METRIC STRUCTURES ON POSSIBILITY DISTRIBUTIONS

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Higashi and Klir have defined a metric on the set of possibility distributions, based on the *U*-uncertainty. We show here that similar metrics can be defined based on Yager's measure of (non) specificity and the imprecision measure of Lamata and Moral. These metrics satisfy almost all properties of the earlier metric, indicating some invariant characteristics of these three measures of (non) specificity. We also disprove some results present in the literature. Finally, we argue in favor of defining these metrics on ordered possibility distributions.

INDEX TERMS: Possibility distributions, uncertainty, nonspecificity metrics

1. INTRODUCTION

Higashi and Klir [1983a] introduced a metric structure on the set of possibility distributions on a finite domain of discourse. The metric is based on the *U*-uncertainty measure. It gives a measure of the separation between two possibility distributions as implied by their *U*-uncertainties. The greater the distance the greater is the dissimilarity between the two possibility distributions. Ramer [1990] elaborated on this metric structure and introduced a new function which is not a metric but is additive with respect to possibility distributions on non-interactive consonant bodies of evidence.

In this paper, first we discuss some inequalities for possibility distributions. Then, as an extension of the treatment in Ramer [1990], we define metrics on possibility distributions based on 1) Yager's [1983] measure of specificity and 2) Lamata and Moral's [1987] measure of imprecision. We then show that properties discussed in Ramer [1990] for the *U*-uncertainty based metric carry over to these metrics as well. This investigation, thus, shows some invariant characteristics of various well known measures of imprecision.

The rest of the paper is organized as follows: Section 2 introduces the notation and various definitions, section 3 discusses three measures of nonspecificity, while section 4 discusses several inequalities on possibility distributions that will be required. Metric definition using various measures of non-specificity and their properties are presented in section 5. Section 6 concludes the paper.

2. DEFINITIONS AND NOTATION

In our subsequent discussions the following notation will be used. Let $\mathcal{X} = \{x_1, x_2, \ldots, x_n\}$ be a finite domain of discourse. A *possibility distribution* on $\mathcal{X}$ is a function $p : \mathcal{X} \to [0,1]$. A *basic probability assignment* (BPA) on $\mathcal{X}$ is defined as a mapping $m : P(\mathcal{X}) \to [0,1]$ such that $m(\emptyset) = 0$ and $\sum_{A \in \mathcal{P}(\mathcal{X})} m(A) = 1$, $P(\mathcal{X})$ is the power set of $\mathcal{X}$. Let $\mathcal{F} = \{A \in P(\mathcal{X}) : m(A) > 0\}$. Then $(\mathcal{F}, m)$ is called a *body of evidence* and the elements of $\mathcal{F}$ are called the *focal sets* of $(\mathcal{F}, m)$. For a *consonant body of evidence*, that is when the focal sets are nested, we can write $A_i = \{x_1, x_2, \ldots, x_i\}$, $i = 1, \ldots, n$, such that $m(A_i) > 0$, $\sum_{i=1}^{n} m(A_i) = 1$. A plausibility measure [Klir and Folger, 1993] for a consonant body of evidence is known as a *possibility measure*. Every possibility measure π on $\mathcal{X}$ can be uniquely characterized by a possibility distribution p: $\mathcal{X} \to [0,1]$ via the formula $\pi(A) = max_{x \in A}\{p(x)\}$.

We write, $p = (p_1, \ldots, p_n)$ where $p_i = p(x_i)$, $i = 1, \ldots, n$. We denote by $\bar{p} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n)$ its descending rearrangement and by $\breve{p} = (\breve{p}_1, \breve{p}_2, \ldots, \breve{p}_n)$ the ascending one. The descending possibility distribution is often called *ordered* possibility distribution. It is easy to see that for an ordered possibility distribution $m(A_i) = p_i - p_{i+1}$.

If $Max\{p_i / i = 1, \ldots, n\} = 1$, then p is a *normalized* possibility distribution. In this investigation we restrict ourselves to normalized possibility distributions.

Let $P_{\mathcal{X}}$ be the set of normalized possibility distributions on $\mathcal{X}$ and $p,q \in P_{\mathcal{X}}$. We say $p = q$ if $p(x_i) = q(x_i) \forall x_i \in \mathcal{X}$, that is, $p_i = q_i \forall i = 1, \ldots, n$. On the other hand, $p \le q$ if $p_i \le q_i$ $\forall i = 1, \ldots, n$; and $p \ge q$ if $q \le p$, i.e., $q_i \le p_i$ $\forall$ $i = 1, \ldots, n$. $p \vee q$ is the distribution $(p \vee q)(x_i) = Max\{p(x_i), q(x_i)\}$. Note that, $p \le p \vee q$ and $q \le p \vee q$.

Given two sequences $a = a_i$, $i = 1,2, \ldots, n$ and $b = b_i$, $i = 1,2, \ldots, n$, we define $a \prec b$ if $\sum_{i=1}^{m} a_i \le \sum_{i=1}^{m} b_i \quad \forall m$. Further $a \succ b$ if $b \prec a$.

If $p \in P_{\mathcal{X}}$ and $q \in P_y$, where $Y = \{y_1, \ldots, y_m\}$, then the cartesian product, $p \otimes q$, of p and q is defined as $p \otimes q$: $\mathcal{X} \times Y \to [0,1]$, such that $p \otimes q(x_i, y_j) = Min\{p(x_i), q(y_j)\}$.

A function f: $P_Z \times P_Z \to \Re$, where Z is any finite domain of discourse is said to be *additive*, if $f(p \otimes r, q \otimes s) = f(p,q) + f(r,s)$, where $p,q \in P_{Z1}$, $r,s \in P_{Z2}$ and Z_1, Z_2 are finite.

A function d: $P_{\mathcal{X}} \times P_{\mathcal{X}} \to \Re$ is a *metric* if for $p,q \in P_{\mathcal{X}}$

1. $d(p,q) \ge 0$ with equality iff $p = q$;
2. $d(p,q) = d(q,p)$;
3. $d(p,q) \le d(p,r) + d(r,q)$ $\forall r \in P_{\mathcal{X}}$.

3. (NON) SPECIFICITY MEASURES

There have been several attempts to quantify nonspecificity [Higashi and Klir, 1983a, 1983b], [Ramer, 1990], [Yager, 1983], [Lamata and Moral, 1987], [Dubois and Prade, 1985] and [Pal et al., 1992, 1993]. We consider here three of them. Let $p = \{p_1, p_2, \ldots, p_n\} \in P_{\mathcal{X}}$ and $\bar{p}$ be its ordered (descending) version.

Higashi and Klir's U-uncertainty. The *U*-uncertainty (a measure of non-specificity) [1983b] associated with $\bar{p}$ is defined by

$$U(p)=\sum_{i=1}^{n}(\bar{p}_i-\bar{p}_{i+1})\log i=\sum_{i=2}^{n}\bar{p}_i\ \log\left(\frac{i}{i-1}\right). \tag{1}$$

Ramer and Lander [1987] have shown that the U-uncertainty is a special case of a wider class $U_{\varnothing}$ defined by

$$U_{\varnothing}(p)=\sum_{i=1}^{n}\left[\varnothing(\bar{p}_i)-\varnothing(\bar{p}_{i-1})\right]\log i \tag{2}$$

where $\varnothing$: $[0,1]\rightarrow[0,1]$ is a continuous non-decreasing mapping.

Yager's Measure of (Non) Specificity. Yager [1983] defined a measure of specificity, s(p) as

$$s(p)=s(\bar{p})=\sum_{i=1}^{n}\frac{\left[\bar{p}_i-\bar{p}_{i+1}\right]}{i}\text{with }\ \bar{p}_{n+1}=0$$
$$=\bar{p}_1+\sum_{i=2}^{n}\bar{p}_i\left[\frac{1}{i}-\frac{1}{i-1}\right]. \tag{3}$$

The specificity measure provides an indication of dispersion of the associated belief function. $N(p)= 1 - s(p)$ is often viewed as a measure of non-specificity.

Lamata and Moral's Measure of Imprecision. Lamata and Moral [1987] defined an index of imprecision, w(p), for a basic probability assignment m as

$$w(m)=\mathrm{Log}\left(\sum_{A\subseteq X}m(A)\,|A|\right). \tag{4}$$

$w(m)$ is the logarithm of the average cardinality of the focal sets, hence it also represents a measure of non-specificity. For a consonant body of evidence $w(m)$ can be interpreted as the imprecision measure of the underlying possibility distribution p and it takes the form

$$\begin{aligned} w(p)&=\log\sum_{i=1}^{n}\left[\bar{p}_i-\bar{p}_{i+1}\right]i\text{ with }\bar{p}_{n+1}=0\\ &=\log\left[p_1\times\ 1-p_2\times 1+p_2\times 2-p_3\ \times 2+\ldots\right.\\ &\quad\left.+p_{n-1}\ \times\ (n-1)-p_n\times(n-1)+p_n\times\ n-p_{n+1}\times\ n\right]\\ &=\log\left[\sum_{i}^{n}p_i\right]. \end{aligned} \tag{5}$$

4. INEQUALITIES FOR POSSIBILITY DISTRIBUTIONS

We first state two results from Ramer [1983]: (1) If $a < b$, then $\bar{a} < \bar{b}$; (2) Let p and q be such that $p > q$ then, $U(p) \geq U(q)$. The following two lemmas show that these are not necessarily true.

LEMMA 1. If $a < b$ then neither $\bar{a} < \bar{b}$ nor $\tilde{a} < \tilde{b}$ is necessarily true. □

Proof. Consider the following counter example: Let $a=$ (0.7,0.2,0.2,1.0) and $b=$ (1.0,0.6,0.4,0.1). Then $\tilde{a} =$ (0.2,0.2,0.7,1.0), $\tilde{b} =$ (0.1,0.4,0.6,1.0), $\bar{a}$ = (1.0,0.7,0.2,0.2) and $\bar{b}$ = (1.0,0.6,0.4,0.1). Clearly $a < b$ but neither $\tilde{a} < \tilde{b}$ nor $\bar{a} < \bar{b}$ is true. □

LEMMA 2. Let p and q be such that $p > q$. Then it is not necessarily true that $U(p) \geq U(q)$.

Proof. Let $p=$ (1.0,0.3,0.3) and $q=$ (0.2,0.4,1.0). Clearly $p > q$. But, $U(p)= 0.3 \log 2 + 0.3 \log (3/2)$, and $U(q)= 0.4 \log 2 + 0.2 \log (3/2)$. And, $U(q)-U(p)= 0.1 \log 2 - 0.1 \log (3/2)= 0.1 \log 4 - 0.1 \log 3 > 0$. □

LEMMA 3. If $p \leq q$ then $\bar{p} \leq \bar{q}$ and $\tilde{p} \leq \tilde{q}$.

Proof. Enough to show $\bar{p}_i \leq \bar{q}_i$. Since $\bar{p}_i$ is the i^{th} largest in the sequence $(p_1, \ldots, p_n)$, we let $\{p_{k1}, p_{k2}, \ldots, p_{ki}\}$ be those $p_k's$ greater than or equal to $\bar{p}_i$. Then we also have $\bar{p}_i$ less than or equal to each element in $\{q_{k1}, q_{k2}, \ldots, q_{ki}\}$, since $p_k \leq q_k$. That is, $\bar{p}_i$ is less than or equal to some i elements in the sequence q, and in particular to the i largest elements in the sequence q. In other words, $\bar{p}_i \leq \bar{q}_i$.

COROLLARY. $p \leq q \Rightarrow \bar{p} < \bar{q}$ and $\tilde{p} < \tilde{q}$.

LEMMA 4. $\bar{p} = \overline{p \vee q} \Rightarrow p = p \vee q$

Proof. Let $S_1 = \{k: p_k < q_k\}$ and $S_2= \{k: p_k \geq q_k\}$. We have,

$$\begin{aligned}\sum_{i=1}^{n} p_i \vee q_i &= \sum_{i=1}^{n} (p \vee q)_i \\ &= \sum_{i=1}^{n} p_i \text{ since } \bar{p} = \overline{p \vee q} \\ &= \sum_{i \in S_1} p_i + \sum_{i \in S2} p_i \\ &= \sum_{i \in S_1} p_i + \sum_{i \in S2} p_i \vee q_i .\end{aligned}$$

Therefore,
$\sum_{i \in S_1} p_i \vee q_i = \sum_{i \in S_1} p_i$; this is possible only if $S_1 = \emptyset$.
Hence we conclude that $p \geq q$ i.e., $p= p \vee q$. □

LEMMA 5. Let p,q,r be finite sequences of the same length n. Then

$$\sum_i \sum_j [p_i \vee q_i][r_j] \leq \sum_i \sum_j [p_i \vee r_i][r_j \vee q_j].$$

Proof. We prove $\forall (i,j)$

$$(p_i \vee q_i) \cdot r_j + (p_j \vee q_j) \cdot r_i \leq (p_i \vee r_i) \cdot (r_j \vee q_j) + (p_j \vee r_j) \cdot (r_i \vee q_i).$$

i.e.,

$$(p_i \cdot r_j \vee q_i \cdot r_j) + (p_j \cdot r_i \vee q_j \cdot r_i) \leq (p_i \cdot r_j \vee p_i \cdot q_j \vee r_i \cdot r_j \vee r_i \cdot q_j) + (p_j \cdot r_i \vee p_j \cdot q_i \vee r_j \cdot r_i \vee r_j \cdot q_i) \tag{6}$$

Let $A = (p_i \cdot r_j \vee q_i \cdot r_j)$, $B = (p_j \cdot r_i \vee q_j \cdot r_i)$, $C = (p_i \cdot r_j \vee p_i \cdot q_j \vee r_i \cdot r_j \vee r_i \cdot q_j)$ and $D = (p_j \cdot r_i \vee p_j \cdot q_i \vee r_j \cdot r_i \vee r_j \cdot q_i)$. If $A \leq C$, $B \leq D$ we are done. So let $A > C$, which implies $p_i r_j < q_i r_j$ and

$$0 < A - C = q_i r_j - C \leq q_i r_j - q_j r_i. \tag{7}$$

But then, we cannot have $B \geq D$, for if $B \geq D$ we would have $p_j r_i \leq q_j r_i$ and $0 < B - D = q_j r_i - D \leq q_j r_i - r_j q_i$, a contradiction to (7). Thus if $A > C$, then $B < D$.

Now note that the exchanges $p_i \leftrightarrow p_j$, $q_i \leftrightarrow q_j$, $r_i \leftrightarrow r_j$, leave (6) unchanged, while effecting the exchanges $A \leftrightarrow B$ and $C \leftrightarrow D$. This symmetry means that, we also have, if $B > D$ then $A < C$. Moreover, it is enough to prove for the case $A > C$, $B < D$; and the other case is proved by this above mentioned symmetry.

$$\text{So , let } A > C,\ B < D.$$

We prove the inequality for the case $q_j \geq p_j$, and by observing that the exchanges $q_i \leftrightarrow p_i$, $q_j \leftrightarrow p_j$ leave (6) unchanged, we deduce that the case $p_j \geq q_j$ is also proved.

$$\text{So , let in addition } q_j \geq p_j.$$

Then,

$$\begin{aligned} 0 < D - B = D - q_j r_i & \qquad \text{since } q_j \geq p_j \\ \geq q_i r_j - q_j r_i & \\ \geq A - C & \qquad \text{by (7)} \end{aligned} \tag{8}$$

and thus the lemma is proved. □

We state, without proof, the following results from Ramer [1990], as they will be used later.

LEMMA 6. Let a, b and c be finite sequences of the same length n. Then

$$\sum_{i=1}^{m} \overline{a_i \vee b_i} + \sum_{i=1}^{m} \bar{c}_i \leq \sum_{i=1}^{m} \overline{a_i \vee c_i} + \sum_{i=1}^{m} \overline{b_i \vee c_i} \quad \forall m = 1, \ldots, n.$$

LEMMA 7. If a and b are finite sequences of the same length, $a > b$ and $(w_1, \ldots, w_n)$ is an arbitrary non-increasing sequence then

$$\sum a_i w_i \geq \sum b_i w_i.$$

5. METRICS BASED ON MEASURES OF (NON) SPECIFICITY

Higashi and Klir [1983a] defined a metric for P_X. This metric is based on the U-uncertainty. They defined a basic function $g(p,q)$, for $p \leq q$ as follows:

$$g(p,q) = U(q) - U(p) \quad \forall\, p, q \in P_X, \quad p \leq$$

Then a metric G(p,q) is defined in terms of this basic function as

$$G(p,q) = g(p, p \vee q) + g(q, p \vee q) \quad \forall\, p,q \in P_X.$$

Here we introduce two new metrics, $D_1(p,q)$ and $D_2(p,q)$, in terms of Yager's measure of (non) specificity and Lamata and Moral's measure of imprecision, respectively.

5.1. Metric Based on Yager's Measure of (Non) Specificity

We define, $\forall\, p,q \in P_X, p \leq q, d_1(p,q) = s(p) - s(q) = N(q) - N(p)$

LEMMA 8. $d_1(p,p \vee q) \geq 0$ with equality iff $p = p \vee q$.

Proof.

$$\begin{aligned} d_1(p, p \vee q) &= s(p) - s(q) \\ &= s(\bar{p}) - s(\overline{p \vee q}) \\ &= \sum_{i=2}^{n} (\bar{p}_i - \overline{p_i \vee q_i})(1/i - 1/(i-1)) \geq 0 \quad \text{by Lemma 3.} \end{aligned}$$

Again by Lemma 3 the equality holds only when

$$\begin{aligned} &\bar{p}_i = \overline{p_i \vee q_i}\ \forall i \\ &\Rightarrow p = p \vee q, \text{ by Lemma 4.} \end{aligned}$$

□

With $d_1(p,q)$ as the basic function, we get our metric $D_1(p,q)$ as follows.
Define $\forall p,q \in P_X$,
$D_1(p,q) = d_1(p,p \vee q) + d_1(q,p \vee q)$.

THEOREM 1. $D_1(p,q)$ is a metric.

Proof. Let $p,q \in P_X$
$D_1(p,q) \geq 0$ by Lemma 8.
Also, $D_1(p,q) = 0 \Rightarrow p = p \vee q$ and $q = p \vee q$ by Lemma 8
Hence $p = q$.
It remains only to prove that $\forall r \in P_X$
$D_1(p,q) \leq D_1(p,r) + D_1(r,q)$.
We use arguments similar to that in Ramer[1990].

$$\begin{aligned} &D_1(p,q) \leq D_1(p,r) + D_1(r,q) \\ &\Leftrightarrow s(p) + s(q) - 2s(p \vee q) \leq s(p) + s(r) + s(r) + s(q) - 2s(p \vee r) - 2s(q \vee r) \\ &\Leftrightarrow s(p \vee q) + s(r) \geq s(p \vee r) + s(r \vee q) \\ &\Leftrightarrow s(\overline{p \vee q}) + s(\bar{r}) \geq s(\overline{p \vee r}) + s(\overline{r \vee q}). \end{aligned}$$

Note that for any ordered, normalized, possibility distribution $s \in P_X$, $\bar{s}_1$ is always 1. So, we can recall Lemma 6 with a,b,c replaced by p,q,r respectively, to get, after deleting the $(i = 1)^{st}$ term,

$$\sum_{i=2}^{m} \overline{p \vee q} + \sum_{i=2}^{m} \bar{r} \leq \sum_{i=2}^{m} \overline{p \vee r} + \sum_{i=2}^{m} \overline{r \vee q} \quad \forall_m = 2, \ldots, n.$$

Using Lemma 7 with $w_i = \left(\frac{1}{i-1} - \frac{1}{i}\right), i = 2, \ldots, n$, and replacing a by $(\overline{p \vee r} + \overline{r \vee q})$, b by $(\overline{p \vee q} + \bar{r})$, and omitting the $(i = 1)^{st}$ term in these sequences, we get

$$\Sigma_{i=2}^{n} \overline{p_i \vee q_i}\left(\frac{1}{i} - \frac{1}{i-1}\right) + \Sigma_{i=2}^{n} \bar{r}_i\left(\frac{1}{i} - \frac{1}{i-1}\right) \geq$$

$$\Sigma_{i=2}^{n} \overline{p_i \vee r_i}\left(\frac{1}{i} - \frac{1}{i-1}\right) + \Sigma_{i=2}^{n} \overline{r_i \vee q_i}\left[\frac{1}{i} - \frac{1}{i-1}\right];$$

$$\text{i.e., } s(\overline{p \vee q}) + s(\bar{r}) \geq s(\overline{p \vee r}) + s(\overline{r \vee q}).$$

Thus, $D_1(p,q)$ is a metric.

THEOREM 2 (MONOTONICITY). If $p \leq q \leq r$ then

$$D_1(p,q) \leq D_1(p,r) \text{ and } D_1(q,r) \leq D_1(p,r).$$

Proof.

$$D_1(p,q) = d_1(p, p \vee q) + d_1(q, p \vee q) = d_1(p,q) + d_1(q,q) = s(p) - s(q)$$

$$D_1(p,r) = s(p) - s(r)$$

$$D_1(p,r) - D_1(p,q) = s(q) - s(r) = d_1(q,r) \geq 0.$$

Similarly, $D_1(q,r) \leq D_1(p,r)$. □

THEOREM 3 (INVARIANCE UNDER JOINT SYMMETRY). Let $\pi : \mathcal{X} \to \mathcal{X}$ be a permutation. Then,

$$D_1(p \circ \pi, q \circ \pi) = D_1(p,q).$$

Proof.

$$\begin{aligned} D_1(p \circ \pi, q \circ \pi) &= d_1(p \circ \pi, (p \circ \pi) \vee (q \circ \pi)) + d_1(q \circ \pi, (p \circ \pi) \vee (q \circ \pi)) \\ &= s(p \circ \pi) + s(q \circ \pi) - 2s((p \circ \pi) \vee (q \circ \pi)) \\ &= s(p) + s(q) - 2s(p \vee q) \\ &= D_1(p,q). \end{aligned}$$

THEOREM 4 (EXPANSIBILITY). Let $p, q \in P_{\mathcal{X}}$, $p', q' \in Py$, where $\mathcal{X} \subseteq y$ and $p'(y) = q'(y) = 0\ \forall y \in Y - X$. Then $D_1(p', q') = D_1(p,q)$.

Proof.

$$\begin{aligned} D_1(p',q') &= s(p') + s(q') - 2s(p' \vee q') \\ &= s(p) + s(q) - 2s(p \vee q) \\ &= D_1(p,q). \end{aligned}$$ □

THEOREM 5 (TRANSLATION PROPERTY). Let p, q be any two possibility distributions, not necessarily normalized, let $s \in \mathfrak{R}$ such that $p + s = (p_i + s) \in P_{\mathcal{X}}$ and $q + s = (q_i + s) \in P_{\mathcal{X}}$. Then $D_1(p,q) = D_1(p + s, q + s)$.

Proof. Immediate. □

For a normalized possibility distribution, the translation property reduces to the trivial case, $s = 0$, and as such is not an important one.

5.2. Metric Based on the Imprecision Measure of Lamata and Moral

We define for $p \leq q, p, q \in P_X$

$d_2(p,q) = w(q) - w(p)$

LEMMA 9. Let $p \leq q$, then $d_2(p,q) \geq 0$ with $d_2(p,q) = 0$ iff $p = q$.

Proof. $d_2(p,q) \geq 0$ is easily seen from equation (5), since $p \leq q$.
$d_2(p,p) = 0$ is clear.
Moreover, $d_2(p,q) = 0$ implies,

$$\begin{aligned} &\log \Sigma p_i = \log \Sigma q_i \\ &\Rightarrow \Sigma p_i = \Sigma q_i \\ &\Rightarrow \Sigma(q_i - p_i) = 0 \\ &\Rightarrow q_i - p_i = 0 \quad \forall_i, \text{ as } q_i - p_i \geq 0 \ \forall_i \\ &\Rightarrow p = q. \end{aligned}$$ □

Now define $\forall p,q \in P_X$,

$$D_2(p,q) = d_2(p, p \vee q) + d_2(q, p \vee q)$$

THEOREM 6. $D_2(p,q)$ is a metric.

Proof. $D_2(p,q) \geq 0$ is clear.

$$\begin{aligned} D_2(p,q) = 0 &\Rightarrow d_2(p, p \vee q) = 0 \text{ and } d_2(q, p \vee q) = 0 \\ &\Rightarrow p = p \vee q \text{ and } q = p \vee q \\ &\Rightarrow p = q. \end{aligned}$$

Now it remains to show $D_2(p,q)$ satisfies the triangle inequality;
i.e., $D_2(p,q) \leq D_2(p,r) + D_2(r,q) \ \forall r \in P_X$

$$\begin{aligned} &\Leftrightarrow w(p \vee q) + w(r) \leq w(p \vee r) + w(r \vee q) \\ &\Leftrightarrow \log(\Sigma p_i \vee q_i) + \log \Sigma r_i \leq \log(\Sigma p_i \vee r_i) + \log(\Sigma r_i \vee q_i) \\ &\Leftrightarrow [\Sigma p_i \vee q_i][\Sigma r_i] \leq [\Sigma p_i \vee r_i][\Sigma r_i \vee q_i] \\ &\Leftrightarrow \sum_i \sum_j [p_i \vee q_i][r_j] \leq \sum_i \sum_j [p_i \vee r_i][r_j \vee q_j] \text{ which follows from Lemma 5.} \end{aligned}$$ □

THEOREM 7 (MONOTONICITY). If $p \lneq q \leq r$ then

$$D_2(p,q) \leq D_2(p,r) \text{ and } D_2(q,r) \leq D_2(p,r).$$

Proof $D_2(p,q) = d_2(p, p \vee q) + d_2(q, p \vee q) = d_2(p,q) + d_2(q,q) = w(q) - w(p)$

$$D_2(p,r) = w(r) - w(p)$$
$$D_2(p,r) - D_2(p,q) = w(r) - w(q) = d_2(q,r) \geq 0$$

Similarly, $D_2(q,r) \leq D_2(p,r)$. □

For this metric too the properties

i) Invariance under joint symmetry and
ii) Expansibility

hold. Their proofs are similar to those given for $D_1(p,q)$.

6. CONCLUSION AND DISCUSSION

In this investigation we proved several inequalities for possibility distributions and then extended the work of Higashi and Klir [1983a] and Ramer [1990]. Higashi and Klir [1983a] defined a metric on possibility distributions in terms of the U-uncertainty—a measure of (non)specificity. Ramer's work [1990] was also restricted to U-uncertainty. We have defined two metrics on $P_{\mathcal{X}} \times P_{\mathcal{X}}$, one based on Yager's measure of (non)specificity and the other based on Lamata and Moral's measure of imprecision. Properties of these metrics have also been investigated. The three measures of nonspecificity have different algebraic forms with different computational overheads. The computation of U—uncertainty is most expensive—it requires $(n-1)$ logarithmic evaluations; while imprecision requires only one and Yager's nonspecificity requires no logarithmic evaluation, yet, we have seen that the mathematical characteristics of the new metrics and the metric of Higashi and Klir are similar. These metrics are also defined in a similar manner. This may be used as indications of two things: different nonspecificity measures have similar characteristics, and the metric structures imposed by them are also similar. Thus, any of the nonspecificity measures or metrics can be used without losing or gaining much over others.

All these metrics are defined on general possibility distributions. But it is clear that if p is a possibility distribution on $\mathcal{X}$ then $\bar{p}$ corresponds to a relabeling of the elements of $\mathcal{X}$. Since a distribution, for the purpose of measuring the uncertainty, is considered independent of the labeling in $\mathcal{X}$, intuitively it is more appealing to have the metric distance between p and $\bar{p}$, as zero. That is, we need to consider only ordered possibility distributions. It is easy to see that, all the above analysis goes through in the case of ordered distributions also.

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