SCATTERING OF WATER WAVES BY A SUBMERGED NEARLY VERTICAL PLATE*

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Abstract. Some new results concerning the scattering of surface water waves by a nearly vertical plate, completely submerged in deep water, have been deduced employing two mathematical methods. The first method concerns an integral equation formulation of the problem obtained by a suitable use of Green's integral theorem in the fluid region, while the second method concerns a simple and straightforward perturbational analysis along with the application of Green's integral theorem. The two methods produce the same result for the first order corrections to the reflection and transmission coefficients. Considering some particular shapes of the curved plate, numerical calculations are also performed.

Key words. water waves, linearised theory, nearly vertical plate, integral equation, perturbational technique, reflection and transmission coefficients

AMS(MOS) subject classification. 76B

1. Introduction. Using linear theory, water wave scattering problems involving obstacles admit exact solutions only when the obstacles are in the form of thin plane barriers in deep water, when the motion is two-dimensional, and when the barriers are either vertical (cf. Ursell [8], Evans [1]) or inclined at special angles (cf. John [3]). The scattering problem involving a partially immersed nearly vertical barrier was considered by Shaw [6], wherein he employed an integral equation formulation and obtained the first-order corrections to the reflection and transmission coefficients—in terms of the shape function of the barrier. Recently, exploiting the idea of Evans [2], along with an appropriately designed perturbational anlysis, Mandal and Chakrabarti [4] deduced the analytical expressions for the first order corrections to the reflection and transmission coefficients of surface waves scattered by a fixed nearly vertical barrier for both cases when (i) the barrier is partially immersed and (ii) the barrier is completely submerged.

The present investigation is concerned with the scattering of surface water waves by a nearly vertical submerged fixed plate in deep water. The corresponding plane vertical plate problem was considered by Evans [1] for a normally incident wave train and by Mandal and Goswami [5] for an obliquely incident wave train. The problem under consideration is attacked for solution by two different mathematical techniques. In the first technique, an integral equation formulation similar to Shaw [6] is employed, while in the other method a suitable exploitation of Evans' [2] idea, along with an appropriately designed perturbational technique used recently by Mandal and Chakrabarti [4], is invoked. Both methods result in the same analytical expression for the first-order correction to the reflection and transmission coefficients. It is verified that, when the depth of the upper edge of the plate tends to zero, known results for a partially immersed nearly vertical barrier given by Shaw [6], as well as by Mandal and Chakrabarti [4], are recovered.

2. Statement of the problem. We consider the motion of a fluid of infinite depth, which is inviscid, incompressible, and homogeneous of density ρ under the action of gravity only. We choose a rectangular coordinate system in which the y-axis is taken vertically downwards into the fluid medium so that the plane $y=0$ is the undisturbed free surface of the fluid and the position of the submerged fixed nearly vertical plate of arbitrary shape is given by $S: x=\varepsilon c(y)$, $a<y<b$, where ε is a small nondimensional number and $c(y)$ is bounded for $a<y<b$ with $c(a)=0$. Assuming the motion of the fluid to be irrotational and simple harmonic in time with circular frequency σ and of small amplitude, there exists a velocity potential which can be expressed as $\operatorname{Re}[\varphi(x,y)\exp(-i\sigma t)]$. Then $\varphi(x,y)$ satisfies the two-dimensional Laplace's equation

$$\nabla^2\varphi=0 \quad \text{in the fluid region,} \tag{2.1}$$

the linearized free surface condition

$$K\varphi+\frac{\partial\varphi}{\partial y}=0 \quad \text{on } y=0, \tag{2.2}$$

with $K=\sigma^2/g$, g being the acceleration due to gravity,

$$\frac{\partial\varphi}{\partial n}=0 \quad \text{on } S, \tag{2.3}$$

where n denotes the outward drawn normal to the surface of the curved plate, the edge condition

$$r^{1/2}\nabla\varphi \quad \text{is bounded as } r\to 0, \tag{2.4}$$

where r is the distance from the two sharp edges of the curved plate, the infinity requirements

$$\varphi, \nabla\varphi\to 0 \quad \text{as } y\to\infty. \tag{2.5}$$

Further, we assume that

$$\begin{aligned} \varphi(x,y)&\sim\varphi^i(x,y)+R\varphi^i(-x,y) && \text{as } x\to-\infty,\\ \varphi(x,y)&\sim T\varphi^i(x,y) && \text{as } x\to+\infty, \end{aligned} \tag{2.6}$$

where $\varphi^i(x,y)=\exp(-Ky+iKx)$ is a train of surface waves incident on the curved plate S from negative infinity, and R and T are the (complex) reflection and transmission coefficients to be determined.

3. Integral equation formulation. In this case the total field $\varphi(x,y)$ is given by

$$\varphi(x,y)=\varphi^i(x,y)+\Phi(x,y) \tag{3.1}$$

where $\Phi(x,y)$ is the scattered velocity potential and satisfies the Laplace's equation in the fluid region, the linearized free surface condition (2.2), the edge condition (2.4), the infinity requirements (2.5), the condition on the plate as

$$\frac{\partial\Phi}{\partial n}=-\frac{\partial\varphi^i}{\partial n} \quad \text{on } S, \tag{3.2}$$

and the radiation condition

$$\begin{aligned} \Phi(x,y)&\sim(T-1)\exp(-Ky+iKx) && \text{as } x\to+\infty,\\ \Phi(x,y)&\sim R\exp(-Ky-iKx) && \text{as } x\to-\infty. \end{aligned} \tag{3.3}$$

In view of (3.3), $\Phi(x,y)$ represents an outgoing wave at infinity.

By an appropriate use of Green's integral theorem in the fluid medium we obtain

$$2\pi\Phi(\xi, \eta) = \int_S f(s) \frac{\partial G}{\partial n}(o, y; \xi, \eta)\bigg|_S ds \tag{3.4}$$

where $\partial/\partial n$ denotes the outward normal derivative on S, $f(s)$ is the difference of the velocity potential across the plate, and $G(x, y; \xi, \eta)$ is the usual Green's function (cf. Thorne [7]) for water waves defined by

$$G(x, y, \xi, \eta) = 2 \int_o^\infty \frac{(k \cos ky - K \sin ky)(k \cos k\eta - K \sin k\eta)}{k(k^2 + K^2)} \cdot \exp\{-k|x-\xi|\}\, dk + 2\pi i \exp\{-K(y+\eta) + iK|x-\xi|\}. \tag{3.5}$$

Now we consider perturbations to the vertical plate solution (see Appendix A) for small values of $\varepsilon > 0$ and assume that

$$\begin{aligned} f(s) &= \Phi(\varepsilon c(y) + o, y) - \Phi(\varepsilon c(y) - o, y) \\ &= f_o(y) + \varepsilon f_1(y) + O(\varepsilon^2). \end{aligned} \tag{3.6}$$

Utilizing (3.2) into (3.4) and using (3.6) we finally obtain the integral equations for $f_o(y)$ and $f_1(y)$ as

$$\int_a^b f_o(y) \frac{\partial^2 G}{\partial \xi\, \partial x}(o, y; o, \eta)\, dy = -2\pi i K \exp(-K\eta) \qquad a < \eta < b, \tag{3.7}$$

and

$$\begin{aligned} &\int_a^b f_1(y) \frac{\partial^2 G}{\partial \xi\, \partial x}(o, y; o, \eta)\, dy \\ &\quad = -\int_a^b f_o(y) \left[\left\{ c(\eta) \frac{\partial^2}{\partial \xi^2} - c'(\eta) \frac{\partial}{\partial \eta} \right\} \frac{\partial}{\partial x} + \left\{ c(y) \frac{\partial^2}{\partial x^2} - c'(y) \frac{\partial}{\partial y} \right\} \frac{\partial}{\partial \xi} \right] \\ &\quad\quad \cdot G(o, y; o, \xi)\, dy - 2\pi K \frac{d}{d\eta}\{c(\eta) \exp(-K\eta)\} \qquad a < \eta < b, \end{aligned} \tag{3.8}$$

respectively. The solution of the integral equation (3.7) is given by (see Mandal and Goswami [5])

$$f_o(y) = \exp(-Ky) \int_a^y \Psi_o(u) \exp(Ku)\, du \tag{3.9}$$

where

$$\Psi_o(y) = \frac{D_o(d_o^2 - y^2)}{(y^2 - a^2)^{1/2}(b^2 - y^2)^{1/2}} \qquad a < y < b. \tag{3.10}$$

Here the constants D_o and d_o^2 are given by

$$D_o = 2i/\Delta_o \tag{3.11}$$

and

$$\int_a^b \frac{(d_o^2 - u^2) \exp(Ku)}{(u^2 - a^2)^{1/2}(b^2 - u^2)^{1/2}}\, du = 0 \tag{3.12}$$

where Δ_o is given in (3.21) below.

Using arguments similar to Shaw [6] (also see Appendix B) the integral equation (3.8) can be reduced to

$$\frac{d}{d\eta}\int_a^b f_1(y)\frac{\partial G}{\partial \eta}(o, y; o, \eta)\, dy = -2\pi K\frac{d}{d\eta}\{c(\eta)\exp(-k\eta)\} \qquad a<\eta<b. \tag{3.13}$$

This implies that the first order correction $f_1(y)$ to $f(y)$ is independent of the vertical plate solution $f_o(y)$.

Now, (3.13) can be reduced to

$$\int_a^b \Psi_1(y)\frac{2y}{y^2-\eta^2}\, dy = KA_1 - 2\pi Kc'(\eta)\exp(-K\eta) \qquad a<\eta<b, \tag{3.14}$$

with

$$\Psi_1(y) = Kf_1(y) + f_1'(y) \tag{3.15}$$

and the integral is in the sense of Cauchy principal value and A_1 is an arbitrary constant. Thus

$$f_1(y) = \exp(-Ky)\int_a^y \Psi_1(u)\exp(Ku)\, du \tag{3.16}$$

where

$$\Psi_1(y) = \frac{D_1(d_1^2-y^2)+\dfrac{4K}{\pi}S(y)}{(y^2-a^2)^{1/2}(b^2-y^2)^{1/2}} \qquad a<y<b, \tag{3.17}$$

with $D_1 = -KA_1/\pi$,

$$S(y) = \int_a^b \frac{(t^2-a^2)^{1/2}(b^2-t^2)^{1/2}}{t^2-y^2}\, tc'(t)\exp(-Kt)\, dt, \qquad a<y<b. \tag{3.18}$$

Expression (3.17) involves the arbitrary constants D_1 and d_1^2. Since $f_1(b)=0$, it follows from (3.16) and (3.17) that

$$\int_a^b \frac{D_1(d_1^2-u^2)+\dfrac{4K}{\pi}S(u)}{(u^2-a^2)^{1/2}(b^2-u^2)^{1/2}}\exp(Ku)\, du = 0 \tag{3.19}$$

giving one equation to determine D_1 and d_1^2. The other equation is obtained from the original integral equation (3.13) after substituting $f_1(u)$ (obtained from (3.17) through (3.16)) and evaluating the different integrals resulting in a factor $\exp(-K\eta)$ in both sides. Thus we obtain

$$D_1\Delta_1 + \frac{4K}{\pi}\Delta_2 - 2K\int_a^b c'(y)\exp(-2Ky)\, dy = 0 \tag{3.20}$$

where

$$\Delta_j = (\alpha_j - \beta_j - i\gamma_j) \tag{3.21}$$

$$\alpha_j = \int_{-a}^a \frac{F_j(u)}{(a^2-u^2)^{1/2}(b^2-u^2)^{1/2}}\, du,$$

$$\beta_j = \int_b^\infty \frac{F_j(u)}{(u^2-a^2)^{1/2}(u^2-b^2)^{1/2}}\, du, \tag{3.22}$$

$$\gamma_j = \int_a^b \frac{F_j(u)}{(u^2-a^2)^{1/2}(b^2-u^2)^{1/2}}\, du, \qquad j=0,1,2 \text{ with}$$

$$F_j(u)=\begin{cases}(d_o^2-u^2)\exp(-Ku) & \text{for } j=0,\\ (d_1^2-u^2)\exp(-Ku) & \text{for } j=1,\\ S(u)\exp(-Ku) & \text{for } j=2.\end{cases} \tag{3.23}$$

Equations (3.19) and (3.20) determine the constants D_1 and d_1^2.

To determine the reflection and transmission coefficients we assume

$$R=R_o+\varepsilon R_1+O(\varepsilon^2)\qquad T=T_o+\varepsilon T_1+O(\varepsilon^2). \tag{3.24}$$

Making $\xi\to+\infty$ and $\xi\to-\infty$, respectively, in (3.4) after using (3.3) and (3.5) we find

$$\begin{aligned} R_o&=-P_o,\\ T_o&=1+P_o,\\ R_1&=P_1+Q_1,\\ T_1&=P_1-Q_1,\end{aligned} \tag{3.25}$$

where

$$\begin{aligned} P_o&=K\int_a^b f_o(y)\exp(-Ky)\,dy,\\ P_1&=iK\int_a^b f_o(y)\frac{d}{dy}\{c(y)\exp(-Ky)\}\,dy,\\ Q_1&=-K\int_a^b f_1(y)\exp(-Ky)\,dy.\end{aligned} \tag{3.26}$$

P_o can be evaluated and we find that

$$R_o=-i\gamma_o/\Delta_o \tag{3.27}$$

and

$$T_o=(\alpha_o-\beta_o)/\Delta_o. \tag{3.28}$$

These expressions have already been obtained by Evans [1]. Similarly, utilizing (3.9), (3.10), and the condition that $f_o(b)=0$, we find P_1 as

$$\begin{aligned} P_1=-iKD_o\Bigg[K\int_a^b c(y)\exp(-2Ky)\left\{\int_y^b \frac{(d_o^2-u^2)\exp(Ku)}{(u^2-a^2)^{1/2}(b^2-u^2)^{1/2}}\,du\right\}dy\\ +\int_a^b \frac{(d_o^2-u^2)c(u)\exp(-Ku)}{(u^2-a^2)^{1/2}(b^2-u^2)^{1/2}}\,du\Bigg].\end{aligned} \tag{3.29}$$

Also using (3.16), (3.17), and $f_1(b)=0$, the expression for Q_1 given by (3.26) reduces to

$$Q_1=-\frac{1}{2}\left(D_1\gamma_1+\frac{4K}{\pi}\gamma_2\right). \tag{3.30}$$

Now we invoke the argument used by Shaw [6] that the transmission coefficient T remains unaltered when a scattering body is reversed but the incident field is left unchanged. We obtain, replacing $\varepsilon c(y)$ by $-\varepsilon c(y)$,

$$\begin{aligned} T&=T_o+\varepsilon T_1+O(\varepsilon^2)\\ &=T_o-\varepsilon T_1+O(\varepsilon^2).\end{aligned}$$

This gives $T_1 = 0$, so that $P_1 = Q_1$. Hence $R_1 = 2P_1$, and utilizing (3.29), we find in this case

$$R_1 = -2iKD_o\left[K\int_a^b c(y)\exp(-2Ky)\left\{\int_y^b \frac{(d_o^2-u^2)\exp(Ku)}{(u^2-a^2)^{1/2}(b^2-u^2)^{1/2}}\,du\right\}dy + \int_a^b \frac{(d_o^2-u^2)c(u)\exp(-Ku)}{(u^2-a^2)^{1/2}(b^2-u^2)^{1/2}}\,du\right]. \tag{3.31}$$

However, it is difficult to prove $P_1 = Q_1$ directly from their integral forms. For the corresponding immersed barrier problem, Shaw [6] also mentioned that a direct proof of this is "very hard" to find.

It can be easily shown that as "a," the depth of the upper edge of the curved plate, approaches the free surface while "b," the depth of the lower edge, is kept fixed (cf. Evans [1]),

$$d_o^2 \to 0, \quad \alpha_o \to 0, \quad \beta_o \to -bK_1(Kb), \quad \gamma_o \to b\pi I_1(Kb).$$

Thus

$$D_o = 2i/b\{K_1(Kb) - i\pi I_1(Kb)\}.$$

Substituting these values into (3.27) and (3.28) we obtain

$$R_o = -i\pi I_1(Kb)/\{K_1(Kb) - i\pi I_1(Kb)\}$$

$$T_o = K_1(Kb)/\{K_1(Kb) - i\pi I_1(Kb)\},$$

respectively, which were obtained by Ursell [8] for a partially immersed vertical barrier.

Similarly, utilizing the approximations of d_o^2, α_o, β_o, γ_o and D_o as $\mu\ (=a/b) \to 0$ (b fixed) in (3.31), we obtain

$$R_1 = \frac{4K}{b\{K_1(Kb) - i\pi I_1(Kb)\}}\left[-K\int_o^b c(y)\exp(-2Ky)\cdot\left\{\int_a^b \frac{u\exp(Ku)}{(b^2-u^2)^{1/2}}\,du\right\}dy - \int_o^b \frac{yc(y)\exp(-Ky)}{(b^2-y^2)^{1/2}}\,dy\right], \tag{3.32}$$

which in the notation of Shaw [5] becomes

$$R_1 = \frac{4Kb}{K_1(Kb) - i\pi I_1(Kb)}\left[Kb\int_{-1}^{o} C(t)\exp(2Kbt)\left\{\int_{-1}^{t} \frac{u\exp(-Kbu)}{(1-u^2)^{1/2}}\,du\right\}dt + \int_{-1}^{o} \frac{C(t)t\exp(Kbt)}{(1-t^2)^{1/2}}\,dt\right]$$

where $C(t) = c(-bt)/b$.

4. Solution by a perturbational analysis. For a nearly vertical plate we can assume ε to be very small. Thus neglecting $O(\varepsilon^2)$ terms, the boundary condition (2.3) can be expressed as (cf. Shaw [6])

$$\frac{\partial\varphi}{\partial x}(\pm o, y) - \varepsilon\frac{\partial}{\partial y}\left\{c(y)\frac{\partial\varphi}{\partial y}(\pm o, y)\right\} = 0 \qquad a < y < b. \tag{4.1}$$

This suggests an expansion for $\varphi(x, y)$ as

$$\varphi(x, y, \varepsilon) = \varphi_0(x, y) + \varepsilon\varphi_1(x, y) + O(\varepsilon^2), \tag{4.2}$$

and a similar type of expansion for R and T given by (3.24). Here we confine our attention to determining the constants R_o, T_o, R_1, and T_1 only, as we are interested in evaluating only the first-order corrections to the reflection and transmission coefficients. Utilizing the expansions given by (4.2) and (3.24) in (2.1), (2.2), (4.1), (2.4), (2.5), and (2.6), after equating coefficients of ε^o and ε from both sides of the results derived thus, we obtain that the functions φ_o and φ_1 must be the solution of the following two independent mixed boundary value problems given by P_o and P_1, respectively.

Problem P_o. The problem is to determine the function $\varphi_o(x, y)$ satisfying

$$\nabla^2\varphi_o = 0 \quad \text{in the fluid region,}$$

$$K\varphi_o + \frac{\partial\varphi_o}{\partial y} = 0 \quad \text{on } y = 0,$$

$$\frac{\partial\varphi_o}{\partial x} = 0 \quad \text{on } x = 0, \qquad a < y < b,$$

$r^{1/2}\nabla\varphi_o$ is bounded as $r \to 0$, where r is the distance from (o, a) and (o, b)

$$\varphi_o, \nabla\varphi_o \to 0 \quad \text{as } y \to \infty,$$

$$\varphi_o \sim \exp(-Ky + iKx) + R_o \exp(-Ky - iKx) \quad \text{as } x \to -\infty,$$

$$\varphi_o \sim T_o \exp(-Ky + iKx) \quad \text{as } x \to +\infty.$$

Problem P_1. The problem is to determine the function $\varphi_1(x, y)$ satisfying

(P1.1) $$\nabla^2\varphi_1 = 0 \quad \text{in the fluid region,}$$

(P1.2) $$K\varphi_1 + \frac{\partial\varphi_1}{\partial y} = 0 \quad \text{on } y = 0,$$

(P1.3) $$\frac{\partial\varphi_1}{\partial x}(\pm o, y) = \frac{\partial}{\partial y}\left\{c(y)\frac{\partial\varphi_o}{\partial y}(\pm o, y)\right\} \quad \text{on } x = 0, \qquad a < y < b,$$

(P1.4) $$r^{1/2}\nabla\varphi_1 \quad \text{is bounded as } r \to 0,$$

(P1.5) $$\varphi_1, \nabla\varphi_1 \to 0 \quad \text{as } y \to \infty,$$

(P.16) $$\varphi_1 \sim R_1 \exp(-Ky - iKx) \quad \text{as } x \to -\infty$$
$$\varphi_1 \sim T_1 \exp(-Ky + iKx) \quad \text{as } x \to +\infty.$$

The function $\varphi_o(x, y)$, which is the solution of the problem P_o, is a discontinuous function on $x = 0$, $a < y < b$, and therefore the boundary condition (P1.3) must be used carefully on the two sides $x = o^+$ and $x = o^-$ of the line $x = 0$.

The explicit solution $\varphi_o(x, y)$ of the problem P_o is given in Appendix A. To find the first order corrections to the reflection and transmission coefficients, the explicit solution $\varphi_1(x, y)$ to the problem P_1 is not necessary. These can be obtained in the following manner using a technique similar to that used by Evans [2]. For this we apply Green's integral theorem to the harmonic functions φ_o and φ_1 in the region bounded by the lines

$$y = 0, \quad -X \le x \le X; \qquad x = -X, \quad 0 \le y \le Y; \qquad y = Y, \quad -X \le x \le X;$$
$$x = X, \quad 0 \le y \le Y; \qquad x = 0^+, \quad a < y < b; \qquad x = 0^-, \quad a < y < b,$$

and circles C_1 and C_2 of small radius δ with centers at (o, b) and (o, a), and we ultimately make X, Y tend to infinity and $\delta \to 0$. Using the same arguments similar to Evans [2], we obtain in this case

$$iR_1 = \int_a^b \left[\varphi_o(+o, y) \frac{\partial}{\partial y} \left\{ c(y) \frac{\partial \varphi_o}{\partial y}(+o, y) \right\} - \varphi_o(-o, y) \frac{\partial}{\partial y} \left\{ c(y) \frac{\partial \varphi_o}{\partial y}(-o, y) \right\} \right] dy, \tag{4.3}$$

in which we have made use of the condition (P1.3).

From (4.3), using the solutions for $\varphi_o(+o, y)$ (see Appendix A), we obtain the analytical expression for the first order correction R_1 to the reflection coefficient as

$$R_1 = \frac{4K}{\Delta_o} \left[K \int_a^b c(y) \exp(-2Ky) \left\{ \int_y^b \frac{(d_o^2 - u^2) \exp(Ku)}{(u^2 - a^2)^{1/2}(b^2 - u^2)^{1/2}} du \right\} dy + \int_a^b \frac{(d_o^2 - u^2) c(u) \exp(-Ku)}{(u^2 - a^2)^{1/2}(b^2 - u^2)^{1/2}} du \right]. \tag{4.4}$$

This is in fact the same expression given by (3.31) obtained by using an integral equation formulation.

Next, in order to obtain the first order correction T_1 to the transmission coefficient, we again utilize Evans' [2] idea, along with the application of Green's integral theorem to the harmonic functions $\Psi_o(x, y)$ and $\varphi_1(x, y)$ in the region mentioned earlier, where $\Psi_o(x, y) = \varphi_o(-x, y)$ to finally obtain

$$iT_1 = \int_a^b \left[\Psi_o(+o, y) \frac{\partial}{\partial y} \left\{ c(y) \frac{\partial \varphi_o}{\partial y}(+o, y) \right\} - \Psi_o(-o, y) \frac{\partial}{\partial y} \left\{ c(y) \frac{\partial \varphi_o}{\partial y}(-o, y) \right\} \right] dy \tag{4.5}$$

in which the relations (P1.3) have been used. Utilizing $\Psi_o(\pm o, y) = \varphi_o(\mp o, y)$ in (4.5), integrating by parts, this produces

$$iT_1 = \left[c(y) \left\{ \varphi_o(-o, y) \frac{\partial \varphi_o}{\partial y}(+o, y) - \varphi_o(+o, y) \frac{\partial \varphi_o}{\partial y}(-o, y) \right\} \right]_{y=a}^{b},$$

and this vanishes after using (A.5). Thus $T_1 = 0$. This result also holds for a partially immersed or completely submerged nearly vertical barrier, as was shown recently by Mandal and Chakrabarti [4] and earlier by Shaw [6] (for the case of a partially immersed plate). Shaw [6], however, used an argument based on symmetry to derive this result for the partially immersed plate, while Mandal and Chakrabarti [4] proved this analytically.

5. Discussion. An analytical expression for the first order correction to the reflection coefficient is obtained here for a surface water wave train incident on a fully submerged fixed nearly vertical plate whose upper and lower edges are at a depth "a" and "b," respectively, below the mean free surface. The problem is attacked for solution by using two methods, one method being based on an integral equation formulation of the problem and the other being a perturbational analysis. The second method seems to be rather simple compared to the first, as the desired results are obtained relatively easily and fairly quickly. The known results for a partially immersed plate are recovered as a special case.

For numerical illustrations we have considered three typical forms of $c(y)$, namely, (i) a fixed vertical sinusoidal plate given by $x = \varepsilon b \sin \omega(y-a)$, $(a < y < b)$; (ii) a fixed slightly curved plate given by $x = \varepsilon(y-a)\exp(-\omega y)$, $(a < y < b)$; and (iii) a fixed slightly inclined straight plate given by $x = \varepsilon b(y-a)/(b-a)$, $a < y < b$. The reflection coefficient $|R|$ $(=|R_o + \varepsilon R_1|)$ is computed correct up to six decimal places for different values of the various parameters, e.g., $Kb = 0.2, 0.4, 0.6, 0.8, 1.0, 1.5, 2.0, 2.5, 3.0, 5.0, 7.0, 9.0$; $\varepsilon = 0.001, 0.005$; $\omega b = 5, 10, 15, 20$, and μ $(= a/b) = 0.01, 0.05, 0.25, 0.50, 0.75$ for each particular type of submerged plate. Some representative numerical results are given in Tables 1 and 2. It is observed that the various values of $|R|$ for most cases differ from $|R_o|$ (i.e., the vertical plate result) only in the sixth decimal place. As terms of the order of ε^2 are neglected in the analysis, this indicates that the influence of ε is not of much significance for these types of nearly vertical plates.

Appendix A. The explicit form of $\varphi_o(x, y)$ can be derived from the general result given by Evans [1]. However, we deduce it here from the result given by Mandal and Goswami [5] in the form

$$\varphi_o(x, y) = \exp(-Ky + iKx) + \chi_o(x, y) \tag{A.1}$$

where

$$\chi_o(x, y) = \frac{1}{2\pi}\int_a^b f_o(\eta) G(o, \eta; x, y)\, d\eta, \tag{A.2}$$

TABLE 1
$x = \varepsilon b \sin \omega(y-a)$, $a < y < b$
$|R| = |R_o + \varepsilon R_1|$

			$\varepsilon = 0.001$		$\varepsilon = 0.005$	
Kb	$\mu = a/b$	$\lvert R_o\rvert$	$\omega b = 5.0$	$\omega b = 15.0$	$\omega b = 5.0$	$\omega b = 15.0$
0.4	0.01	0.131075	0.131077	0.131075	0.131113	0.131077
	0.05	0.098837	0.098838	0.098837	0.098874	0.098838
1.5	0.01	0.647357	0.647365	0.647362	0.647554	0.647481
	0.05	0.464928	0.464937	0.464932	0.465137	0.465031
5.0	0.01	0.666440	0.666468	0.666502	0.667147	0.667965
	0.05	0.407773	0.407799	0.407818	0.408421	0.408898

TABLE 2
$|R| = |R_o + \varepsilon R_1|$

		$\varepsilon = 0.001$			$\varepsilon = 0.005$		
		$x = \varepsilon(y-a)\exp(-\omega y)$, $a < y < b$		$x = \frac{\varepsilon b(y-a)}{(b-a)}$	$x = \varepsilon(y-a)\exp(-\omega y)$, $a < y < b$		$x = \frac{\varepsilon b(y-a)}{(b-a)}$
Kb	$\mu = a/b$	$\omega b = 5.0$	$\omega b = 15.0$	$a < y < b$	$\omega b = 5.0$	$\omega b = 15.0$	$a < y < b$
0.4	0.01	0.131075	0.131075	0.131076	0.131075	0.131075	0.131087
	0.05	0.098837	0.098837	0.098837	0.098837	0.098837	0.098846
1.5	0.01	0.647357	0.647357	0.647357	0.647358	0.647357	0.647357
	0.05	0.464928	0.464928	0.646928	0.464929	0.464928	0.464928
5.0	0.01	0.666440	0.666440	0.666442	0.666449	0.666441	0.666482
	0.05	0.407774	0.407773	0.407775	0.407779	0.407774	0.407817

G, $f_o(y)$ being the same as given in §§ 3 and 4, respectively. Using $G(o, \eta; x, y)$ given by (3.5) in (A.2), we can calculate $\varphi_o(x, y)$ explicitly for $x>0$ as well as for $x<0$. (A.2) involves two integrals which can be simplified. One integral is $2K\int_a^b f_o(y)\exp(-Ky)\,dy$, and this is $2i\gamma_o/\Delta_o$, and the other is $\int_a^b f_o(\eta)\times(k\cos k\eta - K\sin k\eta)\,d\eta$, which is simplified to $-(2i/\Delta_o)J(k)$ where

$$J(k)=\int_a^b \frac{(d_o^2-\eta^2)\sin k\eta}{(\eta^2-a^2)^{1/2}(b^2-\eta^2)^{1/2}}\,d\eta.$$

Hence we finally obtain

$$\text{(A.3)}\quad \begin{aligned}\varphi_o(x,y)=&\frac{\alpha_o-\beta_o}{\Delta_o}\exp(-Ky+iKx)\\ &-\frac{2i}{\pi\Delta_o}\int_o^\infty \frac{J(k)(k\cos ky-K\sin ky)\exp(-kx)}{k^2+K^2}\,dk\quad\text{for } x>0\end{aligned}$$

and

$$\text{(A.4)}\quad \begin{aligned}\varphi_o(x,y)=&\exp(-Ky+iKx)-\frac{i\gamma_o}{\Delta_o}\exp(-Ky-iKx)\\ &+\frac{2i}{\pi\Delta_o}\int_o^\infty \frac{J(k)(k\cos ky-K\sin ky)\exp(kx)}{k^2+K^2}\,dk\quad\text{for } x<0.\end{aligned}$$

From (A.3) and (A.4) it can be shown that

$$\text{(A.5)}\quad \varphi_o(\pm o,y)=\begin{cases}\exp(-Ky) & y\leqq a,\\ \exp(-Ky)\pm\dfrac{i\exp(-Ky)}{\Delta_o}\displaystyle\int_b^y \frac{(d_o^2-u^2)\exp(Ku)}{(u^2-a^2)^{1/2}(b^2-u^2)^{1/2}}\,du & a<y<b,\\ \exp(-Ky) & y\geqq b.\end{cases}$$

Appendix B. It can be shown that

$$\text{(B.1)}\quad \frac{\partial G}{\partial x}(o,y;o,\eta)=-\frac{\partial G}{\partial \xi}(o,y;o,\eta).$$

We proceed as in Shaw [6] to note that

$$\text{(B.2)}\quad \left.\frac{\partial G}{\partial x}\right|_{x=o}\to\frac{\xi}{\xi^2+(y-\eta)^2}=\pi\delta(y-\eta),\quad\text{as } \xi\to 0,$$

where δ is the Dirac delta function. Thus using (B.1) we find

$$\text{(B.3)}\quad \left.\frac{\partial G}{\partial \xi}\right|_{x=o}\to-\pi\delta(y-\eta)\quad\text{as } \xi\to 0.$$

The kernel in the first integral in the right-hand side of (3.8) can be written as

$$-\left[\left\{c(\eta)\frac{\partial^2}{\partial\eta^2}+c'(\eta)\frac{\partial}{\partial\eta}\right\}\frac{\partial}{\partial x}+\left\{c(y)\frac{\partial^2}{\partial y^2}+c'(y)\frac{\partial}{\partial y}\right\}\frac{\partial}{\partial\xi}\right]G(o,y,o,\eta).$$

Utilizing (B.2) and (B.3), this reduces to

$$\begin{aligned}-\pi&\left[\left\{c(\eta)\frac{\partial^2}{\partial\eta^2}+c'(\eta)\frac{\partial}{\partial\eta}\right\}\delta(y-\eta)-\left\{c(y)\frac{\partial^2}{\partial y^2}+c'(y)\frac{\partial}{\partial y}\right\}\delta(y-\eta)\right]\\ &=\pi[\{c(y)-c(\eta)\}\delta''(y-\eta)+\{c'(\eta)+c'(y)\}\delta'(y-\eta)].\end{aligned}$$

After using the properties of the delta function and by integration by parts we find that the first integral in the right-hand side of (3.8) is zero.

Acknowledgments. The authors thank the referees for their valuable comments which improved the paper.

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