Degeneracy Subgraph of the Lemke Complementary Pivot Algorithm and Anticycling Rule[1]

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Communicated by O. L. Mangasarian

Abstract. In this paper, we use the theory of degeneracy graphs recently developed by Gal et al. to introduce a graph for studying the adjacency of almost complementary feasible bases, some of which may be degenerate, which are of interest in the context of the linear complementarity problem. We study the structure of this graph with particular reference to the possibility of cycling and various anticycling rules in the Lemke complementary pivoting algorithm. We consider the transition node pivot rule introduced by Geue and show that this rule helps in avoiding cycling in the Lemke complementary pivoting algorithm under a suitable assumption.

Key Words. Linear complementarity problem, degeneracy graph, cycling, transition node pivot rule.

1. Introduction

Given a square matrix M of order n and a vector $q \in R^n$, the linear complementarity problem is the problem of determining vectors $w \in R^n$ and $z \in R^n$ such that

$$w - Mz = q, \qquad w \geq 0, \qquad z \geq 0, \tag{1}$$

$$w^t z = \sum_{i=1}^{n} w_i z_i = 0, \tag{2}$$

where for any vector x, x^t denotes its row transpose. This problem is well studied in the literature on mathematical programming and has a number

[1]The author is thankful to the anonymous referees for numerous helpful comments on earlier versions of this paper.

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of applications. See the recent books by Cottle, Pang, and Stone (Ref. 1) and Murty (Ref. 2).

The Lemke complementary pivoting algorithm to solve the above problem proceeds by considering the augmented system

$$w - Mz - dz_0 = q, \qquad (w, z, z_0) \geq 0, \tag{3}$$

where d is a given positive vector in R^n. See Ref. 3. The vector d which is artificial to the original system of equations has been called the covering vector by Cottle, Pang, and Stone (Ref. 1), who take it to be merely nonzero and nonnegative.

Given an $n \times n$ matrix M and a vector $q \in R^n$, we call an $n \times n$ submatrix B of $(I, -M, -d)$ an almost complementary matrix if the following hold:

(i) $-d$ is a column of B;
(ii) $-M_{.j}$ is a column of B if and only if $I_{.j}$ is not.

In addition, if such a matrix is nonsingular, then we call it an almost complementary basis matrix. This is abbreviated as ACB matrix. We call B [an $n \times n$ submatrix of $(I, -M, -d)$] an n-complementary matrix if

(i) $-d$ is not a column of B;
(ii) $-M_{.j}$ is a column of B if and only if $I_{.j}$ is not.

If an n-complementary matrix is nonsingular, we call it a complementary basis matrix. This is abbreviated as CB matrix.

Let

$$\bar{X}(q) = \{(w, z, z_0) | w - Mz - dz_0 = q; (w, z, z_0) \geq 0\}.$$

We say that a given (w, z, z_0) is an almost complementary feasible solution if

$$(w, z, z_0) \in \bar{X}(q), \qquad w^t z = 0, \qquad z_0 > 0.$$

We call a $(w, z, z_0) \in \bar{X}(q)$ a complementary feasible solution if

$$z_0 = 0, \qquad w^t z = 0.$$

Suppose that $(w, z, z_0) \in \bar{X}(q)$ is an almost complementary feasible solution. The submatrix formed by the columns of $(I, -M, -d)$ corresponding to the positive coordinates of (w, z, z_0) is said to be the almost complementary matrix that corresponds to (w, z, z_0). In analogy with the practice in the theory of linear programming, one may call an almost complementary feasible solution (w, z, z_0) an almost complementary basic feasible solution if the columns of the almost complementary matrix that corresponds to it are linearly independent. However, unlike in the linear programming context, it may not be possible to extend such an almost complementary matrix, if it contains fewer than n columns, to an ACB matrix by adding more columns to it from among the columns of $(I, -M, -d)$. For an example, see Example 1 of Ref. 4. To avoid such cases, we shall call an almost complementary solution (w, z, z_0) an almost complementary basic feasible solution only if the almost complementary matrix that corresponds to (w, z, z_0) can be extended to an ACB matrix. We denote an almost complementary basic feasible solution by the abbreviation ACBFS. Similarly, a complementary feasible solution is called a complementary basic feasible solution (CBFS) only if the complementary matrix that corresponds to it can be extended to a CB matrix.

The combinatorial structure of a convex polytope S can be represented by a graph $G(S)=(N, E)$, where the set of vertices N is the set of extreme points of S and V, the set of edges, is the set of pairs of adjacent extreme points $\{v^1, v^2\} \subset N$. See Grünbaum (Ref. 5). This can be extended also to unbounded convex polyhedral sets by representing the extreme rays (unbounded edges) of S by artificial vertices. A vertex x and an artificial vertex y are said to be adjacent if the extreme ray represented by the artificial vertex y is incident on the extreme point represented by the vertex x.

An extreme point (w, z, z_0) of $\bar{X}(q)$ is said to be nondegenerate if the number of positive coordinates in (w, z, z_0) is n. If we make the standard nondegeneracy assumption that none of the extreme points of $\bar{X}(q)$ is degenerate, then there is a one–one correspondence between the extreme points of $\bar{X}(q)$ and the basic feasible solutions to (3). Since a basic feasible solution can be represented in the form of a simplex tableau, we can take the node set to be N_1, the set of tableaux corresponding to the extreme points or artificial points representing the extreme rays of $\bar{X}(q)$; and we can take the edge set to be E_1, the set of pairs of tableaux $\{T_u, T_v\}$ such that T_v can be obtained from T_u by a single pivotal transformation with the pivot element as positive and pairs $\{T_i(\infty), T_u\}$ where the tableau T_u corresponds to the extreme point incident on the extreme ray represented by $T_i(\infty)$. We may then consider the graph (N_1, E_1) as the adjacency graph of $\bar{X}(q)$. This way of redefining the graph $G(\bar{X}(q))$ enables us to generalize the concept when the nondegeneracy assumption does not hold. A similar generalization for a convex polytope X (a bounded convex polyhedral set) has already been reported in Ref. 6, and the resulting graph has been called the representation graph $G(X)$ of the polytope X.

We assume that the readers are familiar with the steps of the Lemke complementary pivoting algorithm. We are here concerned with the subgraph G_c^+ of the graph $G(\bar{X}(q))$, the node set of which consists only of those nodes of N_1 which are feasible with respect to the Lemke algorithm. Under the nondegeneracy assumption that all the almost complementary basic feasible solutions in $\bar{X}(q)$ are nondegenerate, the Lemke complementary pivot algorithm traces a path in G_c^+ leading from a $T_i(\infty)$ (primary ray) to either a complementary tableau or to another $T_k(\infty)$, $i \neq k$ (secondary ray); see Ref. 3.

In Section 2, we study the structure of the subgraph G_c^+ when the nondegeneracy assumption does not hold. In Section 3, we consider cycling in the Lemke complementary pivoting algorithm and show, under a certain condition, that a modification of the transition node pivot rule introduced by Geue (Ref. 7, see also Ref. 8) to resolve ties in the selection of the pivot row helps in avoiding cycling in the complementary pivoting algorithm.

2. Graph G_c^+

Given an $n \times n$ matrix M and a vector $q \in R^n$, let B_u be either a feasible ACB or a feasible CB matrix, and let T_u be the associated tableau containing the columns of I, $(B_u)^{-1}\bar{B}_u$ and $y_0 = (B_u)^{-1} q \geq 0$, where for any ACB or CB matrix B, $\bar{B}$ represents the matrix of those columns of $(I, -M, -d)$ not contained in B. Let the entries of the tableau be denoted by $((y_{ij}, 1 \leq i \leq n, 0 \leq j \leq 2n+1))$. The column $y_0 = ((y_{i0}))$ presents $B_u^{-1} q$. Note that $y_{i0} \geq 0$, $\forall i$. Such a tableau is called a feasible tableau, and in what follows we shall consider feasible tableaux only. Without loss of generality, we shall assume that, for an ACB matrix B_u, its first column is $-d$. Thus, y_{10} corresponds to the variable z_0. The tableau also presents the indices $I(r)$, where $I(r)$ is the index of the rth column of B_u in $(I, -M, -d)$. Let $I^c(r)$ denote the index of the column in $(I, -M)$ which is complementary to the column whose index is $I(r)$ for $r > 1$. Let B_u be an ACB matrix. Then, it is clear that there is exactly one complementary pair of column vectors $(I_{.k}, -M_{.k})$ both of which are nonbasic. The Lemke algorithm chooses one of this pair (in fact, the complement of the one which has been eliminated from the basis in the previous iteration) as pivot column for its next iteration. We call this rule of choice the complementary rule. Let B_u and B_v be two ACB matrices. We say that B_u or its associated tableau T_u is adjacent to B_v (T_v) if the tableau T_v can be obtained from T_u by doing one single pivotal transformation of it, where the pivot column k is one of the unique complementary pair of nonbasic columns in T_u and the pivot row is any row i in T_u which is selected by the usual minimum ratio criterion to maintain feasibility.

Note that, if B_u (T_u) is adjacent to B_v (T_v), then B_v (T_v) is adjacent to B_u (T_u). We denote adjacency in this sense by writing $T_u \leftrightarrow T_v$.

Suppose that T_u is a tableau corresponding to an ACB matrix and T_v is a tableau corresponding to a CB matrix. We say that T_u is adjacent to T_v if T_v can be obtained from T_u by doing a single pivotal transformation of it where the pivot column k is chosen by the complementary rule and the pivot row satisfying the minimum ratio criterion is 1 (i.e., the row corresponding to the artificial variable z_0 in T_u which is assumed to be 1). Alternatively, T_u can be obtained from T_v by a single pivotal transformation in which the column corresponding to z_0 is the pivot column and the pivot row i that leads to the tableau T_u from the tableau T_v satisfies the minimum ratio criterion.

Two tableaux corresponding to two different CB matrices are not adjacent.

Note that, if $(B_u)^{-1}q > 0$, then B_u corresponds to a nondegenerate ACBFS x^u of $\bar{X}(q)$ uniquely. Otherwise, B_u corresponds to a degenerate ACBFS, and there may be more than one ACB matrix corresponding to it. In this case, the set of almost complementary tableaux corresponding to the degenerate ACBFS x^u is denoted by T^u and a tableau in this set is denoted by $(T_u)^v$ for some index v. The pivot row in a tableau corresponding to this ACBFS may not be determined uniquely by the minimum ratio criterion. A tableau corresponding to a degenerate ACBFS is called a degenerate tableau.

Let

$$(w^1, z^1, z_0^1) + \theta(w^*, z^*, z_0^*) \in \bar{X}(q), \qquad \forall \theta \geq 0,$$

be an extreme ray, where

$$w^* - Mz^* - dz_0^* = 0,$$

$$(w^*, z^*, z_0^*) \geq 0,$$

$$(w^*)^t(z^*) = (w^*)^t z^1 = (w^1)^t z^* = 0,$$

and (w^1, z^1, z_0^1) is an ACBFS to (3). Then, $(w^1, z^1, z_0^1) + \theta(w^*, z^*, z_0^*)$ is an almost complementary extreme ray in $\bar{X}(q)$. We represent such an extreme ray by the symbol $T_l(\infty)$. Note that, by the characterization theorem for extreme directions of a polyhedral set (see Bazaraa and Shetty, Ref. 9), there are only finitely many almost complementary extreme rays in $\bar{X}(q)$, and it is easy to see that, under the nondegeneracy assumption, each almost complementary extreme ray is incident on a unique ACBFS of $\bar{X}(q)$. However, if nondegeneracy is not assumed, the number of ACB matrices that corresponds to this ACBFS may be more than one. Thus, given $T_l(\infty)$ representing the almost complementary ray $(w^1, z^1, z_0^1) + \theta(w^*, z^*, z_0^*)$, let T_u be a tableau that corresponds to the ACBFS (w^1, z^1, z_0^1) on which the ray $T_l(\infty)$ is incident. We then say that $T_l(\infty)$ and T_u are adjacent. Note that, if the ACBFS (w^1, z^1, z_0^1) is degenerate, then there may be more than one tableau T_u adjacent to $T_l(\infty)$. Let there be m almost complementary extreme rays in $\bar{X}(q)$.

Definition 2.1. Given an $n \times n$ matrix M and a vector $q \in R^n$, let V^* be the set of all feasible almost complementary tableaux, and let $\bar{V}$ be the set of all feasible complementary tableaux associated with $\bar{X}(q)$. Let the nodes $T_i(\infty)$, $1 \leq i \leq m$, represent the almost complementary extreme rays. Let

$$N = \{T_u \mid T_u \in V^* \text{ or } T_u \in \bar{V}\} \cup \{T_i(\infty), 1 \leq i \leq m\},$$

$$E = \{\{T_u, T_v\} \subset N: T_u \leftrightarrow T_v\} \cup \{\{T_i(\infty), T_u\} \subset N \mid T_i(\infty) \leftrightarrow T_u\}.$$

We define the graph $G_c^1 = (N, E)$.

The following is a well known result; see Ref. 3.

Lemma 2.1. Suppose that none of the ACBFSs to (3) is degenerate. Then, the graph G_c^1 is a disjoint union of a finite number of paths or simple cycles. The endpoints of any path consist of one of the following three possible pairs:

(i) T_u and T_v, both complementary;
(ii) T_u and $T_i(\infty)$, where T_u is complementary;
(iii) $T_i(\infty)$ and $T_k(\infty)$, $i \neq k$.

The degree of any vertex is at most two.

In general, we may note the following theorem.

Lemma 2.2. In the graph G_c^1, there is no edge of the form $\{T_i(\infty), T_k(\infty)\}$.

Suppose that x^0 is a degenerate ACBFS in $\bar{X}(q)$. Let B^0 be the set of ACB matrices that corresponds to x^0, and let T^0 be the corresponding set of tableaux. Let us suppose that the set B^0 contains at least two bases. Let

$$N(x^0) = \{x^1, x^2, \ldots, x^r\}$$

be the set of ACBFSs or CBFSs which are the neighbors of x^0 in $\bar{X}(q)$. Let $x^p \in N(x^0)$, so that, if T_p is a tableau corresponding to x^p, then for some $(B_u)^0 \in B^0$, there is a complementary pivoting on a positive pivotal entry that transforms T_p to $(T_u)^0$. It is possible that x^p is also a degenerate ACBFS. Suppose that $x^p \in N(x^0)$ is nondegenerate. Then, if σ is the degree of degeneracy of x^0, there exists a pair of complementary columns $(k, k') \in T_p$ corresponding to the complementary pair (w_k, z_k) such that, when column l (where l is either k or k') is chosen as the pivot column, the cardinality of

$$\Theta = \{r \mid y_{r0}/y_{rl} = \min_{1 \leq i \leq n} [y_{i0}/y_{il} \mid y_{il} > 0]\}$$

is $\sigma+1$. Choosing a row index r from Θ and pivoting on the entry y_{rt}, one obtains a tableau $(T_u)^0$, $1\leq u\leq\sigma$, corresponding to the degenerate almost complementary vertex x^0. Let ${}^pT^0$ denote the set of these tableaux, and let ${}^pB^0$ be the corresponding set of bases. The nodes in the set $\cup\,{}^pT^0$, where the union is over the indices p such that $x^p\in N(x^0)$, are called the outer nodes of T^0 as each member of this set is adjacent to a ACBFS or a CBFS different from x^0, namely, some $x^p\in N(x^0)$. The other nodes $\in T^0$ are called the inner nodes; see Gal (Ref. 10). It is not necessary to assume that each ACBFS or CBFS in the set $N(x^0)$ is nondegenerate. A formal definition of an outer node in G_c^+ is as follows.

Definition 2.2. Let T^0 denote the set of almost complementary tableaux corresponding to a degenerate ACBFS x^0 of $\bar{X}(q)$. We say that a node $(T_u)^0\in T^0$ of G_c^+ is an outer node of G_c^+ if either there exists an ACBFS or CBFS x^p distinct from x^0 in $\bar{X}(q)$ and a tableau T such that $\{(T_u)^0, T\}\in E$, where either $T=(T_v)^p$ for some index v or $T=T_p$ or $T=T_i(\infty)$ for some i.

Lemma 2.3. Let $(T_1)^0, (T_2)^0\in{}^rT^0$. Then, $\{(T_1)^0, (T_2)^0\}\notin E$.

Proof. Let us define a subgraph $G_c^+(x^0)$ of G_c^+ by taking

$$G_c^+(x^0)=(T^0, E^0),$$

where

$$E^0=\{\{(T_u)^0, (T_v)^0\}\,|\,((T_u)^0, (T_v)^0)\in E\}.$$

Now, $G_c^+(x^0)$ is a subgraph of the graph $G^0=(N^*(x^0), E^*)$, where $N^*(x^0)$ is the set of all tableaux (not necessarily the almost complementary or complementary ones) that correspond to the degenerate vertex x^0 and E^* is the set of pairs of tableaux $\{T_u, T_v\}$ such that T_u can be obtained from T_v and vice versa by a single pivotal transformation on a positive entry in these tableaux, where the pivot column is not necessarily determined by the complementary pivot rule. This graph is similar to the one defined by Gal (Ref. 10). Note that $T^0\subset N^*(x^0)$ and that two nodes in $G_c^+(x^0)$ are adjacent only if they are adjacent in G^0. Now, the lemma follows from Lemma 2.1 of Ref. 10. □

Remark 2.1. It may be noted here that in general there may not be a path from a given node $(T_u)^0\in{}^pT^0$ to a node $(T_v)^0\in{}^rT^0$, $r\neq p$, as even in the nondegenerate case the graph G_c^+ may not be connected. See Lemma 2.1 above.

Lemma 2.4. In the graph G_c^+, let $\{T_u, T_v\}$ and $\{T_u, T_s\}$ be two edges. Let the unique nonbasic pair of complementary columns in T_u be $(I_{.k}, -M_{.k})$, and suppose that both the tableaux T_v and T_s are obtained from T_u by inserting the same column $A_{.k}$, where $A_{.k}$ is either $I_{.k}$ or $-M_{.k}$, into the basis. Then T_v and T_s are degenerate tableaux corresponding to the same degenerate ACBFS.

Proof. This is obvious. □

In the following lemma, we use the notion of a walk in a graph. A sequence of nodes of the form $T_1, T_2, T_3, \ldots, T_{n-1}, T_n$ is called a walk in G_c^+ if $\{T_r, T_{r+1}\}$ is an edge in G_c^+ for $r = 1, 2, \ldots, n-1$.

Lemma 2.5. Walks of the form

$$T_i(\infty) \leftrightarrow \cdots \leftrightarrow T_v \leftrightarrow (T_1)^0 \leftrightarrow \cdots \leftrightarrow T_s \leftrightarrow (T_1)^0,$$

which includes a cycle, where T_v, T_s, etc. are nondegenerate tableaux and the only degenerate tableau is $(T_1)^0 \in T^0$, are not contained in G_c^+.

Proof. Suppose there is such a walk. Let $(B_1)^0$ be the ACB corresponding to the tableau $(T_1)^0$. Let $\bar{q}$ be any vector such that $((B_1)^0)^{-1}\bar{q} > 0$. Consider the vector $q' = q + \epsilon\bar{q}$, where $\epsilon > 0$ is sufficiently small so that the minimum ratio which is positive and attained at a unique row in each of the nondegenerate tableaux other than T_v and T_s continues to be attained at the same row. In the tableaux T_v and T_s, although the minimum ratios are positive, they are attained at more than one row for the right-hand side vector q producing the degenerate tableau $(T_1)^0$. Since for small ϵ all of the tableaux T_v, T_s, $(T_1)^0$ are feasible for q', it follows that, in the graph G_c^+ corresponding to $\bar{X}(q')$, this walk containing the cycle is retained, which contradicts Lemma 2.1. This completes the proof. □

Remark 2.2. The implication of this lemma is that, if the sequence of tableaux generated by the Lemke complementary pivoting algorithm contains a cycle, the sequence must include at least two distinct degenerate tableaux corresponding to the same degenerate ACBFS.

Example 2.1. Let

$$M = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}, \qquad q = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}.$$

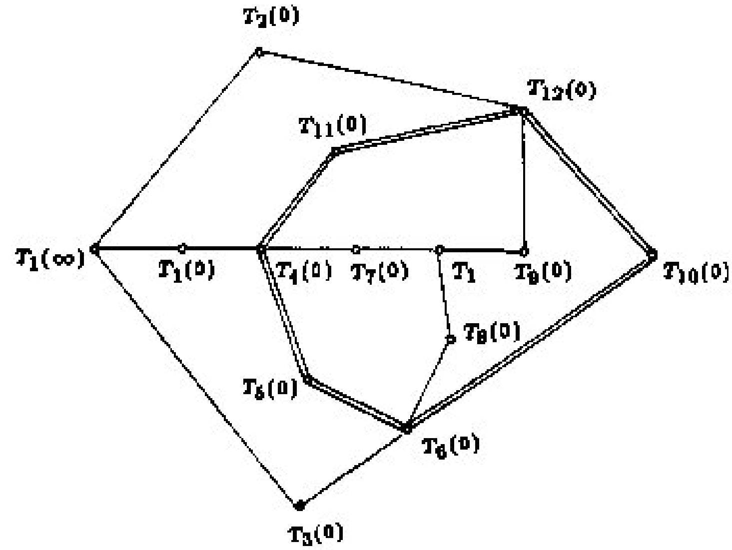

Fig. 1. Graph G_c^+ of Example 2.1.

This example is presented by Kostreva (Ref. 11) to demonstrate cycling in the Lemke complementary pivoting algorithm (Ref. 3). We now consider the graph G_c' for this example. We take the covering vector d to be the same as $-q$, i.e., the vector e each of whose coordinate is 1. Note that M is a P-matrix. There are $2^3 = 8$ CB matrices and $3 \times 2^2 = 12$ ACB matrices. Note that q is contained in each of the 12 nonnegative cones generated by these 12 ACB matrices, and all these correspond to the same ACBFS of $\bar{X}(q)$,

$$(w^0, z^0, z_0^0) = (0, 0, 0, 0, 0, 0, 1).$$

Also, q is contained in exactly one of the 8 complementary cones. Thus, the graph G_c^+ contains 14 nodes, the above 13 nodes and another one representing the primary ray. The graph is shown in Fig. 1. The edges shown in double lines form the cycle traced by the Lemke algorithm with a specific pivot selection rule, as will be pointed out in the next section.

In Fig. 1 the basic columns of the various tableaux are as follows:

(i) $T_1(0) = (-d, I_{.2}, I_{.3})$,
(ii) $T_2(0) = (-d, I_{.1}, I_{.2})$,
(iii) $T_3(0) = (-d, I_{.1}, I_{.3})$,
(iv) $T_4(0) = (-d, -M_{.1}, I_{.3})$,
(v) $T_5(0) = (-d, -M_{.2}, I_{.3})$,
(vi) $T_6(0) = (-d, -M_{.2}, I_{.1})$,

(vii) $T_7(0) = (-d, -M_{.1}, -M_{.2})$,
(viii) $T_8(0) = (-d, -M_{.2}, -M_{.3})$,
(ix) $T_9(0) = (-d, -M_{.1}, -M_{.3})$,
(x) $T_{10}(0) = (-d, I_{.1}, -M_{.3})$,
(xi) $T_{11}(0) = (-d, I_{.2}, -M_{.1})$,
(xii) $T_{12}(0) = (-d, I_{.2}, -M_{.3})$.

All these 12 tableaux correspond to the degenerate ACBFS (w^0, z^0, z_0^0). The tableau T_1 corresponds to the CBFS (w^*, z^*, z_0^*), where

$$w^* = 0, \qquad z^* = (1/3, 1/3, 1/3), \qquad z_0^* = 0.$$

The node $T_1(\infty)$ corresponds to the primary ray. Note that the tableaux $T_1(0)$, $T_2(0)$, $T_3(0)$ and the tableaux $T_7(0)$, $T_8(0)$, $T_9(0)$ are the outer nodes by our definition. The other nodes that correspond to (z_0^0, w^0, z^0) are the inner nodes.

3. Cycling and Anticycling Rules

The phenomenon of cycling in the Lemke complementary pivoting algorithm has been studied by Kostreva (Ref. 11). For anticycling rules in the context of this algorithm, see Murty (Ref. 2), Cottle, Pang, and Stone (Ref. 1), and the references cited in these. We first note the following difference between the phenomenon of cycling in the simplex algorithm for linear programming and that in the Lemke complementary pivoting algorithm.

The simplex algorithm may cycle among the bases corresponding to the same degenerate vertex. Once the algorithm moves away from a degenerate vertex x^0 to an adjacent nondegenerate vertex x^1, the bases corresponding to x^0 do not occur again. This, however, may not be true of cycles in the Lemke complementary pivoting algorithm as the following example quoted in Cottle, Pang, and Stone (Ref. 1) shows.

Example 3.1. Let

$$M = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad q = \begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix}, \qquad d = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

where d is the covering vector. The algorithm goes through the feasible almost complementary bases

$$B_1 = (-d, I_{.1}, I_{.2}),$$
$$B_2 = (-d, I_{.1}, -M_{.3}),$$

$$B_3 = (-d, -M_{.2}, -M_{.3}),$$

$$B_4 = (-d, -M_{.1}, -M_{.3}),$$

$$B_5 = (-d, I_{.2}, -M_{.3}),$$

from where it returns to B_2. The bases B_1 and B_5 correspond to the same degenerate ACBFS,

$$x^1 = (w^1, z^1, z_0^1) = (0, 1, 0, 0, 0, 0, 3),$$

and the bases B_2, B_3, B_4 correspond to distinct nondegenerate ACBFSs in $\bar{X}(q)$.

It is known that cycling in the Lemke complementary pivoting algorithm can be avoided by the use of the lexicographic rule for the selection of a pivot row; see Eaves (Ref. 12). Chang (Ref. 13) has studied the least index rule for the selection of the pivot row in the Lemke algorithm and has observed that, even when M is strictly copositive, this rule is not successful in avoiding cycling. The effectiveness of the least index rule has also been studied recently by Cottle and Chang (Ref. 14). Note that, when the nondegeneracy assumption does not hold, the graph G_c^+ may contain several cycles, and the algorithm requires a rule for the selection of the pivot element which enables it to trace a path in G_c^+ from a node of the form $T_i(\infty)$ to either a node which is a complementary tableau or a node of the form $T_k(\infty)$, $k \neq i$.

In general, suppose that we introduce a rule $\mathfrak{R}$ for the selection of the pivot row which ensures that the tableau generated by the algorithm has a particular property $\mathfrak{P}$. We then observe the following theorem.

Theorem 3.1. Suppose that there are only two edges in the graph G_c^+ incident on any tableau with property $\mathfrak{P}$, each one leading to a tableau with property $\mathfrak{P}$. If the algorithm with rule $\mathfrak{R}$ generates tableaux with property $\mathfrak{P}$ only, then the algorithm does not cycle.

Proof. This is clear from the observation that the subgraph of G_c^+ generated by the algorithm with rule $\mathfrak{R}$ is a graph in which each node has property $\mathfrak{P}$, and hence its degree is at most two. □

Let us now consider two rules which have been tried unsuccessfully. The rules are Rule $\mathfrak{R}_1$, which is to choose the pivot row as the one with the least row index from among the rows that tie for the minimum ratio, and Rule $\mathfrak{R}_2$, which is to choose the row corresponding to the variable which has the least index from among the rows that tie for the minimum ratio.

Rule $\Re_1$ applied to Example 2.1 traces the following cycle:

$$T_1(\infty),\ T_1(0),\ T_4(0),\ T_5(0),\ T_6(0),\ T_{10}(0),\ T_{12}(0),\ T_{11}(0),\ T_4(0).$$

Although at each of the nodes of the graph G_r^+ generated by the algorithm, the edge to traverse through is uniquely determined under this rule, the number of arcs incident on a node (for instance on T_4), either entering or exiting from it under this rule, is more than 2. This perhaps explains why cycling may not be avoided by the use of Rule $\Re_1$. The same difficulty is encountered in the use of Rule $\Re_2$ as well. This may be seen from Example 3.1.

Recently, Gal and Geue have proposed a pivot selection rule which they call transition node pivot rule for the simplex method. Consider first an almost complementary tableau T_u which is a node of G_c^+. Let t be the index of a column which is nonbasic in T_u. Suppose that $T_u=((y_{ij}))$.

Definition 3.1. We call a column in the tableau T_u a transition column if the usual minimum ratio in that column is positive or cannot be determined.

Remark 3.1. Note that the column whose index is t is a transition column of the tableau $T_u=((y_{ij}))$ if and only if

$$y_{i0}=0 \Rightarrow y_{it}\leq 0.$$

Remark 3.2. We shall consider a pivoting rule which produces tableaux all of which are required to have the column with a specified index as a transition column. However, such a rule may not resolve ties uniquely in the selection of a pivot row by the usual minimum ratio criterion. Therefore, it may be necessary to consider a rule that requires a given set of columns as transition columns. This motivates us to introduce the following notion of a transition set.

Definition 3.2. We say that a set of ordered columns with index set $\{t_1<t_2<\cdots<t_p\}$ has the transition property if $y_{i0}=0\Rightarrow y_{it_1}\leq 0$ and $y_{i0}=0,\ y_{it_s}=0,\ \forall 1\leq s\leq r \Rightarrow y_{it_{r+1}}\leq 0,\ 1\leq r<p$. We call an ordered set of indices of the columns with the transition property a transition set.

Let $Y_{.j}$, corresponding to either $I_{.l}$, or $-M_{.l}$, for some $1\leq l\leq n$, be the column chosen as the pivot column in T_u. Let

$$\theta^{(j)}=\min\{y_{i0}/y_{ij}\mid y_{ij}>0\}, \tag{4}$$

$$\Theta^{(j)}=\{i\mid y_{i0}/y_{ij}=\theta^{(j)}\}. \tag{5}$$

Suppose that the column with index t is a nonbasic column in T_u. Then, if we select the pivot row in T_u to be $r \in \Theta^{(j)}$ so that

$$y_{rt}/y_{rj} = \max\{y_{kt}/y_{kj} \mid k \in \Theta^{(j)}\}, \tag{6}$$

it is easy to see that column t is a transition column in the tableau T_v obtained from T_u by pivoting using the pivot element y_{rj}. The reason for the name transition column is that, if we now choose column t as a pivot column in T_v, then a transition occurs to a tableau which corresponds to a BFS of $\bar{X}(q)$ distinct from the ACBFS which corresponds to the tableau T_v. However, such a tableau may not correspond to a ACBFS or CBFS of $\bar{X}(q)$, and hence is not necessarily a tableau in the subgraph G_c^+. Also, the set $\{t_1, t_2, t_3, \ldots, t_r\}$ will be a transition set in the tableau T_u if we select the pivot row as follows. Let $\theta^{(j)}$ and $\Theta^{(j)}$ be as defined in (4) and (5). If $\Theta^{(j)} = \{s\}$ is a singleton set, then the pivot row is selected as s. Else, suppose that

$$\Theta^{(j)} = \{s_1, s_2, \ldots, s_r\}.$$

Let us define the following sets recursively. First, let

$$\theta_1^{(j)} = \max\{y_{kt_1}/y_{kj} \mid k \in \Theta^{(j)}\}, \tag{7}$$

$$\Theta_1^{(j)} = \{p \mid y_{pt_1}/y_{pj} = \theta_1^{(j)}, p \in \Theta^{(j)}\}. \tag{8}$$

In general, having defined $\Theta_{i-1}^{(j)}$, $2 \leq i < r$, if this is not a singleton set, we proceed to define $\theta_i^{(j)}$ and $\Theta_i^{(j)}$ as follows:

$$\theta_i^{(j)} = \max\{y_{kt_i}/y_{kj} \mid k \in \Theta_{i-1}^{(j)}\}, \tag{9}$$

$$\Theta_i^{(j)} = \{p \mid y_{pt_i}/y_{pj} = \theta_i^{(j)}, p \in \Theta_{i-1}^{(j)}\}. \tag{10}$$

We now state a transition node pivot rule for the Lemke algorithm.

Transition Node Pivot Rule (TNP Rule). Suppose that the Lemke algorithm generates a nondegenerate ACBFS x^1 and a corresponding tableau T_1, with column j as the pivot column in T_1 chosen by the complementary rule. Further, suppose that $\Theta^{(j)}$ is not a singleton set. Note that the algorithm generates a degenerate ACBFS x^0 of $\bar{X}(q)$ in the next iteration and a tableau $(T_1)^0$ from the set of tableaux T^0 that corresponds to x^0. Let α be an ordered set of column indices. The rule is:

> The pivot row in each iteration that generates a tableau from the set T^0 is chosen so that the set α remains a transition set in each of these tableaux.

This is done as described above. In particular, we consider the following set β'. Let

$$\Theta^{(j)} = \{k_1, k_2, \ldots, k_s\}.$$

Let

$$\beta' = (t_1, t_2, \ldots, t_s), \quad \text{where } t_i \in \{I^c(k_1), I^c(k_2), \ldots, I^c(k_s)\}, \tag{11}$$

be a fixed ordering of these column indices. We suggest the use of β' as a transition set.

Theorem 3.2. Suppose that the matrix of columns whose indices are in β' as defined in (11) above can be augmented by including $n-s$ more columns from $(I, -M)$, so that the resulting matrix is a basis matrix of the system (3). Consider the Lemke complementary pivoting algorithm with the TNP rule that ensures that the set β' remains a transition set in every degenerate tableau corresponding to the same degenerate ACBFS x^0. Then, the algorithm is finite.

Proof. In Ref. 14, the transition node pivot rule is shown to be a special case of the lexicographic pivot selection rule. For the transition set β' that we have suggested above, let us augment the matrix of columns whose indices are in β' by adding $n-s$ more columns from $(I, -M)$, so that the resulting matrix A is nonsingular and whose first s columns are the columns whose indices are in β'. Preserving lexicographic positivity of the augmented right-hand side $B^{-1}(q \mid (-A))$, where B is any ACB matrix generated from the set B^0 in the course of the algorithm, will ensure that the set of columns whose indices are in β' have the transition property at every iteration. See Ref. 14 for a proof. Now, the finiteness of the Lemke complementary pivoting algorithm with the TNP rule follows from the result of Eaves (Ref. 12). □

Remark 3.3. It may be noted that the almost complementary tableaux generated by the algorithm with the TNP rule are not necessarily the outer nodes of the graph G_c^+ in the sense in which we have defined an outer node. However, they are outer nodes in the larger representation graph of $\bar{X}(q)$ as defined in Ref. 10.

Remark 3.4. We note that both the examples of cycling given by Kostreva (Ref. 11) can be avoided by the use of the TNP rule. In Example 3.1, if we choose the column corresponding to z_3 (i.e., $-M_{.3}$) as the transition column, it is seen that the algorithm moves from the basis B_4 to the basis $B_6 = (d, I_{.2}, -M_{.1})$, at which step it terminates in a secondary ray. Thus, cycling is avoided.

Remark 3.5. The condition stated in Theorem 3.1 is easily seen to be satisfied when the matrix M is nondegenerate, i.e., when all the principal minors of M are nonzero.

References

1. COTTLE, R. W., PANG, J. S., and STONE, R. E., *The Linear Complementarity Problem*, Academic Press, New York, New York, 1992.
2. MURTY, K. G., *Linear Complementarity, Linear and Nonlinear Programming*, Heldermann Verlag, Berlin, Germany, 1988.
3. LEMKE, C. E., *Bimatrix Equilibrium Points and Mathematical Programming*, Management Science, Vol. 11, pp. 681–689, 1965.
4. MOHAN, S. R., *Degeneracy in Linear Complementarity Problems: A Survey*, Annals of Operations Research, Vol. 46, pp. 179–194, 1993.
5. GRÜNBAUM, S., *Convex Polytopes*, John Wiley, New York, New York, 1969.
6. GAL, T., KRUSE, H. J., and ZÖRNIG, P., *Survey of Solved and Open Problems in the Degeneracy Phenomenon*, Mathematical Programming, Vol. 42B, pp. 125–133, 1988.
7. GEUE, F., *Eine Neue Pivotauswahl-Regel und die Durch Sie Induzierten Teilgraphen des Positiven Entartungsgraphen*, Working Paper 141, Fern Universität, Hagen, Germany, 1989.
8. GAL, T., and GEUE, F., *A New Pivoting Rule for Solving Various Degeneracy Problems*, Operations Research Letters, Vol. 11, pp. 23–32, 1992.
9. BAZARAA, M. S., and SHETTY, C. M., *Nonlinear Programming: Theory and Algorithms*, John Wiley, New York, New York, 1979.
10. GAL, T., *On the Structure of the Set Bases of a Degenerate Point*, Journal of Optimization Theory and Applications, Vol. 45, pp. 577–589, 1980.
11. KOSTREVA, M., *Cycling in Linear Complementarity Problems*, Mathematical Programming, Vol. 16, pp. 127–130, 1979.
12. EAVES, B. C., *The Linear Complementarity Problem*, Management Science, Vol. 17, pp. 612–634, 1971.
13. CHANG, Y. Y., *Least Index Resolution of Degeneracy in Linear Complementarity Problems*, Technical Report 79-14, Department of Operations Research, Stanford University, 1979.
14. COTTLE, R. W., and CHANG, Y. Y., *Least Index Resolution of Degeneracy in Linear Complementarity Problems with Sufficient Matrices*, SIAM Journal on Matrix Analysis and Applications, Vol. 13, pp. 1131–1141, 1992.
15. GEUE, F., *An Improved N-Tree Algorithm for the Enumeration of All Neighbors of a Degenerate Vertex*, Annals of Operations Research, Vol. 47, pp. 361–391, 1993.