lambda_k})^2 + (1 - e^{-2\delta\lambda_k})]. \qquad (2.46)$$

Clearly, (2.9) implies $\beta(\delta) \to 0$ as $\delta \to 0$. Using (2.13), it follows that

$$\lim_{\delta \to 0} \sup_{0 \le t - s \le \delta} \|Z_t - Z_s\|^2 = 0 \qquad \text{a.s.}$$

Thus $\{Z_t\}$ is continuous a.s.

It remains to show that $\{Y_t\}$ admits a continuous modification. We achieve this via the Kolmogorov criterion. Choose p such that $(1 - \theta)p > 2$, where θ is as in (2.44). Recall that, by the choice of $\tilde{P}$, $\tilde{E}\|X_0\|^p < \infty$ and hence, by Theorem 2.5, $\sup_{s \le T} \tilde{E}\|X_s\|^p < \infty$. As before, $\tilde{E}$ stands for the integral with respect to $\tilde{P}$. For $s \le t \le T$, writing

$$Y_t - Y_s = \int_0^s (T_{t-u} - T_{s-u}) B(u, X_u)\, dW_u + \int_s^t T_{t-u} B(u, X_u)\, dW_u$$

and using Lemma 2.1, we get

$$\tilde{E}\|Y_t - Y_s\|^p = 2^{p-1} C_p \tilde{E} \left[\left\{ \int_0^s \|(T_{t-u} - T_{s-u}) B^*(u, X_u)\|_{\text{HS}}^2\, du \right\}^{p/2} \right.$$

$$
\begin{aligned}
&\quad + \left\{ \int_s^t \| T_{t-u} B^*(u, X_u) \|_{\mathrm{HS}}^2 \, du \right\}^{p/2} \Bigg] \\
&= 2^{p-1} C_p \tilde{E} \Bigg[\left\{ \int_0^s \sum_k (e^{-\lambda_k (t-u)} - e^{-\lambda_k (s-u)})^2 b_k^2 (1 + \| X_u \|^2) \, du \right\}^{p/2} \\
&\qquad + \left\{ \int_s^t \sum_k e^{-2\lambda_k (t-u)} b_k^2 (1 + \| X_u \|^2) \, du \right\}^{p/2} \Bigg]. \qquad (2.47)
\end{aligned}
$$

We write

$$
\psi_1(u) = \sum_k (e^{-\lambda_k (t-u)} - e^{-\lambda_k (s-u)})^2 b_k^2 \quad \text{and} \quad \psi_2(u) = \sum_k e^{-2\lambda_k (t-u)} b_k^2.
$$

Now

$$
\begin{aligned}
&\tilde{E} \left[\int_0^s \psi_1(u)(1 + \| X_u \|^2) \, du \right]^{p/2} \\
&\quad \le \tilde{E} \left[\left(\int_0^s \psi_1(u) \, du \right)^{p/2-1} \int_0^s \psi_1(u)(1 + \| X_u \|^2)^{p/2} \, du \right] \\
&\quad \le C_p'(1 + \tilde{E} \| X_0 \|^p) \left(\int_0^s \psi_1(u) \, du \right)^{p/2}
\end{aligned}
$$

by Hölder's inequality and (2.39). Similarly, estimating the second term in (2.47), we get

$$
\begin{aligned}
&\tilde{E} \| Y_t - Y_s \|^p \\
&\quad \le C_p'(1 + \tilde{E} \| X_0 \|^p) \left[\left(\int_0^s \psi_1(u) \, du \right)^{p/2} + \left(\int_s^t \psi_2(u) \, du \right)^{p/2} \right]. \qquad (2.48)
\end{aligned}
$$

Evaluating the integrals, we obtain

$$
\begin{aligned}
\tilde{E} \| Y_t - Y_s \|^p \le C_p' \tilde{E}(1 + \| X_0 \|)^p \Bigg[& \left(\sum_k \frac{b_k^2}{2\lambda_k} (1 - e^{-\lambda_k (t-s)})^2 \right)^{p/2} \\
& + \left(\sum_k \frac{b_k^2}{2\lambda_k} (1 - e^{-2\lambda_k (t-s)}) \right)^{p/2} \Bigg].
\end{aligned}
$$

Now using the obvious inequality $1 - e^x \le x \wedge 1 \le x^\delta$ for $x > 0$, $0 < \delta \le 1$, for $\delta = (1-\theta)/2$ and $\delta = 1 - \theta$ respectively, we get

$$
\begin{aligned}
\tilde{E} \| Y_t - Y_s \|^p &\le C_p' \tilde{E}(1 + \| X_0 \|)^p \Bigg[\left(\sum_k \frac{b_k^2}{2\lambda_k} (\lambda_k (t-s))^{1-\theta} \right)^{p/2} \\
&\qquad + \left(\sum_k \frac{b_k^2}{2\lambda_k} (2\lambda_k (t-s))^{1-\theta} \right)^{p/2} \Bigg] \\
&\le C_p' \tilde{E}(1 + \| X_0 \|)^p \left(\frac{1}{2^{p/2}} + \frac{1}{2^{p\theta/2}} \right) \left(\sum_k \frac{b_k^2}{\lambda_k^\theta} \right)^{p/2} (t-s)^{(1-\theta)p/2}.
\end{aligned}
$$

Recalling assumption (2.44) and noting that, by our choice of p, $(p/2)(1-\theta) > 1$, we conclude that

$$\tilde{E}\|Y_t - Y_s\|^p \leq C_{2,5}|t-s|^{1+\delta} \tag{2.49}$$

with $\delta = (p/2)(1-\theta) - 1$, where $C_{2,5}$ depends only on p, $C_{2,4}$. Thus $\{Y_t\}$ has a continuous modification. □

Now the existence and uniqueness result, Theorem 2.4, can be recast as follows.

Theorem 2.7. *There exists a continuous solution X to the SDE* (2.2). *Further, if X' is any other solution to* (2.2) *with continuous paths, then*

$$P(X_t = X'_t \text{ for all } t,\ 0 \leq t \leq T) = 1.$$

Our next step is to prove uniqueness in the law of solutions to (2.2).

Theorem 2.8. *Let $\{X_t\}$ be a solution to* (2.2) [*on* $(\Omega, \mathcal{F}, P)$] *and let $\{X'_t\}$ be a solution to* (2.2) *on* $(\Omega', \mathcal{F}', P')$ *with respect some P'-cylindrical Brownian motion on H. Suppose that X, X' have continuous paths and suppose $P \circ X_0^{-1} = P' \circ X_0'^{-1}$. Then*

$$P \circ X^{-1} = P' \circ X'^{-1}. \tag{2.50}$$

Proof. Let $\{X_t^n\}$ be the approximation constructed in the previous theorem and let $\{V_t^n\}$ be the approximation defined analogously on $(\Omega', \mathcal{F}', P')$ (with X'_0 in place of X_0 and $\{W'_t\}$ in place of $\{W_t\}$ in (2.31)). It is easy to see that the finite-dimensional distributions of $\{X_t^n\}$ and $\{V_t^n\}$ are the same. Now $\tilde{E}\|X_t^n - X_t\|^2 \to 0$ implies that $P(\|X_t^n - X_t\| > \delta) \to 0$ for all $\delta > 0$. Similarly, $P'(\|V_t^n - X'_t\| > \delta) \to 0$. Thus the finite-dimensional distributions of $\{X_t\}$ and $\{X'_t\}$ are the same. Since X, X' have continuous paths, this yields (2.50). □

We now consider the martingale problem corresponding to (2.2).

For $f \in C_0^2(\mathbb{R}^n)$, $n \geq 1$, let $U_n f\colon H \to \mathbb{R}$ be defined by

$$(U_n f)(h) = f((h, \varphi_1), \ldots, (h, \varphi_n)). \tag{2.51}$$

For $f \in C_0^2(\mathbb{R}^n)$, we write $f_i = (\partial/\partial x_i) f$ and $f_{ij} = (\partial/\partial x_j) f_i$. Let

$$\mathcal{D} = \{U_n f\colon f \in C_0^2(\mathbb{R}^n), n \geq 1\}. \tag{2.52}$$

Define $\mathbb{L}_t$ on $\mathcal{D}$ by

$$\begin{aligned}\mathbb{L}_t(U_n f)(h) &= \tfrac{1}{2}\sum_{i,j=1}^{n}(B^*(t,h)\varphi_i, B^*(t,h)\varphi_j)(U_n f_{ij})(h)\\ &\quad + \sum_{i=1}^{n}(A(t,h) - \lambda_i h, \varphi_i)(U_n f_i)(h).\end{aligned} \tag{2.53}$$

If $\{X_t\}$ is a solution to (2.2), then we have seen that (2.27) holds and hence it follows that, for all $g \in \mathcal{D}$,

$$g(X_t) - g(X_0) - \int_0^t (\mathbb{L}_s g)(X_s)\, ds \tag{2.54}$$

is also a martingale. In other words, if $\{X_t\}$ is a solution to (2.2), then $\{X_t\}$ is a solution to the $\{\mathbb{L}_t\}$-martingale problem. That the converse is also true is proved next.

Theorem 2.9. *Let $\{X_t\}$ be a progressively measurable process satisfying* (2.13) *such that* (2.54) *is a martingale for all $g \in \mathcal{D}$. Then on the probability space $(\Omega', \mathcal{F}', P') = (\Omega, \mathcal{F}, P) \otimes ([0, 1], \bar{\mathcal{B}}, \nu)$, there exists a cylindrical Brownian motion (W_t) on H with respect to a family $(\mathcal{G}_t)$ such that* (a) *$\{X_t\}$ is $\mathcal{G}_t$-progressively measurable and* (b) *$\{X_t\}$ is a solution to* (2.2). (*Here ν is the Lebesgue measure on* $[0, 1]$ *and $\bar{\mathcal{B}}$ is the ν-completion of the Borel σ field*).

Proof. Using (2.54) for $g = U_n f$, $f \in C_0^2(\mathbb{R})$, we can first conclude that $(\langle X_t, \varphi_i\rangle, 1 \le i \le n)$ has a right continuous with left limit modification and then further that it has a continuous modification. (This follows using arguments in Theorem IV.3.6 in [4] and Exercise 4.6.3 in [9].) We denote the continuous version of $\langle X_t, \varphi_i\rangle$ by Y^i. Then we also deduce that

$$M_t^i = Y_t^i - Y_0^i - \int_0^t \lambda_i Y_s^i\, ds - \int_0^t \langle A(s, X_s), \varphi_i\rangle\, ds$$

is a continuous local martingale and that

$$\langle M^i, M^j\rangle_t = \int_0^t \langle B^*(s, X_s)\varphi_i, B^*(s, X_s)\varphi_j\rangle\, ds.$$

As a consequence, recalling definition (2.28) of τ_n, and using (2.6) we have

$$E \sup_{t \le \tau_n} |M_t^k|^2 \le 4E\langle M^k, M^k\rangle_{\tau_n} \le b_k^2(1 + n). \tag{2.55}$$

Let $N_t^k := \lambda_k^{-1/2} M_t^k$. Then using (2.44) and (2.55) we get

$$E \sup_{t \le \tau_n} \left\| \sum_{k=m}^{r} N_t^k \varphi_k \right\|^2 \to 0, \qquad m, r \to \infty.$$

Hence $N_t := \sum_{k=1}^{\infty} N_t^k \varphi_k$ is an H-valued continuous local martingale. Here

$$\begin{aligned}\langle N^k, N^j\rangle_t &= \int_0^t \lambda_k^{-1/2}\lambda_j^{-1/2}\langle B^*(s, X_s)\varphi_k, B^*(s, X_s)\varphi_j\rangle\, ds\\ &= \int_0^t \langle G_s^*\varphi_k, G_s^*\varphi_j\rangle\, ds,\end{aligned}$$

where $G_s(w) = L^{-1/2} B(s, X_s(w))$. Note that

$$\int_0^T \|G_s(w)\|_{\mathrm{HS}}^2 \, ds < \infty$$

in view of assumption (2.10). Let $\{\beta_t^j, 1 \leq j < \infty\}$ be a sequence of independent $(\mathcal{F}_t^1)$-Brownian motions on $([0,1], \bar{\mathcal{B}}, \nu)$, where $(\mathcal{F}_t^1)$ satisfy the usual conditions. Let $\mathcal{G}_t^1 = \mathcal{F}_t \otimes \mathcal{F}_t^1$, $t \geq 0$. $\mathcal{G}_t^1$ is a σ-field on $\Omega' = \Omega \times [0,1]$. Let $(\mathcal{G}_t)$ be the smallest family of σ fields on Ω' satisfying the usual conditions such that $\mathcal{G}_t^1 \subseteq \mathcal{G}_t$. Using arguments as in the proof of Theorem IV.3.5. in [12], it can be shown that there exists a cylindrical Brownian motion (W_t) on H with respect to $(\mathcal{G}_t)$ such that

$$N_t = \int_0^t G_s \, dW_s.$$

Then $N_t^k = (N_t, \varphi_k) = \int_0^t \langle \lambda_k^{-1/2} B^*(s, X_s)\varphi_k, dW_s\rangle$ and hence

$$M_t^k = \int_0^t \langle B^*(s, X_s)\varphi_k, dW_s\rangle.$$

From here, it follows that $\{X_t\}$ satisfies (2.27) and hence that $\{X_t\}$ is a solution to (2.2). □

In the light of Theorem 2.7, some of the results concerning (2.2) proved earlier can be recast for the $(\mathbb{L}_t)$-martingale problem as follows.

Theorem 2.10.

(a) *Let $\{X_t\}$ be a progressively measurable process satisfying* (2.13) *and suppose $\{X_t\}$ is a solution to the $(\mathbb{L}_t)$-martingale problem. Then $\{X_t\}$ admits a continuous modification.*

(b) *For all $\mu \in \mathcal{P}(H)$, there exists a continuous process $\{X_t\}$ such that* (2.54) *is a martingale for every $g \in \mathcal{D}$ and such that the law of X_0 is μ. Further, the law of the process X is uniquely determined.*

(c) *For $0 \leq s \leq T$, $x \in H$, there is a unique measure $P_{s,x}$ on $C([0,T], H)$ such that (writing the coordinate process on $C([0,T], H)$ as η_t)*
   (i) $P_{s,x}(\eta(u) = x, 0 \leq u \leq s) = 1$,
   (ii) $g(\eta_t) - \int_s^t (\mathbb{L}_u g)(\eta_u)\, du$, *$t \geq s$ is a $P_{s,x}$ martingale.*

(d) *Further, (η_t) is a time inhomogeneous Markov process on the probability space $(\Omega', \mathcal{F}', P_{s,x})$ (where Ω' is $C([0,T], H)$ and $\mathcal{F}'$ is the Borel σ-field on Ω') for each $(s,x) \in [0,T] \times H$. The (common) transition probability function $P(r, y, t, B)$ is given by*

$$P(r, y, t, B) = P_{r,y}(\eta_t \in B)$$

for $r \leq t \leq T$, $y \in H$, B Borel in H.

Proof. (a) and (b) follow from Theorems 2.4, 2.6, and 2.7. (c) is the same as (b)—with a change of origin from 0 to s in the time variable. For (d), note that if, for each n, $\mathcal{C}_n$ is a countable dense subset of $C_0^2(\mathbb{R}^n)$ (in the norm, $\| f \|_0 = (\| f \| + \sum_i \| f_i \| + \sum_{ij} \| f_{ij} \|)$, $\| \cdot \|$, being sup norm), then

$$\mathcal{D}_0 = \{U_n f\colon f \in \mathcal{C}_n\}$$

is a countable set and for every $g = U_n f \in \mathcal{D}$ we can get $g_k \in \mathcal{D}_0$ such that $g_k \to g$ and $\mathbb{L}_t g_k \to \mathbb{L}_t g$. Just take $g_k = U_n f_k$ where $f_k \in \mathcal{C}_n$ approximate f in $\| \cdot \|_0$ norm. Hence the Markov property of (η_t) under $\{P_{s,x}\}$ and the expression for the transition function follow from well-posedness. (See Theorem 6.2.2 in [9].) □

3. Interacting System of H-Valued SDEs

Let L, A, B, a, and b be as in Section 2 satisfying conditions (2.3)–(2.9) and (2.44). Let $\mathbf{R}\colon H \times H \to H$ be a function satisfying

$$|\langle R(h_1, h_1') - R(h_2, h_2'), \varphi_k\rangle| \le r_k[\|h_1 - h_2\| + \|h_1' - h_2'\|], \tag{3.1}$$

$$|\langle R(h_1, h_1'), \varphi_k\rangle| \le r_k[1 + \|h_1\|^2 + \|h_1'\|^2]^{1/2}. \tag{3.2}$$

Here r_k are assumed to satisfy

$$\sum_k r_k^2 \lambda_k^{-\theta} \le C_{3,\theta} < \infty \tag{3.3}$$

for some θ, $0 < \theta < 1$. Let $(W_t^1, W_t^2, \ldots, W_t^N)$ be N-independent cylindrical Brownian motions on H and let $X_0^{N,1}, \ldots, X_0^{N,N}$ be H-valued random variables independent of $\{W_t^j, 1 \le j \le N\}$. Consider the following equations for an interacting system $X_t^N = (X_t^{N,1}, \ldots, X_t^{N,N})$:

$$\begin{aligned} dX_t^{N,j} &= -LX_t^{N,j}\,dt + B(t, X_t^{N,j})\,dW_t^j + A(t, X_t^{N,j})\,dt \\ &\quad + \frac{1}{N}\sum_{k=1}^{N} R(X_t^{N,j}, X_t^{N,k})\,dt. \end{aligned} \tag{3.4}$$

Letting $\mathcal{H}$ denote the N-fold product of H, define

$$\tilde{L}(h_1, \ldots, h_N) = (Lh_1, \ldots, Lh_N).$$

$\tilde{W}_t := (W_t^1, \ldots, W_t^N)$ becomes a cylindrical Brownian motion on $\mathcal{H}$ and (3.4) can be recast as

$$dX_t^N = -\tilde{L}X_t^N\,dt + \tilde{B}(t, X_t^N)\,d\tilde{W}_t + \tilde{A}(t, X_t^N)\,dt$$

for appropriate choices of $\tilde{B}$, $\tilde{A}$, $\tilde{b}$, and $\tilde{a}$ which satisfy (2.3)–(2.9) and (2.44). Hence, it follows that the system of equations (3.4) admits a unique solution with continuous paths.

Proceeding as in the proof of Theorem 2.5, we can show that for a constant C'_p, depending only on p (and on L, A, B, and R) but not depending on N,

$$E\|X_t^{N,j}\|^p \le C'_p \left[E\|X_0^{N,j}\|^p + \int_0^t f_1(t-s) \times \left\{ 1 + E\|X_s^{N,j}\|^p + \frac{1}{N}\sum_{k=1}^N E\|X_s^{N,k}\|^p \right\} ds \right]. \tag{3.5}$$

Here $f_1(u) := f_A(u) + Tf_B(u) + Tf_R(u)$, $f_R(u) := \sum_k e^{-2\lambda_k s} r_k^2$. Now summing over j, we get

$$\frac{1}{N}\sum_{j=1}^N E\|X_t^{N,j}\|^p \le C'_p \left[\frac{1}{N}\sum_{j=1}^N E\|X_0^{N,j}\|^p + \int_0^t f_1(t-s)\left\{ 1 + 2\frac{1}{N}\sum_{j=1}^N E\|X_s^{N,j}\|^p \right\} ds \right].$$

Let $g(t) = (1/N)\sum_{j=1}^N E\|X_t^{N,j}\|^p$, $c = C'_p(1/N)\sum_{j=1}^N E\|X_0^{N,j}\|^p$, $f(t) = 2C'_p f_1(t)$, and $\delta(t) = \frac{1}{2}$. It follows from similar arguments as in the proof of Theorem 2.3 that the conditions of Lemma 2.2(i) are satisfied for the present case. By Lemma 2.2(i), it is easy to see that there exists a constant C''_p such that

$$\frac{1}{N}\sum_{j=1}^N E\|X_t^{N,j}\|^p \le C''_p \left[1 + \frac{1}{N}\sum_{j=1}^N E\|X_0^{N,j}\|^p \right]. \tag{3.6}$$

Using (3.6) in (3.6) and once again applying Lemma 2.2, we get

$$\sup_{t\le T} E\|X_t^{N,j}\|^p \le C'''_p \left[1 + E\|X_0^{N,j}\|^p + \frac{1}{N}\sum_{k=1}^N E\|X_0^{N,k}\|^p \right], \tag{3.7}$$

where the constant C'''_p does not depend on N.

We fix a sequence of initial random variables $X_0^N = (X_0^{N,1}, \ldots, X_0^{N,N})$ for the interacting system with N-components satisfying the following conditions:

$$\text{the law of} \quad X_0^N := (X_0^{N,1}, \ldots, X_0^{N,N}) \quad \text{is symmetric on} \quad H \times \cdots \times H, \tag{3.8}$$

$$\nu_0^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_0^{N,j}} \to \mu_0 \qquad \text{in probability}, \tag{3.9}$$

$$\sup_N E\|X_0^{N,1}\|^p \le C_{3.2}, \tag{3.10}$$

for a constant $C_{3.2}$, where $p \ge 2$ satisfies $(1-\theta)p > 2$, θ as in (2.44). Our problem is to investigate the asymptotics of

$$\Gamma^N := \frac{1}{N}\sum_{j=1}^N \delta_{X^{N,j}} \in \mathcal{P}(\mathcal{C}).$$

Here, $\mathcal{C} = C([0,T], H)$ and $\mathcal{P}(\mathcal{C})$ is the space of probability measures on $\mathcal{C}$. $\mathcal{C}$ is equipped with sup norm and $\mathcal{P}(\mathcal{C})$ with the topology of weak convergence.

The problem for finite-dimensional diffusions has been investigated by, among others, Sznitman [10] and for nuclear space-valued diffusions by Chiang et al. [2].

Note that assumptions (3.8) and (3.10) and inequality (3.7) implies that, for some constant $C_{3.3}$, we have

$$\sup_{N\geq 1}\sup_{j\leq N}\sup_{t\leq T} E\|X_t^{N,j}\|^p \leq C_{3.3}. \tag{3.11}$$

Also, symmetry in the law of the initial random vector X_0^N and uniqueness in the law of solutions to system (3.4) implies that the law of $X_\cdot^N = (X_\cdot^{N,1}, \ldots, X_\cdot^{N,N})$ is symmetric.

We first prove

Theorem 3.1. *The sequence of probability measures $P \circ (X_\cdot^{N,1})^{-1}$ is tight in $\mathcal{P}(\mathcal{C})$.*

Proof. We write

$$X_t^{N,j} = T_t X_0^{N,j} + U_t^{N,j} + V_t^{N,j} + \frac{1}{N}\sum_{k=1}^{N} Y_t^{N,j,k},$$

where

$$\begin{aligned} U_t^{N,j} &= \int_0^t T_{t-s} B(s, X_s^{N,j})\, dW_s^j, \\ V_t^{N,j} &= \int_0^t T_{t-s} A(s, X_s^{N,j})\, ds, \\ Y_t^{N,j,k} &= \int_0^t T_{t-s} R(X_s^{N,j}, X_s^{N,k})\, ds. \end{aligned}$$

First note that (3.8) and (3.9) imply that $X_0^{N,1}$ converges in law to μ_0 in $\mathcal{P}(H)$, and hence $\{\mathcal{L}\{X_0^{N,1}\} : N \geq 1\}$ is tight. (We denote the law of a random variable Z by $\mathcal{L}(Z)$.) Since $h \to T_t h$ is continuous from H into $\mathcal{C}$, it follows that $\{\mathcal{L}\{T_\cdot X_0^{N,1}\}, N \geq 1\}$ is tight.

The same computations as in the proof of (2.49) yield

$$E\|U_t^{N,j} - U_s^{N,j}\|^p \leq C_{3.4}|t-s|^{1+\delta},$$

where $C_{3.4}$ depends on p, $C_{2.1}$, and the trace of $B^2 L^{-\theta}$. Since $U_0^{N,1} = 0$, this implies tightness of $\{\mathcal{L}(U^{N,1}) : N \geq 1\}$. Computations similar to those leading to (2.48) yield

$$\|V_t^{N,j} - V_s^{N,j}\|^2 \leq 4C_{2.1}^2\left[\int_0^T (1 + \|X_u^{N,j}\|^2)\, du\right]\beta(t-s), \tag{3.12}$$

where $\beta(\delta) \to 0$ as $\delta \to 0$. For $M \geq 1$, let

$$G_M := \{\eta \in C([0,T], H)\colon\ \eta_0 = 0,\ \|\eta_t - \eta_s\| \leq M\beta(t-s),\ \forall s,t\}.$$

By the Ascoli–Arzela theorem, G_M is compact. Given $\varepsilon > 0$, using (3.11) and (3.12) we can get M such that

$$P(V_{\cdot}^{N,j} \notin G_M) < \varepsilon.$$

Thus $\{\mathcal{L}(V_{\cdot}^{N,1})\}$ is tight.

Similar computations yield

$$\|Y_t^{N,j,k} - Y_s^{N,j,k}\|^2 \leq 4C_{3.1}^2 \int_0^T (1 + \|X_s^{N,j}\|^2 + \|X_s^{N,k}\|^2)\, ds\beta'(\delta),$$

where

$$\begin{aligned}\beta'(\delta) &= \sum_{k=1}^{\infty} \frac{r_k^2}{2\lambda_k}[(1 - e^{-\delta\lambda_k})^2 + (1 - e^{-2\delta\lambda_k})] \\ &\to 0 \qquad \text{as} \quad \delta \to 0.\end{aligned}$$

Thus

$$\begin{aligned}&\left\|(1/N)\sum_{k=1}^{N} Y_t^{N,1,k} - \frac{1}{N}\sum_{k=1}^{N} Y_s^{N,1,k}\right\|^2 \\ &\quad \leq 4C_{3.1}^2 \int_0^T \left(1 + \|X_s^{N,1}\|^2 + \frac{1}{N}\sum_{k=1}^{N}\|X_s^{N,k}\|^2\right) ds\beta'(\delta).\end{aligned}$$

Tightness of $\{\mathcal{L}(\frac{1}{N}\sum_{k=1}^{N} Y^{N,1,k}) : N \geq 1\}$ follows from this using exactly the same arguments as used to prove the tightness of $V_{\cdot}^{N,1}$ above. Thus each of the four components of $X^{N,1}$ is tight and hence $\{\mathcal{L}(X^{N,1}) : N \geq 1\}$ is tight. □

We can now deduce

Theorem 3.2. $\{P \circ (\Gamma^N)^{-1} : N \geq 1\}$ *is tight in* $\mathcal{P}(\mathcal{P}(\mathcal{C}))$.

Proof. For each $\varepsilon > 0$, let K_ε be a compact set in $\mathcal{C}$ such that $P(X_{\cdot}^{N,1} \in K_\varepsilon^c) < \varepsilon^2$. This choice is possible in view of Theorem 3.1. Then using the symmetry of the law of $(X^{N,1}, \ldots, X^{N,N})$ we have

$$\begin{aligned}E[\Gamma^N(K_\varepsilon^c)] &= \frac{1}{N}\sum_{j=1}^{N} P(X_{\cdot}^{N,j} \in K_\varepsilon^c) \\ &\leq \varepsilon^2.\end{aligned}$$

Let $\mathcal{K}_\varepsilon = \{\Lambda \in \mathcal{P}(\mathcal{C}) : \Lambda(K_{\varepsilon\cdot 2^{-m}}) \geq 1 - \varepsilon \cdot 2^{-m}, m \geq 1\}$. Then $\mathcal{K}_\varepsilon$ is compact in $\mathcal{P}(\mathcal{C})$ and

$$P(\Gamma^N \notin \mathcal{K}_\varepsilon) \leq \sum_{m=1}^{\infty} P(\Gamma^N(K_{\varepsilon\cdot 2^{-m}}^c) > \varepsilon \cdot 2^{-m})$$

$$\leq \sum_{m=1}^{\infty} (\varepsilon \cdot 2^{-m})^{-1} E(\Gamma^N(K^c_{\varepsilon \cdot 2^{-m}}))$$
$$\leq \sum_{m=1}^{\infty} \varepsilon \cdot 2^{-m} = \varepsilon. \qquad \square$$

Before we identify the limit of Γ^N, we need to study the equation which will characterize the limit.

4. The McKean–Vlasov Equation

For $\mu_1, \mu_2 \in \mathcal{P}(H)$, let $\mathcal{M}(\mu_1, \mu_2)$ be the class of probability measures λ on $H \times H$ with marginals μ_1, μ_2 respectively. For $p \geq 1$, let $\mathcal{P}_p(H) = \{\mu \in \mathcal{P}(H)\colon \int \|h\|^p \mu(dh) < \infty\}$. For $\mu_1, \mu_2 \in \mathcal{P}_p(H)$, let

$$\rho_p(\mu_1, \mu_2) := \inf_{\lambda \in \mathcal{M}(\mu_1, \mu_2)} \left\{ \int \|h_1 - h_2\|^p \lambda(dh_1\, dh_2) \right\}^{1/p}.$$

Then $\mathcal{P}_p$ is a metric space with metric ρ_p. Note that $\rho_1 \leq \rho_2$.

Let $\hat{R}\colon H \times \mathcal{P}_1(H) \to H$ be defined by

$$\hat{R}(h, \mu) = \int R(h, h')\mu(dh'). \tag{4.1}$$

Using (3.1), we can deduce that

$$\|\hat{R}(h_1, \mu_1) - \hat{R}(h_2, \mu_2)\| \leq C_{3.1}[\|h_1 - h_2\| + \rho_1(\mu_1, \mu_2)]. \tag{4.2}$$

We now consider the McKean–Vlasov equation

$$dZ_t = -LZ_t\, dt + B(t, Z_t)\, dW_t + A(t, Z_t)\, dt + \hat{R}(Z_t, \mathcal{L}(Z_t))\, dt \tag{4.3}$$

(where $\mathcal{L}(Z_t)$ denotes the law of Z_t). A solution to (4.3) is defined as in Section 2. Arguments as in Theorem 2.3 would give the following: If (Z_t) is a solution to (4.3) with $E \int_0^T \|Z_t\|^2\, dt < \infty$ and $E\|Z_0\|^2 < \infty$, then

$$\sup_{t \leq T} E\|Z_t\|^2 < \infty \tag{4.4}$$

and those in Theorem 2.6 would give that $\{Z_t\}$ admits a continuous modification.

We now prove the existence and uniqueness of the solution to (4.3) with $\mathcal{L}(Z_0) = \mu_0$ for every $\mu_0 \in \mathcal{P}_2(H)$.

Theorem 4.1.

(a) *Let Z_0 be an H-valued $\mathcal{F}_0$ measurable random variable such that $E\|Z_0\|^2 < \infty$ and let (W_t) be an $(\mathcal{F}_t)$-cylindrical Brownian motion on H. Then there exists a solution (Z_t) to (4.3) with continuous paths such that $E \int_0^T \|Z_t\|^2\, dt < \infty$.*

(b) *Let* (Z_t^1), (Z_t^2) *be solutions to* (4.3) (*on the same probability space, with respect to the same cylindrical Brownian motion*). *Suppose that* Z_t^1, Z_t^2 *have continuous paths with* $Z_0^1 = Z_0^2$ *and*

$$E\|Z_0^i\|^2 < \infty, \qquad E\int_0^T \|Z_t^i\|^2\,dt < \infty. \tag{4.5}$$

Then $P(Z_t^1 = Z_t^2 \text{ for all } t) = 1$.

(c) *Further, the law of the solution* (Z_t) *in* (a) *above is uniquely determined* (*by* L, A, B, $\hat{R}$, *and* $\mathcal{L}(Z_0)$).

Proof. The proof closely follows the proof of Theorem 2.4. For $n \ge 1$, let $t_i^n = (i/n)\cdot T$, $0 \le i \le n$. Define $(X_t^n, t_i^n < t \le t_{i+1}^n)$, $i \ge 0$, inductively as follows. Let $X_0^n = Z_0^1$ and

$$X_t^n = X_{t_k^n}^n + (T_t - T_{t_k^n})Z_0^1 + \int_{t_k^n}^t T_{t-u}B(u, X_{t_k^n}^n)\,dW_u$$
$$+ \int_{t_k^n}^t [T_{t-u}A(u, X_{t_k^n}^n) + T_{t-u}\hat{R}(X_{t_k^n}^n, \mathcal{L}(X_{t_k^n}^n))]\,du.$$

Note that, for any random variables U_1, U_2,

$$\rho_1^2(\mathcal{L}(U_1), \mathcal{L}(U_2)) \le \rho_2^2(\mathcal{L}(U_1), \mathcal{L}(U_2)) \le E\|U_1 - U_2\|^2.$$

Hence, using (3.1),

$$\begin{aligned} E|\hat{R}(U_1, \mathcal{L}(U_1))\varphi_k - \hat{R}(U_2, \mathcal{L}(U_2))\varphi_k|^2 \\ \le 2r_k^2[E\|U_1 - U_2\|^2 + \rho_2^2(\mathcal{L}(U_1), \mathcal{L}(U_2))] \\ \le 4r_k^2 E\|U_1 - U_2\|^2. \end{aligned} \tag{4.6}$$

Using (4.6) and proceeding as in Theorem 2.4, it can be shown that, as $n, m \to \infty$,

$$\sup_{t\le T} E\|X_t^n - X_t^m\|^2 \to 0$$

and that there exists a progressively measurable process X such that

$$\sup_{t\le T} E\|X_t^n - X_t\|^2 \to 0.$$

Then it follows that (X_t) is a solution to (4.3) and that $\sup_{t\le T} E\|X_t\|^2 < \infty$. As noted earlier, it then follows that (X_t) admits a continuous modification (Z_t) which is the required solution. This proves (a). Part (b), namely, pathwise uniqueness of the solution, is proved as in Theorem 2.7 using (4.6). For (c), we note that the law of (X_t^n) defined above is uniquely determined by L, A, B, R, and $\mathcal{L}(Z_0)$ and the hence the law of (Z_t) is uniquely determined as well. □

The Martingale Problem. Let $\mathcal{D}$ and $(\mathbb{L}_t)$ be given by (2.52) and (2.53) respectively. (In (2.53), λ_i, φ_i are eigenvalues and eigenfunctions of L respectively.) For $U_n f \in \mathcal{D}$,

$f \in C_0^2(\mathbb{R}^n)$, let

$$\mathcal{R}(U_n f)(h_1, h_2) = \sum_{i=1}^{n} \langle R(h_1, h_2), \varphi_i \rangle f_i(h_1),$$

$$\hat{\mathcal{R}}(U_n f)(h_1, \mu) = \sum_{i=1}^{n} \langle \hat{R}(h_1, \mu), \varphi_i \rangle f_i(h_1)$$

for $h_1, h_2 \in H, \mu \in \mathcal{P}_1(H)$, where $f, U_n f$ are related via (2.51) and $f_i = (\partial/\partial x_i) f$.

As in Section 2, it follows that if (Z_t) is a solution to (4.3), then

$$g(Z_t) - g(Z_0) - \int_0^t \mathbb{L}_s g(Z_s)\, ds - \int_0^t \hat{\mathcal{R}} g(Z_s, \mathcal{L}(Z_s))\, ds \tag{4.7}$$

is a martingale for all $g \in \mathcal{D}$. Conversely, if (4.7) is a martingale for all $g \in \mathcal{D}$, then (Z_t) is a solution to (4.3) with respect to some cylindrical Brownian motion (W_t') defined perhaps on an extended probability space.

This leads us to the following definition. A continuous process (Z_t) is said to be a solution to the McKean–Vlasov martingale problem if it satisfies (4.5) and if (4.7) is a martingale for all $g \in \mathcal{D}$.

A probability measure Λ on $\mathcal{C}$ is said to be a solution to the McKean–Vlasov martingale problem if the coordinate process (η_t) on $\mathcal{C} = C([0, T], H)$ is a solution to the martingale problem under Λ. The measure $\Lambda \circ \eta_0^{-1}$ is called the initial condition for the martingale problem.

As a consequence of Theorem 4.1 we get the following:

Theorem 4.2.

(a) *Let $\mu_0 \in \mathcal{P}_2(H)$. Then there exists a solution Λ to the McKean–Vlasov martingale problem with $\Lambda \circ \eta_0^{-1} = \mu_0$ and $\int \int_0^T \|\eta_s\|\, ds\, \Lambda(d\eta) < \infty$.*

(b) *Let $\Lambda_1, \Lambda_2 \in \mathcal{P}(\mathcal{C})$ be solutions to the McKean–Vlasov martingale problem with*

$$\Lambda_1 \circ \eta_0^{-1} = \Lambda_2 \circ \eta_0^{-1} \in \mathcal{P}_2(H), \tag{4.8}$$

$$\int \int_0^T \|\eta_s\|^2\, ds \Lambda_i(d\eta) < \infty \tag{4.9}$$

for $i = 1, 2$. Then $\Lambda_1 = \Lambda_2$.

Remark 4.1. We can choose a countable subset $\mathcal{D}_0 \subseteq \mathcal{D}$ such that (4.7) is a martingale for all g in $\mathcal{D}$ iff it is a martingale for all g in $\mathcal{D}_0$. Indeed, for $n \geq 1$, let $\mathcal{U}_n$ be a countable dense subset of $C_0^2(\mathbb{R}^n)$ (in C^2-norm) and let

$$\mathcal{D}_0 = \{U_n f\colon\ f \in \mathcal{U}_n, n \geq 1\}.$$

Let $\hat{\mathcal{P}}(\mathcal{C}) = \{\Lambda \in \mathcal{P}(\mathcal{C})\colon\ \rho(\Lambda) = \int_0^T \int \|\eta_t\|^2 \Lambda(d\eta)\, dt < \infty\}$. It is easy to see that, for $g \in \mathcal{D}_0$, $\Lambda \in \hat{\mathcal{P}}(\mathcal{C})$

$$g(\eta_t) - g(\eta_0) - \int_0^t \mathbb{L}_s g(\eta_s)\, ds - \int_0^t \hat{\mathcal{R}} g(\eta_s, \mathcal{L}(\eta_s))\, ds$$

is a Λ-martingale iff

$$F(\Lambda) := \int \left[g(\eta_t) - g(\eta_s) - \int_s^t \mathbb{L}_u g(\eta_u)\, du - \int_s^t \hat{\mathcal{R}} g(\eta_u, \Lambda \circ \eta_u^{-1})\, du \right] \times g_1(\eta_{r_1}) g_2(\eta_{r_2}) \cdots g_k(\eta_{r_k})\, d\Lambda(\eta) \tag{4.10}$$

is zero for all $r_1 \le r_2 \le \cdots \le r_k \le s \le t$, $g_1, \ldots, g_k \in \mathcal{D}_0$, $k \ge 1$. Also, we can restrict $r_1, \ldots, r_k, s, t$ to rationals.

Let $\mathcal{E}$ be the class of functionals $F\colon \hat{\mathcal{P}}(\mathcal{C}) \to \mathbb{R}$ defined by (4.10) for $g_1, g_2, \ldots, g_k \in \mathcal{D}_0, r_1 \le r_2 \cdots r_k \le s \le t$ rationals. Then $\mathcal{E}$ is countable and we have $\Lambda \in \hat{\mathcal{P}}(\mathcal{C})$ is a solution to the McKean–Vlasov martingale problem if and only if $F(\Lambda) = 0$ for all $F \in \mathcal{E}$.

Remark 4.2. Let $\mu_t = \Lambda \circ \eta_t^{-1}$, where Λ is a solution to the McKean–Vlasov martingale problem. Then it follows that $\{\mu_t\}$ satisfies the following nonlinear equation for $g \in \mathcal{D}$:

$$\langle \mu_t, g\rangle = \langle \mu_0, g\rangle + \int_0^t \langle \mu_s, \mathbb{L}_s g\rangle\, ds + \int_0^t \langle \mu_s \otimes \mu_s, \mathcal{R} g\rangle\, ds \tag{4.11}$$

for $g \in \mathcal{D}$. Here $\langle \mu, g\rangle = \int g\, d\mu$. If instead of (2.5), (2.6), and (3.2) A, B, and R satisfy $|\langle A(t,h), \varphi_k\rangle| \le a_k$, $\|B(t,h)\varphi_k\| \le b_k$, and $|\langle R(h_1,h_2), \varphi_k\rangle| \le r_k$, then $\{\mu_t\}$ is the only solution to (4.11). We give an outline of the proof. First, for any continuous function $t \to \nu_t$ from $[0, T]$ into $\mathcal{P}_1(H)$, we can consider the martingale problem for

$$\mathcal{K}_t g := \mathbb{L}_t g + \hat{\mathcal{R}} g(\cdot, \nu_t).$$

Then Theorem 2.10 implies that the $\mathcal{K}_t$-martingale problem is well posed, and any progressively measurable solution admits a continuous modification. Here, since A, B, and R are assumed to be bounded, (2.13) holds for any solution to the $\mathcal{K}_t$-martingale problem. Thus, Theorem 5.2 in [1] implies that the forward equation for $\{\mathcal{K}_t\}$, namely,

$$\langle \mu_t, g\rangle = \langle \mu_0, g\rangle + \int_0^t \langle \mu_s, \mathbb{L}_s g\rangle\, ds + \int_0^t \langle \mu_s \otimes \nu_s, \mathcal{R} g\rangle\, ds \tag{4.12}$$

for $g \in \mathcal{D}$ admits a unique solution, in the class of probability measures, and the unique solution is $\mu_t = \mathcal{L}(Z_t)$, where (Z_t) is the solution to the $\mathcal{K}_t$-martingale problem.

The uniqueness of solution $\{\mu_t\}$ to (4.11) in the class of probability measures $\{\nu_t\}$ such that $t \to \nu_t$ is continuous from $[0, T] \to \mathcal{P}_1(H)$ and $\nu_0 \in \mathcal{P}_2(H)$ can be proved as follows. Let $\{\mu_t^1\}$ and $\{\mu_t^2\}$ be solutions to (4.11). Let Z_t^i be the solution to the $\mathcal{K}_t$-martingale problem with $\nu_t = \mu_t^i$ and let $\tilde{\mu}_t^i = \mathcal{L}(Z_t^i)$. Then, for $i = 1, 2$, $\{\mu_t^i\}$, $\{\tilde{\mu}_t^i\}$ are solutions to (4.12) with $\nu_t = \mu_t^i$. Hence $\mu_t^i = \tilde{\mu}_t^i$, and hence Z^1, Z^2 are solutions to the McKean–Vlasov martingale problem. That Z^1, Z^2 satisfy (4.5) can be checked using the additional conditions imposed on the coefficients. Hence $\mathcal{L}(Z^1) = \mathcal{L}(Z^2)$ by Theorem 4.2 and as a consequence $\mu_t^1 = \mu_t^2$.

5. Propagation of Chaos in $C([0, T], H)$

We return to the setup of Section 3. We assume conditions (2.3)–(2.9), (2.44), and (3.1)–(3.10). Thus, by Theorem 3.2, $P \circ (\Gamma^N)^{-1}$ is tight. We need to identify the limit points of this sequence. We fix a subsequence N' such that $P \circ (\Gamma^{N'})^{-1}$ converges, i.e., $(\Gamma^{N'})$ converges in distribution to say Γ, which then is a $\mathcal{P}(\mathcal{C})$-valued random variable.

We show that $\Gamma = \Lambda_0$, where Λ_0 is the unique solution to the McKean–Vlasov martingale problem with $\Lambda_0 \circ (\eta_0)^{-1} = \mu_0$.

Recall that $\hat{\mathcal{P}}(\mathcal{C})$ is the class of $\Lambda \in \mathcal{P}(\mathcal{C})$ with $\rho(\Lambda) < \infty$ where

$$\rho(\Lambda) = \int_{\mathcal{C}} \int_0^T \|\eta_t\|^2 \, dt \, d\Lambda(\eta).$$

Using (3.11), we have

$$\begin{aligned} E[\rho(\Gamma^N)] &= E\left[\frac{1}{N}\sum_{j=1}^N \int_0^T \|X_t^{N,j}\|^2 \, dt\right] \\ &\le T C_{3.3}, \end{aligned}$$

and hence, by Fatou's lemma,

$$E[\rho(\Gamma)] \le T C_{3.3}. \tag{5.1}$$

In particular, $\Gamma \in \hat{\mathcal{P}}(\mathcal{C})$ a.s. and thus $F(\Gamma)$ is well defined for $F \in \mathcal{E}$. Fix $F \in \mathcal{E}$, given by (4.10) our aim is first to prove $F(\Gamma) = 0$ a.s. Then we can write, for $\Lambda \in \hat{\mathcal{P}}(\mathcal{C})$,

$$F(\Lambda) = \int_{\mathcal{C}} \int_{\mathcal{C}} G(\eta, \eta') \, d\Lambda(\eta) \, d\Lambda(\eta'),$$

where

$$\begin{aligned} G(\eta, \eta') &= \left[g(\eta_t) - g(\eta_s) - \int_s^t \mathbb{L}_u g(\eta_u) \, du - \int_s^t \mathcal{R} g(\eta_u, \eta'_u) \, du\right] \\ &\quad \times g_1(\eta_{r_1}) g_2(\eta_{r_2}) \cdots g_k(\eta_{r_k}). \end{aligned}$$

It is easy to see that G is a continuous function on $\mathcal{C} \times \mathcal{C}$ and that

$$|G(\eta, \eta')|^2 \le C_F \left[1 + \int_0^T \|\eta_u\|^2 \, du + \int_0^T \|\eta'_u\|^2 \, du\right] \tag{5.2}$$

for a constant C_F depending on F.

Lemma 5.1. $E|F(\Gamma^{N'})| \to E|F(\Gamma))|$.

Proof. For $k \geq 1$, let

$$F_k(\Lambda) = \int_{\mathcal{C}} \int_{\mathcal{C}} [\{G \vee (-k)\} \wedge k] \, d\Lambda \, d\Lambda.$$

Then F_k is a continuous function on $\mathcal{P}(\mathcal{C})$ and thus $F_k(\Gamma^{N'}) \rightarrow F_k(\Gamma)$ in distribution. Since $|F_k| \leq k$, we get

$$E|F_k(\Gamma^{N'})| \rightarrow E|F_k(\Gamma)|.$$

Moreover,

$$\begin{aligned} E|F_k(\Gamma^{N'}) - F(\Gamma^{N'})| &\leq E \int\int |G| 1_{\{|G| \geq k\}} \, d\Gamma^{N'} \, d\Gamma^{N'} \\ &\leq \frac{1}{k} E \int\int |G|^2 \, d\Gamma^{N'} \, d\Gamma^{N'} \\ &\leq \frac{1}{k} C_F \, E[1 + 2\rho(\Gamma^{N'})] \\ &\leq \frac{1}{k} C_F [1 + 2T C_{3.3}]. \end{aligned}$$

Similarly, using (5.1) it follows that $E|F_k(\Gamma) - F(\Gamma)| \leq k^{-1}[1 + 2TC_{3.3}]$. The familiar $(\varepsilon/3)$ argument would now yield the result. □

Lemma 5.2. $EF^2(\Gamma^N) \rightarrow 0$.

Proof. Note that

$$F(\Gamma^N) = \frac{1}{N} \sum_{i=1}^{N} [M_t^i - M_s^i] g_1(X_{r_1}^{N,i}) \cdots g_k(X_{r_k}^{N,i}),$$

where

$$M_t^i := g(X_t^{N,i}) - g(X_0^{N,i}) - \int_0^t \mathbb{L}_u g(X_u^{N,i}) \, du - \frac{1}{N} \sum_{j=1}^{N} \int_0^t \mathcal{R} g(X_u^{N,i}, X_u^{N,j}) \, du.$$

Recall that X_t^N is the solution to system (3.4). It follows by Ito's formula that M_t^i is a martingale and, for $i \neq j$, M_t^i, M_t^j are orthogonal martingales (i.e., $M_t^i M_t^j$ is a martingale). In particular, $E[(M_t^i - M_s^i)(M_t^j - M_s^j)|\mathcal{F}_s^N] = 0$, where $\mathcal{F}_s^N = \sigma(X_u^{N,i}\colon u \leq s, i \leq N)$. Thus for a constant C_F depending on F,

$$EF^2(\Gamma^N) \leq C_F \frac{1}{N^2} \sum_{j=1}^{N} E(M_t^i - M_s^i)^2.$$

Moreover, it can be shown that

$$\langle M^i, M^i\rangle_t = \int_0^t \sum_{k,j=1}^{N} (U_n f_k)(X_s^{N,i})(U_n f_j)(X_s^{N,i})\langle B^*(s, X_s^{N,i})\varphi_k, B^*(s, X_s^{N,i})\varphi_j\rangle\, ds,$$

where $g = U_n f$ and $f_k = (\partial/\partial x_k) f$. Hence

$$\langle M^i, M^i\rangle_t \le C'_F t,$$

where C'_F depends on F. As a consequence

$$EF^2(\Gamma^N) \le \frac{1}{N} C_F C'_F T. \qquad \square$$

Together, the two preceding lemmas yield the propagation of chaos result.

Theorem 5.3. *Let Λ_0 be the solution to the McKean–Vlasov martingale problem with initial condition μ_0. Then*

$$\Gamma^N \to \Lambda_0 \qquad \textit{in probability.}$$

Proof. We have noted that Γ^N is tight and that if $\Gamma^{N'}$ is any weakly convergent subsequence converging to Γ, then $E|F(\Gamma^{N'})| \to E|F(\Gamma)|$. This and $EF^2(\Gamma^N) \to 0$ together imply that $E|F(\Gamma)| = 0$. Since $\mathcal{E}$ is countable, we get $P(F(\Gamma) = 0$ for all $F \in \mathcal{E}) = 1$. We have already noted $\rho(\Gamma) < \infty$ a.s. Let

$$\tilde{\Omega} = \{\omega\colon\ \rho(\Gamma(\omega)) < \infty \text{ and } F(\Gamma) = 0 \text{ for all } F \in \mathcal{E}\}.$$

Then $P(\tilde{\Omega}) = 1$ and, for all $\omega \in \tilde{\Omega}$, we have $\Gamma(\omega)$ is a solution to the McKean–Vlasov martingale problem. Since

$$\Gamma^N(\omega) \circ (\eta_0)^{-1} = \nu_0^N(\omega)$$

and $\nu_0^N \to \mu_0$ in probability, we have

$$\Gamma(\omega) \circ (\eta_0)^{-1} = \mu_0 \qquad \text{a.s.}$$

By Theorem 4.2(b), it follows that $\Gamma(\omega) = \Lambda_0$. Since all subsequential limits of $\{P \circ (\Gamma^N)^{-1}\}$ are identified as δ_{Λ_0} it follows that $\Gamma^N \to \Lambda_0$ in probability. $\square$

As a consequence of the preceding theorem, we have the following result, also called propagation of chaos.

Theorem 5.4. *Let*

$$\nu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{N,i}}$$

and

$$\mu_t = \Lambda_0 \circ (\eta_t)^{-1},$$

Λ_0 *being the unique solution to the McKean–Vlasov martingale problem with initial condition* μ_0. *Then* $\nu_\cdot^N \to \mu_\cdot$ *in probability (in the space* $C([0,T],\mathcal{P}(H))$*).*

Proof. Let $\Psi\colon \mathcal{P}(\mathcal{C}) \to C([0,T],\mathcal{P}(H))$ be defined by

$$\Psi(\Gamma)(t) = \Gamma \circ (\eta_t)^{-1}.$$

Then it follows that Ψ is a continuous mapping. It is easy to observe that $\Psi(\Gamma^N)(t) = \nu_t^N$ and that $\Psi(\Lambda_0)(t) = \mu_t$ for all t. Thus the required conclusion, namely, the convergence of $\Psi(\Gamma^N)$ to $\Psi(\Lambda_0)$ in probability follows from the previous theorem. □

6. Approximations to Generalized Solutions

There are many examples of SPDEs which only have distributions or generalized functions as solutions. Making use of the results obtained in the previous sections, we now give a systematic procedure for approximating such equations by "smoother" SPDEs which have ordinary random fields as solutions. Then we demonstrate this method by considering the voltage potential model for a two-dimensional neuron studied by Walsh [11].

Let H be a separable Hilbertian space and let L be an unbounded self-adjoint positive definite operator on H such that $T_t \equiv e^{-tL}$ is a contraction semigroup on H and L^{-1} is a bounded self-adjoint positive definite operator on H with discrete spectrum $\{\lambda_j^{-1}\}$ and corresponding eigenvectors $\{\xi_j\}$, $j = 0,1,2,\ldots$. Let $A\colon [0,T]\times H \to H$ and $B\colon [0,T]\times H \to \mathcal{L}(H,H)$ be continuous functions satisfying the following condition:

(C1) There exists a constant K such that

$$\|A(t,h_1) - A(t,h_2)\|_H^2 + \|B(t,h_1) - B(t,h_2)\|_{\mathcal{L}(H,H)}^2 \le K\|h_1 - h_2\|_H^2 \tag{6.1}$$

and

$$\|A(t,h)\|_H^2 + \|B(t,h)\|_{\mathcal{L}(H,H)}^2 \le K(1 + \|h\|_H^2) \tag{6.2}$$

for any $h, h_1, h_2 \in H$.

Consider the following SDE:

$$X_t = X_0 - \int_0^t LX_s\,ds + \int_0^t A(s,X_s)\,ds + \int_0^t B(s,X_s)\,dW_s, \tag{6.3}$$

where X_0 is an H-valued random variable such that $E\|X_0\|_0^2 < \infty$ and W is a cylindrical Brownian motion on H.

Remark 6.1. As L^{-1} is not assumed to be nuclear and B is not necessarily in $\mathcal{L}_{(2)}(H, H)$, the formal SDE (6.3) does not necessarily have a solution on H.

Suppose that $\{a^n\}$ and $\{b^n\}$ are two sequences in $\mathcal{L}(H, H)$ which satisfy the following condition:

(C2) For each n, there exists a constant $\theta_n \in (0, 1)$ such that

$$\sum_j \frac{\|(a^n)^*\xi_j\|^2}{\lambda_j} + \sum_j \frac{\|(b^n)^*\xi_j\|^2}{\lambda_j^{\theta_n}} < \infty. \tag{6.4}$$

Let A^n: $[0, T] \times H \to H$ and B^n: $[0, T] \times H \to \mathcal{L}(H, H)$ be given by

$$A^n(t, h) = a^n A(t, h) \quad \text{and} \quad B^n(t, h) = b^n B(t, h). \tag{6.5}$$

Consider the following SDE:

$$X_t^n = X_0 - \int_0^t LX_s^n\, ds + \int_0^t A^n(s, X_s^n)\, ds + \int_0^t B^n(s, X_s^n)\, dW_s \tag{6.6}$$

on H.

Theorem 6.1. *Under conditions* (C1) *and* (C2), (6.6) *has a unique solution* $X^n \in C([0, T], H)$ *for each* n.

Proof. It is easy to show that, for each n, conditions (2.3)–(2.9) and (2.44) are satisfied by (L, A^n, B^n). Hence it follows from the results of Section 2 that (6.6) has a unique solution $X^n \in C([0, T], H)$. □

Based on Remark 6.1, we seek a solution of (6.3) in a larger space. Suppose that we have an index $r_1 > 0$ such that L^{-r_1} is a Hilbert–Schmidt operator on H and let

$$\Phi \equiv \{\varphi \in H\colon \|\varphi\|_r < \infty,\ \forall r \in R\}, \tag{6.7}$$

where

$$\|\varphi\|_r^2 \equiv \sum_j \langle \varphi, \xi_j\rangle^2 \lambda_j^{2r}. \tag{6.8}$$

For each r, let H_r be the completion of Φ with respect to the norm $\|\cdot\|_r$. Let Φ' be the union of all H_r, $r \in \mathbf{R}$. Then Φ is a countably Hilbertian nuclear space and Φ' its dual space.

Suppose that there exists an index $p_0(T) \geq 0$ which satisfies the following conditions:

(C3) $\forall p \geq p_0(T)$, a^n, b^n can be extended to $\tilde{a}^n, \tilde{b}^n \in \mathcal{L}(H_{-p}, H_{-p})$ such that both $\tilde{a}^n$ and $\tilde{b}^n$ tend to the identity of H_{-p} strongly.

(C4) $\forall p \geq p_0(T)$, A and B can be extended to $\tilde{A}$: $[0, T] \times H_{-p} \to H_{-p}$ and $\tilde{B}$: $[0, T] \times H_{-p} \to \mathcal{L}_{(2)}(H, H_{-p})$ respectively and there exists a constant K such that

$$\|\tilde{A}(t, v_1) - \tilde{A}(t, v_2)\|_{-p}^2 + \|\tilde{B}(t, v_1) - \tilde{B}(t, v_2)\|_{\mathcal{L}_{(2)}(H, H_{-p})}^2 \leq K\|v_1 - v_2\|_{-p}^2 \tag{6.9}$$

and

$$\|\tilde{A}(t, v)\|_{-p}^2 + \|\tilde{B}(t, v)\|_{\mathcal{L}_{(2)}(H, H_{-p})}^2 \leq K(1 + \|v\|_{-p}^2) \tag{6.10}$$

for any $v, v_1, v_2 \in H_{-p}$.

Now consider the following SDEs on Φ':

$$\tilde{X}_t = X_0 - \int_0^t L\tilde{X}_s\, ds + \int_0^t \tilde{A}(s, \tilde{X}_s)\, ds + \int_0^t \tilde{B}(s, \tilde{X}_s)\, dW_s \tag{6.11}$$

and

$$\tilde{X}_t^n = X_0 - \int_0^t L\tilde{X}_s^n\, ds + \int_0^t \tilde{A}^n(s, \tilde{X}_s^n)\, ds + \int_0^t \tilde{B}^n(s, \tilde{X}_s^n)\, dW_s, \tag{6.12}$$

where $\tilde{A}^n = \tilde{a}^n \tilde{A}$ and $\tilde{B}^n = \tilde{b}^n \tilde{B}$.

To obtain the existence and uniqueness for (6.11) and (6.12), we need a general result from [6]. We state this result in Theorem 6.2 below without proof.

Let Φ be a countably Hilbertian nuclear space. Consider the following diffusion equation on Φ':

$$X_t = X_0 + \int_0^t C(s, X_s)\, ds + \int_0^t D(s, X_s)\, dW_s, \tag{6.13}$$

where C: $\mathbf{R}_+ \times \Phi' \to \Phi'$ and D: $\mathbf{R}_+ \times \Phi' \to \mathcal{L}(\Phi', \Phi')$ are two measurable mappings and W is a Φ'-valued Wiener process with covariance Q.

To solve the SDE (6.13), we make the following assumptions:

(D) For any $T > 0$ there exists an index $p_0 = p_0(T)$ such that, $\forall p \geq p_0, \exists q \geq p$ and a constant $K = K(p, q, T)$ such that

(D1) (Continuity) $\forall t \in [0, T]$, $v \in H_{-p}$, and $v_1, v_2 \in H_{-p}$, $C(t, v) \in H_{-q}$ and $D(t, v_1)(v_2) \in H_{-p}$. Furthermore, for t fixed, $C(t, v)$ and $|Q_{D_t(v_1) - D_t(v_2)}|_{-p,-p}$ are continuous in v, v_1, and v_2, where

$$|Q_{D_t(v_1) - D_t(v_2)}|_{-p,-p} = \sum_j Q((D_s^*(v_1) - D_s^*(v_2))h_j^p, (D_s^*(v_1) - D_s^*(v_2))h_j^p). \tag{6.14}$$

(D2) (Coercivity) $\forall t \in [0, T]$ and $\varphi \in \Phi$,

$$2C(t, \varphi)[\theta_p(\varphi)] \leq K(1 + \|\varphi\|_{-p}^2). \tag{6.15}$$

(D3) (Growth) $\forall t \in [0, T]$ and $v \in H_{-p}$,

$$\|C(t, v)\|_{-q}^2 \leq K(1 + \|v\|_{-p}^2) \text{ and } |Q_{D_t(v)}|_{-p,-p} \leq K(1 + \|v\|_{-p}^2). \quad (6.16)$$

(D4) (Monotonicity) $\forall t \in [0, T]$ and $v_1, v_2 \in H_{-p}$,

$$\begin{aligned} &2\langle C(t, v_1) - C(t, v_2), v_1 - v_2\rangle_{-q} + |Q_{D_t(v_1) - D_t(v_2)}|_{-q,-q} \\ &\quad \leq K\|v_1 - v_2\|_{-q}^2. \end{aligned} \quad (6.17)$$

(D5) (Initial) There exists an index r_0 such that

$$E((1 + \|X_0\|_{-r_0}^2)[\log(3 + \|X_0\|_{-r_0}^2)]^2) < \infty.$$

Theorem 6.2. *Under assumptions* (D), *the SDE* (6.13) *has a unique $H_{-p_1(T)}$-valued solution where $p_1(T)$ is an index such that the canonical map from $H_{-p(T)}$ to $H_{-p_1(T)}$ is Hilbert–Schmidt and $p(T) = \max(p_0(T), r_0)$. Further,*

$$E\left(\sup_{t \leq T}(1 + \|X_t\|_{-p_1(T)}^2)[\log(3 + \|X_t\|_{-p_1(T)}^2)]^2\right) < \infty. \quad (6.18)$$

Now we apply Theorem 6.2 to the present setup. As $r_0 = 0$, we have $p_1 = p_1(T) = p_0(T) + r_1$.

Theorem 6.3. *Suppose that $E((1 + \|X_0\|_0^2)[\log(3 + \|X_0\|_0^2)]^2) < \infty$. Then*:

(1) *Under condition* (C4), *the SDE* (6.11) *has a unique H_{-p_1}-valued solution $\tilde{X}$.*

(2) *Under conditions* (C3) *and* (C4), *the SDE* (6.12) *has a unique H_{-p_1}-valued solution $\tilde{X}^n$. Furthermore,*

$$E\sup_{t \leq T}\|\tilde{X}_t^n - \tilde{X}_t\|^2 \to 0 \qquad as \quad n \to \infty. \quad (6.19)$$

(3) *Under conditions* (C1)–(C4), *we have*

$$P(\tilde{X}_t^n = X_t^n,\ \forall t \in [0, T]) = 1, \quad (6.20)$$

where X^n is given by Theorem 6.1.

Proof. (1) Let

$$Q(\varphi, \psi) = \langle \varphi, \psi\rangle_0, \quad C(t, v) = -Lv + \tilde{A}(t, v), \text{ and } D(t, v) = \tilde{B}(t, v). \quad (6.21)$$

Then (Q, C, D, X_0) satisfies assumptions (D) with $q = p + r_1$.

(2) The existence and uniqueness of (6.12) follow from (1) by replacing $\tilde{A}$ and $\tilde{B}$ by $\tilde{A}^n$ and $\tilde{B}^n$.

We only need to verify (6.19). Note that

$$\begin{aligned} \tilde{X}_t^n - \tilde{X}_t = &-\int_0^t L(\tilde{X}_s^n - \tilde{X}_s)\,ds + \int_0^t (\tilde{a}^n\tilde{A}(s, \tilde{X}_s^n) - \tilde{A}(s, \tilde{X}_s))\,ds \\ &+ \int_0^t (\tilde{b}^n\tilde{B}(s, \tilde{X}_s^n) - \tilde{B}(s, \tilde{X}_s))\,dW_s. \end{aligned} \quad (6.22)$$

Then

$$
\begin{aligned}
(\tilde{X}_t^n - \tilde{X}_t)[\xi_j^{p_1}] = -\lambda_j \int_0^t (\tilde{X}_s^n - \tilde{X}_s)[\xi_j^{p_1}]\, ds \\
+ \int_0^t (\tilde{a}^n \tilde{A}(s, \tilde{X}_s^n) - \tilde{A}(s, \tilde{X}_s))[\xi_j^{p_1}]\, ds \\
+ \int_0^t \langle (\tilde{b}^n \tilde{B}(s, \tilde{X}_s^n) - \tilde{B}(s, \tilde{X}_s))^* \xi_j^{p_1}, dW_s \rangle_0.
\end{aligned} \tag{6.23}
$$

Making use of Itô's formula, we have

$$
\begin{aligned}
(\tilde{X}_t^n - \tilde{X}_t)[\xi_j^{p_1}]^2 = -2\lambda_j \int_0^t (\tilde{X}_s^n - \tilde{X}_s)[\xi_j^{p_1}]^2\, ds \\
+ \int_0^t 2(\tilde{a}^n \tilde{A}(s, \tilde{X}_s^n) - \tilde{A}(s, \tilde{X}_s))[\xi_j^{p_1}](\tilde{X}_s^n - \tilde{X}_s)[\xi_j^{p_1}]\, ds \\
+ \int_0^t 2(\tilde{X}_s^n - \tilde{X}_s)[\xi_j^{p_1}]\langle (\tilde{b}^n \tilde{B}(s, \tilde{X}_s^n) - \tilde{B}(s, \tilde{X}_s))^* \xi_j^{p_1}, dW_s \rangle_0 \\
+ \int_0^t \|(\tilde{b}^n \tilde{B}(s, \tilde{X}_s^n) - \tilde{B}(s, \tilde{X}_s))^* \xi_j^{p_1}\|_0^2\, ds.
\end{aligned} \tag{6.24}
$$

Summing up both sides of (6.24) over j and using the Burkholder–Davis–Gundy inequality, we have

$$
\begin{aligned}
f(r) &= E \sup_{t \le r} \|\tilde{X}_t^n - \tilde{X}_t\|_{-p_1}^2 \qquad (6.25) \\
&\le 0 + E \int_0^r 2\langle \tilde{a}^n \tilde{A}(s, \tilde{X}_s^n) - \tilde{A}(s, \tilde{X}_s), \tilde{X}_s^n - \tilde{X}_s \rangle_{-p_1}\, ds \\
&\quad + 4E \left(\int_0^r \|2 \sum_j (\tilde{X}_s^n - \tilde{X}_s)[\xi_j^{p_1}](\tilde{b}(s, \tilde{X}_s^n) - \tilde{B}(s, \tilde{X}_s))^* \xi_j^{p_1}\|_0^2\, ds \right)^{1/2} \\
&\quad + E \int_0^r \|\tilde{b}^n \tilde{B}(s, \tilde{X}_s^n) - \tilde{B}(s, \tilde{X}_s)\|_{\mathcal{L}_{(2)}(H_0, H_{-p_1})}^2\, ds \\
&\le E \int_0^r (\|\tilde{a}^n \tilde{A}(s, \tilde{X}_s^n) - \tilde{A}(s, \tilde{X}_s)\|_{-p_1}^2 + \|\tilde{X}_s^n - \tilde{X}_s\|_{-p_1}^2)\, ds \\
&\quad + 4E \left(4 \int_0^r \|\tilde{X}_s^n - \tilde{X}_s\|_{-p_1}^2 \|\tilde{b}^n \tilde{B}(s, \tilde{X}_s^n) - \tilde{B}(s, \tilde{X}_s)\|_{\mathcal{L}_{(2)}(H_0, H_{-p_1})}^2\, ds \right)^{1/2} \\
&\quad + E \int_0^r \|\tilde{b}^n \tilde{B}(s, \tilde{X}_s^n) - \tilde{B}(s, \tilde{X}_s)\|_{\mathcal{L}_{(2)}(H_0, H_{-p_1})}^2\, ds \\
&\le E \int_0^r (\|\tilde{a}^n \tilde{A}(s, \tilde{A}(s, \tilde{X}_s)\|_{-p_1}^2 + \|\tilde{X}_s^n - \tilde{X}_s\|_{-p_1}^2)\, ds \\
&\quad + 8E \left(\left(\sup_{t \le r} \|\tilde{X}_t^n - \tilde{X}_t\|_{-p_1} \right) \right. \\
&\qquad \left. \times \left(\int_0^r \|\tilde{b}^n \tilde{B}(s, \tilde{X}_s^n) - \tilde{B}(s, \tilde{X}_s)\|_{\mathcal{L}_{(2)}(H_0, H_{-p_1}}^2 ds \right)^{1/2} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+E\int_0^r \|\tilde{b}^n\tilde{B}(s\tilde{X}_s^n)-\tilde{B}(s,\tilde{X}_s)\|^2_{\mathcal{L}_{(2)}(H_0,H_{-p_1})}\,ds\\
&\le \int_0^r E\|\tilde{a}^n\tilde{A}(s,\tilde{X}_s^n)-\tilde{A}(s,\tilde{X}_s)\|^2_{-p_1}ds+\int_0^r E\|\tilde{X}_s^n-\tilde{X}_s\|^2_{-p_1}\,ds\\
&\quad+\tfrac{1}{2}f(r)+33\int_0^r E\|\tilde{b}^n\tilde{B}(s,\tilde{X}_s^n)-\tilde{B}(s,\tilde{X}_s)\|^2_{\mathcal{L}_{(2)}(H_0,H_{-p_1})}ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
f(r)&\le 4\|\tilde{a}^n\|^2_{\mathcal{L}(H_{-p_1})}\int_0^r E\|\tilde{A}(s,\tilde{X}_s^n)-\tilde{A}(s,\tilde{X}_s)\|^2_{-p_1}ds\\
&\quad+4\int_0^r E\|\tilde{a}^n\tilde{A}(s,\tilde{X}_s)-\tilde{A}(s,\tilde{X}_s)\|^2_{-p_1}ds+2E\int_0^r \|\tilde{X}_s^n-\tilde{X}_s\|^2_{-p_1}ds\\
&\quad+132\|\tilde{b}^n\|^2_{\mathcal{L}(H_{-p_1})}\int_0^t E\|\tilde{B}(s,\tilde{X}_s^n)-\tilde{B}(s,\tilde{X}_s)\|^2_{\mathcal{L}_{(2)}(H_0,H_{-p_1})}\,ds\\
&\quad+132\int_0^t E\|\tilde{b}^n\tilde{B}(s,\tilde{X}_s)-\tilde{B}(s,\tilde{X}_s)\|^2_{\mathcal{L}_{(2)}(H_0,H_{-p_1})}\,ds\\
&\le M\int_0^r f(s)\,ds+4\int_0^r E\|\tilde{a}^n\tilde{A}(s,\tilde{X}_s)-\tilde{A}(s,\tilde{X}_s)\|^2_{-p_1}\,ds\\
&\quad+132\int_0^t E\|\tilde{b}^n\tilde{B}(s,\tilde{X}_s)-\tilde{B}(s,\tilde{X}_s)\|^2_{\mathcal{L}_{(2)}(H_0,H_{-p_1})}\,ds,
\end{aligned}
$$

where

$$
M=4K\sup_n\|\tilde{a}^n\|^2_{\mathcal{L}(H_{-p_1})}+2+132K\sup_n\|\tilde{b}^n\|^2_{\mathcal{L}(H_{-p_1})} \tag{6.26}
$$

is a finite constant as both $\tilde{a}^n$ and $\tilde{b}^n$ tend to the identity of H_{-p_1} strongly. It follows from Gronwall's inequality that

$$
\begin{aligned}
&E_t\sup_{t\le T}\|\tilde{X}_t^n-\tilde{X}_t\|^2_{-p_1}\\
&\le 4e^{MT}\int_0^T e^{-Ms}E\|\tilde{a}^n\tilde{A}(s,\tilde{X}_s)-\tilde{A}(s,\tilde{X}_s)\|^2_{-p_1}ds\\
&\quad+132e^{MT}\int_0^t e^{-Ms}E\|\tilde{b}^n\tilde{B}(s,\tilde{X}_s)-\tilde{B}(s,\tilde{X}_s)\|^2_{\mathcal{L}_{(2)}(H_0,H_{-p_1})}\,ds.
\end{aligned} \tag{6.27}
$$

By (6.18), (C3), and (C4) we only need to show that

$$
\|\tilde{a}^n\tilde{A}(s,\tilde{X}_s)-\tilde{A}(s,\tilde{X}_s)\|^2_{-p_1}\to 0 \tag{6.28}
$$

and

$$
\|\tilde{b}^n\tilde{B}(s,\tilde{X}_s)-\tilde{X}(s,\tilde{X}_s)\|^2_{\mathcal{L}_{(2)}(H_0,H_{-p_1})}\to 0 \tag{6.29}
$$

almost surely. Equation (6.28) follows from (C3) directly. For (6.29), we note that

$$
\begin{aligned}
&\|\tilde{b}^n\tilde{B}(s,\tilde{X}_s)-\tilde{B}(s,\tilde{X}_s)\|^2_{\mathcal{L}_{(2)}(H_0,H_{-p_1})}|\\
&\quad=\sum_j\|(\tilde{b}^n\tilde{B}(s,\tilde{X}_s)-\tilde{B}(s,\tilde{X}_s))\xi_j\|^2_{-p_1}
\end{aligned} \tag{6.30}
$$

and, for each j,

$$(\tilde{b}^n \tilde{B}(s, \tilde{X}_s) - \tilde{B}(s, \tilde{X}_s))\xi_j \to 0, \tag{6.31}$$

$$\|(\tilde{b}^n \tilde{B}(s, \tilde{X}_s) - \tilde{B}(s, \tilde{X}_s))\xi_j\|^2_{-p_1} \leq \left(\sup_n \|\tilde{b}^n\|^2_{\mathcal{L}(H_{-p_1})} + 1\right)\|(\tilde{B}(s, \tilde{X}_s) - \tilde{B}(s, \tilde{X}_s))\xi_j\|^2_{-p_1}. \tag{6.32}$$

The right-hand side of (6.32) is summable and hence (6.9) then follows from the dominated convergence theorem.

(3) As X^n is an H-valued solution of (6.6), it is easy to show that X^n is an H_{-p_1}-valued solution of (6.12). Equation (6.20) then follows from the uniqueness of solution of (6.12). □

Example 6.1. Consider the following formal SPDE:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2 u}{\partial x_2^2} - \alpha u + \dot{W}_{t,x_1,x_2}, \qquad (t, x_1, x_2) \in [0, T] \times [0, \pi]^2 \tag{6.33}$$

with Neumann boundary conditions and $u(0, x_1, x_2) = u_0(x_1, x_2)$ and W is the Gaussian white noise in space-time. This equation describes the stochastic behavior of the voltage potential of a neuron whose cell membrane is regarded as a two-dimensional (rectangular) patch (see [11] and [7] for details). It is known that (6.33) does not have a solution which is a random field. We write it as the following integral equation:

$$X_t = X_0 - \int_0^t LX_s\, ds + W_t, \tag{6.34}$$

where $L = -\frac{1}{2}\Delta + \alpha$ and X_0 is an initial random vector determined by the initial random field $u_0(x_1, x_2)$. Let $\lambda_0 = 0$ and $\varphi_0(x) = 1/\sqrt{\pi}$. For $j = 1, 2, \ldots,$ write $\lambda_j = \frac{1}{2}j^2$ and $\varphi_j(x) = \sqrt{(2/\pi)}\cos jx$. For $i,\ j = 1, 2, \ldots$, let

$$\lambda_{ij} = \lambda_i + \lambda_j + \alpha \quad \text{and} \quad \xi_{ij}(x, y) = \varphi_i(x)\varphi_j(y). \tag{6.35}$$

For each p, we define a norm $\|\cdot\|_p$ on $H = L^2([0, \pi]^2)$ by

$$\|h\|_p^2 = \sum_{i,j=0}^{\infty} \lambda_{ij}^{2p}\langle h, \xi_{ij}\rangle_H^2. \tag{6.36}$$

Let H_p be the completion of H with respect to $\|\cdot\|_p$.

Let $p_0(T) > \frac{1}{2}$. Let A: $[0, T] \times H \to H$ and B: $[0, T] \times H \to \mathcal{L}(H, H)$ be given by

$$A(t, v) = -\sum_{i,j=0}^{\infty} \lambda_{ij}\langle v, \xi_{ij}\rangle\xi_{ij} \quad \text{and} \quad B(t, v)(h) = h, \quad \text{for} \quad v, h \in H. \tag{6.37}$$

Then A and B satisfy conditions (C3) and (C4). Hence (6.34) has a unique H_{-p_1}-valued solution ($p_1 \geq \frac{1}{2}$). Let $\{b^n\}$ be a sequence in $\mathcal{L}(H, H)$ such that

$$\sum_{ij} \frac{\|(b^n)^* \xi_{ij}\|^2}{\lambda_{ij}^{\theta_n}} < \infty \tag{6.38}$$

for some $\theta_n \in (0, 1)$.

Consider the following SPDE:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2 u}{\partial x_2^2} - \alpha u + b^n \overset{\bullet}{W}_{t,x_1,x_2}, \qquad (t, x_1, x_2) \in [0, T] \times [0, \pi]^2 \tag{6.39}$$

with Neumann boundary conditions and $u(0, x_1, x_2) = u_0(x_1, x_2)$. Then (6.39) has a random field solution $u^n(t, x)$ given by

$$\begin{aligned} u^n(t, x) &= \sum_{i,j=1}^{\infty} e^{-\lambda_{ij} t} \left(\int_{[0,\pi]^2} u_0(y)\xi_{ij}(y)\, dy \right) \xi_{ij}(x) \\ &\quad + \sum_{i,j=1}^{\infty} \int_0^t \int_{[0,\pi]^2} \left((b^n)^* \xi_{ij} \right)(y) e^{-\lambda_{ij}(t-s)} W(ds\, dy) \xi_{ij}(x). \end{aligned} \tag{6.40}$$

Furthermore, let $X_t^n(x) = u^n(t, x)$, then $X^n \in C([0, T], H)$.

Consider X^n as H_{-p_1}-valued processes denoted by $\tilde{X}^n$. Suppose that b^n also satisfies (C3), then $\tilde{X}^n$ converges to X in $C([0, T], H_{-p_1})$ where X is the unique H_{-p_1}-valued solution of (6.34).

Remark 6.2. Typical examples of operators $\{b^n\}$ which satisfy our conditions are of the form

$$b^n h = \sum_{i,j} C_{ij}^n \langle \xi_{ij}, h \rangle \xi_{ij}, \qquad \forall h \in H, \tag{6.41}$$

where $\{C_{ij}^n\}$ satisfies the following conditions:

(a) For each n, there exists a constant $\theta_n \in (0, 1)$ such that

$$\sum_{i,j} \frac{|C_{ij}^n|^2}{\lambda_{ij}^{\theta_n}} < \infty. \tag{6.42}$$

(b) There exists a constant M and a sequence $\alpha_n \to \infty$ such that

$$\sup_{i,j \leq \alpha_n} |C_{ij}^n - 1| \to 0 \qquad \text{as} \quad n \to \infty \tag{6.43}$$

and

$$\sup_{i,j} |C_{ij}^n| \leq M. \tag{6.44}$$

Proof. Condition (C2) follows from (a) directly. Now we prove condition (C3). b^n can be extended to $\tilde{b}^n$ on H_{-p} for any $p > 0$. In fact, $\forall v \in H_{-p}$, we have

$$\tilde{b}^{n_v} = \sum_{ij} C_{ij}^n \langle \xi_{ij}^{-p}, v \rangle_{-p} \xi_{ij}^p. \tag{6.45}$$

It follows from (6.44) that the right-hand side of (6.45) is well defined. By (b), it is easy to see that $\tilde{b}^n v \to v$ in H_{-p}. □

Remark 6.3. A more specific class of examples of operators $\{b^n\}$ is of the form

$$C^n_{ij} = 0 \qquad \text{if} \quad i \geq \alpha_n \quad \text{or} \quad j \geq \alpha_n. \tag{6.46}$$

In this case, condition (b) implies condition (a) in the last remark.

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