Oblique wave scattering by submerged thin wall with gap in finite-depth water

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The problem of oblique wave scattering by a submerged thin vertical wall with a gap in finite-depth water and its modification when another identical wall is introduced, are investigated in this paper. The techniques of both one-term and multi-term Galerkin approximations have been utilized in the mathematical analysis. The multi-term approximations in terms of appropriate Chebyshev polynomials provide extremely accurate numerical estimates for the reflection coefficient. The reflection coefficient is depicted graphically for a number of geometries. It is found that by the introduction of another identical wall, there occurs zero reflection for certain wave numbers. This may have some bearings on the modelling of a breakwater.

Key words: linearised theory, oblique wave scattering, galerkin approximations, Chebyshev polynomials, reflection coefficient, finite-depth water.

1 INTRODUCTION

Breakwaters are useful to protect a harbour from the rough sea. Their models in the form of thin vertical barriers are important mostly because of their simplicity in the engineering design and also in the related mathematical analysis. A thin vertical barrier is perhaps the simplest model of a breakwater. The problems of wave scattering by a thin vertical barrier in deep water have received much attention in the literature on the linearised theory of water waves due to the ability to solve them explicitly for the case of normal incidence of a train of surface water waves (cf. Ursell[1] for a partially immersed vertical plate on a submerged vertical barrier extending infinitely downwards, Evans[2] for a submerged vertical plate, Porter[3] for a wall with a submerged gap, Banerjea[4] for a surface piercing wall with multiple submerged gaps, Banerjea and Mandal[5] for a submerged wall with a gap, and others). However, for oblique incidence as well as for finite-depth water, these problems do not possess explicit solutions and can be handled by some approximate methods to estimate quantities of physical interest such as the reflection and transmission coefficients. For example, for oblique wave scattering by a surface-piercing vertical barrier in deep water, Evans and Morris[6] used one-term Galerkin approximation for two integral equations obtained for the problem, one in terms of the difference of velocity potential across the barrier and the other in terms of the horizontal component of velocity across the gap below the barrier. They then used Ursell's[1] exact solutions for normally incident waves in deep water as the one-term Galerkin approximations and obtained reflection coefficient in terms of definite integrals which can be evaluated numerically. The average of the upper and lower bounds then produce very accurate estimates for the reflection coefficient. Mandal and Das[7] used this technique to tackle the problem of oblique wave scattering by a plate submerged in deep water, where the one-term approximations are taken in terms of the exact results for normally incident waves given by Mandal and Kundu[8] (for the difference of potential across the plate) and Banerjea and Mandal[9] (for the horizontal velocity above and below the plate). Earlier Mandal and Dolai[10] obtained very accurate upper and lower bounds for the reflection coefficients for oblique wave scattering problems involving four types of vertical barrier configurations in finite-depth water by the same technique. For each configuration the one-term approximations are taken in terms of the corresponding known results for normally incident waves in deep water. Recently, Kanoria and Mandal[11]

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used this technique to investigate the oblique wave scattering problem involving two thin vertical walls with submerged symmetrical gaps in finite-depth water, choosing the one-term approximations in terms of exact solutions for normally incident waves in the presence of a single wall with a gap in deep water.

However, when the above *one-term technique* is applied to the problem of oblique wave scattering by a submerged wall with a gap in finite-depth water by utilizing the known exact solutions for normally incident waves in deep water recently obtained by Banerjea and Mandal,[5] it is observed that the numerical bounds for the reflection coefficient are not very close and as such their averages cannot serve as good estimates for the reflection coefficient. Also these exact results involve complicated expressions which require careful handling for numerical computations. It is then felt that perhaps the technique of multi-term Galerkin approximations will provide accurate bounds. Recently Porter and Evans[12] used suitable multi-term Galerkin approximations involving Chebyshev polynomials to obtain very accurate bounds for the reflection coefficients for a number of water wave scattering problems involving vertical barriers in finite-depth water. The theory of multi-term Galerkin approximations is best described in Porter and Evans.[12] In the present paper this technique of multi-term Galerkin approximations is successfully utilized to investigate the problem of oblique water wave scattering by a submerged vertical wall with a gap in finite-depth water, and also its modification when an identical wall is introduced. Very accurate upper and lower bounds of the reflection coefficient for a number of geometries involving one wall or two walls have been computed numerically. Their averages produce very accurate numerical estimates for the reflection coefficient. It is observed that a six-figure accuracy in the results is achieved by taking only four terms in the Galerkin approximations. The reflection coefficient is depicted graphically against the wave number for a number of geometries. The figures show that the behaviour of the reflection coefficient for the problem involving a single wall differs significantly from that when an identical wall is introduced. In the double-wall case, zeros of the reflection coefficient occurs at certain wave numbers.

In Section 2 we consider in some detail the problem of oblique wave scattering by a submerged wall with a gap in finite-depthwater and in Section 3 its modification when an identical wall is introduced. The numerical results are given in Section 4.

2 SCATTERING BY A SUBMERGED WALL WITH A GAP

A submerged thin vertical wall with a gap occupies the portion $x=0$, $y\in L=(a,b)+(c,h)$, $-\infty<z<\infty(a<b<c<h)$ where $y=0$ is the mean free surface, a,b and c are the depths of the submerged edges of the upper and lower parts of the wall respectively below the mean free surface and h is the depth of water. The position of the gaps of the wall is represented by $x=0$, $y\in L'=(0,a)+(b,c)$ so that $L+L'=(0,h)$. Assuming linear theory, a train of progressive waves represented by $Re\{\phi_0(x,y)e^{i\nu z-i\sigma t}\}$ is obliquely incident from very large distances on the right side of the wall, where

$$\phi_0(x,y)=\psi_0(y)e^{-i\mu x}$$

with

$$\psi_0(y)=N_0^{-1/2}\cosh k_0(h-y),\ N_0=\frac{1}{2}\left(1+\frac{\sinh 2k_0h}{2k_0h}\right).$$

and k_0 being the unique real positive root of

$$k\tanh kh=K,$$

$\mu=k_0\cos\alpha$, $\nu=k_0\sin\alpha$, $K=\sigma^2/g$, g being the acceleration due to gravity, α being the angle of incidence of the wave train. Due to the geometrical symmetry of the problem, the z-dependence can be eliminated by assuming the velocity potential to be of the form $Re\{\phi(x,y)e^{i\nu z-i\sigma t}\}$. Then $\phi(x,y)$ satisfies the following boundary value problem:

$$(\nabla^2-\nu^2)\phi=0 \text{ in } 0\leq y\leq h, \tag{1}$$

$$K\phi+\phi_y=0 \text{ on } y=0 \tag{2}$$

$$\phi_x=0 \text{ on } x=0 \text{ for } y\in L, \tag{3}$$

$$\phi_y=0 \text{ on } y=h, \tag{4}$$

$$r^{1/2}\nabla\phi \text{ is bounded as } r\to 0 \tag{5}$$

where r is the distance from the submerged edges of the wall.

$$\phi(x,y)\to\begin{cases}\phi_0(x,y)+R\phi_0(-x,y) & \text{as } x\to\infty\\ T\phi_0(x,y) & \text{as } x\to-\infty\end{cases} \tag{6}$$

where R and T are respectively the reflection and transmission coefficients and are part of the BVP. These will be determined in the course of the analysis. We follow the method of Porter and Evans[12] to tackle the problem.

A representation of $\phi(x,y)$ satisfying (1), (2), (4) and (6) is given by

$$\phi(x,y)=\begin{cases}(e^{-i\mu x}+Re^{i\mu x})\psi_0(y)+\sum_{n=1}^{\infty}A_ne^{-s_nx}\psi_n(y) & \text{for } x>0,\\ Te^{-i\mu x}\psi_0(y)+\sum_{n=1}^{\infty}B_ne^{s_nx}\psi_n(y) & \text{for } x<0\end{cases} \tag{7}$$

where

$$\psi_n(y)=N_n^{-1/2}\cos k_n(h-y),$$

$$N_n=\frac{1}{2}\left(1+\frac{\sin 2k_nh}{2k_nh}\right),\ n\geq 1,$$

k_n being the positive roots of

$$K+k\tan kh=0$$

and $s_n=(k_n^2+\nu^2)^{1/2}$ with $s_n=k_n$ when $\nu=0$.

Here $\psi_n(y)$ ($n = 1,2\ldots$) are orthonormal over $(0, h)$ and A_n, B_n ($n = 1,2,\ldots$) are unknown constants.

Let us denote

$$U(y) = \phi_x(\pm 0, y) \text{ for } 0 < y < h, \tag{8}$$

$$V(y) = \phi(+0, y) - \phi(-0, y) \text{ for } 0 < y < h, \tag{9}$$

then

$$U(y) = 0 \text{ for } y \in L, \tag{10}$$

$$V(y) = 0 \text{ for } y \in L' \tag{11}$$

It is easy to see that the unknown constants R,T,A_n and $B_n (n = 1,2,\ldots)$ are related to $U(y)$ and $V(y)$ given by

$$-i\mu(1-R) = -i\mu T = \frac{1}{h}\int_{L'} U(y)\psi_0(y)\mathrm{d}y, \tag{12}$$

$$-s_n A_n = s_n B_n = \frac{1}{h}\int_{L'} U(y)\psi_n(y)\mathrm{d}y \tag{13}$$

$$2R = \frac{1}{h}\int_{L} V(y)\psi_0(y)\mathrm{d}y, \tag{14}$$

$$2A_n = \frac{1}{h}\int_{L'} V(y)\psi_n(y)\mathrm{d}y. \tag{15}$$

Also, the functions $U(y)$ and $V(y)$ can be shown to satisfy the integral equations

$$\int_{L'} U(t)\mathcal{L}(y,t)\mathrm{d}t = R\psi_0(y), \;\; y \in L' \tag{16}$$

$$\int_{L} V(t)\mathcal{M}(y,t)\mathrm{d}t = -2i\mu h^2(1-R)\psi_0(y), \;\; y \in L \tag{17}$$

where

$$\mathcal{L}(y,t) = \frac{1}{h}\sum_{n=1}^{\infty} \frac{\psi_n(y)\psi_n(t)}{s_n} y, t \in L' \tag{18}$$

$$\mathcal{M}(y,t) = h\sum_{n=1}^{\infty} s_n\psi_n(y)\psi_n(t), \;\; y, t \in L \tag{19}$$

so that $\mathcal{L}(y,t)$ and $\mathcal{M}(y,t)$ are symmetric in y and t.

Now, we define the inner products

$$(f, g), ((f, g)) = \int_{L, L'} f(t)g(t)\mathrm{d}t, \tag{20}$$

when the functions $f(y), g(y)$ are defined for $y \in L, L'$. Obviously, these inner products are linear and symmetric. We also define the operators

$$\left.\begin{aligned} (\mathcal{L}f)(y) &= \int_{L'} \mathcal{L}(y,t)f(t)\mathrm{d}t = <\mathcal{L}(\cdot,t), f(t)>(y) \\ (\mathcal{M}f)(y) &= \int_{L} \mathcal{M}(y,t)f(t)\mathrm{d}t = ((\mathcal{M}(\cdot,t), f(t)))(y) \end{aligned}\right\} \tag{21}$$

It is easy to verify that these operators are symmetric, self adjoint and positive semi-definite. If we now define $u(y)$ ($y \in L'$), $v(y)$ ($y \in L$) and the constant A as

$$u(y) = \frac{1}{R}U(y), \;\; y \in L' \tag{22}$$

$$v(y) = -\frac{1}{2i\mu h^2(1-R)}V(y), \;\; y \in L \tag{23}$$

$$A = \frac{\mu h(1-R)}{iR}, \tag{24}$$

then $u(y)$, $v(y)$ satisfy the integral equations

$$(\mathcal{L}u)(y) = \psi_0(y), \;\; y \in L' \tag{25}$$

$$(\mathcal{M}v)(y) = \psi_0(y), \;\; y \in L \tag{26}$$

and A is related to u and v by

$$(u, \psi_0) = A \tag{27}$$

$$((v, \psi_0)) = \frac{1}{A} \tag{28}$$

It is important to note that the functions $u(y)$, $v(y)$ and the constant A are real.

For the solution of the integral eqn (25), if $u(y)$ is approximated by $F(y)$ for $y \in L'$, then due to symmetry and positive semi-definiteness of the operator $\mathcal{L}$, it can be shown that (cf. Jones,[13] p. 269)

$$(F, \psi_0) \leq (u, \psi_0) = A, \tag{29}$$

Similarly for the solution of the integral eqn (26) if $v(y)$ is approximated by $G(y)$ for $y \in L$, then

$$((G, \psi_0)) \leq ((v, \psi_0)) = \frac{1}{A}. \tag{30}$$

Combining (20) and (30) we find

$$A_1 \leq A \leq A_2$$

where

$$A_1 = (F, \psi_0), \;\; A_2^{-1} = ((G, \psi_0)) \tag{31}$$

so that A_1 and A_2 are lower and upper bounds for the real quantity A. Then using (24), upper and lower bounds R_1 and R_2 for the reflection coefficient $|R|$ can be obtained once F and G are chosen suitably for the solutions of the integral Eqs. (25) and (26) respectively. Here F and G are chosen as suitable one-term or multi-term Galerkin approximations.

2.1 One-term Galerkin approximation

We choose $F(y)$ and $G(y)$ in terms of exact solutions for the horizontal velocity $f(y)$ in the gaps and $g(y)$, the difference of potential across the wall, respectively for the problem of water wave scattering by a submerged wall with a gap in deep water when a train of progressive waves is normally incident on the wall from a large distance on its right.

These exact solutions are given by (cf. Banerjea and Mandal[7])

$$f(y) = \begin{cases} f_1(y) = \lambda_1'(y) & \text{for } 0 < y < a \\ f_2(y) = \lambda_2'(y) & \text{for } b < y < c \end{cases} \tag{32}$$

and

$$g(y)=\begin{cases} g_1(y) & \text{for} \quad a<y<b \\ g_2(y) & \text{for} \quad y>c \end{cases} \tag{33}$$

where

$$\lambda_1(y)=e^{-Ky}\left[\frac{e^{Ka}}{K}-\int_a^y \frac{e^{Ku}}{|\rho(u)|^{1/2}}\left\{\delta_1+\delta_2 u^2 - \frac{2}{\pi}\int_a^b \frac{v|\rho(v)|^{1/2}}{v^2-u^2}\mathrm{d}v\right\}\mathrm{d}u\right] \text{for } 0<y<a, \tag{34}$$

$$\lambda_2(y)=e^{-Ky}\left[\int_c^y \frac{e^{Ku}}{|\rho(u)|^{1/2}}\left\{\delta_1+\delta_2 u^2 - \frac{2}{\pi}\int_a^b \frac{v|\rho(v)|^{1/2}}{v^2-u^2}\mathrm{d}v\right\}\mathrm{d}u\right] \text{for } b<y<c,$$

with $\rho(y)=(u^2-a^2)(u^2-b^2)(u^2-c^2)$, and

$$g_j(y)=(-1)^{j+1}e^{-Ky}\int_{l_j}^y \frac{e^{Ku}}{|\rho(u)|^{1/2}} \times \left\{\delta_1+\delta_2 u^2-\frac{2}{\pi}\int_a^b \frac{v|\rho(v)|^{1/2}}{v^2-u^2}\mathrm{d}v\right\}\mathrm{d}u, \ j=1,2 \tag{35}$$

with $l_1=a$, $l_2=c$, the inner integral in $g_1(y)$ being taken in the sense of Cauchy principal value. The constant δ_1 and δ_2 appearing in (34) and (35) are defined by

$$\delta_1=\frac{(\gamma_2-\gamma_1)\alpha_1+\frac{e^{Kb}}{K}\gamma_1}{\beta_1\gamma_2-\gamma_1\beta_2}, \ \delta_2=\frac{\beta_1\alpha_2-\alpha_1\beta_2-\frac{e^{Kb}}{K}\beta_1}{\beta_1\gamma_2-\gamma_1\beta_2} \tag{36}$$

where

$$\alpha_1,\alpha_2=\frac{2}{\pi}\int_a^b\int_b^c \frac{e^{Ku}}{|\rho(u)|^{1/2}}\left\{\int_a^b \frac{v|\rho(v)|^{1/2}}{v^2-u^2}\mathrm{d}v\right\}\mathrm{d}u,$$

the inner integral being in the sense of Cauchy principal value for α_1,

$$\beta_1,\beta_2=\int_a^b\int_b^c \frac{e^{Ku}}{|\rho(u)|^{1/2}}\mathrm{d}u, \tag{37}$$

$$\gamma_1,\gamma_2=\int_a^b\int_b^c \frac{u^2e^{Ku}}{|\rho(u)|^{1/2}}\mathrm{d}u.$$

It may be noted that while the expression for $f(y)$ is given by Banerjea and Mandal,[5] the expression for $g(y)$ can be deduced from their results.

The one-term Galerkin approximations $F(y)$ $y\in L'$ and $G(y)$ $y\in L$ are chosen as

$$F(y)=\begin{cases} p_1 f_1(y), & 0<y<a \\ p_2 f_2(y), & b<y<c \end{cases} \tag{38}$$

and

$$G(y)=\begin{cases} q_1 g_1(y), & a<y<b \\ q_2 g_2(y), & c<y<h \end{cases} \tag{39}$$

where p_1, p_2 and q_1, q_2 are constants. Now using (38) in the integral eqns (25) and (39) in (26) respectively, we find

$$p_1\int_0^a f_1(t)\mathcal{L}(y,t)\mathrm{d}t+p_2\int_b^c f_2(t)\mathcal{L}(y,t)\mathrm{d}t=\psi_0(y), \ y\in L' \tag{40}$$

$$q_1\int_b^c g_1(t)\mathcal{M}(y,t)\mathrm{d}t+q_2\int_c^h g_2(t)\mathcal{M}(y,t)\mathrm{d}t=\psi_0(y), \ y\in L. \tag{41}$$

Multiplication of both sides of (40) by $f_j(y)$ ($j=1,2$) and integration over L_j' ($j=1,2$) where $L_1'=(0,a)$ and $L_2'=(b,c)$, produces two linear equations for p_1 and p_2 as

$$\sum_{i=1}^{2} p_i P_{ij}=P_j, \ j=1,2 \tag{42}$$

where

$$P_{ij}=\int_{L_i'}\int_{L_j'} f_i(t)f_j(y)\mathcal{L}(y,t)\mathrm{d}y\mathrm{d}t, i,j=1,2 \tag{43}$$

$$P_j=\int_{L_j'}\psi_0(y)f_j(y)\mathrm{d}y, \ j=1,2.$$

Similarly, (41) produces the two linear equations for q_1 and q_2 as

$$\sum_{i=1}^{2} q_i Q_{ij}=Q_j, \ j=1,2 \tag{44}$$

where

$$Q_{ij}=\int_{L_i}\int_{L_j} g_i(t)g_j(y)\mathcal{M}(y,t)\mathrm{d}y\mathrm{d}t, i,j=1,2 \tag{45}$$

$$Q_j=\int_{L_j}\psi_0(y)g_j(y)\mathrm{d}y, \ j=1,2.$$

with $L_1=(a,b)$, $L_2=(c,h)$.

The constants p_j and q_j ($j=1,2$) are obtained by solving the linear Eqs. (42) and (44). Then the bounds A_1 and A_2 for A are obtained as

$$A_1=\sum_{j=1}^{2} p_j P_j, \ A_2=\left[\sum_{j=1}^{2} q_j Q_j\right]^{-1}. \tag{46}$$

Once these are evaluated numerically, the bounds R_1 and R_2 for $|R|$ can be obtained numerically.

2.2 Multi-term Galerkin approximation

For the solution of the integral eqn (25) an appropriate multi-term form for $F(y)$, following Porter and Evans,[12] involving Chebyshev polynomials, is chosen as

$$F(y)=\begin{cases}\sum_{n=0}^{N} b_n f_n^{(1)}(y) & \text{for} \quad 0<y<a,\\ \sum_{n=0}^{N} c_n f_n^{(2)}(y) & \text{for} \quad b<y<c\end{cases} \tag{47}$$

where

$$f_n^{(1)}=-\frac{d}{dy}\left[e^{-Ky}\int_y^a \hat{f}_n^{(1)}(t)e^{Kt}dt\right]$$

with

$$\hat{f}_n^{(1)}(y)=\frac{2(-1)^n}{\pi(a^2-y^2)^{1/2}}T_{2n}\left(\frac{y}{a}\right),\ 0<y<a, \tag{48}$$

and

$$f_n^{(2)}(y)=\frac{1}{\pi\{(y-b)(c-y)\}^{1/2}}T_n\left(\frac{2y-b-c}{c-b}\right),\quad b<y<c. \tag{49}$$

Here $T_n(x)$ $(n = 0,1,2,\ldots,N)$ denote the Chebyshev polynomials of first kind, and b_n, c_n $(n=0,1,\ldots N)$ are unknown constants to be determined. To find these constants we first substitute (47) in the integral eqn (25) to obtain

$$\sum_{n=0}^{N} b_n\int_0^a f_n^{(1)}(t)\mathcal{L}(y,t)dt+\sum_{n=0}^{N} c_n\int_b^c f_n^{(2)}(t)\mathcal{L}(y,t)dt = \psi_0(y),\ \text{for } y\in L'. \tag{50}$$

Multiplying both sides of (50) by $f_m^{(1)}(y)$ $(f_m^{(2)}(y))(m = 0,1,\ldots,N)$ and integrating over $(0,a)((b,c))$ we obtain the linear system

$$\sum_{n=0}^{N} L_{mn}^{(i,1)}b_n+\sum_{n=0}^{N} L_{mn}^{(i,2)}c_n=F_{m0}^{(i)},\quad i=1,2;\ m=0,1,\ldots,N \tag{51}$$

where

$$L_{mn}^{(i,j)}=\sum_{r=1}^{\infty}\frac{F_{mr}^{(i)}F_{nr}^{(j)}}{hs_r},\quad i,j=1,2;\ m,n=0,1,\ldots N \tag{52}$$

$$F_{mn}^{(1)}=\int_0^a f_m^{(1)}(y)\psi_n(y)dy=N_n^{-1/2}\cos k_nhJ_{2m}(k_na), \tag{53}$$

$$F_{mn}^{(2)}=\int_b^c f_m^{(2)}(y)\psi_n(y)dy =N_n^{-1/2}\cos\left\{k_n\left(h-\frac{b+c}{2}\right)-\frac{m\pi}{2}\right\}J_m\left(k_n\frac{c-b}{2}\right)\quad m,n=0,1,2\ldots N$$

The linear system (51) can be solved by standard numerical methods. Then A_1, a lower bound for A, is obtained as

$$A_1=(F,\psi_0)=\sum_{n=0}^{N}(b_nF_{n0}^{(1)}+c_nF_{n0}^{(2)}). \tag{54}$$

Similarly, to solve the integral eqn (26), an appropriate multi-term form of $G(y)$ is chosen as (cf. Porter and Evans[12])

$$G(y)=\begin{cases}\sum_{n=0}^{N} d_ng_n^{(1)}(y) \text{ for } a<y<b,\\ \sum_{n=0}^{N} e_ng_n^{(2)}(y) \text{ for } c<y<h\end{cases} \tag{55}$$

where

$$g_n^{(1)}(y)=\frac{2}{\pi}\frac{\{(y-a)(b-y)\}^{1/2}}{(n+1)(b-a)h}U_n\left(\frac{2y-a-b}{b-a}\right),\quad a<y<b, \tag{56}$$

$$g_n^{(2)}(y)=\frac{2}{\pi}\frac{(-1)^n\{(h-c)^2-(h-y)^2\}}{(2n+1)(h-c)h}U_{2n}\left(\frac{h-y}{h-c}\right),\quad c<y<h,$$

$U_m(x)$ being Chebyshev polynomials of second kind, and d_n, $e_n(n=0,1,\ldots N)$ are unknown constants. Proceeding as in the case of b_n and c_n, these constants d_n, $e_n(n = 0,1,\ldots N)$ satisfy the linear system

$$\sum_{n=0}^{N} M_{mn}^{(i,1)}d_n+\sum_{n=0}^{N} M_{mn}^{(i,2)}e_n=G_{m0}^{(i)},\quad i=1,2;\ m=0,1,\ldots N \tag{57}$$

where

$$M_{mn}^{(i,j)}=\sum_{r=1}^{\infty} hs_rG_{mr}^{(i)}G_{nr}^{(j)},\quad i,j=1,2;\ m,n=0,1,2\ldots N \tag{58}$$

$$G_{mn}^{(1)}=\int_a^b g_m^{(1)}(y)\psi_n(y)dy =N_n^{-1/2}(k_nh)^{-1}\cos\left\{k_n\left(h-\frac{a+b}{2}\right)-\frac{m\pi}{2}\right\}J_{m+1}\times\left(k_n\frac{b-a}{2}\right) \tag{59}$$

$$G_{mn}^{(2)}=\int_c^h g_m^{(2)}(y)\psi_n(y)dy =N_n^{-1/2}(k_nh)^{-1}J_{2m+1}(k_n(h-c)),\quad m,n=0,1,\ldots N.$$

The system of $(2N + 2)$ linear equations in d_n and $e_n(n = 0,1,\ldots N)$ can be solved by standard methods and then an upper bound A_2 for A is obtained as

$$A_2=[((G,\psi_0))]^{-1}=\left[\sum_{n=0}^{N}(d_nG_{n0}^{(1)}+e_nG_{n0}^{(2)})\right]^{-1} \tag{60}$$

Since $A_1 < A_2$, we find that

$$R_1<|R|<R_2$$

where

$$R_1 = \frac{\mu h}{\{(\mu h)^2 + A_2^2\}^{1/2}},\ R_2 = \frac{\mu h}{\{(\mu h)^2 + A_1^2\}^{1/2}} \tag{61}$$

so that R_1 and R_2 can be easily computed.

3 SCATTERING BY TWO SUBMERGED IDENTICAL WALLS WITH GAPS

In this section we consider oblique wave scattering by two identical submerged vertical walls, each having a gap at the same level, in finite depth water. The walls occupy the positions $x = \pm d, y \in L$, $2d$ being the distance between the two walls. Due to geometrical symmetry, it is possible to split $\phi(x,y)$ in this case into its symmetric and asymmetric parts such that

$$\phi(x, y) = \phi^s(x, y) + \phi^a(x, y) \tag{62}$$

where

$$\phi^s(x, y) = \phi^s(-x, y), \phi^a(x, y) = -\phi^a(-x, y). \tag{63}$$

We need to consider only the region $x \geq 0$, and use of (63) will extend the solution into $x < 0$. Then $\phi^{s,a}(x,y)$ satisfy (1) to (5) with the additional condition

$$\phi_x^s(0, y) = \phi^a(0, y) = 0 \tag{64}$$

and the requirement that

$$\phi^{s,a}(x, y) \to \{e^{-i\mu(x-d)} + R^{s,a}e^{i\mu(x-d)}\}\psi_0(y) \text{ as } x \to \infty \tag{65}$$

where R^s and R^a are related to the reflection and transmission coefficients R and T (cf. eqn (67) below).

An appropriate representation for $\phi^s(x,y)(x \geq 0)$ is

$$\phi^s(x, y) = \begin{cases} \psi_0(y)\{e^{-i\mu(x-d)} + R^s e^{i\mu(x+d)}\} + \sum_{n=1}^{\infty} A_n^s \psi_n(y) e^{-s_n(x-d)}, & x > d, \\ A_0^s \psi_0(y)\cos \mu x + \sum_{n=1}^{\infty} B_n^s \psi_n(y)\cosh s_n x, & 0 < x < d, \end{cases} \tag{66}$$

where A_0^s, A_n^s, B_n^s $(n = 1,2,\ldots)$ are unknown constants. A representation for $\phi^a(x,y)$ differs from $\phi^s(x,y)$ only in replacing R^s by R^a, $\cos \mu x$ and $\cosh s_n x$ by $\sin \mu x$ and $\sin hs_n x$ respectively with different coefficients A_0^a, A_n^a and B_n^a. It is now easy to see that

$$\phi(x, y) \to \begin{cases} \phi_0(x,y) + \frac{1}{2}(R_s + R_a)\phi_0(-x, y) & as\ x \to \infty \\ \frac{1}{2}(R_s - R_a)\phi_0(x, y) & as\ x \to -\infty. \end{cases}$$

It thus follows that

$$R = \tfrac{1}{2}(R_s + R_a),\ T = \tfrac{1}{2}(R_s - R_a) \tag{67}$$

The A in (24) is replaced by A^s and A^a in the symmetric and asymmetric case respectively where

$$A^s = \frac{i\mu h(R^s e^{2i\mu d} - 1)}{(R^s e^{2i\mu d} + 1) + i(R^s e^{2i\mu d} - 1)\cot \mu d}. \tag{68}$$

$$A^a = \frac{i\mu h(R^a e^{2i\mu d} - 1)}{(R^a e^{2i\mu d} + 1) - i(R^a e^{2i\mu d} - 1)\tan \mu d} \tag{69}$$

The quantities $A^{s,a}$ are real and their upper and lower bounds can be obtained in the same way as for A in Section 2. It may be noted that for the symmetric case the factors $(1 + \coth s_n d)$ and $(1 + \coth s_n d)^{-1}$ are to be included respectively in the expressions for $\mathcal{L}(y,t)$ in (18) and $\mathcal{M}(y,t)$ in (19). For the asymmetric case the corresponding factors are $(1 + \tanh s_n d)$ and $(1 + \tanh s_n d)^{-1}$. It is then easy to modify the expressions for P_{ij} in (42), Q_{ij} in (45), $L_{mn}^{(i,j)}$ in (52) and $M_{mn}^{(i,j)}$ in (58) for the symmetric and asymmetric cases. Now $R^{s,a}$ can be expressed in terms of $A^{s,a}$ as

$$R^s = -e^{-2i\mu d}\left(\frac{1 - iS_1}{1 + iS_1}\right),\ R^a = -e^{-2i\mu d}\left(\frac{1 + iS_2}{1 - iS_2}\right) \tag{70}$$

where

$$S_1 = \cot \mu d - \frac{\mu h}{A^s},\ S_2 = \tan \mu d + \frac{\mu h}{A^a}. \tag{71}$$

Thus

$$|R| = \frac{1}{2}|R^s + R^a| = \frac{|1 - S_1 S_2|}{(1 + S_1^2 + S_2^2 + S_1^2 S_2^2)^{1/2}} \tag{72}$$

$$|T| = \frac{1}{2}|R^s - R^a| = \frac{|S_1 + S_2|}{(1 + S_1^2 + S_2^2 + S_1^2 S_2^2)^{1/2}}$$

so that

$$|R|^2 + |T|^2 = 1$$

which also follows from other principle of conservation of energy.

Using the upper and lower bounds for $A^{s,a}$ with appropriate case, upper and lower bounds R_1 and R_2 for $|R|$ can be obtained.

4 NUMERICAL RESULTS

It is sufficient to only concentrate on obtaining numerical estimates for the reflection coefficient $|R|$. For the one-term technique, the various integrals occuring in P_{ij}, P_j and Q_{ij}, $Q_j (i, j = 1,2)$ in (43) and (45) respectively for the single wall problem are numerically evaluated by standard techniques for various values of the different parameters after taking into account the Cauchy principal value integrals. Then the coefficients p_j and $q_j (j = 1,2)$ are computed by solving the linear equations (42) and (44) respectively. Once the bounds A_1 and A_2 for A are computed from (46), the bounds R_1 and R_2 for $|R|$ are evaluated numerically. For the double wall problem similar procedures have been adopted. In the evaluation of the bounds for $|R|$, proper care has been

Table 1. Reflection coefficient $a/h = 0.2$, $b/h = 0.4$, $c/h = 0.6$, $Kh = 0.6$

	Single-wall		Double-wall	($d/h = .3$)
α	R_1	R_2	R_1	R_2
0°	0.114525	0.165642	0.140684	0.275302
30°	0.098924	0.143023	0.134951	0.246698
60°	0.056827	0.082099	0.094000	0.152289

taken. In Table 1, a representative set of values of R_1 and R_2 for both single-wall and double-wall problems computed through the 'one-term technique' are presented taking $a/h = 0{\cdot}2$, $b/h = 0{\cdot}4$, $c/h = 0{\cdot}6$ ($d/h = 0{\cdot}3$ for the double-wall case), $Kh = 0{\cdot}6$ and $\alpha = 0°$, 30° and 60°. It is observed from this table that for both the problems the numerical bounds for $|R|$ are not very close, particularly for the double-wall case, and as such their averages cannot serve as good estimates for $|R|$.

For the multi-term technique with $(N + 1)$-term approximation, we have to first compute $L_{mn}^{(i,j)}$ and $M_{mn}^{(i,j)}$ ($i, j = 1,2; m,n = 0,1,2,\ldots N$) given respectively by (52) and (58) (with appropriate modifications for the two-wall problem). These are computed by truncation of the corresponding infinite series. The accuracy in the computations of these types of series has been discussed by Porter and Evans[12] in some detail. Each series is first computed taking 500 terms. Then, these are improved by using the asymptotic values of the Bessel functions and the k_n in each case. It is found that by taking 10^4 terms in the computations of these improvements, a six-figure accuracy is finally achieved. Because of the limitations in the computational facilities at our disposal, we could not go beyond 10^4 terms. However, eight-figure accuracy could have been achieved had we taken 10^6 terms in these computations, as has been followed by Porter and Evans.[7] In the single-wall case, each set of the $2(N+1)$ constants b_n, c_n and d_n, e_n ($n = 0,1,\ldots N$) satisfies $2(N + 1)$ linear equations given in (51) and (57), which have been solved numerically taking $N = 0,1,2,\ldots 5$. The computed values of the constants, b_n, c_n and d_n, e_n ($n = 0,1,\ldots N$) are used to compute the bounds A_1, A_2 for A from (54) and (60), and the bounds R_1, R_2 for $|R|$ from (61). In Table 2 a representative set of values of R_1 and R_2 for $N = 0,1,2,\ldots 5$ is presented taking $a/h = 0{\cdot}2$, $b/h = 0{\cdot}4$, $c/h = 0{\cdot}6$, $Kh = 0{\cdot}6$ for $\alpha = 0°$, 30°, 60°. It is observed

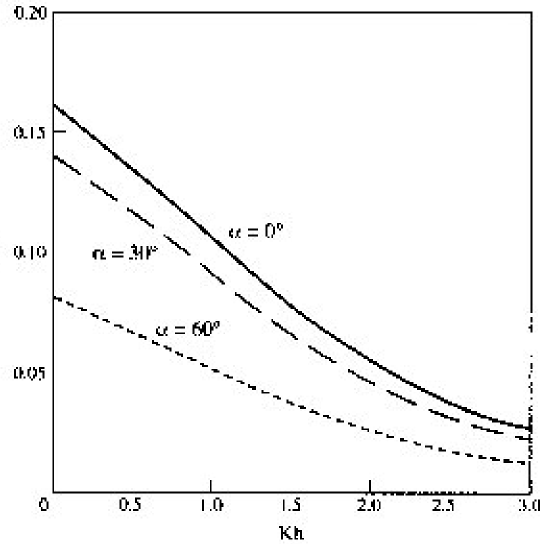

Fig. 1. Reflection coefficient vs wave number Kh. Single wall ($a/h = 0{\cdot}2$, $b/h = 0{\cdot}4$, $c/h = 0{\cdot}6$).

from this table that a six-figure accuracy in the numerical estimates for $|R|$ is achieved when $N = 4$.

Similarly, for the double-wall case, upper and lower bounds for $A^{s,a}$ are computed. Then upper and lower bounds for $|R|$ are computed using (72), taking extra care since upper (lower) bounds for $A^{s,a}$ do not necessarily correspond to upper (lower) bounds for $|R|$. In Table 3, a representative set of values of R_1 and R_2 for $N = 0,1,\ldots 5$ is presented for $a/h = 0{\cdot}2$, $b/h = 0{\cdot}4$, $c/h = 0{\cdot}6$, $d/h = 0{\cdot}3$, $Kh = 0{\cdot}6$ for $\alpha = 0°$, 30°, 60°. From this table it is observed that in this case also, a six-figure accuracy in the numerical estimates for $|R|$ has been achieved for $N = 4$. Tables 2 and 3 show that the bounds for $|R|$ become extremely accurate with the increase of N. For $N = 4$, a six-figure accuracy in the numerical estimates for $|R|$ is achieved in all the cases.

The results presented in Tables 2 and 3 correspond to some particular geometries of the single and double walls, a particular value of the wave number and particular values of the angle of incidence. For other geometries of the single and double walls, other values of the wavenumber and angle of incidence, very accurate estimates (six-figure) for $|R|$ are obtained and these are illustrated by plotting curves for $|R|$ against the wave number Kh. Figures 1–3 depict $|R|$ against

Table 2. Reflection coefficient for single wall $a/h = 0.2$, $b/h = 0.4$, $c/h = 0.6$, $Kh = 0.6$

	$\alpha = 0°$		$\alpha = 30°$		$\alpha = 60°$	
N	R_1	R_2	R_1	R_2	R_1	R_2
0	0.129695	0.131414	0.112015	0.113491	0.064333	0.065172
1	0.129624	0.129684	0.111957	0.112009	0.064303	0.064334
2	0.129625	0.129629	0.111958	0.111961	0.064304	0.064306
3	0.129626	0.129628	0.111959	0.111961	0.064305	0.064305
4	0.129628	0.129628	0.111960	0.111960	0.064305	0.064305
5	0.129628	0.129628	0.111960	0.111960	0.064305	0.064305

Table 3. Reflection coefficient for double wall $a/h = 0.2$, $b/h = 0.4$, $c/h = 0.6$, $Kh = 0.6$

	$\alpha = 0°$		$\alpha = 30°$		$\alpha = 60°$	
N	R_1	R_2	R_1	R_2	R_1	R_2
0	0.190175	0.193676	0.175169	0.178046	0.113703	0.115179
1	0.190996	0.191072	0.175663	0.176067	0.113825	0.113833
2	0.191016	0.191022	0.175729	0.175735	0.113827	0.113829
3	0.191019	0.191021	0.175731	0.175734	0.113827	0.113829
4	0.191021	0.191021	0.175733	0.175733	0.113829	0.113829
5	0.191021	0.191021	0.175733	0.175733	0.113829	0.113829

Kh for a single submerged wall with a gap. The Fig. 1 shows the variation of $|R|$ against Kh for $\alpha = 0°$, $30°$ and $60°$. It is observed from this figure that for a fixed wave number, the reflection coefficient decreases with the increase of the incident angle, which is plausible since more energy is transmitted through the gaps as the incident angle increases. Figure 2 depicts $|R|$ against Kh for $b/h = 0{\cdot}2$, $0{\cdot}4$ and $0{\cdot}6$ keeping other parameter fixed. This figure shows that for a fixed Kh, $|R|$ increases with the increase of b/h, demonstrating that more energy is reflected with the decrease of the length of the gap in the wall, which is also plausible. In Fig. 3 $|R|$ is shown against Kh for $a/h = 0{\cdot}01$, $0{\cdot}1$, $0{\cdot}2$ keeping b/h, c/h and α fixed. This figure shows that $|R|$ decreases with the increase in the depth of the upper edge of the wall below the free surface which is equivalent to the fact that more energy in transmitted as the depth of submergence of the upper edge of the wall increases. Also, the qualitative features of the curves for $|R|$ depicted in Figs. 1–3 are almost similar in the sense that they steady decrease with the increase of the wave number and asymptotically become zero for large Kh. The latter is plausible since for large wave number, the waves are confined to a thin layer below the free surface and the submerged wall does not 'feel' them.

When another identical wall with a gap of same length at the same level of the first wall is introduced, the aforesaid qualitative feature of steady decrease of $|R|$ with wave number in moderate wave number range is no longer visible. Figure 4 depicts $|R|$ against Kh for a double wall configuration for $a/h = 0{\cdot}2$, $b/h = 0{\cdot}4$, $c/h = 0{\cdot}6$, $d/h = 0{\cdot}3$ and $\alpha = 0°, 30°, 60°$. This figure shows the onset of change in the above qualitative features, particularly the appearance of zeros values of the reflection coefficient in the moderate wave number range. This is more apparent from Fig. 5 which depicts $|R|$ against Kh for different separation parameters d/h with $a/h = 0{\cdot}2$, $b/h = 0{\cdot}4$ and $c/h = 0{\cdot}6$ and $\alpha = 30°$. It is observed from Fig. 5 that the number of zeros of the reflection coefficient increases with the increase of separation parameter d/h. It is interesting to note that when the separation length is small ($d/h = 0{\cdot}01$), the qualitative feature of the curve for $|R|$ is similar to that for a single wall. The appearance of zeros of $|R|$ is attributed to multiple reflections by the two walls, a phenomenon always associated with a double barrier configuration. The first fundamental zero of $|R|$ can be obtained with six-digit accuracy using the same method used by Porter and Evans.[7] However, this is not pursued here, although from Fig. 5 these can be estimated to at least one decimal place.

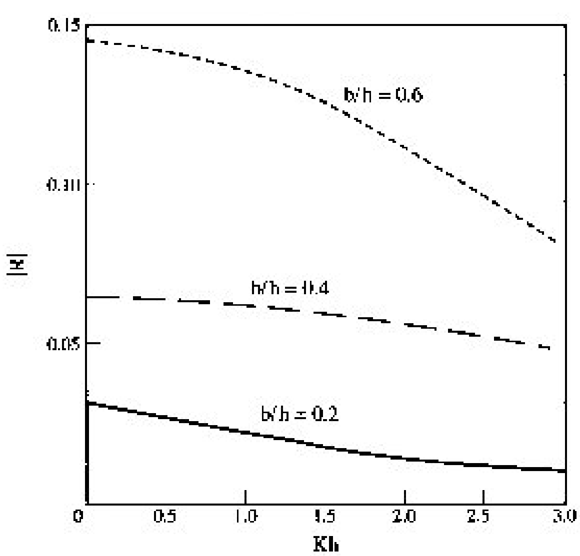

Fig. 2. Reflection coefficient vs wave number Kh. Single wall ($a/h = 0{\cdot}1$, $c/h = 0{\cdot}8$, $\alpha = 30°$).

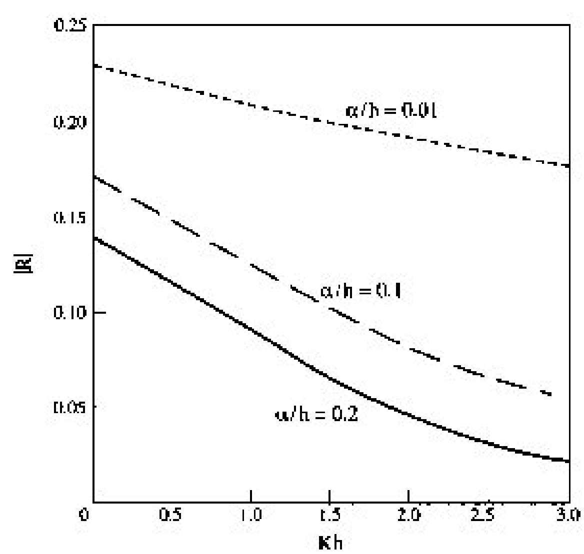

Fig. 3. Reflection coefficient vs wave number Kh. Single wall ($b/h = 0{\cdot}4$, $c/h = 0{\cdot}6$, $\alpha = 30°$).

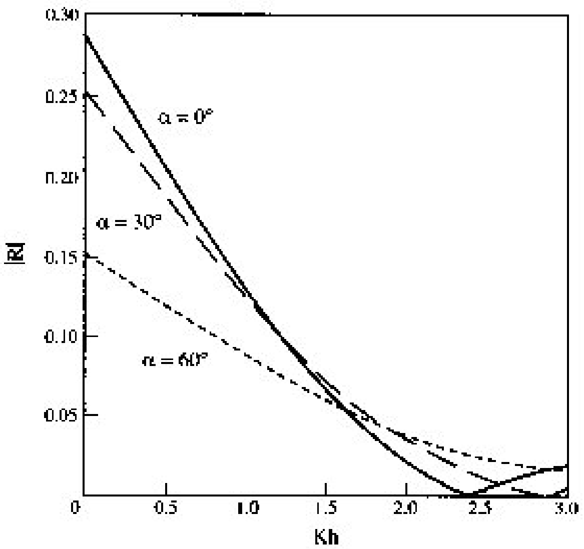

Fig. 4. Reflection coefficient vs wave number *Kh*. Double wall ($a/h = 0{\cdot}2$, $b/h = 0{\cdot}4$, $c/h = 0{\cdot}6$, $d/h = 0{\cdot}3$).

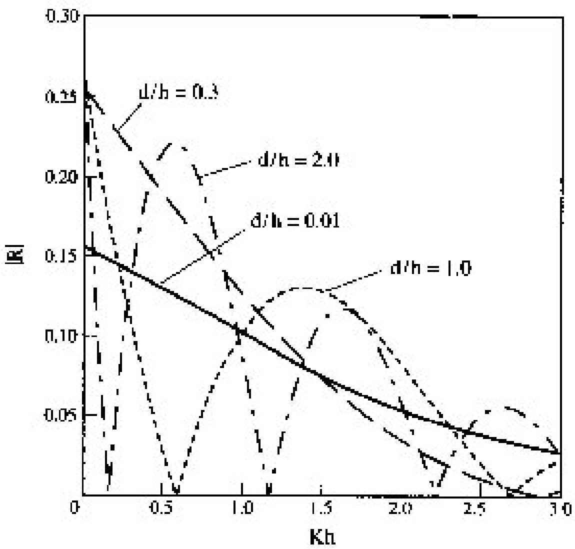

Fig. 5. Reflection coefficient vs wave number *Kh*. Double wall ($a/h = 0{\cdot}2$, $b/h = 0{\cdot}4$, $c/h = 0{\cdot}6$, $\alpha = 30°$).

5 CONCLUSIONS

One-term, as well as multi-term Galerkin, approximations have been utilized to obtain upper and lower bounds for the reflection coefficient for two problems of oblique wave scattering by a submerged single wall with a gap and two submerged double walls each with a gap at the same level in finite-depth water. The one-term approximations involve exact known results for the corresponding deep water problems for normal incidence, and the numerical bounds for the reflection coefficient obtained therein are seen to be not very close. However, the multi-term Galerkin approximations involving appropriate Chebyshev polynomials produce very accurate upper and lower bounds for the reflection coefficient for both the problems. In fact by choosing four terms a six-figure accuracy in the numerical estimation for the reflection coefficient is achieved.

The feature of the curves for the reflection coefficient for the double wall problem is seen to be significantly different from the corresponding curves for the single wall problem in the moderate wave number range. The double wall configuration results in the occurance of zero reflection for certain wave numbers. This may have some bearings on the modelling of breakwaters by thin vertical barriers.

It should be mentioned that because of complete submergence of the single or double walls, wave energy will always be transmitted through the gap below the free surface and no zero transmission occurs for any wave number. This is also apparent from the different curves for the reflection coefficient, which show that $|R|$ never becomes unity.

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