PART 2: AUTOREGRESSIVE SERIES

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1. Introduction

In his classical paper of 1927, where the autoregressive scheme was introduced, Yule developed a method[1] for fitting the scheme to a given time series. In later contributions by Wold (1938), Kendall (1947), Quenouille (1947) and others, the problems of fitting and parameter estimation were given further consideration. In the present paper an attempt is made with the help of model samples to see how far the methods given by these authors hold good when the length of the time series is small. In this part, models conforming to stationary time series of autoregressive type have been considered. Quenouille (1947) has given a test for testing the goodness of fit of an *a priori* known model to an observational time series when the length of the series is large. The suitability or otherwise of this test for time series of small length is considered in this paper. He has also given a method of fitting an autoregressive model and to test for the fitted model. In this paper this method is applied to time series of small length and alternative methods of fitting autoregressive models are considered. The possibility of fitting a suitable moving average model to a stationary time series of autoregressive type is also considered.

These investigations were undertaken primarily to get material for further studies, and to gain instructive insight into the problems at issue than to arrive at definite conclusions. Nevertheless, the conclusions drawn here are pointers to further studies.

2. Models for study

2.0. *Construction of models*: The autoregressive models chosen for study are :

Model I $\qquad \xi_t + a\xi_{t-1} = \eta_t$ ... (1)

Model II $\qquad \xi_t + a_1\xi_{t-1} + a_2\xi_{t-2} = \eta_t$ ... (2)

where ξ_t is the variable at time point t,

$$a = -.8; \qquad a_1 = -.7; \qquad a_2 = .0125\,;$$

and η_t is a random normal variable with

$$E(\eta_t) = 0, \quad E(\eta_t^2) = 1 \text{ and } E(\eta_t\, \eta_{t'}) = 0 \quad \text{for} \quad t \neq t'.$$

These models were constructed by using random normal deviates given in Tracts for Computers No. XXV by Herman Wold. Twenty five samples of length 35 and fifty samples of length 15 were constructed for both the models. In building up the models it was necessary to know ξ_1 for model I and ξ_1 and ξ_2 for model II with the help of which other values of ξ_t for $t \geqslant 2$ could be built up from equations (1) and (2).

ξ_t can also be written in the form

$$\xi_t = \eta_t + b_1\eta_{t-1} + b_2\eta_{t-2} + \dots \qquad \dots \quad (3)$$

where b_i is given by

Model I $\qquad ab_{i-1} + b_i = 0$

Model II $\qquad a_2b_{i-2} + a_1b_{i-1} + b_i = 0$

where $b_i = 0$ for $i < 0$, and $b_0 = 1$.

The initial values of ξ_t were built up from equation (3). Values of ξ_t for $t > 2$ were then built up from equations (1) and (2).

Values of random normal deviates given in pages 4, 5, 6, 7 and 8 for model I and pages 7, 14 and 21 for model II were used for calculating the initial values of ξ_t in the model samples. These satisfy almost all tests given in the introduction to Tracts for Computers No. XXV.

2.1. *Serial correlation coefficients*: Yule (1927) has shown that the serial correlation coefficients for the *a priori* known model, say ρ_s, should satisfy the recursive relation

$$\text{Model I} \qquad \rho_s + a\rho_{s-1} = 0 \qquad \dots \quad (4)$$

$$\text{Model II} \qquad \rho_s + a_1\rho_{s-1} + a_2\rho_{s-2} = 0 \qquad \dots \quad (5)$$

when $\rho_0 = 1$ and $\rho_{-1} = \rho_1$.

But Wold (1938) showed that equations (4) and (5) are valid only for $s > 0$; for $s \leqslant 0$, the zero in the right hand side of the equations should be replaced by $\frac{b_s}{\Sigma b_s^2}$. The equations (4) and (5) may therefore be called Yule-Wold relations.

The values of ρ_s for s equal to 1 to 12 in the case of samples of length 35 and s equal to 1 to 8 in the case of samples of length 15 were worked out with the help of these recursive relations. The value of ρ_s are shown in Tables 1.1 to 1.4 at the end of this Part.

The corresponding serial correlation coefficients for the samples, say r_s, where r_s represents the product moment correlation coefficient between ξ_t and ξ_{t-s}, were worked out with the help of Hollerith machines. The values of r_s are shown in Tables 1.1 to 1.4.

2.2. *Bias in the serial correlation coefficients from small samples* : Since the values of r_s were calculated from the series of length 35 and 15 only, these are biased estimates of ρ_s. The bias can be calculated by working out the expectation of r_s, which is

$$E(r_s) = E\left\{\frac{\sum_{t=1}^{T-s}\xi_t\xi_{t+s} - \left(\sum_{t=1}^{T-s}\xi_t \sum_{t=1}^{T-s}\xi_{t+s}\right)/(T-S)}{\sqrt{D_1^2 \cdot D_2^2}}\right\} \qquad \dots \quad (6)$$

where $$D_1^2 = \sum_{t=1}^{T-s} \xi_t^2 - \left(\sum_{t=1}^{T-s} \xi_t\right)^2 \Big/ (T-S)$$

$$D_2^2 = \sum_{t=s+1}^{T} \xi_t^2 - \left(\sum_{t=s+1}^{T} \xi_t\right)^2 \Big/ (T-S)$$

and T=length of the time series. From the above,

$$E(r_s) \sim \frac{E\left(\sum_{t=1}^{T-s} \xi_t \xi_{t+s}\right) - E\left\{\left(\sum_{t=1}^{T-s} \xi_t \sum_{t=1}^{T-s} \xi_{t+s}\right) \Big/ (T-S)\right\}}{E(\sqrt{D_1^2 D_2^2})}. \quad \ldots \quad (7)$$

When T is fairly large D_1^2 can be taken as approximately equal to D_2^2.

It can be shown that

$$E(D_1^2) \sim E(D_2^2) = \sigma^2\left[(T-S) - \frac{1}{(T-S)}\left\{(T-S) + 2(T-S-1)\rho_1 + \ldots 2\rho_{T-s-1}\right\}\right] \quad \ldots \quad (8)$$

where $\sigma^2 = E(\eta_t^2) = 1$.

The numerator in equation (7) can be written as $C-D$ where

$$C = (T-S)\rho_s \quad \ldots \quad (9)$$

and $$D = \frac{E-F}{T-S}$$ where

$$E = \left\{\sum_{t=1}^{i+s-1} (i + \overline{i-t} - \overline{t-s})\rho_t\right\} + i \qquad \text{where} \quad i = T-S \quad \ldots \quad (10)$$

$$F = \left\{\sum_{t=1}^{i_1+s_1-1} (i_1 + \overline{i_1-t} - \overline{t-s_1})\rho_t\right\} + i_1 \text{ where } i_1 = S, \text{ and } S_1 = T-2S \quad \ldots \quad (11)$$

and $(i-t)=0$ for $i \leqslant t$; $(i_1-t)=0$ for $i_1 \leqslant t$; $t-s=0$ for $t \leqslant s$; $(t-s_1)=0$ for $t \leqslant s_1$.

Values of $E(r_s)$ worked out by the above formulae are given in Tables 1.1 to 1.4. Comparing $E(r_s)$ with ρ_s it is seen that the bias is of a small order in the case of the models considered in this paper.

The average correlograms are shown in Figs. 1.1 to 1.4. It is seen that the observed correlograms follow the theoretical correlograms fairly closely.

3. Test for the goodness of fit of a priori known models

Quenouille (1947) has given a test criterion for testing the goodness of fit of *a priori* known models to time series of large length. His test is to build up R_s given by

Model I $$R_s = r_s + A_1 r_{s-1} + A_2 r_{s-2} \quad \ldots \quad (12)$$
where $A_1 = 2a$; $A_2 = a^2$,

a being known and equal to -0.8 in our case;

Model II $$R_s = r_s + A_1 r_{s-1} + A_2 r_{s-2} + A_3 r_{s-3} + A_4 r_{s-4} \quad \ldots \quad (13)$$
where $A_1 = 2a_1$; $A_2 = a_1^2 + 2a_2$; $A_3 = 2a_1 a_2$; $A_4 = a_2^2$,

a_1 and a_2 being known and equal to -0.7 and 0.0125 respectively.

Quenouille has shown that R_s are independently and normally distributed about zero with variance approximately equal to $\frac{(1-a^2)^2}{T-S}$ in the case of model (I) and to

$$\frac{1}{T-S}\left[\frac{(1-a_2)\{(1+a_2)^2-a_1^2\}}{(1+a_2)}\right]^2$$

in the case of model II, and hence $\frac{R_s^2}{V(R_s)}$ are distributed like χ^2 with 1 degree of freedom. Further he has shown that $(r_1-\rho_1)$ is distributed normally with zero mean and variance equal to $\frac{1-\rho^2}{T}$ in the case of model I and hence $\frac{(r_1-\rho_1)^2}{V(r_1)}$ is distributed like χ^2 with 1 degree of freedom. In the case of model II it was assumed that $(r_1-\rho_1)$ and $(r_2-\rho_2)$ are distributed normally with zero mean and variance approximately equal to $\frac{1-\rho_1^2}{T-1}$ and $\frac{1-\rho_2^2}{T-2}$.

The values of χ^2 were worked out by these formulae. The frequency distribution of χ^2 with 1 degree of freedom and also of total χ^2 for R_s are shown in Table 2.1. It is seen that a large percentage of χ^2 are significant at 5% level and that the observed frequency distributions of χ^2 are not in good agreement with the theoretical distribution of χ^2. The observed frequencies in different class intervals are smallar than the corresponding theoretical frequencies upto $P(\chi^2)$ equal to 0.50 and larger beyond.

Quenouille's test for goodness of fit of *a priori* known models was applied to the average serial correlations with and without correction for bias and the values of χ^2 obtained are as shown in Tables 2.2 and 2.3. It is seen that only 3 out of the 16 χ^2's with correction for bias in r_s are significant at 5% level in the case of model II.

Since there was a preponderence of χ^2 significant at 5% level, an attempt was made to see if significant χ^2's correspond to larger values of S. Table (2.4) shows the number of χ^2 significant at 5% level for different values of S. It is seen that there is no such tendency.

TABLE 2.4. NO. OF χ^2 WITH 1 D.F. SIGNIFICANT AT 5% LEVEL

lag (s)	model: $\xi_t-0.7\,\xi_{t-1}+0.6125\,\xi_{t-2}=\eta_t$	
	T = 35	T = 15
(1)	(2)	(3)
1	5	12
2	4	4
3	2	9
4	1	7
5	2	5
6	1	14
7	2	..
8	0	..
9	5	..
10	..	..
11	..	..
12	..	..
total	22	51

4. Fitting of models

4.0. *Fitting of autoregressive models to time series*: In fitting autoregressive models to time series, the method adopted by Yule (1927), Wold (1938), and Kendall (1947) is to equate the first serial correlation r_1 to ρ_1 in the model I and to equate r_1 and r_2 to ρ_1 and ρ_2 in the model II. For model II, a_1 and a_2 can be calculated from the relation

$$\rho_1+a_1+a_2\rho_1 = 0 \qquad \ldots \quad (14)$$

$$\rho_2+a_1\rho_1+a_2 = 0 \qquad \ldots \quad (15)$$

where ρ_1 and ρ_2 are equated to r_1 and r_2.

This method was adopted for model II and the values of a_1 and a_2 for the twenty five samples of length 35 are shown in Table 3. The absolute percentage differences of these estimated values from the theoretical values range from 1.2% to 75.4%.

Kendall (1947) suggested that a_1 and a_2 can be obtained by minimising

$$\sum_{s=1} (r_s+a_1r_{s-1}+a_2r_{s-2})^2.$$

The values of a_1 and a_2 obtained by this method are shown in Table 3. The absolute percentage difference of these estimated values from theoretical values range from 1.4% to 61.1% and are generally less than in the case of earlier method of fitting.

A third alternative method was also tried. The values of A_1, A_2, A_3 and A_4 were determined by minimising $\sum_{s=2}^{12} R_s^2$ where

$$R_s = r_s+A_1r_{s-1}+A_2r_{s-2}+A_3r_{s-3}+A_4r_{s-4}. \qquad \ldots \quad (16)$$

This method suggests itself if we consider Quenouille's test.

Having got A_1, A_2, A_3 and A_4 the best value of a_1 and a_2 were estimated by the following procedure. These constants can be written as

$$A_1 = 2a_1 = 2a'_1+2\delta a_1, \qquad \ldots \quad (17)$$

$$A_2 = a_1^2+2a_2 = a'^2_1+2a'_1\delta a_1+2a'_2+2\delta a_2, \qquad \ldots \quad (18)$$

$$A_3 = 2a_1a_2 = 2a'_1a'_2+2a'_1\delta a_2+2a'_2\delta a_1, \qquad \ldots \quad (19)$$

$$A_4 = a_2^2 = a'^2_2+2\delta a_2, \qquad \ldots \quad (20)$$

where $a_1 = a'_1+\delta a_1$ and $a_2 = a'_2+\delta a_2$, and second and higher powers and products of δ's are omitted as negligible. a'_1 and a'_2 are first approximations to a_1 and a_2 and these are taken from the second method of fitting. The values of a_1 and a_2 were worked out for 5 samples only by this method. These values are shown in Table 3 and they differ very much from the theoretical values.

4.1. *Test for goodness of fit of fitted autoregressive models*: Quenouille's test of goodness of fit for fitted values of a_1 and a_2 by the first two methods was carried out and the distribution of χ^2 with 1 d.f. and of total χ^2 with 10 d.f. are shown in

Table 4. The observed frequencies are more than the theoretical frequencies upto $P(\chi^2)$ equal to 0.50 and less beyond.

4.2. *Fitting of moving average models to stationary time series of autoregressive type*: The method of fitting moving average schemes to time series data is described in Part 1. The same method was adopted in fitting a three-constant moving average scheme to a model time series of autoregressive type with two constants. Herman Wold (1949) has given a large sample test for the goodness of fit of moving averages to time series data. This test was adopted for testing the goodness of fit. The study was carried out with 6 samples only and the values of χ^2 obtained with 1 d.f. are shown in Table 5. It is seen that none out of 54 have turned out to be significant. This indicates that the moving average scheme with three constants fits an autoregressive time series with two constants fairly well. It may be that a moving average scheme with more number of constants fits an autoregressive time series with less number of constants fairly well.

5. Summary of conclusions

(i) Quenouille's test of goodness of fit of *a priori* known models to time series data is not very satisfactory when the length of the time series is short.

(ii) Kendall's method of fitting autoregressive models with h constants to time series data by minimising

$$\sum_{s=1} (r_s + a_1 r_{s-1} + \dots + a_h r_{s-h})^2$$

appears to be better than that by equating $r_1, r_2, \dots, r_h$ to $\rho_1, \rho_2, \rho_3 \dots, \rho_h$ respectively.

(iii) Moving average schemes with more than h constants appear to fit the autoregressive time series with h constants fairly well.

References

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TABLE 1.1 VALUES OF SERIAL CORRELATION COEFFICIENTS (r_k)

MODEL I—AUTOREGRESSIVE MODEL: $\xi_t = 0.8\ \xi_{t-1} + \eta_t$ T=15

sample no.	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8
1	2	3	4	5	6	7	8	9
1	.6083	.1271	−.3547	−.7789	−.6276	.1214	.5789	.7065
2	.0055	.0741	−.0416	−.3506	−.5730	−.0072	−.6076	−.6540
3	.7420	.7209	.4902	.5444	.4231	.0931	.2123	−.3822
4	.4042	.1006	−.2140	−.2891	−.5204	−.4390	.0445	.7618
5	.8012	.4819	.0634	−.4283	−.7524	−.6611	−.6593	−.3493
6	.3931	−.2709	−.5635	−.4330	.0430	.2903	.4194	.4567
7	.6561	.1980	−.3067	−.4333	−.4877	−.6370	−.6739	−.9473
8	.7259	.7399	.4447	.0784	.1590	−.6194	−.3686	−.5633
9	.9330	.6266	.3022	−.0909	−.5419	−.7921	−.7024	−.9556
10	.3715	.1119	−.2367	−.6151	−.5540	−.7178	.1049	.4380
11	.3276	−.1610	−.1748	−.3857	−.3910	−.1359	.2844	.0769
12	.0860	.5206	.4563	.5148	.2962	−.1684	−.4237	−.1706
13	.2353	.4352	.0443	−.0135	−.2091	−.2918	−.3440	−.2612
14	.8264	.7881	.7783	.7075	.7260	.3598	.6090	.4560
15	.3825	.0480	−.1031	.0493	−.4162	−.6808	−.0206	.4353
16	.5117	.0115	−.1029	−.1046	−.5430	−.7748	−.4769	.0958
17	.8855	.8036	.7048	.0212	.6243	.6792	.2912	−.2518
18	.6544	.4161	−.1304	−.3912	−.5269	−.0979	−.7691	−.0153
19	.4843	−.2003	−.2769	−.1529	−.5478	−.6467	−.2208	.0044
20	.3179	.0168	−.2973	−.1112	−.2442	−.7370	−.1178	.2954
21	.6939	.5477	.1928	−.1109	−.5490	−.7091	−.6153	.1285
22	.6866	.2348	−.4005	−.3655	.0271	.1510	−.1563	−.7082
23	.7726	.3838	.0718	−.1482	−.5544	−.9316	−.8773	−.4867
24	.4733	.0082	−.1047	−.5509	−.6163	−.0333	.2357	.6388
25	.6343	.1264	−.0770	−.2913	−.5020	−.6976	−.6000	−.6358
26	.6945	.2721	.0229	−.2702	−.4529	−.0453	−.5621	−.6350
27	.7926	.6245	.5214	.3080	.1390	−.7024	−.6334	−.2314
28	.5808	.2813	.3348	.5366	.7219	.2118	−.1785	.1409
29	.4174	−.1192	−.3554	−.2470	−.0630	.3718	.6151	−.1012
30	.8043	.6571	.5590	.6550	.0668	.5051	.2830	.0398
31	.4849	.2042	.0990	−.0625	.3530	.6747	.1589	−.2565
32	.8240	.6983	.5887	.4578	.0670	−.1502	−.5273	−.6304
33	.0118	.1600	−.2508	−.5871	−.6930	−.6496	.2370	.7217
34	.3509	−.3901	−.7004	−.2503	.0972	.3476	.2104	.1377
35	.1326	−.4290	−.3710	−.0889	.0975	.0501	−.4095	−.2157
36	.1602	.0901	−.1820	−.4090	−.1434	.0279	−.4352	−.3010
37	.2388	−.3266	−.3055	−.1545	.2843	.3009	.0568	−.4240
38	.6902	.3807	.0039	−.2800	−.0991	.3655	.6985	.5715
39	.7050	.4792	.1504	−.0003	−.0773	−.2769	−.2707	−.6206
40	.7307	.4151	.1605	.0755	.2671	.1343	−.0457	−.1957
41	.0015	−.0366	.1109	−.3358	−.2600	−.3133	−.5200	.4078
42	.5418	.5483	.3090	−.0074	−.3748	−.5073	−.6050	−.5047
43	.7378	.5340	.2835	−.0768	−.1904	−.4080	−.5827	−.6553
44	.7329	.3054	.1120	−.0010	−.3925	−.8508	−.8516	−.4365
45	.4238	.2837	.2345	−.1389	−.2605	−.0031	−.0587	−.0898
46	.4462	−.0789	−.3298	−.4700	−.5706	−.3084	.1068	.8165
47	.8593	.8887	.8784	.8208	.8589	.7809	.7157	.7478
48	−.0115	−.1105	.1602	−.1042	−.0120	.4323	−.3705	−.3478
49	.6189	.2016	.0268	−.1063	−.2394	−.0881	−.0078	−.1840
50	.7819	.5547	.5969	.0083	.4066	.2438	−.4909	−.4434
average	.6027	.2630	.0568	−.0760	−.1333	−.1081	−.1670	−.1157
$E(r_k)$	.6467	.3605	.1343	−.0334	−.1396	−.1779	−.1541	−.1005
ρ_k	.8000	.6400	.5120	.4096	.3277	.2621	.2097	.1678
$B_k = \rho_k - E(r_k)$	.1533	.2795	.3777	.4430	.4673	.4400	.3638	.2683
average $+ B_k$	.7160	.5425	.4345	.3004	.3340	.2419	.1962	.1520

TABLE 1.2. VALUES OF SERIAL CORRELATION COEFFICIENTS (r_s).
Model I—AUTOREGRESSIVE MODEL: $\xi_t = 0.8\,\xi_{t-1} + \eta_t$ T=35

sample no.	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
1	2	3	4	5	6	7	8	9	10	11	12	13
1	.8225	.6105	.3336	.0918	.0550	.1957	.3043	.4935	.5863	.5613	.4427	.2113
2	.6850	.3711	.2327	.2403	.1050	.1294	.0288	−.2002	−.1995	−.1760	−.1604	−.3679
3	.7306	.4840	.1856	.0273	−.0903	−.0600	−.1054	.2552	.3843	.4224	.4236	.2294
4	.3005	−.0671	−.3145	−.2000	−.2131	−.0597	.1331	.1621	.2429	−.3035	−.4147	−.3377
5	.8140	.0095	.3750	.0891	−.1869	−.4267	−.4044	−.3975	−.2090	−.0837	−.0601	−.0481
6	.0042	.1825	−.0151	−.0785	−.0755	−.1098	−.1017	−.0346	.0008	−.1221	−.2748	−.1557
7	.7792	.5110	.2201	−.0264	−.1868	−.2318	−.3351	−.3091	−.3218	−.4210	−.4112	−.4273
8	.7439	.6968	.4045	.3655	−.2890	.1220	.1363	.0243	−.0078	−.0759	−.1547	−.1758
9	.7413	.5888	.3663	.2101	.0935	.0089	−.1052	−.2371	−.3303	−.5425	−.6407	−.7221
10	.5132	.1642	−.1582	−.3444	−.3354	−.3720	−.0944	−.0248	.0905	.3393	.4380	.3793
11	.3980	−.0944	−.0765	−.0380	.0562	.0102	−.2373	−.1195	.1131	.2681	.0260	−.1770
12	.8305	.0808	.5361	.4079	.2276	.0082	−.1112	−.2310	−.3407	−.4704	−.5430	−.5990
13	.6513	.5896	.3825	.2508	.0704	.0961	−.0494	.0404	−.1070	.0708	−.1067	−.1773
14	.8395	.0837	.5354	.4363	.3387	.1450	.0150	−.1517	−.3204	−.3166	−.2141	−.0452
15	.7158	.4724	.1956	−.0354	−.2086	−.3079	−.4214	−.4697	−.4384	−.2135	.0700	.2548
16	.7860	.5663	.4133	.2400	.0635	−.0020	−.2481	−.3238	−.5130	−.6397	−.6830	−.7302
17	.8366	.0646	.9304	.2504	.1558	.0332	−.0863	−.2160	−.2878	−.3270	−.3440	−.3133
18	.8156	.7490	.6477	.5836	.5854	.4540	.4599	.3481	.3549	.4802	.3804	.6001
19	.8542	.6876	.6400	.6207	.4802	.3403	.3353	.4633	.5167	.5255	.6107	.6900
20	.4997	.1067	−.0558	−.1760	−.2049	−.4042	−.3226	−.3748	−.1409	.0016	.1448	.2221
21	.7302	.5964	.4046	.1932	−.0332	−.0257	−.0960	−.1487	.0112	.1208	.1827	.4188
22	.8123	.5730	.4006	.2866	.2328	.1642	−.0002	−.1500	−.2236	−.2790	−.3017	−.4015
23	.7272	.4205	.3304	.2800	.1787	−.0037	−.1316	.1082	.3307	.0890	−.0399	.0946
24	.8452	.0974	.5836	.4281	.3164	.3408	.2000	.2565	.2087	.0367	−.0000	−.1070
25	.8021	.5096	.2070	.0879	−.0768	−.1071	.0167	.3710	.5751	.5761	.4868	.2045
average	.7156	.4852	.3154	.1822	.0078	−.0170	−.0407	−.0326	−.0027	−.0156	−.0503	−.0571
$E(r_s)$	.7413	.5331	.3657	.2312	.1239	.0384	−.0287	−.0804	−.1191	−.1402	−.1631	−.1727
ρ_s	.8000	.6400	.5120	.4096	.3277	.2621	.2097	.1678	.1342	.1073	.0859	.0687
$B_s = \rho_s - E(r_s)$	.0587	.1069	.1463	.1784	.2038	.2237	.2384	.2482	.2533	.2535	.2490	.2414
average $+ B_s$	.7743	.5921	.4617	.3606	.2717	.2067	.1977	.2156	.2506	.2379	.1987	.1843

TABLE 1.3. VALUES OF SERIAL CORRELATION COEFFICIENTS (r_s)
MODEL II—AUTOREGRESSIVE MODEL: $\xi_t = 0.7\,\xi_{t-1} - 0.6125\,\xi_{t-2} + \eta_t$ T = 15

sample no.	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8
1	2	3	4	5	6	7	8	9
1	.2173	−.5403	−.6618	−.0113	.2973	.5173	.5277	−.2332
2	.4106	−.1726	−.1026	−.1543	−.4620	−.0896	.2523	−.2044
3	.3797	−.2825	−.0299	.2069	−.0828	−.6886	−.7767	−.2076
4	.2777	−.6370	−.7511	.1182	.7082	.4324	−.5495	−.0062
5	.6419	−.1209	−.7872	−.7484	−.5530	−.1565	3694	.7710
6	.5006	−.0323	−.1741	.1072	.2904	.1425	.1000	−.1935
7	.3742	−.1676	−.0354	.1740	−.3976	−.8619	−.3987	.6236
8	.5645	−.1589	−.1811	.0406	−.5087	−.7638	−.2386	−.1726
9	.4254	−.3621	−.3830	−.0170	.3128	.2407	.0360	−.4759
10	.3497	−.5272	−.6007	−.2720	.2769	.5600	.4540	−.0935
11	.3881	−.5841	−.7969	−.0727	.7293	.5782	−.2258	−.7646
12	.3654	−.3958	−.2050	.1189	.1112	−.2805	−.5212	−.2651
13	.3665	−.5455	−.7589	−.0056	.7705	.4403	−.3357	−.7719
14	.3714	−.2019	−.1071	.3374	.0074	−.2355	.1180	.3391
15	.3917	−.4709	−.4213	.6120	.7292	−.0128	−.8368	−.2998
16	.4585	−.4040	−.7005	−.2927	.4495	.6961	.0419	−.4567
17	.0191	−.6626	−.0757	.3524	.0272	−.0715	.3010	−.4773
18	.2188	−.6867	−.8203	.3483	.7657	−.0113	−.8126	−.3704
19	.3741	−.5010	−.7411	−.1987	.5345	.4477	−.1245	−.5126
20	.4291	−.2556	−.5718	−.5076	.1013	.4780	.5420	.2603
21	.4604	−.1570	−.2980	−.3574	−.2731	.1788	.4725	.5913
22	.5124	−.4057	−.5657	−.0551	.2767	.1984	.0873	−.1109
23	.1009	−.5160	.0139	.1359	−.3915	−.3147	.7306	.2386
24	.4380	−.3199	−.4362	.1082	.2749	.0721	−.0418	.3906
25	.2303	−.3441	−.5038	−.1058	.1006	−.0018	−.0847	.1658
26	.3348	−.5247	−.4838	.2782	.5493	−.1194	−.5685	−.0771
27	.4075	−.1097	−.1808	−.1672	−.4218	−.6132	−.4243	.1878
28	.3901	−.2864	−.5694	−.4165	.0307	.2532	.3079	.2333
29	.3087	−.4026	−.2236	.0006	−.1426	−.0566	.0074	−.3738
30	.4161	−.3910	−.7017	−.4212	−.0841	.3234	.5692	.6040
31	.2644	−.1996	−.2283	−.1478	.2279	.3075	−.2930	−.2942
32	.3911	−.5256	−.8416	−.3380	.5094	.7736	.1112	−.6669
33	.5096	−.1010	−.7127	−.7886	−.2045	.3903	.5078	.2817
34	.5420	−.2037	−.4277	−.0923	−.1117	−.1299	.0680	.3353
35	.3217	−.1926	.3750	.2844	.1329	.3190	.3084	−.5999
36	.5298	−.1262	−.5218	−.6331	−.5686	−.3364	.0640	.2127
37	.4411	−.3208	−.6707	−.2104	.3367	.4947	.4118	−.0718
38	.3058	−.4731	−.7216	−.1980	.4990	.4822	.0799	−.3298
39	.4311	−.2312	−.6014	−.2794	.0398	.0249	.1796	.5254
40	.1063	−.8351	−.3555	.6383	.4893	−.4490	−.6680	.2339
41	.4162	−.1419	−.6440	−.7117	−.1384	.4404	.5910	.3666
42	.5949	−.0252	−.6319	−.7130	−.5923	−.1406	.2755	.5984
43	.1701	−.7562	−.4201	.3326	.5532	−.1245	−.6958	−.0589
44	.4015	−.1518	.1249	.1490	−.3040	−.4976	−.2099	−.3780
45	.0159	−.2330	−.1290	−.2624	−.3657	.4660	.5013	.0291
46	.2081	−.3156	−.3396	.0416	.0989	.0489	−.1703	−.4310
47	.3263	−.4114	−.6386	−.4436	.3497	.4709	.1320	−.1816
48	.2564	−.5768	−.5430	.0956	.4897	.1195	−.2787	−.0414
49	.4372	−.4386	−.8030	−.3701	.2045	.7040	.5840	.1237
50	.2025	−.7049	−.5543	.1086	.6030	.6655	−.1469	−.8427
average	.3617	−.3644	−.4387	−.0693	.1274	.1046	.0151	−.0750
$E(r_s)$	.4000	−.3078	−.5460	−.1921	.1732	.2110	.0404	−.1056
ρ_s	.4341	−.3086	−.4809	−.1483	.1014	.2248	.0402	−.1096
$B_s = \rho_s - E(r_s)$	.0251	.0592	.0677	.0438	.0182	.0138	−.0002	−.0010
average + B_s	.3868	−.3052	−.3710	−.0455	.1456	.1184	.0149	−.0760

TABLE 1.4. VALUES OF SERIAL CORRELATION COEFFICIENTS (r_s).
MODEL II—AUTOREGRESSIVE MODEL: $\xi_t = 0.7\xi_{t-1} - 0.0125\,\xi_{t-2} + v_t$

T=35

sample no.	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}
1	2	3	4	5	6	7	8	9	10	11	12	13
1	.3162	−.3728	−.3430	.1526	.2830	−.1033	−.2024	.0529	.2053	−.0555	−.2072	−.2565
2	.4406	−.3459	−.4439	−.0857	.2691	.2543	−.2891	−.0306	−.1144	.6045	.6516	.1155
3	.5460	−.0140	−.2250	−.1354	−.1413	−.1632	−.0763	.2262	.2183	−.0805	−.2258	−.0400
4	.3097	−.4944	−.0505	−.0856	.3994	.3731	−.1508	−.5458	−.2092	.2331	.5825	.3280
5	.5972	.1738	−.1661	−.2913	−.1096	−.0566	−.1243	−.2807	−.2840	−.3086	−.2544	−.0601
6	.5068	−.0597	−.2523	−.0832	.0540	−.0701	−.2761	−.2913	−.1497	−.2477	−.2110	.1543
7	.4336	−.1227	.0398	.3707	.0393	−.3777	−.1530	.1373	.0203	−.0979	.1418	.0427
8	.5541	−.0441	−.1998	−.1764	−.2486	−.1434	.0046	.1052	.0254	.0154	−.1231	−.3032
9	.3792	−.2898	−.3850	−.2018	.1280	.1693	.1337	−.1004	−.5232	−.4393	.2777	.5138
10	.3394	−.5387	−.6451	−.0746	.4005	.3650	−.0571	−.2182	−.0213	.0608	−.1610	−.3705
11	.4911	−.1604	−.3171	.0130	.3359	.2773	−.0894	−.2038	.0513	.1932	−.0530	−.3911
12	.2368	−.4043	−.2489	.0247	.1213	−.2514	−.3511	−.2049	.1297	.4049	.1400	.0391
13	.3687	−.5702	−.7045	.0148	.6134	.4105	−.2246	−.4098	−.1201	.3710	.3148	−.1564
14	.4299	−.4327	−.0645	−.1521	.3549	.4587	.1951	−.2167	−.4150	−.1006	.2263	.2058
15	.2148	−.1394	−.0038	.1308	.0083	−.0343	.0023	.0585	−.0760	−.1060	−.0632	.0267
16	.2356	−.0867	−.2311	−.1492	.0000	.0205	−.0499	−.0560	−.00[illegible]	.0053	−.0384	−.0715
17	.2020	−.4377	−.3634	.0346	.0820	.0212	.0507	−.0240	−.0750	.0820	.0486	−.0877
18	.1950	−.2347	−.2784	−.0153	.1831	.1082	−.1027	−.1501	.0490	.1826	.1216	−.1809
19	.3830	−.3946	−.6300	−.2335	.2460	.1809	−.0565	−.1042	.1035	.2767	.0217	−.1942
20	.4702	−.1874	−.5234	−.4081	.0604	.1721	.0311	−.1184	−.2086	−.0805	.1254	.3381
21	.3787	−.3618	−.5041	−.2816	.1422	.3952	.1793	−.2613	−.3350	−.0167	.2552	.0795
22	.6275	−.2578	−.5851	−.2067	.3509	.4440	.0523	−.3330	−.2316	.2045	.4129	−.0427
23	.3935	−.3567	−.4837	−.3420	−.2679	.0500	.4980	.3944	−.0621	−.2574	−.1219	−.1209
24	.4146	−.3021	−.4664	−.0402	.2282	−.0168	−.4255	−.1042	.4158	.3580	−.2412	−.4909
25	.2789	−.3813	−.3605	.0317	.3261	.2748	−.1910	−.4111	−.0701	.3300	.2424	−.2015
average	.3921	−.2750	−.3811	−.0877	.1565	.1079	−.0060	−.1720	−.0906	.0608	.0545	−.0492
$E(r_s)$	.4241	−.3320	−.5084	−.1682	.1784	.2127	.0245	−.1281	−.1191	−.0185	.0472	.0324
ρ_s	.4341	−.3086	−.4819	−.1483	.1914	.2248	.0402	−.1096	−.1013	−.0038	.0594	.0439
$B_s = \rho_s - E(r_s)$	.0100	.0234	.0265	.0199	.0130	.0121	.0157	.0185	.0178	.0147	.0122	.0115
average + B_s	.4021	−.2516	−.3546	−.0678	.1695	.1200	−.0512	−.1541	−.0728	.0755	.0667	−.0377

TABLE 2.1. FREQUENCY DISTRIBUTION OF χ^2 FOR TESTING THE SIGNIFICANCE OF DIFFERENCE BETWEEN OBSERVED AND THEORETICAL SERIAL CORRELATIONS

$P(\chi^2)$	expected percentage frequency	Autoregressive Model: $\xi_t - 0.7\,\xi_{t-1} + 0.0125\,\xi_{t-2} = \eta_t$ — length of model sample = 35				length of model sample = 15				Autoregressive Model: $\xi_t - .8\xi_{t-1} = \eta_t$ — length of model sample = 35		length of model sample = 15	
		χ^2 for r_1 d.f. = 1	χ^2 for r_2 d.f. = 1	χ^2 for R_s, $s=3,4..$ d.f. = 1	$\Sigma\chi^2$ for R_s d.f. = 10	χ^2 for r_1 d.f. = 1	χ^2 for r_2 d.f. = 1	χ^2 for R_s, $s=3,4..$ d.f. = 1	$\Sigma\chi^2$ for R_s d.f. = 9	χ^2 for R_s d.f. = 1	$\Sigma\chi^2$ d.f. = 12	χ^2 for R_s d.f. = 1	$\Sigma\chi^2$ d.f. = 8
1	2	3	4	5	6	7	8	9	10	11	12	13	14
.99	1	—	—	1	—	2	—	2	—	4	—	3	—
.98	1	—	—	4	1	—	—	1	—	4	—	2	—
.95	3	2	—	9	—		1	9	—	10	1	7	—
.90	5	1	1	16	2	4	—	12	2	7	—	12	—
.80	10	2	4	19	—	10	6	18	4	25	1	19	—
.70	10	6	3	21	2	5	4	18	1	18	1	26	—
.50	20	4	5	35	4	8	6	46	1	41	3	37	2
.30	20	5	3	48	3	7	10	54	7	48	—	43	2
.20	10	3	3	26	2	4	8	28	8	28	—	32	—
.10	10	2	3	33	—	3	8	44	7	35	2	47	4
.05	5	—	2	16	1	1	2	17	6	16	2	33	2
.02	3	—	—	6	3	1	3	19	2	22	—	22	4
.01	1	—	—	5	2	2	1	8	1	9	2	21	1
.01	1	—	1	11	5	—	1	24	11	33	13	96	35
total	100	25	25	250	25	50	50	300	50	300	25	400	50

TABLE 2.2. VALUES OF χ^2 GIVEN BY QUENOUILLE'S TEST FOR GOODNESS OF FIT OF A PRIORI KNOWN MODELS TO AVERAGE CORRELOGRAM

Model: $\xi_t - 0.8\,\xi_{t-1} = \eta_t$

s	length 35		length 15	
	R_s	χ^2	R_s	χ^2
1	2	3	4	5
		with correction for bias		
1	−.0257	1.56	−.0840	13.72*
2	.0009	.02	.0247	3.00
3	.0008	.00	.0184	1.56
4	−.0098	.57	.0258	2.83
5	.0028	.04	−.0580	12.98*
6	.0409	9.34*	.0220	1.82
7	.0316	5.37*	−.0065	.13
8	.0322	5.39*	..	..
9	−.0250	3.13	..	..
10	−.0217	2.27	..	..
11	.0187	1.62	..	..
total		29.91		36.10
		without correction for bias		
1	−.0844	16.82*	−.2373	109.40*
2	−.0029	.05	−.0039	.07
3	−.0110	.88	.0008	.00
4	−.0217	2.83	.0256	2.78
5	−.0090	.47	−.0338	4.41*
6	.0300	5.02*	.0640	14.24*
7	.0210	2.53	.0267	2.03
8	.0234	2.87	..	..
9	−.0321	5.18*	..	..
10	−.0271	3.53	..	..
11	.0134	.83	..	..
total		41.01		133.02

* significant at 5% level

TABLE 2.3. VALUES OF χ^2 GIVEN BY QUENOUILLE'S TEST FOR GOODNESS OF FIT OF A PRIORI KNOWN MODELS TO AVERAGE CORRELOGRAM

Model: $\xi_t - 0.7\,\xi_{t-1} + 0.8125\,\xi_{t-2} = \eta_t$

$s-2$	length 35		length 15	
	R_s	χ^2	R_s	χ^2
1	2	3	4	5
		with correction for bias		
1	−.0193	1.24	.0082	.10
2	.0275	2.43	−.0046	.05
3	.0229	1.63	−.0218	1.11
4	−.0230	1.73	.0410	3.59
5	−.0034	.03	−.0017	.00
6	−.0474	6.34*	−.0355	2.21
7	.0158	.68	..	..
8	.0021	.01	..	..
9	−.0509	6.56*	..	..
10	.0030	.02	..	..
total		20.60		7.06
		without correction for bias		
1	−.0340	3.83*	−.0300	2.45
2	.0132	.36	−.0350	3.10
3	.0060	.23	−.0518	6.25*
4	−.0380	4.35*	.0126	.34
5	−.0173	.88	−.0009	.00
6	−.0610	10.40*	−.0593	5.15*
7	.0025	.01	..	..
8	−.0105	.29	..	..
9	−.0831	10.08*	..	..
10	−.0083	.17	..	..
total		30.67		18.29

* significant at 5% level

TABLE 4. FREQUENCY DISTRIBUTION OF χ^2 AS GIVEN BY QUENOUILLE'S TEST FOR SERIAL CORRELATION MODEL FITTED BY DIFFERENT METHODS; LENGTH OF THE SAMPLE = 35.

Model: $\xi_t + a_1\,\xi_{t-1} + a_2\,\xi_{t-2} = \eta_t$

$P(\chi^2)$	expected percentage frequency	fitted from r_1 and r_2		fitted by minimising $\sum_{s=1}^{11} (r_s + a_1 r_{s-1} + a_2 r_{s-2})^2$	
		χ^2 for R_s $s=3,4$ .. d.f. = 1	cumulative χ^2 for R_s d.f. = 10	χ^2 for R_s $s=3,4,$.. d.f. = 1	cumulative χ^2 for R_s d.f. = 10
1	2	3	4	5	6
.99	1	3	3	1	3
.98	1	1	1	—	1
.95	3	13	2	13	1
.90	5	15	—	16	—
.80	10	30	2	27	1
.70	10	24	1	22	1
.50	20	45	3	38	3
.30	20	44	4	51	4
.20	10	19	—	20	2
.10	10	27	4	21	3
.05	5	13	2	13	1
.02	3	8	1	14	—
.01	1	5	—	3	—
.01	1	3	2	11	5
total	100	250	25	250	25

Table 3. Values of a_1 and a_2 obtained by different methods of fitting time series from model: $\xi_t - 0.7\,\xi_{t-1} + 0.6125\,\xi_{t-2} = \eta_t$

sample	from r_1 and r_2				minimising $\Sigma(r_s + a_1 r_{s-1} - a_2 r_{s-2})^2$				minimising ΣR_s^2			
	a_1	% diff.	a_2	% diff	a_1	% diff	a_2	% diff.	a_1	% diff.	a_2	% diff.
1	2	3	4	5	6	7	8	9	10	11	12	13
model	−0.7		0.6125		−0.7		0.6125		−0.7		0.6125	
1	−.4823	31.1	.5253	14.2	−.4589	34.4	.6120	0.0				
2	−.7584	8.3	.6860	12.1	−.7846	12.1	.8104	32.3				
3	−.7888	12.7	.4447	27.4	−.7649	9.3	.4460	27.2	−.3729	46.7	.7944	29.7
4	−.6399	8.6	.7310	10.3	−.7637	9.1	.8435	37.7				
5	−.7600	9.0	.2842	53.6	−.9629	37.6	.4290	30.0				
6	−.7227	3.2	.4259	30.5	−.7411	5.9	.4038	34.1				
7	−.5995	14.4	.3826	37.5	−.4437	30.6	.3203	47.7				
8	−.8340	19.3	.5067	17.3	−.8299	18.6	.4517	26.3	−.6143	12.2	.3161	48.4
9	−.5712	18.4	.5064	17.3	−.7166	2.4	.6820	11.5				
10	−.5902	15.7	.7390	20.7	−.6526	6.8	.7300	19.2				
11	−.7510	7.3	.5292	13.6	−.7100	1.4	.5934	3.1				
12	−.3520	49.7	.4870	20.4	−.4573	34.7	.4844	20.9				
13	−.6700	4.3	.8172	33.4	−.6690	4.4	.8467	38.2	−1.0608	51.5	.6454	5.3
14	−.7556	7.9	.7575	23.7	−.7931	13.3	.7710	26.0				
15	−.2566	63.3	.1945	68.2	−.2724	61.1	.2308	62.3				
16	−.2711	61.3	.1506	75.4	−.3289	53.0	.2047	66.6				
17	−.4606	34.2	.5726	6.5	−.4338	38.0	.5130	16.2				
18	−.2515	64.1	.2840	53.6	−.3330	52.4	.4138	32.4	−.2613	62.7	.4181	31.7
19	−.6260	10.6	.6343	3.6	−.6712	4.1	.6663	8.8				
20	−.7168	2.4	.5244	14.4	−.8430	20.4	.6421	4.8				
21	−.6021	14.0	.5898	3.7	−.7036	0.5	.6850	11.8				
22	−.9193	31.3	.7427	21.3	−.7908	13.0	.7720	28.0				
23	−.6317	9.8	.6053	1.2	−.7731	10.4	.5882	4.0	−.6701	4.2	.5513	10.0
24	−.6819	2.6	.6448	5.3	−.6105	12.8	.7298	19.2				
25	−.4177	40.3	.4078	18.7	−.5221	25.4	.6817	11.3				
average	−.6047	13.6	.5306	13.4	−.6412	8.4	.5821	4.0				

TABLE 5. VALUES OF χ^2 GOODNESS OF FIT OF A MOVING AVERAGE SCHEME WITH THREE CONSTANTS FITTED TO AUTOGRESSIVE SERIES OF LENGTH 35

sample no. 1			sample no. 2			sample no. 14			sample no. 15			sample no. 16			sample no. 17		
lag no.	χ_i^2	$\Sigma\chi^2$	lag no.	χ_i^2	$\Sigma\chi^2$	lag no.	χ_i^2	$\Sigma\chi^2$	lag no.	χ_i^2	$\Sigma\chi^2$	lag no.	χ_i^2	$\Sigma\chi^2$	lag no.	χ_i^2	$\Sigma\chi^2$
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
4	.4213	.4213	4	.1117	.1117	4	.2730	.2730	4	.4616	.4616	4	.5597	.5597	4	.0204	.0204
5	1.0272	1.4485	5	1.7237	1.8354	5	2.3701	2.6521	5	.0817	.5433	5	.1273	.6870	5	.0912	.1116
6	2.6058	4.0543	6	.0373	1.8727	6	.2068	2.8589	6	.0269	.5720	6	.0021	.7401	6	.0007	.1123
7	.3083	4.3026	7	2.8053	4.6780	7	.6325	3.4914	7	.1058	.6700	7	.2782	1.0273	7	.5011	.6134
8	.0018	4.3044	8	1.5345	6.2125	8	.0011	3.4925	8	.0497	.7257	8	.0014	1.0267	8	.3974	1.0108
9	.0773	4.4417	9	.6558	6.8683	9	.3083	3.8008	9	.2779	1.0036	9	.0210	1.0497	9	.1218	1.1326
10	.0750	4.5170	10	1.7026	8.5709	10	.1070	3.9678	10	.2940	1.2976	10	.0300	1.0797	10	.0704	1.8030
11	2.1806	6.6982	11	.1291	8.7000	11	.2680	4.2358	11	.0079	1.3055	11	.0344	1.1141	11	.8196	2.0226
12	.6586	7.3568	12	1.3454	10.0454	12	.0408	4.2766	12	.0033	1.3088	12	.1130	1.2271	12	.4947	3.1173

Theoretical and average correlograms—Figs. (1.1) to (1.4)

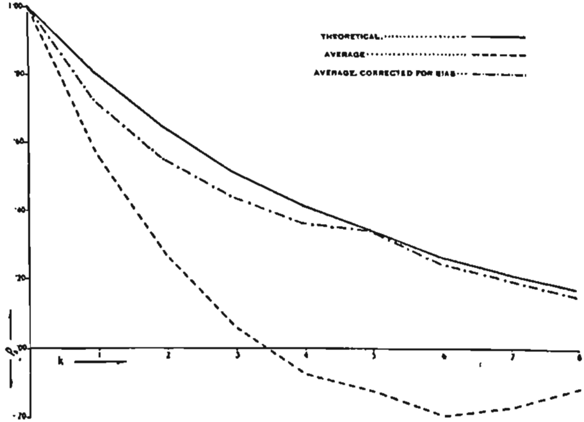

Fig. (1.1). Autoregressive Model I. $T=15$

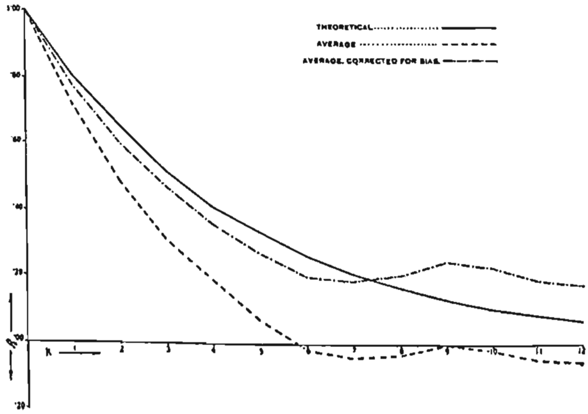

Fig. (1.2). Autoregressive Model I. $T=35$

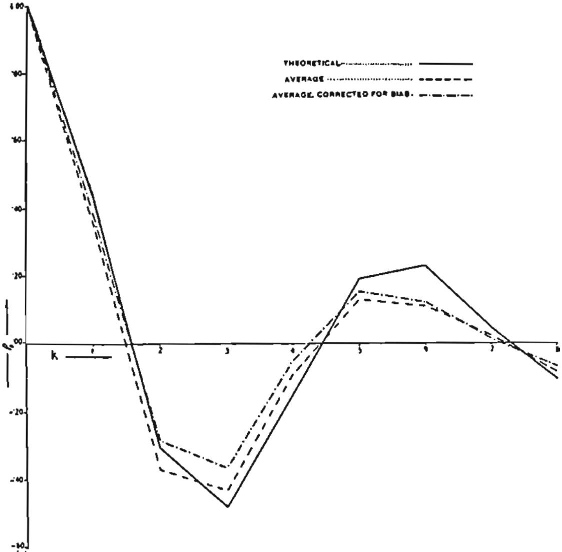

Fig. (1.3). Autoregressive Model II : $T=15$

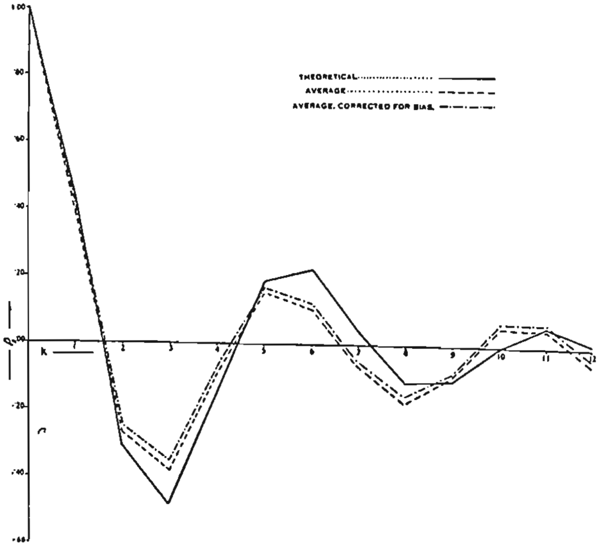

Fig. (1.4). Autoregressive Model II : $T=35$