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The hierarchy of fractional quantum Hall states and the Z_p spin system

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Abstract

The hierarchy of fractional quantum Hall states has been studied here in the framework of a Z_p spin system. It is shown that we can derive an equivalence relation between the anyon statistical formulation of the hierarchical states depicted in the continued fraction scheme and the Jain classification scheme based on the composite fermion model or its variant in the Berry phase formulation. Also the FQH states in the even denominator filling factor can be realized in this framework in special cases when electrons appear in pairs in concurrence with the results of the Berry phase formulation. © 1998 Published by Elsevier Science B.V.

Fractional quantum Hall (FQH) states are formed in a magnetic field by the strongly correlated system of electrons representing incompressible fluids, at filling factor $\nu = 1/m, m$ being an odd integer, and can be described by the Laughlin wave function [1]

$$\psi_0 = \prod_{i<j} (z_i - z_j)^m \exp\left(-\tfrac{1}{4} \sum |z_i|^2 \right), \tag{1}$$

where $z_i = x_i + \mathrm{i}y_i$ is the coordinate of the ith electron. This wave function suggests that this has a zero of the mth order as any pair of electrons approach each other and hence electrons try to stay away from each other as much as possible. The effective theory of such a state is described by the Lagrangian

$$L = -m\frac{1}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} - \frac{e}{2\pi} A_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}, \tag{2}$$

where a $U(1)$ gauge field a_μ is introduced to describe the conserved particle number current

$$J^\mu = \frac{1}{2\pi} \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} \tag{3}$$

and A_μ describes the vector potential of the electromagnetic field. The hierarchy of FQH states are generally studied by the following two schemes:

(1) one proposed by Haldane [2] and Halperin [3] based on anyon statistics;

(2) another proposed by Jain [4] based on a composite fermion model.

The filling fraction ν in the first proposal based on anyon statistics in a 2 + 1-dimensional system is described by a continued fraction

$$\nu = \cfrac{1}{m + \cfrac{\alpha_1}{p_2 - \cfrac{\alpha_2}{p_3 - \cfrac{\alpha_3}{p_4 \ldots}}}} \tag{4}$$

where m is the inverse of the parent filling factor, m being an odd integer and $\alpha_i = \pm 1, +(-)$ corresponding

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to condensation of quasiholes (quasielectrons) and p_i an even integer. Since all fractions are obtained starting from $1/m$ the latter has come to be known as the fundamental fraction. The physical basis of this continued fraction scheme follows from the following considerations. It is assumed that as the magnetic field is varied away from the $1/m$ (m odd) filling factor, quasiparticles (quasiholes or quasielectrons) having fractional charge and fractional statistics are created. At certain filling factors these anyonic quasiparticles themselves form a Laughlin-type correlated state. Now two equivalent pictures emerge.

(1) In a mean field theory approach, we may view the gauge field a_μ as a fixed background and the quasiparticle gas behaves like bosons in the magnetic field $b = \epsilon_{ij}\partial_i a_j$. These bosons do not carry any electric charge and when the boson density satisfies

$$J_0 = \frac{1}{p_2}\frac{b}{2\pi}, \tag{5}$$

where p_2 is even, the bosons have a filling function $1/p_2$. The ground state of the bosons can again be described by a Laughlin state. The final electronic state is just a second level hierarchical FQH state constructed by Haldane [2].

(2) If we let a_μ interact with the current of the quasiparticles, quasiparticles will be dressed by the a_μ flux. The dressed quasiparticles carry an electric charge e/m and a statistics of $\theta = \pi/m$, where the quasiparticles have the density

$$J_0 = \frac{1}{(p_2 + \theta/\pi)}\frac{eB}{2\pi m}, \tag{6}$$

where p_2 is even, the quasiparticle will have a filling fraction $1/(p_2 + \theta/\pi)$. In this case the quasiparticle system can form a Laughlin state described by the wave function

$$\prod_{i<j}(z_i - z_j)^{p_2+\theta/\pi}. \tag{7}$$

The final electronic state obtained this way is again a second level hierarchical FQH state. This scheme was proposed by Halperin [3].

Haldane's scheme of hierarchical states can be generalized to have an effective theory of FQH states as has been described in detail by Wen [5] and Zee [6]. Introducing a new $U(1)$ gauge field $\tilde{a}_\mu$ to describe the boson current, the total effective theory for the second level hierarchical state has the form

$$L = \left(\frac{-m}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda} - \frac{-e}{2\pi}A_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}\right) + \left(\frac{-p_2}{4\pi}\tilde{a}_\mu\partial_\nu\tilde{a}_\lambda\epsilon^{\mu\nu\lambda} + \frac{1}{2\pi}a_\mu\partial_\nu\tilde{a}_\lambda\epsilon^{\mu\nu\lambda}\right). \tag{8}$$

The filling fraction ν is given by

$$\nu = \frac{1}{m - 1/p_2}. \tag{9}$$

This equation (8) can be written in a more compact form by introducing $(a_\mu, \tilde{a}_\mu = (a_{1\mu}, a_{2\mu}))$

$$L = \sum_{I,I'}\frac{1}{4\pi}K_{II'}a_{I\mu}\partial_\nu a_{I'\lambda}\epsilon^{\mu\nu\lambda} - \frac{e}{2\pi}A_\mu\partial_\nu t_I a_{I\lambda}\,\epsilon^{\mu\nu\lambda}, \tag{10}$$

where the matrix K has integer elements given by

$$K = \begin{pmatrix} m & -1 \\ -1 & p_2 \end{pmatrix} \tag{11}$$

and $t^T = (t_1, t_2) = (1, 0)$ is called the charge vector. The filling fraction can be written as $\nu = t^T K^{-1} t$. The above construction can be easily generalized to higher order hierarchical states. Indeed, the effective theory of the nth level hierarchical state will be given by (10) with n gauge fields. An nth level hierarchical state is obtained by condensation of quasiparticles with the $a_{I\mu}$ charge $l_I|_{I=1,\dots,n-1}$. The matrix K is given by

$$K^{(n)} = \begin{pmatrix} K^{(n-1)} & -l \\ -l^T & p_n \end{pmatrix} \tag{12}$$

with p_n even. Now with the charge vector given by $t_I = \delta_{1I}$, the filling factor is given by

$$\nu = t^T K^{-1} t. \tag{13}$$

Alternatively, Jain [4] proposed a composite fermion theory by suggesting that the FQHE of electrons is a manifestation of the IQHE of some more complicated fermionic objects called composite fermions where a composite fermion is formed by the bound state of an electron and even number of flux tubes. The inverse of the filling factor is given by the number of flux quanta per particle. The same external flux is available to each composite fermion in the composite fermion state (not counting the flux which is a part of the composite fermion) as to each electron in the original electron state. For a filling factor n of the composite fermion state, this leads to an electron state such that $(n^{-1}+2m)$ flux quanta are available to each electron. Thus the composite fermion state of filling factor n is equivalent, in a mean field sense, to an electron state at a filling factor

$$\nu = \frac{n}{2mn+1}\,. \tag{14}$$

Incompressibility is obtained when the composite fermions occupy an incompressible state, e.g. when they fill an integer number of Landau levels. However, as we know, electron–electron interaction is responsible for FQH states where the IQHE represents a noninteracting electron state. Jain has pointed out that interactions are needed to stabilize the composite fermions but once the composite fermions are generated, they are treated as weakly interacting objects. However, Jain himself has admitted that why composite fermions are generated is a priori difficult to justify. Greiter and Wilczek [7] have pointed out that flux tubes carrying any integral multiple of basic flux can be gauged away. Besides the notion of composite fermions is quite peculiar as we have the same statistical phase. These authors have suggested that the acquisition of flux tubes by an electron can be considered as *processes* rather than completed acts so that one can carry the adiabatic evolution to have at the end an even number of flux quanta localized on the electrons.

In a recent paper [8], Basu and Bandyopadhyay have studied the hierarchical states in the framework of chiral anomaly and Berry phase. It has been shown that the unambiguously observed FQH states with filling factor $\nu = p/q$ with p even or odd and q an odd integer can be considered from the viewpoint that the Berry phase associated with even number of vortices can be removed to the dynamical phase and the corresponding fermionic state attains a higher Landau level. Indeed, in a spherical geometry when we consider quantum Hall states on the $2D$ surface of a sphere with a magnetic monopole of strength μ at the center, the angular momentum relation is given by

$$\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{p} - \mu\boldsymbol{r},$$
$$\mu = 0,\ \pm 1/2,\ \pm 1,\ \pm 3/2\ldots \tag{15}$$

It has been pointed out earlier that μ is related to the Berry phase factor $e^{i\phi}$, where $\phi = 2\pi\mu$ and is associated with the chiral anomaly through the relation [9]

$$q = 2\mu = -\tfrac{1}{2}\int \partial_\mu J^5_\mu\, d^4x\,, \tag{16}$$

where J^5_μ is the axial vector current related to a chiral fermion. Indeed, it has been pointed out that the strong external magnetic field causes a chiral symmetry breaking of fermions (Hall particles) and as a result anomaly is realized in association with the quantization of Hall conductivity. This helps to study the behaviour of quantum Hall fluid from the viewpoint of the Berry phase which is linked with chiral anomaly [10].

From the angular momentum relation (15), we note that when we consider the ground state $\boldsymbol{r}\times\boldsymbol{p} = 0$, $\mu = 1/2$, the Dirac quantization condition $e\mu = 1/2$ suggests that $e = 1$, which will exhibit the IQH with fermion number 1. However, if we consider the next excited states with $\boldsymbol{r}\times\boldsymbol{p} = 1$, which exhibits explicitly an interacting system, the respective angular momentum is changed to $J = 3/2$ for $\mu = 1/2$. This can be viewed as a system with $\mu_{\text{eff}} = 3/2$ having $\boldsymbol{r}\times\boldsymbol{p} = 0$. In this excited state, the quantization condition $e\mu_{\text{eff}} = 1/2$ suggests that the quasiparticle will have fermion number $1/3$ indicating that the filling factor is $1/3$. For $\boldsymbol{r}\times\boldsymbol{p} = 2$ indicating $\mu_{\text{eff}} = 5/2$ with $\boldsymbol{r}\times\boldsymbol{p} = 0$, we have the filling factor $1/5$ FQH state. Now we note that when μ is an integer, we come across a peculiar situation where we can use a transformation which effectively suggests that we have a dynamical relation of the form

$$\boldsymbol{j} = \boldsymbol{r}\times\boldsymbol{p} - \mu\boldsymbol{r} = \boldsymbol{r}'\times\boldsymbol{p}'\,.$$

This indicates that the Berry phase, which is associated with μ, may be unitarily removed to the dynamical phase. This implies that the average magnetic field may be taken to be vanishing in these states. So in FQH states, where we have the relation $2\mu_{\text{eff}} = 2m+1$, where m is an integer, these can be viewed as if one vortex is attached to an *electron* as the attachment of $2m$ vortices to an electron effectively leads to the removal of Berry phase to the dynamical phase. Now we note that for a higher Landau level, we can consider the Dirac quantization condition $e\mu_{\text{eff}} = \frac{1}{2}n$, where n

can be viewed as a vortex of strength $2\ell+1$. This can generate FQH states having the filling factor $n/2\mu_{\rm eff}$ where both n and $2\mu_{\rm eff}$ are odd integers with the Berry phase factor $2\mu_{\rm eff} = 2m'+1$, m' being an integer. Since the Berry phase factor related to $2m'$ vortices or its multiple can be removed to the dynamical phase, we can view these FQH states being depicted by the following relation for the filling factor [8],

$$\frac{n}{2\mu_{\rm eff}} = \frac{1}{(2\mu_{\rm eff}\mp 1)/n \pm \frac{1}{n}} = \frac{n}{2mn\pm 1}, \qquad (17)$$

where $2\mu_{\rm eff}\mp 1$ is an even integer given by $2m' = 2mn$. Here the $+(-)$ sign indicates the orientations of the vertex line. This is essentially the Jain classification scheme excepting the fact that here n is an odd integer and we have taken both $\pm$ sign indicating the orientation of the vortex. This is not surprising as in the Jain scheme a composite fermion is a bound state of an electron with $2m$ flux quanta and a flux quantum is topologically equivalent to a vortex line. In the present scheme, the FQH states having the form $n'/(2mn'\pm 1)$ with n' an even integer can be generated through particle hole conjugate states

$$1-\frac{n}{2mn\pm 1} = \frac{n(2m-1)\pm 1}{2mn\pm 1}, \qquad (18)$$

where n is an odd integer. It is noted that for $m=1$, we have

$$1-\frac{n}{2n\pm 1} = \frac{n\pm 1}{2n\pm 1} = \frac{n'}{2n'\mp 1}, \qquad (19)$$

where $n' = n\pm 1$, n being an odd integer. This suggests that for $m=1$ in relations (17) and (18) particle–hole conjugate states can be accommodated in a unified relation of the form

$$\nu = \frac{n}{2n\pm 1} \qquad (20)$$

indicating that this is valid for both n odd and even integers.

In this note, we shall show that FQH states with filling factor p/q, p even or odd and q odd, both the continued fraction scheme based on anyon statistics and the composite fermion scheme or its variant in Berry phase formulation can be related *in an equivalent way* when we consider that FQH states can be described in the framework of a Z_p spin system.

In an earlier paper [11] it has been shown that fractional statistics has its relavance in a Z_p spin system. It is well known that a Z_2 system represents an Ising model, whereas a Z_p system with $p\to\infty$ represents a plane rotor model (XY model). The planar model exhibits certain specific characteristics which are significantly different from the Ising model. We can go gradually from the Ising model (Z_2 model) to the XY model by considering a series of the Z_p model and sending $p\to\infty$. Indeed, the Z_p model is one in which each classical spin can form only one of the p discrete angles $\theta_m = 2\pi m/p$ with some fixed direction in the space of internal degrees of freedom. Elitzer, Pearson and Shigemetsu [12] have shown that for $p\leqslant 4$ we have two massive spin wave phases with a conventional singularity behaviour at the transition. However, for $p\geqslant 5$ we have three phases, i.e. a massless phase appears in between two massive phases. Again, in the XY model (Z_p with $p\to\infty$) we have two phases characterized by the fact that, at high temperature, we have massive spin waves with finite correlation length similar to the high temperature behaviour of other spin systems, but below a critical temperature T_k, we have massless spin waves with power behaved correlation function and continuously varying exponents. This implies that as $p\to\infty$, the lowest ordered phase shrinks down to zero temperature forbidding the existence of an ordered phase in a continuous symmetry model. A Z_2 system (Ising model) can be described by a system of fermionic gas. In an earlier paper [13] it has been pointed out that in three space dimension, such a Z_2 spin system (fermionic gas) can be well described by a system of scalar particles, each having half orbital angular momentum with a specific l_z value ($+1/2$ or $-1/2$) in an anisotropic space or in the field of a magnetic monopole. It can be shown that a two-dimensional Z_p spin system can be well described by a system of scalar particles, each having an orbital angular momentum $l = 1/p$ and this gives rise to the behaviour of such particles having fractional statistics [11]. Indeed, in a $(2+1)$-dimensional system, the angular momentum can take any arbitrary value. The Chern–Simons term in the $(2+1)$ dimension given by

$$L_{\rm CS} = \frac{\mu}{2}\epsilon^{\rho\nu\lambda}a_\rho\partial_\nu a_\lambda, \qquad (21)$$

where a_ρ is a gauge field, corresponds to any arbitrary fractional value of the angular momentum. In fact, this corresponds to the field equation

$$eJ^{\rho} = \frac{\mu}{2}\epsilon^{\rho\nu\lambda}F_{\nu\lambda} \tag{22}$$

and integrating the 0-component of this equation, we find

$$eN = \mu\phi , \tag{23}$$

where N is the particle number and ϕ is the magnetic flux. Thus the effect of the Chern–Simons term is to associate with each particle a magnetic flux e/μ. This corresponds to the change in statistics characterized by the parameter

$$\Delta\theta = e^2/2\mu \tag{24}$$

in terms of the phase factor $e^{i\theta}$ characterizing the statistics of the particle. This phase factor $e^{i\theta}$ is again related to the angular momentum through the relation

$$e^{2i\pi J} = e^{i\theta} . \tag{25}$$

Now just as a fermion gas represented by a scalar particle moving with $l = 1/2$ having a fixed l_z value corresponds to a Z_2 system, we can associate a Z_p spin system characterized by the angle $\theta_m = 2\pi m/p$, which the classical spin forms with some fixed direction with the angular momentum $J = 1/p$. Now, from the statistical parameter

$$\theta = \pi\left[1 - \frac{1}{n}\right] , \tag{26}$$

where $n = \infty$ corresponds to a fermion and $n = 1$ a boson, we find from relations (25) and (26) that for $J = 1/p$

$$n = \frac{p}{p-2} . \tag{27}$$

Thus a two-dimensional Z_p spin system can be taken to represent a gas of particles having fractional statistics given by the statistical parameter (26), where n is related to p through relation (27).

Now from relation (27) we note that for a Z_2 system, we have $n = \alpha$ implying that it is equivalent to a fermionic gas. Again for $p = 3$ and 4, we find $n = 3$ and 2, respectively, implying that the system can be represented by a system of fermions having fractional fermion number. However, for $p \geqslant 5$, we note that n is not an integer and hence will not represent a system of any specific fractional fermion number. Indeed, the quasiparticles corresponding to these systems will represent a hedgehog mixture of bosons and fermions having no specific fermionic and bosonic property. Again as $p \to \alpha$, we have $n = 1$ implying that the system represents a gas of bosons. This analysis suggests that although Z_p systems with finite p represent discrete symmetries, all discrete symmetries cannot be associated with a pure fermionic character. It is only for $p = 2, 3, 4$ that we can get a pure fermionic property when $p = 2$ corresponds to a conventional fermion with a fermion number 1 and $p = 3$ and 4 corresponds to a fractional fermion number. For $p \geqslant 5$, we have a hedgehog mixture of bosons and fermions with no specific fermionic property. This is the reason why we have different phase structures for $p \leqslant 4$ and $p \geqslant 5$ in a Z_p spin system.

Now to study the relationship between hierarchical FQH states with the Z_p spin system, we note that the effective theory of hierarchical FQH states governed by the Laughlin wave function can be constructed by considering a charged anyon system in a magnetic field [14]

$$L = \psi^{\dagger}_{\text{anyon}}\, i(\partial_0 - i\,eA_0)\psi_{\text{anyon}} + \frac{1}{2M}\psi^{\dagger}_{\text{anyon}}(\partial_i - i\,eA_i)^2\psi_{\text{anyon}} . \tag{28}$$

ψ_{anyon} is the field that describes anyons with fractional statistics θ. At filling fraction $\nu = (\theta/\pi + \tilde{m})^{-1}$, where $\tilde{m}$ is an even integer, the anyon ground state is given by the Laughlin wave function

$$\prod (z_i - z_j)^{\theta/\pi+\tilde{m}}\, e^{-\frac{1}{4}\sum |z_i|^2} . \tag{29}$$

We note that the state is an incompressible fluid and the anyon number current J_μ has the following response to a change of electromagnetic fields,

$$\delta J^{\mu} = \frac{\nu}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \delta A_\lambda . \tag{30}$$

Introducing a $U(1)$ gauge field a_μ to describe the conserved anyon number current

$$J^{\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda , \tag{31}$$

the effective Lagrangian can be written as

$$L = -(\theta/\pi + \tilde{m})\frac{1}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda} + \frac{1}{g^2}(f_{\mu\nu})^2 + \frac{e}{2\pi}A_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}. \quad (32)$$

A quasiparticle excitation at position ξ in the Laughlin state is created by multiplying $\prod_i(\xi - z_i)$ with the groundstate wave function in the electron condensate carrying electric charge $1/(\tilde{m} + \theta/\pi)$ and can be created in the effective theory by inserting a unit charge of the a_μ gauge field, i.e. by adding a term $\int d^2z\, a_0(z)\delta(z-\xi)$ to the effective theory. Introducing a bosonic field to describe the source that creates the quasiparticles, the full Lagrangian of the effective theory with (second quantized) quasiparticle excitations is then given by

$$L = -\left(\frac{\theta}{\pi} + \tilde{m}\right)\frac{1}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda} + \frac{1}{g^2}(f_{\mu\nu})^2 + \frac{e}{2\pi}A_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda} + \phi^\dagger \mathrm{i}(\partial_0 - \mathrm{i}a_0)\phi + \frac{1}{2M}\phi^\dagger(\partial_i - \mathrm{i}a_i)^2\phi, \quad (33)$$

where the bosonic field ϕ describes the quasiparticle in the Laughlin state.

From the equation of motion $\partial L/\partial a_0 = 0$, we find the filling fraction to be

$$\nu = \frac{2\pi J_0}{eB} = (\theta/\pi + \tilde{m})^{-1}. \quad (34)$$

In the case $\theta/\pi = 1$, we have $\nu = 1/(\tilde{m}+1)$. We can increase the filling fraction by creating quasiparticles. As mentioned earlier in the mean field theory sense, the quasiparticles behave like bosons in the magnetic field $b = \partial_i a_j \epsilon_{ij}$. When the boson density satisfies

$$\phi^\dagger\phi = \frac{1}{p_2}\frac{b}{2\pi}, \quad (35)$$

where p_2 is an even integer, the bosons have a filling fraction $1/p_2$. The final state represent a second level hierarchical FQH state as constructed by Haldane and Halperin. In this way, we can construct the nth level hierarchical state. This analysis of Wen [14] shows how we can arrive at the continued fraction scheme from the anyonic formalism in a direct way.

Now we note that from the expressions (26) and (27) showing the relationship of fractional statistics and Z_p spin system $\theta = \pi(1 - \frac{1}{n})$ and $n = p/p-2$, we find

$$\theta/\pi = 1 - \frac{1}{n} = 1 - \frac{p-2}{p} = 2/p. \quad (36)$$

So from the relation for the filling factor in terms of fractional statistics

$$\nu^{-1} = \frac{\theta}{\pi} + \tilde{m} \quad (\tilde{m}\text{ even integer}), \quad (37)$$

we can write

$$\nu^{-1} = \frac{2}{p} + \tilde{m} \quad (38)$$

so that

$$\nu = \frac{p}{p\tilde{m}+2}. \quad (39)$$

Now for p an even integer given by $p = 2n$, $\tilde{m} = 2m$, we find

$$\nu = \frac{2n}{4mn+2} = \frac{n}{2mn+1}. \quad (40)$$

Thus we arrive at the Jain classification scheme from the relationship between fractional statistics and Z_p spin system.

This is not surprising. Indeed, we have noted that for a Z_p spin system with $p > 4$, the system represents a hedgehog behaviour showing a peculiar mixture state of fermions and bosons having fractional statistics and this is similar to the analysis of a filling factor in the anyon statistical formulation of a continued fraction. Again, as in a two-dimensional Z_p system, we can consider that a particle is moving with angular momentum $J = 1/p$, where a fermion is represented by a scalar particle moving with $l = 1/2$, which can be represented by a vortex, the angular momentum $l = 1/p$ $(p > 2)$ just shows the departure from the pure fermionic character which can be characterized by the strength of a vortex added to a fermion when an even number of vortices added to a fermion will not alter its statistics. So in the expression for filling factors (39) and (40), where we have taken $p = 2n$, we note that for $n = 1, 2, 3, \ldots$, representing $p = 2, 4, 6, \ldots$, a vortex of strength $1, 1/2, 1/3, \ldots$ is attached to an *electron* which is already attached to $2m$ vortices. As we know topologically a vortex is equivalent to a flux quantum, this effectively leads to the fact that $(n^{-1}+2m)$ flux quanta are available to each electron.

As mentioned earlier, a variant of this formulation can be derived when we consider that in case a fermion is attached to an even number of vortices, the Berry phase can be removed to the dynamical phase and hence the concept of a composite fermion as proposed by Jain is not necessary. In view of this, we find that when the hierarchical FQH states are analysed in the framework of a Z_p spin system, we find an equivalence relationship between the anyon statistical analysis incorporating a continued fraction and the classification scheme proposed by Jain as well as that based on the analysis of the Berry phase proposed by Basu and Bandyopadhyay [8].

We may point out here that both the anyon quasiparticle condensation approach of Haldane and Halperin and the composite fermion approach of Jain cannot explain the even denominator filling factor recently observed in special cases. However, the Berry phase approach can explain it in a nice way when electrons appear in pairs giving rise to a non-Abelian Berry phase [15]. In view of this, these are termed non-Abelian Hall fluids. In terms of the Z_p spin system, we note that from relation (39) when p is odd given by $p=2n+1$ we have a filling factor of the form $\nu = n'/(2mn'+2)$ with $n'=2n+1$. It is noted that for $m=0$, n' an odd integer, we have an even denominator filling factor given by $\nu = n'/2$. With $n'=1$ and 5, we have $\nu = 1/2$ and $5/2$, where the latter is known as the Haldene–Rezayi state [16]. However we have a peculiar situation here when the average magnetic field vanishes. In terms of the Berry phase, this suggests that the Berry phase in this case can be removed to the dynamical phase. These FQH states can only be observed when the state is split into a pair of electrons. For $\nu = 1/2$, the pair is a p-wave state of spin polarized (or spinless) electrons. The splitting of the state into spin polarized states suggests that $\nu = 1/2$ can be observed in a double layer or thick layer system [17]. The Haldane–Reazayi state ($\nu = 5/2$) can be viewed as a state in the second Landau level when the pair may be viewed as a d-wave state having even parity representing a spin singlet state. Since the spin states are unpolarized, we can observe it in a single layer system [18]. Since these pairs will give rise to the $SU(2)$ symmetry as we can consider these two electrons as a $SU(2)$ doublet, these represent non-Abelian Hall fluids, characterising a non-Abelian Berry phase.

From this analysis, it appears that when the hierarchical FQH states are analysed in the framework of a Z_p spin system, we have a unified description of all the classification schemes and an equivalence relationship can be established. The fact that some of the FQH states which generally arise in the continued fraction scheme have not been observed unambiguously but are automatically forbidden in the Jain or Berry phase scheme is generally considered a point against the Haldane–Halperin scheme. Indeed, the naive formulation of the continued fraction scheme cannot explain why we have not observed the FQH states with $\nu = 2/5$ and $2/7$ with equal prominence, $\nu = 5/13$ at the third level is not observed, whereas $\nu = 6/13$ at the sixth level is observed and similar other features [4]. Generally, these unobserved or unsharp states are forbidden in the Jain scheme or its variant in the Berry phase scheme. However, the equivalence relationship of different classification schemes obtained in the framework of the Z_p spin system suggests the exclusion of these undesirable states in the continued fraction scheme and we can have a formal explanation of this feature.

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