CONTRIBUTIONS TO THE THEORY OF DIRECTED AND UNDIRECTED GRAPHS

by

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CONTENTS

INTRODUCTION

Graph theory has become such a well known and widely applied subject with numerous applications in operations research, coding theory, game theory, physical and social sciences (to mention only a few), that it is not necessary to give a general introduction to it. Instead we give below a summary of the results contained in this thesis chapterwise.

This thesis contains five chapters which are, more or less, independent of each other. In Chapter 1, we study the existence of locally restricted graphs, that is graphs having a prescribed property with given degrees. In Section 1.1, we obtain necessary and sufficient conditions for the existence of a p-connected graph with given degrees for $p = 3$ and state two conjectures in the general case. The concept of k-factorable sequences in introduced in Section 1.2. A k-factor of a graph G is a partial graph of G in which every vertex has degree k. A sequence is called k-factorable (connected k-factorable) if there exists a graph with a k-factor (a graph with a connected k-factor) with the given degree sequence. We obtain a necessary and sufficient condition for a k-factorable sequence $(k \geq 2)$ to be connected k-factorable which turns out to be independent of k. Since a connected 2-factor is a Hamiltonian cycle, this condition is necessary and sufficient for a 2-factorable sequence to be the degree sequence of a Hamiltonian graph. We prove that every k-factorable sequence $(k \geq 2)$ is $(k-2)$-factorable, 2-factorable and 1-factorable. We further show that a 4-factorable sequence is 3-factorable provided n is even. If $\{d_i\}$ is a k-factorable sequence then there exists a graph G with degree sequence $\{d_i\}$ having a connected partial graph in which two vertices have degrees $k-1$ and the rest have degrees k. This proves, in particular, that every 2-factorable sequence is realisable by a graph with a Hamiltonian **chain**. Although, the general problem of characterising the k-factorable sequences is left unanswered we present two conjectures, one of which for $k = 2$ was mentioned by Prof. B. Grünbaum at the Combinatorics Conference, Calgary 1969 and prove that the truth of conjecture 1 implies the truth of conjecture 2 by proving that if $\{d_i\}$ and $\{d_i - k\}$ are graphical so is $\{d_i - r\}$ provided rn is even and

$0 \leq r < k$. In Section 1.3, we solve the following problem posed by A. M. Hobbs in the book entitled, 'Recent Progress in Combinatorics' edited by W. T. Tutte [24]. For what values of n is there a planar graph on n vertices without loops or multiple edges which has 12 vertices of degree 5 and n-12 vertices of degree 6? Some other related problems are also solved.

Chapter 2 is strongly connected with the work of Ramachandra Rao [19] who determined, among other things, the ranges of the number of cut vertices and cut edges in an undirected graph on n vertices with m edges. In Sections 2.1, 2.3, we solve the corresponding problems for strongly connected digraphs and characterise some extremal graphs. This generalises some of the results of Gupta [6]. In Section 2.2, we prove that every strong complete graph on $n > 3$ vertices has at most n-2 cut vertices, without using Camion's theorem, which solves a problem posed by Korwin [12] in Rome Conference. In Sections 2.4, 2.5, we find the maximum number of edges in an undirected graph on n vertices with r cut vertices (s cut edges) in which minimum degree $\geq d$ and give partial solutions to the problems of the determination of the maximum number of cut vertices (cut edges) in an undirected graph on n vertices with m edges in which minimum degree $\geq$ d, considered by Ramachandra Rao [19] and solved in [19] for all d $\leq$ 4.

In Chapter 3, we consider the k-th power of a graph, G, denoted by G^k, defined by Ross, Harary, Karp, Tutte in [23], [10], as a graph with same set of vertices as G and two vertices are joined in G^k if and only if there is a chain of length $\leq$ k joining them in G. In Sections 3.1, 3.2, we find necessary and sufficient conditions for a graph to be the cube of tree and prove that T^3 determines the tree T uniquely, up to isomorphism and give an algorithm to construct the tree cube root of a graph (if it exists). In Section 3.3, some necessary conditions for a graph to be the (2k)-th power of a tree are obtained which generalise some of the results of Ross and Harary [23]. Using these, in Section 3.4, we obtain a criterion for a graph to be the fourth power of a tree. In general, the tree fourth root of a graph is not unique. We give an algorithm to construct all the tree fourth roots of a graph and characterise graphs with a unique tree fourth root. The algorithms aforementioned utilise a result of Harary and Ross [8] for determining the set of all cliques in a graph.

In Chapter 4, we solve a problem posed by Ore [15]. A graph G is said to have the property P_0 (after Ore who posed the problem) if for every maximal tree T (i.e. spanning tree) of G there exists a vertex $a_T \in V(G)$ such that $d_T(a_T, x) = d_G(a_T, x)$ for every $x \in V(G)$, where $d_G(x_1, x_2)$ denotes the distance in G between x_1 and x_2. We show that, in fact, there are only two classes of biconnected graphs having the property P_0. Further, we determine the structure of all finite connected graphs having the property P_0.

The last chapter is concerned with some extremal problems concerning radius and diameter in digraphs. In Section 5.1, we find the maximum number of arcs in a digraph (not necessarily strong) on n vertices with radius r and characterise all extremal graphs. The analogous results in the undirected case were obtained by Vizing [25]. Further, we obtain an expression for the maximum number of arcs in a strong digraph on n vertices with radius r for $r \leq 3$ and state a conjecture in the general case. In Section 5.2, we extend the results of Ore [16] to digraphs. A digraph is called diameter critical if the addition of a new arc decreases its diameter. We characterise all k-connected diameter critical digraphs, determine the maximum number of arcs in a k-connected digraph on n vertices with diameter d and characterise all extremal graphs. Finally, in Section 5.3, we give a partial solution to the following problem of Murty [14]. For what values of n is it possible to orient the complete graph on n vertices in such a way that the resulting tournament has diameter ≤ 2 and the tournament obtained by removing s or fewer vertices has also diameter ≤ 2, where s is a fixed non-negative integer.?

CHAPTER 1

LOCALLY RESTRICTED GRAPHS

Throughout this thesis we consider only finite graphs (undirected or directed) without loops or multiple edges (arcs). If G is a graph, then V(G), E(G) denote the vertex set, edge set of G. For notation and terminology Berge [2] is generally followed.

The degree sequence of a graph G is the (finite) sequence of degrees of the vertices of G. A sequence of n non-negative integers is said to be graphical if it is the degree sequence of some graph G.

Call a graph p - connected (p-coherent) if it is connected and the removal of any p-1 or fewer vertices (edges) does not disconnect the graph. In Section 1, we obtain necessary and sufficient conditions for a sequence of positive integers to be the degree sequence of a 3-connected graph.

A k-factor of a graph G is a partial graph of G in which every vertex has degree k. A graphical sequence is called k-factorable (connected k-factorable) if there exists a graph with a k-factor (connected k-factor) and with the given degree sequence. In Section 2, we give a necessary and sufficient condition for a k-factorable sequence to be connected k-factorable. We also prove that every k-factorable sequence is (k-2) - factorable, 2-factorable and 1-factorable, we further show that every 4-factorable sequence is 3-factorable. If $\{d_i\}$ is a k-factorable sequence $(k \geq 2)$, then there exists a graph with degree sequence $\{d_i\}$ and having a connected partial graph in which two vertices are of degree k-1 and the rest have degrees k. Further some conjectures are presented and it is also proved that if $\{d_i\}$ and $\{d_i - k\}$ are graphical then so is $\{d_i - r\}$ for $0 \leq r \leq k$.

In the book entitled, 'Recent Progress in Combinatorics' edited by W. T. Tutte [24] the following problem was posed by A. M. Hobbs. For what values of n is there a planar graph without loops or multiple edges with 12 vertices of degree 5 and n-12 vertices of degree 6? In Section 3 we solve this and some other related problems.

1.1 Existence of triconnected graphs with prescribed degrees

Necessary and sufficient conditions for the existence of a p-coherent graph with prescribed degrees were obtained by Edmonds [4]. Necessary and sufficient conditions for the existence of a p-connected graph with prescribed degrees are known for $p = 1, 2$ (see [7] and [19]). In this section we solve this problem for $p = 3$. Further some conjectures in the general case are presented. We start with a simple result.

Let $\{d_i\} = \{d_1, d_2, \ldots, d_n\}$ where each d_i is a positive integer and let $d_1 \leq d_2 \leq \ldots \leq d_n$.

<u>Lemma 1.1.1.</u> If a triconnected graph exists with degree sequence $\{d_i\}$, then

(1) $d_i \geq 3$.

(2) $\{d_i\}$ is a graphical sequence.

(3) $d_n + d_{n-1} \leq m - n + 4$ where $2m = \sum_{i=1}^{n} d_i$.

<u>Proof.</u> (1) and (2) are evident. To prove (3), let x_n, x_{n-1} be the vertices of G with degrees d_n and d_{n-1} respectively. Then the number of edges in $G - \{x_n, x_{n-1}\}$ is $m - (d_n + d_{n-1} - 1)$ or $m - (d_n + d_{n-1})$ according as x_n, x_{n-1}

are adjacent or not adjacent in G. Also, since G is triconnected, $G - \{x_n, x_{n-1}\}$ is connected, so (3) follows. This completes the proof of the lemma.

Theorem 1.1.2. Conditions (1) to (3) of Lemma 1.1.1 are necessary and sufficient for the existence of a triconnected graph with degree sequence $\{d_i\}$.

Proof. Necessity was proved in Lemma 1.1.1. To prove sufficiency, first let the conditions (1) and (3) be satisfied and let $d_n + d_{n-1} = m - n + 4 = n + \lambda$. Since

$2m = \sum_{i=1}^{n} d_i \geq d_n + d_{n-1} + 3(n-2) = m + 2n - 2$, and

$d_i \leq n-1$ it follows that $2 \leq \lambda \leq n-2$. Let k be the number of d_i's such that $1 \leq i \leq n-2$ and $d_i = 3$. Define

$$e_i = d_i - 2 \quad \text{for} \quad k+1 \leq i \leq n-2.$$

Then we have

$$\sum_{i=1}^{n-2} d_i = 2m - d_n - d_{n-1} = 3n + \lambda - 8.$$

$$\sum_{i=k+1}^{n-2} e_i = 3n + \lambda - 8 - 3k - 2(n-2-k)$$
$$= n + \lambda - k - 4.$$

Define now $\eta = n - 2 - \lambda$ and $\epsilon = k - \eta$. Then $\eta \geq 0$, and $\epsilon \geq 2$ since

$$2m \geq m - n + 4 + 3k + 4\,(n - 2 - k)$$
$$= m + 3n - k - 4$$

and so $\lambda = m - 2n + 4 \geq n - k$.

Write now

$$c_i = \begin{cases} 1 & \text{for } 1 \leq i \leq \epsilon \\ 2 & \text{for } \epsilon + 1 \leq i \leq k \\ d_i - 2 & \text{for } k+1 \leq i \leq n-2 \; . \end{cases}$$

Then $\sum_{i=1}^{n-2} c_i = 2(n-3)$ and so there exists a tree T with degrees $c_1, c_2, \ldots, c_{n-2}$, attained by the vertices $x_1, x_2, \ldots x_{n-2}$, (say), in that order [19]. Take two more vertices x_{n-1}, x_n and join them. Also join each of x_{n-1}, x_n to x_i for $i = 1, \ldots, \epsilon, k+1, \ldots, n-2$. Of the η vertices $x_{\epsilon+1}, \ldots, x_k$, join $d_{n-1} - 1 - \epsilon - n + 2 + k$ to x_{n-1} and the rest ($d_n - 1 - \epsilon - n + 2 + k$ in number) to x_n. Note that

$$d_{n-1} - 1 - \epsilon - n + 2 + k = d_{n-1} - \lambda - 1 \geq 0.$$

The graph thus obtained has degree sequence $\{ d_i \}$ and is triconnected since any vertex of T with degree in T less than three is joined to either x_{n-1} or x_n.

Next let conditions (1) and (2) be satisfied and let $d_n + d_{n-1} \leq m - n + 3$. Then $d_n < m - n + 2$, so there exists a biconnected graph G with degree sequence $\{d_i\}$, (see [7], [19]). If G is not triconnected, let x_i, x_j be two vertices such that $G - \{x_i, x_j\}$ is disconnected. Let $C_1, C_2, \ldots,$ be the connected components of $G - \{x_i, x_j\}$. By (1), $|C_g| \geq 2$ for $g = 1,2,\ldots$. Also by hypothesis, $m - d_i - d_j \geq n - 3$, so it follows that one of the components, say C_1, contains a cycle.

We first prove that there exists an edge (x,y) in C_1 and two chains μ_1, μ_1' of G connecting x and y such that (x,y), μ_1, μ_1' are disjoint except for x and y, and μ_1 is contained in C_1. Observe that, since G is biconnected, there always exists a chain connecting x_i and x_j with all intermediate vertices in C_2. If now two vertices x, y with degree 2 in C_1 are adjacent and belong to a cycle of C_1, the required edge is (x, y). So we may assume that no two vertices of degree two in C_1 can belong to a block (on more than two vertices) and be adjacent. Let B be any block of C_1 which is not an edge. If some cycle of B has a chord (x,y), then (x,y) is the required edge. Otherwise, by the results of [17], two vertices y, z of degree two in B

will be adjacent to a vertex x of degree three in B. If w is another vertex of B adjacent to x, then there is a chain connecting w to y in $B - \{x\}$. This chain together with (x, w) may be taken as μ_1. To get μ_1', go from x to z along (x, z), from z to x_i or x_j (through another block of C_1 at z if necessary) then to y (through another block of C_1 at y if necessary). Thus (x, y) is the required edge.

Let now (x, y) be an edge of C_1 chosen as explained above. If C_2 is a tree, take any edge (u, v) of C_2. Then (u, v) is a chord of a cycle of G. If C_2 is not a tree, choose an edge (u, v) of C_2 such that there are chains μ_2, μ_2' of G connecting u and v, such that (u, v), μ_2, μ_2' are disjoint except for u, v and μ_2 is contained in C_2.

We define $f_G(s, t)$ to be the number of components of $G - \{s, t\}$. Now we will make a modification on G so that the degrees of the vertices are unaltered, $f(x_i, x_j)$ decreases and $f(s, t)$ does not increase for any two vertices s and t.

First we associate with x, a subset $A(x)$ of $\{x_i, x_j\}$ by the following rule: $x_i \in A(x)$ if and only if there is a chain η connecting x to x_i with all intermediate vertices in C_1 such that η is disjoint with (x,y) and μ_1 except for x. Similarly $A(y)$ is defined. If C_2 is a tree, put $A(u) = A(v) = \{x_i, x_j\}$. Otherwise $A(u)$, $A(v)$ are defined in a manner similar to that of $A(x)$ and $A(y)$. Now $A(x)$, $A(y)$ are made nonempty by a proper choice of μ_1, and $A(u)$, $A(v)$ are made nonempty by a proper choice of μ_2 (in case C_2 is not a tree).

Now suppress the edges (x,y), (u, v) and join x to one of u, v and y to the other as follows. Join x to u if $A(x) \neq A(u)$ and $A(y) \neq A(v)$ whenever such a choice is possible. Let the new graph thus obtained be H. To be specific we take that x is joined to u in H.

We prove that H is biconnected. Obviously $G_1 = G - (x,y)$ is biconnected. Now we show that (u, v) is a chord of a cycle of $H + (u, v)$. If C_2 is a tree, then the cycle is

$$(u, x) + \mu_1[x, y] + (y, v) + [v, \ldots, p_1] + (p_1, x_i) + (x_i, p_2) + + [p_2, \ldots, u],$$

where p_1, p_2 are suitable pendant vertices of C_2. Otherwise the cycle is $\mu_2[u, v] + \mu_2'[v, u]$ where if μ_2' contains the edge (x, y), then (x, y) is replaced by $\mu_1[x, y]$ and the resulting cycle is made elementary. Thus H is biconnected.

Trivially now $f_G(x_i, x_j) = f_H(x_i, x_j) + 1$. Next we will show that

$$f_G(s, t) \geq f_H(s, t) \tag{4}$$

for any two vertices s and t. For this it is enough to show that x, y are connected and u, v are connected in $H - \{s, t\}$.

First let $s = x_i$. Now x, y, u, v belong to a cycle in $H - \{x_i\}$, so (4) follows. So we may assume that $\{s, t\} \cap \{x_i, x_j\} = \emptyset$.

Now let $s = x$. Then to prove (4) it is enough to show that u, v are connected in $H - \{x, t\}$ when $t \neq u$ and $t \neq v$. This is evident if C_2 is a tree or $t \notin \mu_2$. So let $t \in \mu_2$ and C_2 be not a tree. If $A(u) \cap A(v) \neq \emptyset$, there is a chain connecting u, v in $H - \{x, t\}$. So we may take without loss of generality and write without brackets that $A(u) = x_j$ and $A(v) = x_i$. If now $x_j \in A(y)$, then u, v are connected through x_j and y in $H - \{x, t\}$. So we take $A(y) = x_i$. If $x_j \in A(x)$, then y would not have been joined to v, so $A(x) = x_i$. Now in G, x_j is connected to some vertex z of μ_1 by a chain with all intermediate vertices belonging to C_1 but no vertex to μ_1. Now we obtain a chain connecting u, v in $H - \{x, t\}$ by going from u to x_j, x_j to z, z to y along μ_1, y to v. Thus we may take that $\{s, t\} \cap \{x_i, x_j, x, y\} = \emptyset$.

Next let $s = u$. If $t \notin \mu_1$, then (4) is trivial, so let $t \in \mu_1$. Suppose first that C_2 is a tree. Then we obtain a chain connecting x, y in $H - \{u, t\}$ by going from x to x_i or x_j, then to v through a suitable pendant vertex of C_2 and then to y. If C_2 is not a tree, the situation is similar to that of the preceding paragraph. Thus we take $\{s, t\} \cap \{x_i, x_j, x, y, u, v\} = \emptyset$.

If none of s, t belongs to μ_1, then (4) is trivial. So let $s \in \mu_1$. Suppose now that C_2 is a tree. Then for any fixed t, there are chains in $H - \{s, t\}$ from one of u, v to both x_i and x_j, and a chain from the other (of the vertices u, v) to x_i or x_j. Hence u, v are connected in $H - \{s, t\}$ and (4) follows. Suppose next that C_2 is not a tree. Obviously we may take that $s \in \mu_1$ and $t \in \mu_2$. If now $A(x) \cap A(y) \neq \emptyset$ or $A(u) \cap A(v) \neq \emptyset$, then again (4) follows. So we may take that $A(x) = x_i$, $A(y) = x_j$, $A(u) = x_j$ and $A(v) = x_i$. Now we obtain a chain connecting x, y in $H - \{s, t\}$ by going from x to u, u to x_j, x_j to y. This proves (4) completely.

Now by a repeated application of the above procedure we reduce the graph until finally $f(s,t) = 1$ for any two vertices. The final graph thus obtained has degree sequence $\{d_i\}$ and is triconnected and this completes the proof of the theorem.

The results of this section appeared in [21].

The method used above does not seem to work in the general case. However, we take the risk of conjecturing the following.

Conjecture 1. If $\{d_i\}$ is a graphical sequence with $d_i \geq p+1$ for every i, $1 \leq i \leq n$, then there exists a p-connected graph with degree sequence $\{d_i\}$.

Conjecture 2. Let $d_1 \leq d_2 \leq \ldots \leq d_n$. Then there exists a p-connected graph $(p \geq 2)$ with degree sequence $\{d_i\}$ if and only if

i) $d_i \geq p$, $1 \leq i \leq n$.

ii) $\{d_i\}$ is a graphical sequence.

iii) $\sum_{j=0}^{p-2} d_{n-j} \leq \bar{m} - n + \binom{p}{2} + 1$.

It can be easily seen that the truth of conjecture 2 implies the truth of conjecture 1. Further if $p = 2$ or 3 then conjecture 2 is true by previously known result and our Theorem 1.1.2.

1.2 On factorable degree sequences

A k-factor of a graph G is a partial subgraph of G in which every vertex has degree k. A graphical sequence is called k-factorable (connected k-factorable) if there exists a graph with a k-factor (connected k-factor) and with the given degree sequence.

If C is a connected graph, denote by $A(x, C)$ (respectively $B(x, C)$) the set of all vertices at even (respectively odd) distance in C from x. If C_1 is a connected partial subgraph of G and C_2 is a partial subgraph of G with $V(C_1) \cap V(C_2) = \emptyset$, then we write $C_1 \rightarrow C_2$ if there exists $x \in V(C_1)$ such that the subgraph of G spanned by $A(x, C_1)$ is complete, every vertex of $A(x, C_1)$ is joined in G to all vertices of C_2, $B(x, C_1)$ is an independent set in G and no vertex of $B(x, C_1)$ is joined in G to any vertex of C_2.

A cut A of G is a nonempty proper subset of V(G). m(A, B) denotes the number of edges of G with one end vertex in A and the other in B where $A \cap B = \emptyset$. The value of a cut A of G is equal to m(A, V(G) - A). A connected graph is p-coherent if and only if there is no cut with value $\leq$ p-1.

A vertex x (edge u) of a graph G is called a cut vertex (cut edge) if the graph G - x (G - u) has more components than G.

We start our investigation on k-factorable degree sequences by the following

Lemma 1.2.1. Let G be a graph with a k-factor F consisting of two components C_1 and C_2, let $k \geq 2$ and C_1, C_2 be bicoherent. If the degree sequence of G is not connected k-factorable then either $C_1 \to C_2$ or $C_2 \to C_1$.

Proof. Let G be a graph satisfying the hypothesis of the lemma. If (x, y) and (u, v) are edges of C_1 and C_2 respectively then, one of the vertices x, y, u, v is joined to the remaining three vertices in G for otherwise, by a simple interchange of edges C_1 and C_2 can be combined into a single component. So let without loss of generality x be adjacent to u and v. If now y is joined to u or v, again C_1 and C_2 can be combined, so y is joined to neither u nor v. If z is any vertex adjacent to x in C_1 then z is joined to neither u nor v. Proceeding further we get that every vertex of $A(x,C_1)$ is joined to u, v and no vertex of $B(x,C_1)$ is joined to u or v. If w is any vertex adjacent to u in C_2 then by the same argument, every vertex of $A(x,C_1)$ is joined to w and no vertex of $B(x,C_1)$ is joined to w. Proceeding further we finally get that every vertex of $A(x,C_1)$ is joined to all vertices of C_2 and no vertex of $B(x,C_1)$ is joined to any vertex of C_2. Also $A(x,C_1)$ and $B(x,C_1)$ are independent sets in C_1. Since C_1 is regular of degree $k(\geq 2)$ it follows that $|A(x,C_1)| = |B(x,C_1)|$. Thus $|V(C_1)|$ is even.

If now two vertices z_1 and z_2 of $A(x,C_1)$ are not adjacent in G, then let z_3 be a vertex adjacent in C_1 to z_1. Then $z_3 \in B(x,C_1)$. Now choose an edge (u,v) of C_2, then remove the edges (z_1,z_3) and (z_2,u) from G and add the new edges (z_1,z_2) and (z_3,u). Then we get a graph with a connected k-factor obtained from the original k-factor by deleting the edges (z_1,z_3), (u,v) and adding the edges (z_1,v) and (z_3,u). This contradiction shows that the subgraph of G spanned by $A(x,C_1)$ is complete.

If two vertices z_1, z_2 of $B(x,C_1)$ are adjacent in G, then let z_3 be a vertex adjacent in C_1 to z_1, let

z_4, z_5 be vertices adjacent in C_1 to z_2, and let (u, v) be an edge of C_2. Consider the k-factor F' of G obtained from the original k-factor F by deleting the edges (z_1, z_3), (z_2, z_4), (u, v) and adding the edges (z_3, u), (z_4, u), (z_1, z_2). By hypothesis, F' is not connected. Hence $C_1' = C_1 + (z_1, z_2) - (z_1, z_3) - (z_2, z_4)$ is also not connected and since C_1 is bicoherent, C_1' has exactly two components A_1, A_2. Also z_1, z_2 belong to A_1 (say) and then z_3, z_4 belong to A_2 (because C_1' is not connected). Now evidently the k-factor F'' of G obtained from F by deleting the edges (z_1, z_3), (z_2, z_5), (u, v) and adding the edges (z_3, u), (z_5, v), (z_1, z_2) is connected, a contradiction which shows that $B(x, C_1)$ is an independent set in G. Thus $C_1 \rightarrow C_2$ and the lemma is proved.

Lemma 1.2.2. Let $k \geq 2$, $\{d_i\}$ be k-factorable and p be the minimum number of components in a k-factor of a graph with degree sequence $\{d_i\}$. Then a graph with the degree sequence $\{d_i\}$ can be chosen such that it has a k-factor with p components each of which is bicoherent.

Proof. If k is even the lemma is trivial. So let k be odd and let G be a graph with degree sequence $\{d_i\}$ and with a k-factor consisting of p components $C_1, C_2, \ldots, C_p$. If C_i is not bicoherent we will show that the subgraph of

G spanned by $V(C_i)$ can be modified so that the degrees are unaltered and the new subgraph has a bicoherent k-factor. Let D_1, D_2 be two terminal blocks of C_i separated by some cut edge of C_i and let a_1, a_2 be the cut vertices of C_i belonging to D_1, D_2 respectively. Now let (x, y), (u, v) be edges in D_1, D_2 respectively where x, y are different from a_1 and u, v are different from a_2. Then one of x, y, u, v is joined in G to the remaining three vertices for otherwise a simple interchange reduces the number of cut edges of C_i. Let without loss of generality x be adjacent in G to u, v. Then as in the proof of Lemma 1.2.1, we can show that if the number of cut edges in C_i can not be reduced then $A(x, D_1)$, $B(x, D_1)$ are independent sets in D_1. Thus D_1 is bipartite, in D_1 all vertices except a_1 have degrees k and the degree of a_1 is $\leq k-1$, a contradiction which proves that the subgraph can be modified until C_i becomes bicoherent. This completes the proof of the lemma.

It may be remarked that Lemma 1.2.2 is the best possible with respect to the cohersiveness of a k-factor of a graph with degree sequence $\{ d_i \}$. To show this we give an example. Let n be an integer $\geq 3k+1$ and n even if k is odd. Let $V = \{ 1, 2, \ldots, n \}$, $A = \{ 1, 2, \ldots, k \}$, $B = \{ k+1, \ldots, 2k \}$, $C = \{ 2k+1, \ldots, n \}$. Define a graph

G with $V(G) = V$ in which $A \cup C$ is complete, B is an independent set, every vertex i, $2 \leq i \leq k$, of A is joined to all vertices of B, 1 is joined to all j, $k+2 \leq j \leq 2k$; and $k+1$ is joined to $2k+1$. It is easy to see that if H is any graph with the degree sequence $\{d_i\}$, same as the degree sequence of G, then H is isomorphic to G. Further $\{d_i\}$ is connected k-factorable and in any k-factor of G, the cut C has value equal to 2. Thus no k-factor of the graph G is h-coherent with $h > 2$.

In what follows we assume that $\{d_i\} = \{d_1, d_2 \ldots, d_n\}$ and $d_1 \geq d_2 \geq \ldots \geq d_n$.

Theorem 1.2.3. $\{d_i\}$ is connected k-factorable if and only if $\{d_i\}$ is k-factorable and the following condition is satisfied whenever $s < \frac{n}{2}$,

$$\sum_{i=1}^{s} d_i < s(n-s-1) + \sum_{j=0}^{s-1} d_{n-j} \qquad (6)$$

Proof. If $\{d_i\}$ is graphical, let G be any graph with degree sequence $\{d_i\}$. Let A be a set of s vertices of G with degrees $d_1, d_2, \ldots, d_s$ and let B be a set of s vertices disjoint with A and with degrees $d_{n-s+1}, \ldots, d_n$. Now observe that the right hand expression of (6) is

$$|A|.|V(G) - A - B| + |A|.(|A| - 1) + \sum_{j=0}^{s-1} d_{n-j},$$

hence the left hand expression of (6) does not exceed the right hand expression. If now equality holds for some $s < \frac{n}{2}$, then every edge with one end vertex in B has the other end vertex in A and since $|A| = |B| < \frac{n}{2}$ it follows that G does not have a connected k-factor. This proves the necessity of (6).

To prove sufficiency, let $\{d_i\}$ be k-factorable and let (6) be satisfied whenever $s < \frac{n}{2}$. Let G be a graph with degree sequence $\{d_i\}$ and with a k-factor having the minimum number of components. Let $C_1, C_2, \ldots, C_p$ be the components in this k-factor of G. By Lemma 1.2.2, we may and do assure that each C_i is bicoherent. We will prove the theorem by showing that $p \geq 2$ leads to a contradiction. So let $p \geq 2$.

Construct a directed graph D with $C_1, C_2, \ldots, C_p$ as its vertices, an arc going from C_i to C_j if $C_i \rightarrow C_j$ in G. By the definition of p and from the proof of Lemma 1.2.1, it follows that D is a complete directed graph. Hence the radius of $D \leq 2$ (see p.121, [2]). Thus either there is a circuit on 3 vertices in D or there is a C_i

such that $C_i \to C_j$ whenever $j \neq i$. We consider these two cases separately.

Case (i). $C_1 \to C_2 \to C_3 \to C_1$. Then edges (u, v), (w, x), (y, z) can be found in C_1, C_2, C_3 respectively such that (v, w), (x, y), (z, u) are edges in G. Thus the components C_1, C_2, C_3 can be combined into a single component, a contradiction to the definition of p.

Case (ii). $C_1 \to C_i$ for all $i = 2, \ldots, p$. First let a vertex x of C_1 be adjacent to all vertices of C_i and to no vertex of C_j, for some i and j. Now either $C_i \to C_j$ or $C_j \to C_i$ and we can combine C_1, C_i, C_j into a single component as in case (i). Next let a vertex x of C_1 be adjacent to all vertices of C_i, $i = 2, \ldots, p$. Then if we write $A = A(x, C_1)$, $B = B(x, C_1)$ and $C = V(G) - A - B$, we get from the proof of Lemma 1.2.1 that $|A| = |B| < \frac{n}{2}$, the subgraph spanned by A is complete, every vertex of A is joined in G to all vertices of C, B is an independent set in G and no vertex of [illegible] is joined in G to any vertex of C. But this gives a contradiction to condition (6) and the theorem is proved.

It may be pointed out that condition (6) above is necessary and sufficient for a 2-factorable degree sequence to be realisable as the degree sequence of a Hamiltonian graph.

<u>Theorem 1.2.4.</u> If $\{d_i\}$ is k-factorable with $k \geq 2$, then $\{d_i\}$ is (k-2)-factorable.

<u>Proof.</u> If k is even the theorem follows from the fact that any regular graph of even degree has a 2-factor (see p.189, [2]). If k is odd, let p be the minimum number of components in a k-factor of a graph with degree sequence $\{d_i\}$. By Lemma 1.2.2, there exists a graph G with degree sequence $\{d_i\}$ and with a k-factor consisting of p components $C_1, C_2, \ldots, C_p$, each C_i being bicoherent. By a result of Baebler (see p. 189, [2]), each C_i has a 2-factor. Hence the k-factor of G has a 2-factor and therefore a (k-2)-factor. This proves the theorem.

The argument in the proof of Theorem 1.2.4 proves the following

<u>Corollary.</u> If $\{d_i\}$ is k-factorable with $k \geq 2$, then $\{d_i\}$ is 2-factorable.

<u>Theorem 1.2.5.</u> If $\{d_i\}$ is k-factorable, then $\{d_i\}$ is 1-factorable provided n is even.

<u>Proof.</u> By the corollary to Theorem 1.2.4 we may take $k = 2$. Let then G be a graph with degree sequence $\{d_i\}$ and with a 2-factor having the minimum number of components.

Let $C_1, C_2, \ldots, C_p$ be the cycles in this 2-factor. Now by Lemma 1.2.1, either $C_i \to C_j$ and C_i is an even cycle or $C_j \to C_i$ and C_j is an even cycle. Hence all C_i except possibly one are even and since n is even each C_i is even. Thus the 2-factor has a 1-factor and the theorem is proved.

Theorem 1.2.6. If $\{ d_i \}$ is 4-factorable then $\{ d_i \}$ is 3-factorable provided n is even.

Proof. Let p be the minimum number of components in a 4-factor of a graph G with degree sequence $\{ d_i \}$. Let $C_1, C_2, \ldots, C_p$ be the components in this 4-factor of G. As in the proof of Lemma 1.2.1 (see also Theorem 1.2.5) it can be proved that each C_i is even. Replace the subgraph of G spanned by $V(C_i)$ by a graph with the same degree sequence and having a bicoherent 4-factor with minimum number, ∂_i (say), of cuts of value 2. Now if $\partial_i = 0$ for some i, then C_i is 3-coherent so this C_i has a 1-factor by Theorem 6, p. 182 of [2]. If $\partial_i > 0$ for some i, without loss of generality assume that $i = 1$. We show that C_1 has a 1-factor by using Tutte's theorem (see p. 182, [2]): The necessary and sufficient condition for a graph H to possess a perfect matching (1-factor) is that $p_i(S) \leq |S|$ for every $S \subseteq V(H)$, where $p_i(S)$ denotes the number of odd components of the subgraph of H spanned by $V(H) - S$. To prove this for C_1 we proceed as follows. Let S be a non-empty proper subset of $V(C_1)$ and $D_1, D_2, \ldots, D_{p_i}(S)$ be the components of odd order in the subgraph of C_1 spanned by $V(C_1) - S$. Further let θ be the number of j's, $1 \leq j \leq p_i(S)$, such that $m(V(D_j), S) = 2$. We first show that $\theta \leq 1$. Suppose $\theta > 1$ and assume that $m(V(D_1), S) = m(V(D_2), S) = 2$. Let $D_{i1}, D_{i2}, \ldots, D_{is_i}$ be the maximal bicoherent subgraphs of D_i, $i = 1,2$. Clearly $V(D_{ij})$'s, $1 \leq j \leq s_i$, are pairwise disjoint. Since $m(V(D_i), S) = 2$ and C_1 is bicoherent there are at most two cut edges of D_i containing a fixed vertex of $V(D_i)$, so $V(D_{ij})$'s, $1 \leq j \leq s_i$, partition the vertices $V(D_i)$. Further the value of the cut $V(D_{ij})$ in C_1 is 2, $1 \leq j \leq s_i$, $i = 1,2$. Now since $V(D_i)$'s are odd we can without loss of generality assume that $V(D_{11})$, $V(D_{21})$ are odd. Choose minimal cuts $E_i \subseteq V(D_{i1})$, $i = 1,2$, of value 2 in C_1. Since C_1 is regular of degree 4 the value of any cut in C_1 is even. Now it can be easily proved that the subgraph of C_1 spanned by E_i is 3-coherent. Let (x, y), (u, v) be edges in the subgraphs of C_1 spanned by E_1, E_2 respectively. Then one of x, y, u, v is joined in G to the other three for otherwise by a simple interchange we can get a graph with degree sequence same as the degree sequence of the subgraph of G spanned by $V(C_1)$ and having a bicoherent 4-factor with number of cuts of value $2 < \partial_1$, a contradiction. Without loss of generality assume that x is joined in G to u, v. Now as in the proof of Lemma 1.2.1, it can be proved that $A(x, D_{11})$, $B(x, D_{11})$ are independent sets in C_1. Since the value of the cut $V(D_{11})$ in C_1 is 2 and C_1 is regular of degree 4 it follows that $|A(x, D_{11})| = |B(x, D_{11})|$, thus $V(D_{11})$ is even, a contradiction. This proves that $\theta \leq 1$.

Now it follows that

$$4\,|S| \geq \sum_{i=1}^{p_i(S)} m(V(D_i), S) \geq 4\,(p_i(S) - 1) + 2,$$

which shows that $p_i(S) \leq |S|$, By Tutte's theorem C_1 has a 1-factor, so C_1 has a 3-factor. Thus each C_i has a 3-factor. Hence $\{d_i\}$ is 3-factorable and this completes the proof of the theorem.

<u>Lemma 1.2.7.</u> Let C_1 be a connected partial subgraph of G in which two vertices have degrees $k-1$ and the rest have degrees k and let C_2 be a bicoherent partial subgraph of G which is regular of degree k. Also let $V(C_1)$ and $V(C_2)$ partition the vertices of G. Then if the degree

sequence $\{d_i\}$ of G is k-factorable, there exists a graph H with degree sequence $\{d_i\}$ and having a connected partial graph in which two vertices have degree k-1 and the rest have degrees k.

<u>Proof.</u> Suppose, if possible, that H does not exist. If some vertex t of C_2 is joined in G to both vertices of some edge of C_1, it can be proved as in Lemma 1.2.1 that t is joined in G to all vertices of C_1. This is a contradiction since if u is a vertex with degree k-1 in C_1 and z is a vertex adjacent to t in C_2, then $C_1 + C_2 + (u, t) - (t, z)$ is a connected partial subgraph of G in which two vertices have degrees k-1 and the rest have degrees k. Thus we may take that some vertex x of C_1 is adjacent to both vertices of some edge of C_2 in G. Then as before we can show that every vertex of $A(x, C_1)$ is joined in G to all vertices of C_2, no vertex of $B(x, C_1)$ is joined in G to any vertex of C_2, $A(x, C_1)$ and $B(x, C_1)$ are independent sets in C_1. Further the two vertices of C_1 with degree k-1 belong to $B(x, C_1)$. Now it is evident that $k = 2$ and $|A(x, C_1)| = |B(x, C_1)| - 1$. Further it can also be proved that $C_1 \rightarrow C_2$. As in the proof of Theorem 1.2.3 it follows that, if G* is any graph with degree sequence $\{d_i\}$ then its vertices can be partitioned

into nonempty sets A, B, C such that $|A| = |B| - 1$ and every edge with one end vertex in B has the other end vertex in A. Thus $\{d_i\}$ is not 2-factorable, a contradiction which proves the lemma.

<u>Theorem 1.2.8.</u> If $\{d_i\}$ is k-factorable, then there exists a graph H with degree sequence $\{d_i\}$ having a connected partial graph in which two vertices are of degree k-1 and the rest have degrees k.

<u>Proof.</u> This theorem follows easily from Lemma 1.2.2 and Lemma 1.2.7.

Taking $k=2$ we see that if $\{d_i\}$ is 2-factorable, then it is traceable, i.e., there exists a graph with degree sequence $\{d_i\}$ and having a Hamiltonian chain.

The problem of finding necessary and sufficient conditions for a graphical degree sequence to be k-factorable seems to be much deeper. In this connection we make the following conjectures.

<u>Conjecture 1.</u> A graphical degree sequence $\{d_i\}$ is k-factorable if and only if $\{d_i - k\}$ is graphical.

<u>Conjecture 2.</u> If $\{d_i\}$ is k-factorable then $\{d_i\}$ is (k-1)-factorable provided n is even.

Dr. J. A. Bondy kindly informed that conjecture 1 for $k = 2$ was also mentioned by Prof. B. Grünbaum at the combinatorics conference held in Calgary in June 1969. We prove below that the truth of conjecture 1 implies the truth of conjecture 2. Further, by corollary to Theorem 1.2.4 and Theorems 1.2.5, 1.2.6, it follows that conjecture 2 is true for all $k \leq 4$.

Theorem 1.2.9. Let $\{d_i\}$ and $\{d_i - k\}$ be graphical sequences. Then $\{d_i - r\}$ is a graphical sequence provided $0 \leq r < k$ and rn is even.

Proof. Let $b_i = d_i - k$ and $c_i = d_i - r$ for $1 \leq i \leq n$. That $\sum_{i=1}^{n} c_i$ is even is evident. So by a theorem of Erdős and Gallai [5] (see also Beineke and Harary [1]), we have only to show that $\sum_{i=1}^{s} c_i^{**} \geq \sum_{i=1}^{s} c_i$ for $1 \leq s \leq n$, where c_j^{**} is the number of i such that $i < j$ and $c_i \geq j-1$ plus the number of i such that $i > j$ and $c_i \geq j$. For this it is enough to show that

$$\sum_{i=1}^{s} c_i^{**} + rs \geq \min \left\{ \sum_{i=1}^{s} d_i^{**}, \sum_{i=1}^{s} b_i^{**} + ks \right\}. \qquad (7)$$

Construct now the (0, 1)-matrix $A = ((a_{ij}))$ of order n where $a_{ij} = 1$ if and only if $i < j$ and $d_i \geq j-1$ or $i > j$ and $d_i \geq j$. Evidently then d_i^{**} is the i-th column sum of A. Let us now put in the i-th row of A two marks, a red mark immediately after the (d_i-k)-th 1 and a blue mark immediately after the (d_i-r)-th 1 for $1 \leq i \leq n$. For any s with $1 \leq s \leq n$, let A_s denote the $n \times s$ matrix consisting of the first s columns of A.

<u>Case (i)</u>. $\sum_{i=1}^{s} d_i^{**} \leq \sum_{i=1}^{s} b_i^{**} + ks.$

Then the number of 1's to the right of red marks in $A_s \leq ks$. Hence the number of 1's to the right of blue marks in $A_s \leq rs$ for otherwise, a 1 occurs to the right of blue marks in at least s+1 rows of A_s, hence the number of 1's between the red and blue marks in $A_s \geq (k-r)(s+1)$ and the number of 1's to the right of red marks in $A_s \geq rs + 1 + (k-r)(s+1) = ks + k - r + 1 > ks$, a contradiction. This proves (7) in this case.

<u>Case (ii)</u>. $\sum_{i=1}^{s} d_i^{**} > \sum_{i=1}^{s} b_i^{**} + ks.$

Then the number of 1's to the right of red marks in $A_s > ks$. Hence the number of 1's between the red and blue marks in $A_s > (k-r)s$ for otherwise, the number of 1's to the right of blue marks in $A_s > rs$, hence a 1 occurs to the right of blue marks in at least $s+1$ rows of A_S and the number of 1's between the red and blue marks in A_S is not less than $(k-r)(s+1) > (k-r)s$, a contradiction. This proves (7) in this case and the theorem is proved.

The results of this section are to appear in [20].

Remarks: A k-factor of a digraph G is a partial graph of G in which the indegree and outdegree of every vertex is k. Call a demi-degree sequence $\{d_i^+, d_i^-\}$ k-factorable (strongly connected k-factorable), if there exists a digraph with a k-factor (strongly connected k-factor) and with the given demi-degree sequence. With these definitions, it would be interesting to extend the results of this section to digraphs. Are the following statements true?

1. A graphical demi-degree sequence $\{d_i^+, d_i^-\}$ is k-factorable if and only if $\{d_i^+ - k, d_i^- - k\}$ is graphical.

2. A k-factorable demi-degree sequence is $(k-1)$-factorable.

1.3. On planar degree sequences

In the book entitled, 'Recent Progress in combinatorics', edited by W. T. Tutte [24] the following problem was posed by A. M. Hobbs (see problem 9, P. 344 of [24]). For what values of n is there a planar graph on n vertices without loops or multiple edges which has 12 vertices of degree 5 and $n-12$ vertices of degree 6? He also observed that such a graph exists for $n = 12$ and none exists for $n = 13$. In this section we solve this and some other related problems.

A graph is said to be embeddable on a surface S if it can be drawn on the surface S such that no two edges intersect. A planar graph is a graph which can be embeddable on the plane. An embedding of a planar graph on the plane is called a planar realisation of the graph.

Let G be a graph, (u, v) be an edge of G and $w_1, w_2, \ldots, w_s$ be vertices not in $V(G)$. A subdivision of the edge (u, v) of G by $w_1, w_2, \ldots, w_s$ consists of omitting the edge (u, v) of G and including the edges (u, w_1), (w_i, w_{i+1}), $1 \leq i \leq s-1$, and (w_s, v). A double wheel on $n+2$ vertices consists of a circuit C on n vertices with two additional vertices adjacent to every $v \in V(G)$.

A planar graph on n vertices is said to have the <u>property $P(i, j)$,</u> $i \leq \min\{5, j\}$, if it has k vertices of degree i and $n-k$ vertices of degree j, where i, j, k are connected by the relation

$$ki + (n-k)j = 2(3n-6).$$

In this section we consider the following problem. For what values of n is there a planar graph on n vertices with property $P(i,j)$? If $i = 5$ and $j = 6$ then $k = 12$ and this is the problem of Hobbs. Clearly a planar graph on n vertices with property $P(i, j)$ triangulates the plane.

We solve the problem of Hobbs by proving the following

<u>Theorem 1.3.1.</u> For all values of $n \geq 12$ and $n \neq 13$, there exists a planar graph on n vertices with property $P(5, 6)$.

<u>Proof.</u> We construct the required planar graphs inductively. For $n = 12$, 14 and 16 the graphs G_1, G_2 and G_3 of Figures 1, 2 and 3 respectively are the required graphs.

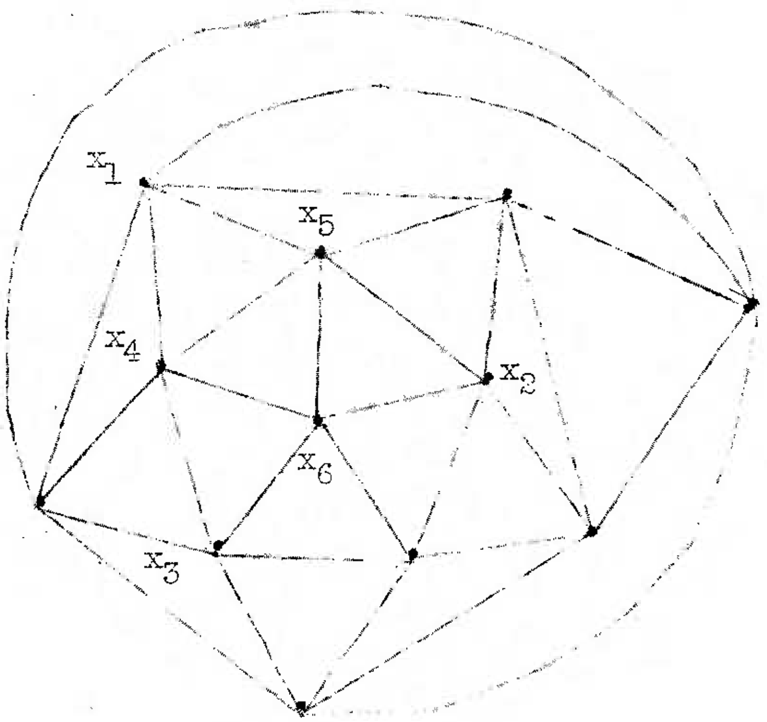


Fig. 1. G_1

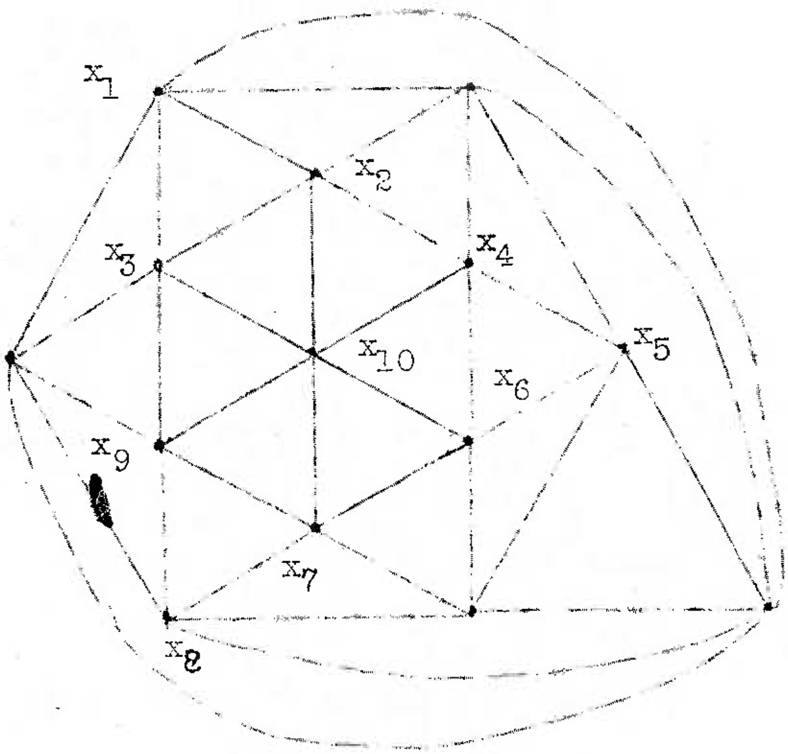


Fig. 2. G_2

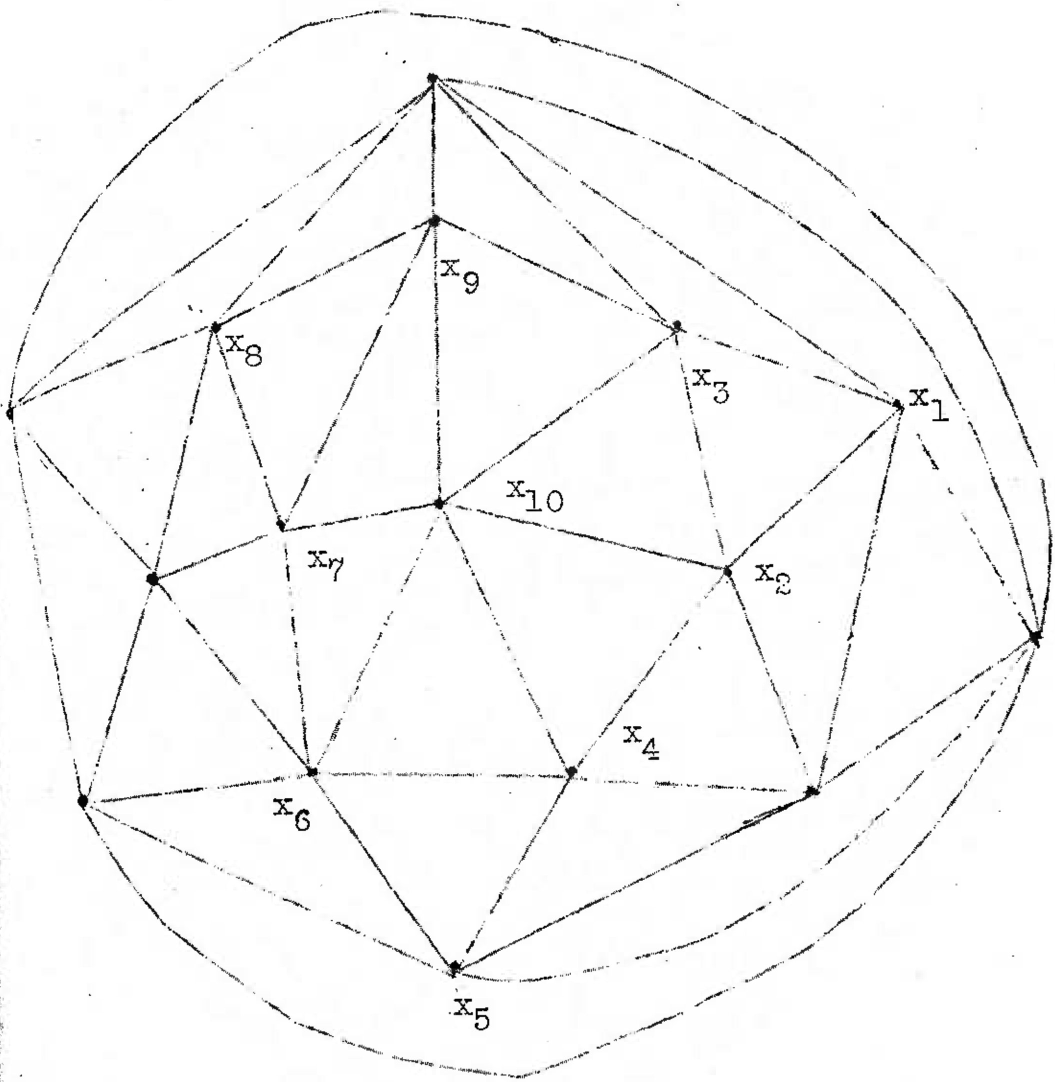


Fig. 3. G_3

We observe that in the planar realisation Fig. 1 of G_1 there are four faces having the structure described in Fig. 4 in which each vertex x_i, $1 \leq i \leq 6$, incident with them has degree 5 in G_1 (see Fig. 1).

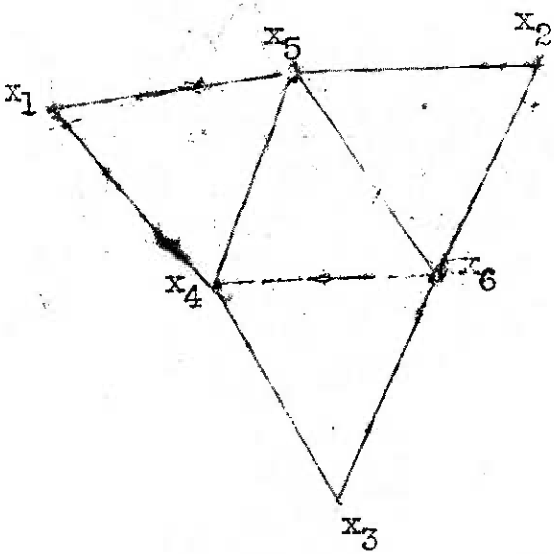


Fig. 4.

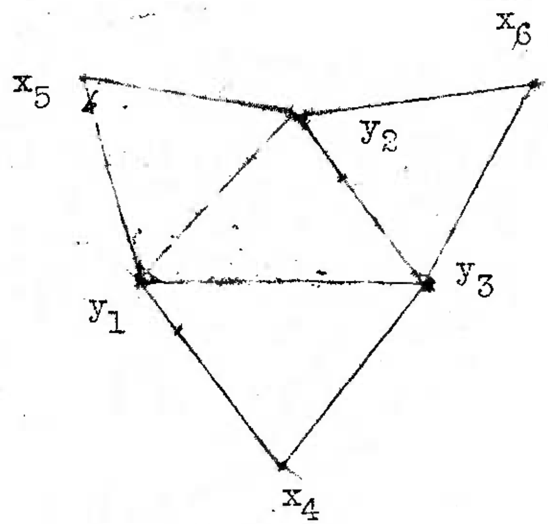


Fig. 5.

Now we construct a planar graph H on 15 vertices with property P(5, 6) having four faces with structure described in Fig. 4. To get H from G_1, subdivide the edges (x_4, x_5), (x_5, x_6), (x_6, x_4) of G_1 by y_1, y_2, y_3 respectively and join y_1 to x_1, y_2, y_3; join y_2 to x_2, y_3; and join y_3 to x_3. Now in H there are four faces (see Fig. 5) having the structure described in Fig. 4 in which each vertex incident with them has degree 5 in H. Now repeat the construction with these four faces on H. Thus inductively it follows that for all n of the form 12+3s, where s is a nonnegative integer, there is a planar graph on n vertices with property P(5, 6).

In the planar realisations Fig. 2 and Fig. 3 of G_2 and G_3 there are nine faces having the structure described in Fig. 6 in which each of the vertices x_1, x_4, x_5, x_7, x_8, x_9 has degree 5 in G_2 and G_3.

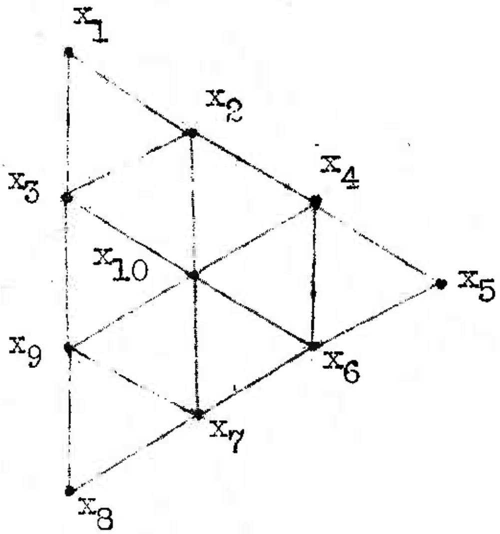


Fig. 6.

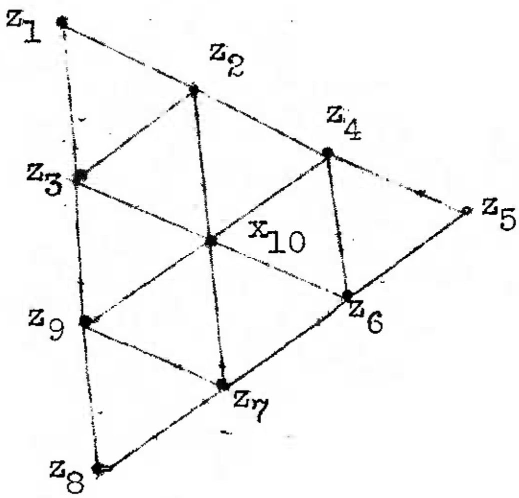


Fig. 7.

Now we describe a procedure by which we can get planar graphs H_1, H_2 and H_3 on 17, 20 and 23 vertices respectively each having the property P(5, 6) and H_3 having nine faces with structure described in Fig. 6. To get H_1 from G_2, subdivide the edges (x_3, x_2), (x_2, x_{10}), (x_{10}, x_3) by z_1, z_2, z_3 respectively and join z_1 to x_1, z_2, z_3; join z_2 to x_4, z_3; and join z_3 to x_9. Since the degree of each of the vertices x_1, x_4, x_9 in G_2 is 5, H_1 is a planar graph on 17 vertices with property P(5, 6). We construct H_2 from H_1 as follows. Subdivide the edges (x_{10}, x_4), (x_4, x_6), (x_6, x_{10}) of H_1 by z_4, z_5, z_6 respectively and join z_4 to z_2, z_5, z_6; join z_5 to x_5, z_6; and join z_6 to x_7. Then, since the degree of each of the vertices z_2, x_5, x_7 in H_1 is 5, H_2 is a planar graph on 20 vertices with property P(5, 6). To get H_3 from H_2, subdivide the edges (x_{10}, x_7), (x_7, x_9), (x_9, x_{10}) of H_2 by z_7, z_8, z_9 respectively and join z_7 to z_6, z_8, z_9; join z_8 to x_8, z_9; and join z_9 to z_3. Now since degree of each of the vertices z_6, x_8, z_8 in H_2 is 5, H_3 is a planar graph on 23 vertices with property P(5,6). Further in H_3 there are nine faces (see Fig. 7) having the structure described in Fig. 6. Now repeat the construction with these nine faces of H_3. Thus for all n of the form 14 + 3s, where s is a non-negative integer, there is a planar graph on n vertices with property P(5,6). Starting with the nine faces in Fig. 3 of G_3 having the structure described in Fig. 6, we can similarly construct planar graphs on n vertices with property P(5, 6) for every n of the form 16 + 3s as well. Thus for all $n \geq 12$ and $n \neq 13$, there exists a planar graph on n vertices with property P(5, 6) and this completes the proof of the theorem.

Remark. If $n(\geq 12)$ is an even integer and $n \neq 14$ let G be the planar graph with property $P(5, 6)$ constructed above. It can be easily verified that in the subgraph of G generated by degree 6 vertices of G there is a perfect matching (1- factor). Removing this 1-factor from G we get a planar regular graph of degree 5. Thus a planar regular graph of degree 5 exists for all even n (≥ 12) and $n \neq 14$. This result was earlier proved by Chvátal [3].

Theorem 1.3.2. Let $j(\geq 6)$ be odd. Then a planar graph on n vertices with property $P(4, j)$ exists if and only if $n \geq 6$ and $k = \dfrac{(j-6)n+12}{(j-4)}$ is a positive integer.

Proof. The necessity that $n \geq 6$ and k is an integer is trivial. To prove sufficiency, we construct the required graphs inductively. For $n = 6$ the double wheel G on 6 vertices has the property $P(4, j)$ and has two faces with structure described in Fig. 8 in which each of the vertices x_1, x_2, x_3 has degree 4 in G.

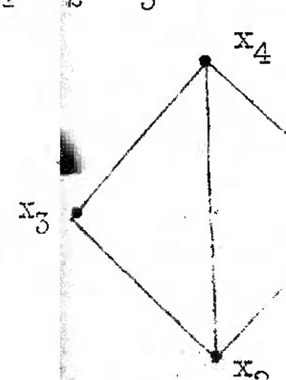


Fig. 8

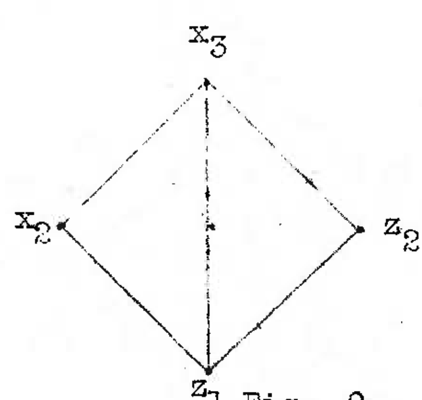


Fig. 9.

To get a planar graph H on $j+2$ vertices (the next admissible value of n, i.e., the next value for which k is a positive integer) with property $P(4, j)$, subdivide the edge (x_2, x_4) of G by $z_1, z_2, \ldots, z_{j-4}$ and join each of them to x_1, x_3. Clearly then H has two faces with structure described in Fig. 8 (see Fig. 9). Thus inductively we can construct planar graphs on n vertices with property $P(4, j)$ for all $n \geq 6$ whenever k is a positive integer and this completes the proof of the theorem.

An argument similar to the argument used in the proof of Theorem 1.3.2 proves the following

<u>Corollary</u>. If j (≥ 6) is even then a planar graph on n vertices with property $P(4, j)$ exists whenever n is even and $n \geq 6$ provided $k = \frac{(j-6)n+12}{(j-4)}$ is a positive integer.

<u>Theorem 1.3.3.</u> Let $j \equiv 2 \pmod 4$ and $j \geq 4$. Then a planar graph on n vertices with property $P(4, j)$ exists whenever n is even and $n \geq 6$, or n is odd and $n \geq \frac{j^2}{4} + 2$ provided $k = \frac{(j-6)n+12}{(j-4)}$ is a positive integer.

<u>Proof</u>. By Corollary to Theorem 1.3.2 it is enough to consider the case in which n is odd. So let n be odd and $n \geq \frac{j^2}{4} + 2$. Now we construct a planar graph G on $\frac{j^2}{4} + 2$

(which is obviously odd since $j \equiv 2 \pmod 4$) vertices having two faces with structure described in Fig. 8 as follows: Take a wheel on $j+1$ vertices, $1, 2, \ldots, j+1$ (say), with $j+1$ being the centre of the wheel, attach chains of length $\frac{j-4}{2}$, on $a_{s,1}, a_{s,2}, \ldots, a_{s,\frac{j-4}{2}}$ (say), at each of the vertices $s \in \{2, 4, \ldots, j\}$, join every vertex of the chain attached at s to $s-1$, $s+1 \pmod j$, $s \in \{2, 4, \ldots, j\}$ and let H be this graph. Then take a new vertex x and join it to $1, 3, \ldots, j-1$; $a_{2,\frac{j-4}{2}}, a_{4,\frac{j-4}{2}}, \ldots, a_{j,\frac{j-4}{2}}$. Let G be the resulting graph (see Fig. 10 for the case $j = 10$, in which only H is given).

Then G is a planar graph and has $\frac{j^2}{4} + 2$ vertices out of which $\frac{j}{2} + 2$ are of degree j, namely $1, 3, \ldots, j+1, x$ and the remaining are of degree 4. So G is a planar graph on $\frac{j^2}{4}+2$

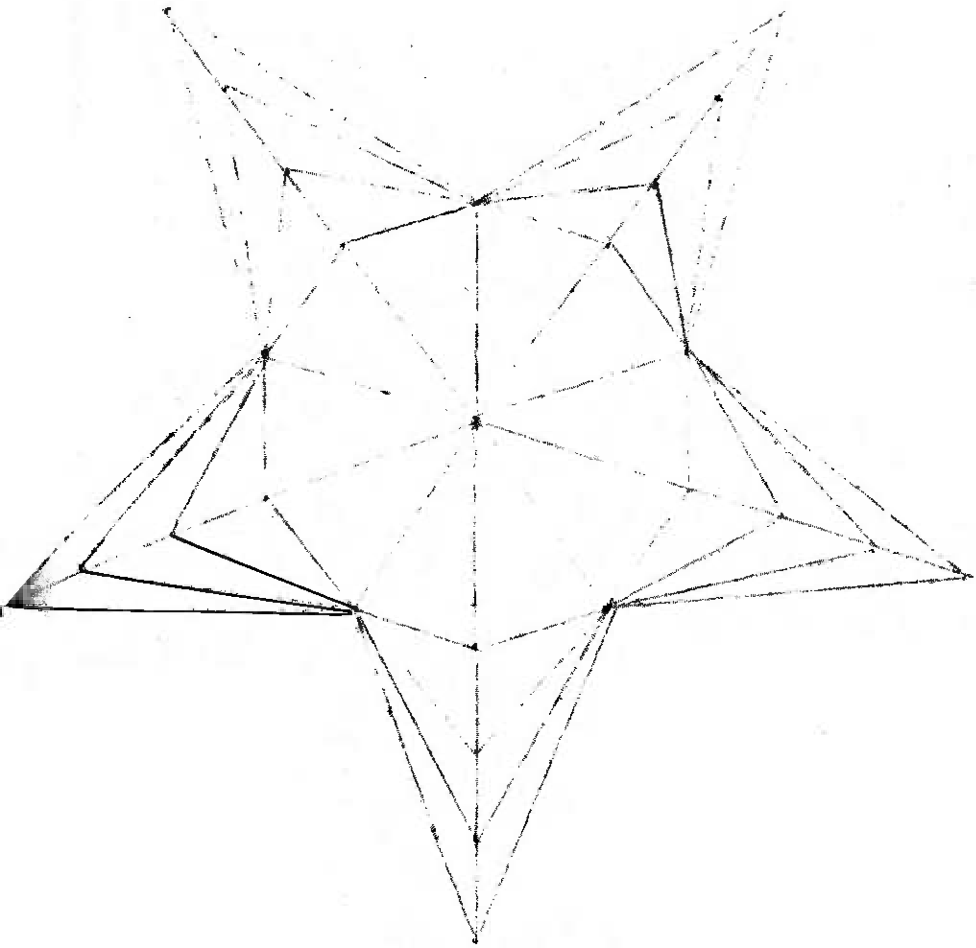


Fig. 10. H.

vertices with property $P(4, j)$. Further G has two faces with structure described in Fig. 8. Now, as in the proof of Theorem 1.3.2, planar graphs on n vertices with property $P(4, j)$ can be constructed for all n odd and $n \geq \frac{j^2}{4} + 2$ provided k is an integer and this completes the proof of the theorem.

Perhaps these are the only values of n for which there exist planar graphs on n vertices with property $P(4,j)$ whenever $j \equiv 2 \pmod 4$ and $j > 6$.

When $j = 6$, the planar graph G_4, of Fig. 11 on 9 vertices has the property $P(4, 6)$ and it can be easily proved that there is no planar graph on 7 vertices with property $P(4, 6)$.

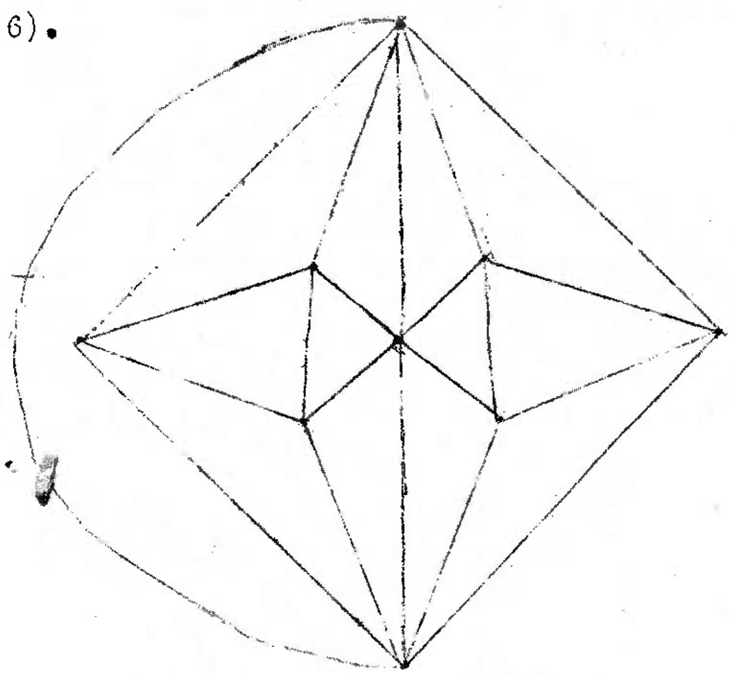


Fig. 11. G_4

This observation coupled with Theorem 1.3.3 for $j = 6$ yields the following

Theorem 1.3.4. A planar graph on n vertices with property $P(4, 6)$ exists if and only if $n \geq 6$ and $n \neq 7$.

Theorem 1.3.5. A planar graph on n vertices with property $P(3, 6)$ exists for all even $n \geq 4$ and $n \neq 6$.

Proof. For $n = 4$, the complete graph on 4 vertices and for $n = 8, 10$ the graphs G_5, G_6 of Figures 12, 13 are the required graphs.

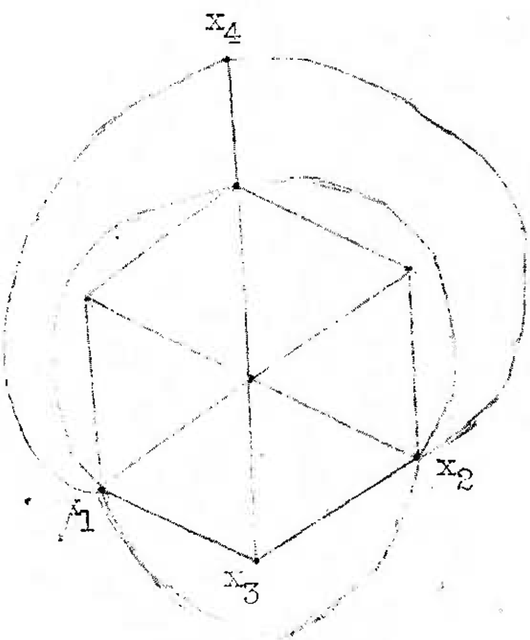


Fig. 12. G_5

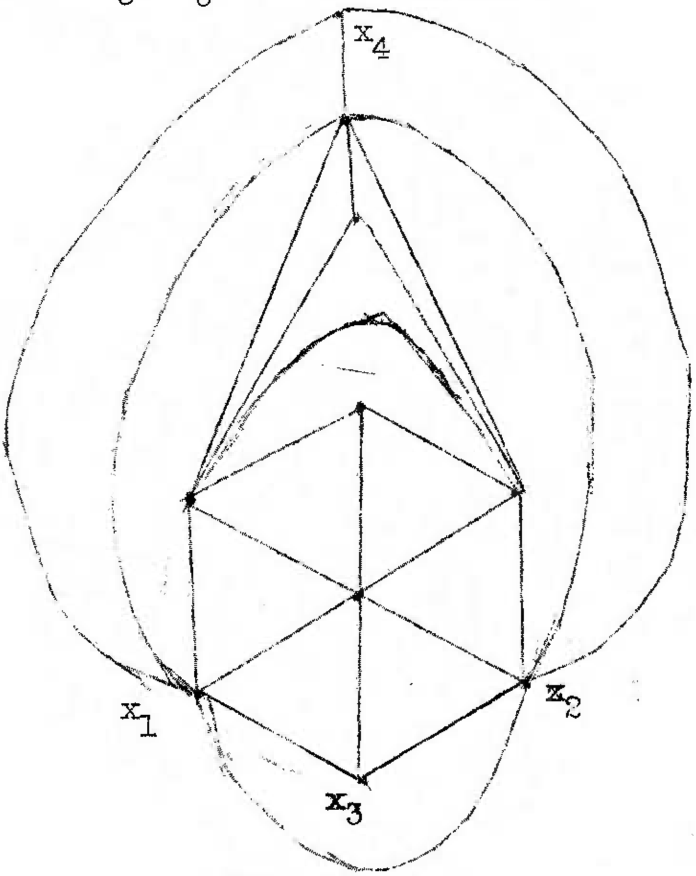


Fig. 13. G_6

Let x_1, x_2 be the vertices of degree 6 on the infinite face of Fig. 12 of G_5. Then the infinite face of $H = G_5-(x_1,x_2)$ is a 4-cycle with two nonadjacent vertices of degree 3, x_3, x_4 (say), and two vertices of degree 5. Now to get a planar graph G on 12 vertices with property $P(3,6)$ from H, take three new vertices z_1, z_2, z_3 and join each of them to x_3; join z_1 to x_1, x_4, z_2, z_3; join z_2 to x_2, x_4, z_3; and take one more vertex z_4 and join it to x_4, z_1, z_2. Now the infinite face of $G-(z_1, z_2)$ is a 4-cycle with two nonadjacent vertices z_3, z_4 of degree 3 and two vertices of degree 5. Similarly we can construct a planar graph K on 14 vertices with property $P(3, 6)$ from Fig. 13 of G_6 with the infinite face of $K-(z_1,z_2)$ having the structure described above. Thus inductively we can construct planar graphs on n vertices with property $P(3, 6)$ whenever n is even, $n \geq 4$ and $n \neq 6$.

Now we conjecture that there is no planar graph on n vertices with property $P(3, 6)$ whenever n is odd.

We conclude this section with the observation that there is no planar graph on n (≥ 4) vertices with property $P(i,j)$ if $i = 1, 2$ and the following list in whose third column the only values of n for which there are planar graphs on n vertices with property $P(i,j)$ are given.

i	j	n
5	5	12
4	5	6, 7, 8, 9, 10, 12
3	5	8, 12
4	4	6
3	4	4, 5, 6
3	3	4

CHAPTER 2

STUDIES ON CUT VERTICES, CUT ARCS AND CUT EDGES IN GRAPHS

In [19], A. Ramachandra Rao determined, among other things, the ranges of the number of cut vertices and cut edges in an undirected graph on n vertices with m edges. In this chapter we solve the corresponding problems for strong directed graphs. We also give partial solutions to the problems of the determination of the number of cut vertices and cut edges in an undirected graph on n vertices and m edges in which minimum degree $\geq d$, considered by Ramachandra Rao [19]. More precisely, in Sections 2.1, 2.3, we determine the ranges of the number of cut vertices, cut arcs in a strong digraph on n vertices with m arcs and characterise some extremal graphs. This also generalises some of the results of Gupta [6]. In Section 2.2, we prove that a strong complete digraph on $n(>3)$ vertices has at most $n-2$ cut vertices, without using the existence of a Hamiltonian circuit and this solves a problem raised by Korvin [12] at the Rome Conference. In Sections 2.4, 2.5, we find the maximum number of edges in a connected undirected graph on n vertices with r cut vertices (s cut edges) in which minimum degree $\geq d$ and give partial solutions to the problems of Ramachandra Rao [19].

A vertex (arc) of a strong digraph is called a cut vertex (cut arc) if its deletion makes the graph not strong. $n(G)$, $m(G)$, $r(G)$, $s(G)$ denote respectively the number of vertices, arcs, cut vertices, cut arcs of G. Also $R(G)$ denotes the set of all cut vertices of G.

An arc (x,y) of a digraph G is symmetric if $(y,x) \in G$. A symmetric graph whose associated undirected graph is a tree is called a symmetric tree. G^S denotes the partial subgraph of G consisting of all the symmetric arcs of G. $d_G^+(x)$, $d_G^-(x)$ denotes the outdegree and the indegree of x in G respectively. The degree of $x = d_G^+(x) + d_G^-(x)$. The converse of G has the same set of vertices as G and (x,y) is an arc in the converse if and only if $(y,x) \in G$.

A vertex (edge) of an undirected graph G is called a cut vertex (cut edge) if its deletion increases the number of components of G. A pendent block of a graph G is a block of G which is incident with exactly one other block of G. Let x be a cut vertex of G and C a component of $G-x$, then the subgraph of G generated by $V(C) \cup \{x\}$ is called a piece of G with respect to x.

2.1 The number of cut vertices in a strong directed graph

In this section we determine the maximum number of cut vertices in a strong directed graph on n vertices with m arcs. Throughout this section by a maximal graph on n vertices with r cut vertices, we mean a strong directed graph such that the addition of any arc decreases the number of cut vertices. An extremal graph is a graph with n vertices, r cut vertices and with maximum number of arcs.

Lemma 2.1.1. If G is a maximal graph and there are two vertex disjoint paths from x to y, then $(x, y) \in G$.

Proof. If $(x, y) \notin G$, add the arc (x,y) to G. If this converts some cut vertex u of G into a noncut vertex, then u belongs to every path from x to y in G, a contradiction.

Lemma 2.1.2. Let G be a maximal graph and x a noncut vertex of G. Then $R(G - x) \subseteq R(G)$. Further if $y \in R(G) - R(G - x)$, then either $d^+(x) = 1$ and $(x,y) \in G$ or $d^-(x) = 1$ and $(y, x) \in G$.

Proof. Suppose some noncut vertex z of G is a cut vertex of $G-x$. Then there exist vertices u, v such that every path from u to v in $G-x$ passes through z. Since z is a noncut vertex of G, there is a path η from u to v in $G-z$. Let $\mu = [u, u_1, \ldots, u_{k-1}, z, u_{k+1}, \ldots, v]$ be a path from u to v in $G-x$. If μ and η are vertex disjoint then $(u, v) \in G$ since G is a maximal graph. Even otherwise $(u_i, u_j) \in G$ for some $i \leq k-1$ and some $j \geq k+1$. Thus there is a path from u to v not passing through z in $G-x$, a contradiction.

Now let $y \in R(G) - R(G-x)$. We consider two cases.

Case (i). Every path from x to some vertex v in G passes through y. Now if u is any vertex other than y and $(x, u) \in G$, then since $G_1 = G - \{x, y\}$ is strong, there is a path from u to v in G_1. Thus there is a path from x to v, not passing through y, in G. This contradiction shows that $d^+(x) = 1$ and $(x, y) \in G$.

Case (ii). Every path from some vertex v to x in G passes through y. As in case (i), it follows that $d^-(x) = 1$ and $(y, x) \in G$. This completes the proof of the lemma.

Lemma 2.1.3. Let G be a maximal graph and x a noncut vertex of G such that $R(G) = R(G-x)$. If the degree of x in G is at least $2n-2-k$, then $r(G) \leq k$.

Proof. If $k = n-1$, the lemma is trivial. So let $k \leq n-2$. Evidently now there exists a path $[x, y, z]$ in G such that $(x, z) \notin G$. Since G is maximal, it follows that y is a cut vertex of G and is a noncut vertex of the graph G* obtained from G by adding the arc (x,z). If G* is not maximal let H be a maximal graph containing G* such that $R(G^*) = R(H) = R(G) - y$. By Lemma 2.1.2, $R(H-x) \subseteq R(H)$. Let now u be a noncut vertex of H-x. Then $H - \{x, u\}$ is strong. So if u is a cut vertex of H, then either $d_H^+(x) = 1$ or $d_H^-(x) = 1$. Now $d_H^+(x)$ is not possible, so $d_H^-(x) = 1 = d_G^-(x)$. Since $k \leq n-2$, by hypothesis, $d_G^+(x) \geq n-1$, a contradiction since $(x,z) \notin G$. Thus u is a noncut vertex of H and $R(H-x) = R(H)$. Now H satisfies the hypothesis of the lemma with k replaced by $k-1$ and the proof is completed by using induction on k.

Lemma 2.1.4. Let G be a maximal graph, x a noncut vertex of G and let $r(G) = n-1$. Then the degree of x is at most $n-1$.

Proof. This lemma follows from Lemma 2.1.3 if $R(G-x) = R(G)$. If $r(G) = r(G-x) + 2$, then by Lemma 2.1.2, $d^+(x) = d^-(x) = 1$. So we may take that $R(G-x) = R(G) - y$

where $y \in R(G)$. Now we prove the lemma by induction on n. The lemma is trivial for $n = 3$, so assume the result for $n-1$ and let G be a graph with n vertices and satisfying the hypothesis of the lemma. Without loss of generality we take $d^+(x) = 1$ and $(x,y) \in G$. If possible let the degree of $x \geq n$. Then evidently $d^-(x) = n-1$. Since G is maximal and $G-x$ is strong, it follows, by Lemma 2.1.1, that $d^-(y) = n-1$.

First we show that $G-x$ is maximal. Otherwise add an arc α to $G-x$ such that the resulting graph has the same cut vertices as $G-x$. Let H be the graph $G+\alpha$. Now $R(H) \subseteq R(G)$. If possible let $u \in R(G) - R(H)$. Since $d^+_H(x) = 1$, $y \in R(H)$ and $u \neq y$. Suppose now that u is a cut vertex of $H-x$ and v_1, v_2 are two vertices of $H-x$ such that every path from v_1 to v_2 in $H-x$ posses through u. Since $u \notin R(H)$, there is a path μ from v_1 to v_2 not containing u in H. Evidently now $x \in \mu$. Now the vertex succeeding x on μ is y and if z is the vertex preceding x on μ, then $(z, y) \in H$ since $d^-_H(y) = n-1$. Thus there is a path from v_1 to v_2 in $H-x$ and not containing u. This contradiction shows that u is a noncut vertex of $H-x$. By the definition of α, u is a noncut vertex of $G-x$ and so of G. This contradiction shows that $R(H) = R(G)$. But then G is not maximal. This finally proves that $G-x$ is maximal.

Since $r(G) = n-1$, $r(G-x) = n-2$ and y is a noncut vertex of $G-x$. Hence, by induction hypothesis, the degree of y in $G-x$ is at most $n-2$. This is a contradiction since $G-x$ is strong and the indegree of y in $G-x$ is $n-2$. Thus our supposition that the degree of x in $G \geq n$ leads to a contradiction and the lemma is proved.

Theorem 2.1.5. The maximum number of arcs in a strong digraph with $n \geq 4$ vertices and r cut vertices is

$$A(n, r) = \begin{cases} \binom{n-1}{2} + 3 & \text{if } r = n \\ \binom{n}{2} + 1 & \text{if } r = n-1 \\ \binom{n}{2} + \binom{n-r}{2} + r & \text{if } r \leq n-2 \end{cases}$$

Further the extremal graphs with n vertices and r cut vertices are $G_i(n)$, $i = 1,2,\ldots, 7$ described below: $G_1(n)$ has the vertices $1, 2, \ldots, n$.

In $G_1(n)$, (i,j) is an arc if either $1 \leq i < j \leq n-2$ or $i = j+1$ or $i = 1$ and $j = n$ or $i = n-1$ and $j = n$.

In $G_2(n)$, (i,j) is an arc if either $1 \leq i < j \leq n-4$ or $i = j+1$ or $1 \leq i \leq n-3$ and $j = n-2$ or $i = n-1$ and $1 \leq j \leq n-4$ or $i = n-1$ and $j = n$ or $i = 1$ and $j = n$.

$G_3(n)$ is the converse of $G_2(n)$.

In $G_4(n)$, (i, j) is an arc if either $1 \leq i < j \leq n-1$ or $i = j+1$ or $i = 1$ and $j = n$.

In $G_5(n)$, (i,j) is an arc if either $1 \leq i < j \leq n-2$ or $i = j+1$ or $i = n-1$ or $i = 1$ and $j = n$.

$G_6(n)$ is the converse of $G_5(n)$.

In $G_7(n,r)$, (i,j) is an arc if either $1 \leq i < j \leq n$ or $i = j+1$ or $k \leq j < i \leq k+n-r-1$, where k is a fixed integer such that $1 \leq k \leq r+1$.

It may be noted that $G_i(n)$ has n or $n-1$ cut vertices according as $1 \leq i \leq 3$ or $4 \leq i \leq 6$ and $G_7(n,r)$ has r cut vertices.

Proof. We prove the theorem by induction on n. The theorem is easily verified when $n = 4$ using the list of digraphs on 4 vertices given in [9],[11]. So assume the result for $n-1$ and let G be an extremal graph on $n \geq 5$ vertices with r cut vertices.

The graphs described in the statement of the theorem have $A(n,r)$ arcs where n is the number of vertices and r is the number of cut vertices. Thus G has at least $A(n,r)$ arcs. We now prove that G has at most $A(n,r)$ arcs and is one of the graphs described in the theorem.

First let $r = n$. *Let* x *be a vertex of* G. *Then* there exist vertices y and z such that (y, x) and (x, z) are arcs of G, and every path in G from y to z passes through x. Let H be the graph obtained from G by adding the arc (y, z) and dropping the vertex x. Since G is maximal, H is strong. Evidently any vertex of G other than x, y, z is a cut vertex of H also.

If $r(H) = n - 1$, then consider the graph $G' = G + (y,z)$. If G^* is a maximal graph containing G' with $R(G^*) = R(G')$, then by Lemma 2.1.4, the degree of x in G^* and hence in G is at most $n-1$. Now by induction hypothesis, H has at most $\binom{n-2}{2} + 3$ arcs. Thus G has at most $\binom{n-2}{2} + 3 + (n-1) - 1 = A(n, n)$ arcs. Now it also follows that H is an extremal graph and the degree of x in G is $n-1$. We consider the case $H = G_2(n-1)$, the other two cases can be disposed similarly. Now the only cut arcs of $G_2(n-1)$ are $(1, n-1)$ and $(i+1, i)$ where $i \neq n-3$. Thus (y,z) is $(1, n-1)$ or $(i+1, i)$ with $i \neq n-3$.

If $i = n - 2$, then x is not adjacent to any of the vertices $1, 2, \ldots, n-3$, for otherwise $n-2$ or $n-1$ will be a noncut vertex of G. This is a contradiction since the degree of x in G is $n-1$. The case $i = n-4$ is similar. If (y, z) is $(1, n-1)$ or $(i+1, i)$ with $1 \leq i \leq n-5$, then the symmetric path $[1, 2, \ldots, n-5]$ extends in length by unity with the vertex x. Thus $G = G_2(n)$. This proves that if $r(H) = n-1$, then $G = G_i(n)$ for some $i \leq 3$.

Now let $r(H) \leq n-2$. Then we may take that y is a noncut vertex of H, for, the case z is a noncut vertex of H is similar. Evidently then $G - \{x, y\}$ is strong. If $d_G^-(x) \geq 2$, then y is a noncut vertex of G. If $d_G^+(y) \geq 2$, then there is a path in G from y to z and not using x. Thus $d^+(x) = d^+(y) = 1$. Now let G_0 be the graph obtained from G by amalgamating x and y. Let x_0 denote the vertex of G_0 obtained by amalgamating x and y of G. If any vertex u of G_0 other than x_0 is a noncut vertex of G_0, then u is a noncut vertex of G also (since (y, x) is the only arc incident into x and is the only arc incident out from y). This contradiction shows that $r(G_0) = n-2$. We now show that G_0 is maximal. If G_0 is not maximal, let the addition of an arc (a, b) not affect the cut vertices of G_0. Then consider the graph $G + (a,b)$, $G + (a,y)$ or $G + (x,b)$ according as $a \neq x_0$ and $b \neq x_0$, $b = x_0$ or $a = x_0$. This graph has the same cut vertices as G, a contradiction. Thus G_0 is maximal and by Lemma 2.1.4, the degree of x_o in G_o is at most n-2.

If now $d^-_{G_o}(x_o) = 1$ and $d^+_{G_o}(x_o) = 1$, then $G-\{x,y\}$ has at least $n-4$ cut vertices, hence by induction hypothesis, $G - \{x, y\}$ has at most $\binom{n-2}{2} + n-3$ arcs. Thus G has at most $\binom{n-2}{2} + n-3+4 = A(n,n)$ arcs. It also follows that $G - \{x,y\}$ has exactly $n-4$ cut vertices and is extremal and $G = G_1(n)$.

If $d^-_{G_o}(x_o) \geq 2$ or $d^+_{G_o}(x_o) \geq 2$, then $G - \{x,y\}$ has at least n-3 cut vertices, hence by induction hypothesis, $G-\{x,y\}$ has at most $\binom{n-2}{2} + 1$ arcs. Thus G has at most $\binom{n-2}{2} + 1 + (n-2) + 2 = A(n,n)$ arcs. It also follows that $G - \{x,y\}$ has n-3 cut vertices and is extremal. To be specific, let $d^-_{G_o}(x_o) \geq 2$. If $G-\{x,y\}$ is $G_5(n-2)$, then it follows that $d^+_{G_o}(x_o) = 1$ and $(x, n-4) \in G$. If $(n-2, y) \in G$, then n-3 is a noncut vertex of G. So $(i,y) \in G$ for $1 \leq i \leq n-3$. Thus $G = G_2(n)$. If $G-\{x,y\} = G_4(n-2)$, then again $d^+_{G_o}(x_o) = 1$ and $(x, n-2) \in G$. If $n \geq 6$, then $(i,y) \in G$ for some i with $2 \leq i \leq n-3$, hence 1 is a noncut vertex of G. Thus $n = 5$ and $G = G_2(5)$. The case $G-\{x,y\} = G_6(n-2)$ is similar to the case $G - \{x,y\} = G_5(n-2)$. Further when $d^+_{G_o}(x_o) \geq 2$ also the proof is similar. Thus we have proved that when $r = n$, $G = G_i(n)$ for some $i \leq 3$.

Next let $r \leq n-1$. Then G has a noncut vertex x. By Lemma 2.1.2, $r(G) \geq r(G-x) \geq r(G) - 2$. Thus we have three cases.

Case (i). $r(G-x) = r(G)$. Then by Lemma 2.1.3, there are at most $2n-2-r$ arcs of G incident with x. By induction hypothesis, $G-x$ has at most $A(n-1, r)$ arcs. So G has at most $A(n, r)$ arcs. It also follows that $r \leq n-3$, $G-x$ is extremal and the degree of x in G is $2n-2-r$. By induction hypothesis, $G-x$ is $G_7(n-1, r)$. Now observe that i is a cut vertex of $G_7(n-1, r)$ if and only if $2 \leq i \leq k$ or $k+n-r-2 \leq i \leq n-2$. Also if (x, i), (x, j) are symmetric arcs of G, then all vertices λ with $i < \lambda < j$ are noncut vertices of $G-x$. Now since the degree of x in G is $2n-2-r$, there are at least $n-1-r$ symmetric arcs incident with x.

If $r = n-3$, evidently there are exactly two symmetric arcs (x, i_o) and (x, i_o+1) incident with x and since G is extremal, $(j, x) \in G$ for $1 \leq j \leq i_o-1$ and $(x, j) \in G$ for $i_o+2 \leq j \leq n-1$. Thus $G = G_7(n,r)$.

If $r \leq n-4$, then there is no symmetric arc (x,i) whenever $i < k$ or $i > k+n-r-2$. Thus (x,i) is a symmetric arc for $k \leq i \leq k+n-r-2$, $(i, x) \in G$ for

$1 \leq i \leq k-1$, and $(x, i) \in G$ for $k+n-r-1 \leq i \leq n-1$. Thus $G = G_7(n,r)$.

Case (ii). $r(G-x) = r(G) - 1$. Let $y \in R(G) - R(G-x)$. Then by induction hypothesis, $G-x$ has at most $A(n-1, r-1)$ arcs. By Lemma 2.1.2, we may take, without loss of generality, that $d^+(x) = 1$ and $(x,y) \in G$. So there are at most n arcs of G incident with x.

If now $r \leq n-2$, then G has at most $A(n,r)$ arcs. It also follows that the degree of x is n and $G-x$ is extremal. So $G-x = G_7(n-1, r-1)$ and evidently either $y = n-1$ or $k = r < y$. Thus $G = G_7(n,r)$.

If $r = n-1$, then by Lemma 2.1.4, the degree of x in G is at most $n-1$. By induction hypothesis, $G-x$ has at most $A(n-1, n-2)$ arcs. So G has at most $A(n, n-1)$ arcs and $G-x$ is extremal. If now $G-x = G_4(n-1)$, then the vertex 1 of $G-x$ is a noncut vertex of G, a contradiction. If $G-x = G_5(n-1)$, then $(n-1, x) \notin G$, hence $(i,x) \in G$ for $i \leq n-2$. Thus $G = G_5(n)$. If $G-x = G_6(n-1)$, then $(n-2, x) \notin G$, hence $(i,x) \in G$ for all $i \neq n-2$. But then 1 is a noncut vertex of G, a contradiction. This completes the case (ii).

Case (iii). $r(G-x) = r(G)-2$. Let $y, z \in R(G) - R(G-x)$. Then by induction hypothesis, $G-x$ has at least $A(n-1, r-2)$ arcs. Also $G-x$ has at most $A(n-1, r-2)$ arcs. Also by Lemma 2.1.2, $d^+(x) = d^-(x) = 1$. Since G has at least $A(n, r)$ arcs, it follows that $r = n-1$ and $G-x$ is extremal. Thus $G-x$ is $G_7(n-1, n-3)$ and $G = G_4(n)$. This completes the proof of the theorem.

Theorem 2.1.6. The maximum number of cut vertices in a strong graph with n vertices and m arcs is $r = r(n, m)$ where

$$r(n, m) = \max \{ q : m \leq A(n, q) \}.$$

Proof. By Theorem 2.1.5, it follows that the number of cut vertices in a graph with n vertices and m arcs is at most $r(n, m)$. To show that the bound is the best possible, consider the following graph. Take a graph on n vertices with $r = r(n, m)$ cut vertices and with $A(n,r)$ arcs. By Theorem 2.1.5, it has a Hamiltonian circuit and the deletion of $A(n, r) - m$ arcs not belonging to the Hamiltonian circuit gives the required graph. This completes the proof of the theorem.

Now it is not difficult to prove that the range r, the number of cut vertices in a graph on n vertices with m arcs is

$$0 \leq r \leq r(n, m) \qquad \text{if} \quad m \geq 2n,$$

$$2n - m \leq r \leq r(n, m) \qquad \text{if} \quad n \leq m \leq 2n - 1 \text{ and } m \neq 2n - 2,$$

$$1 \leq r \leq r(n, m) \qquad \text{if} \quad m = 2n - 2.$$

It may be remarked that if $r \leq n - 2$ then in each of the extremal graphs $G_7(n, r)$, the indegree and outdegree of every vertex is greater than 1. If $r = n$ or $n - 1$ then in each of the extremal graphs $G_i(n)$, $1 \leq i \leq 6$, there are vertices with indegree equal to 1 and there are vertices with outdegree equal to 1. In view of these remarks it would be interesting to know whether in any strong graph G in which the indegree and outdegree of every vertex is greater than one, there is a noncut vertex or not. If in addition the graph G has a Hamiltonian circuit then it is easy to see that it has a noncut vertex.

2.2 The number of cut vertices in a strong complete graph

In [12], Korvin has asked for a proof of the fact that any strong complete graph on $n > 3$ vertices has at most $n-2$ cut vertices without using Camion's theorem. In this section we give a simple proof of the result using induction on n.

The result is easily verified for $n = 4$, so assume it for $n-1$ and let G be a strong complete graph with n vertices where $n \geq 5$. We may assume that G is a maximal graph, i.e., the addition of any arc to G converts some cut vertex into a noncut vertex. We consider two cases.

Case (i). G has a noncut vertex x. Then by induction hypothesis, $G-x$ has two noncut vertices y, z. Since $d_G^+(x) > 1$ or $d_G^-(x) > 1$, it follows that one of y, z is a noncut vertex of G. Thus G has at least two noncut vertices.

Case (ii). All vertices of G are cut vertices. Let x be any vertex of G. Then there exist vertices y and z such that every path from y to z in G passes though x and $(y,x), (x,z) \in G$. Let $G_1 = G + (y,z)$. Since G is maximal, x is a noncut vertex of G_1. By induction hypothesis

$G_1 - x$ has at least two noncut vertices. Evidently, y, z are noncut vertices of $G_1 - x$ and as in case (i) it follows that one of y, z is a noncut vertex of G, a contradiction. Thus case (ii) does not occur and the proof is complete.

2.3 The number of cut arcs in a strong directed graph

In this section, by a maximal graph on n vertices with s cut arcs we mean a graph such that the addition of any arc decreases the number of cut arcs. An extremal graph is a graph with n vertices, s cut arcs and with the maximum number of arcs.

It is convenient to note down the following facts which will be used repeatedly. If G is a strong graph and the addition of a new arc (x, y) converts some cut arc (u, v) into a noncut arc then (u, v) belongs to every path from x to y in G. Further in G, there are paths from u to x and y to v not including the arc (u, v). If G is a strong graph and the removal of a noncut arc (x, y) converts some noncut arc (u, v) into a cut arc, then (u, v) belongs to every path from x to y in $G-(x, y)$ and (x, y) belongs to every path from u to v in $G-(u, v)$. An arc (u, v) is a cut arc of G if and only if (v, u) is a cut arc of the converse of G.

Lemma 2.3.1. If there are two arc disjoint paths from a vertex x to another vertex y in a maximal graph G, then (x, y) belongs to G.

Proof. This lemma is an immediate consequence of the first observation made above.

Lemma 2.3.2. If G is a maximal graph there is no circuit of length > 2 without chords.

Proof. If possible let $C = [x_1, x_2, \ldots, x_k, x_{k+1} = x_1]$ be a circuit without chords and $k \geq 3$. Then add the arc (x_2, x_1). Since G is maximal this converts some cut arc (u, v) of G into a noncut arc. Then $(u, v) \in C$ and there exist paths in G from u to x_2 and x_1 to v not using the arc (u, v). Hence by Lemma 2.3.1, C has a chord, a contradiction.

Now we prove the following important lemma.

Lemma 2.3.3. There exists an extremal graph containing a symmetric tree on n or $n-1$ vertices according as $s \neq 2n-3$ or $s = 2n-3$.

Proof. We start with any extremal graph with n vertices and s cut arcs. If G does not contain a symmetric tree as stated in the lemma, we describe a procedure by which we can increase the maximum size of a symmetric tree contained in G. Let T be a maximal symmetric tree contained in G.

Suppose there is a circuit containing exactly one vertex u of T. Then we can choose the circuit such that it has no chord incident with u. Let this circuit be $C = [u, x_1, \ldots, x_k, u]$.

First let there be a chain $\eta = [x_j, y_1, \ldots, y_p, v]$ connecting some vertex of C and a vertex $v \neq u$ of T. If η is not a path, we may assume without loss of generality that $(y_{p-1}, y_p) \in G$ and $(v, y_p) \in G$. We call vertices like y_p distinguished vertices of η. Now consider a path μ from y_p to u. Let z be the first vertex on this path belonging to $V(C) \cup V(T)$. If $z \in V(C) - u$, then we have the path $(v, y_p) + \mu[y_p, z]$ instead of the chain η. If $z \in V(T) - u$, we have the chain $\eta[x_j, y_p] + \mu[y_p, z]$ and this has fewer distinguished vertices than η. Proceeding further we see that we can take η to be a path from x_j to v. Then by Lemma 2.3.1, $(x_j, u) \in G$, hence $j = k$. Now add the new arc

(x_1, u) to G. If any cut arc (x_i, x_{i+1}), $1 \leq i \leq k-1$, converts into a noncut arc, then $(u, x_{i+1}) \in G$, a contradiction. Also (x_k, u) is a noncut arc of G, so the new graph has the same cut arcs as G, a contradiction.

If there is no chain connecting a vertex of $V(G)-u$ and a vertex of $T-u$, then u is a cut vertex of the undirected graph H associated with G. Let L be the piece of H with respect to u containing the vertices of C. Then by Lemma 2.3.2, L contains a symmetric arc. Now turn the piece of L (in G) at u such that a symmetric arc becomes incident with u. Then the size of the tree T is increased.

Thus we may take that there is no circuit containing exactly one vertex of the tree T. Let $\mu = [u_1, u_2, \ldots, u_p]$ be the chain in T connecting the vertices u_1 and u_p of T and $\eta = [u_1, x_1, x_2, \ldots, x_k, u_p]$ a path from u_1 to u_p with all intermediate vertices outside T. We can choose the paths μ and η such that the only arcs of the type (u_i, x_j) or (x_j, u_i) are (u_1, x_1) and (x_k, u_p).

Now consider the graph $G_1 = G + (u_p, x_k)$. If some cut arc of G belonging to μ becomes a noncut arc in G_1, we get a contradiction. If one of the arcs (x_i, x_{i+1}), $1 \leq i \leq k-1$, is a cut arc in G and a noncut arc in G_1, then

$(x_i, u_p) \in G$, a contradiction. Since G is extremal, it follows that (u_1, x_1) is a cut arc of G and a noncut arc of G_1.

Now let $k+1$ = length of $\eta \geq 3$. Then $(x_k, x_1) \in G$. Now construct the graph G^* from G by deleting the arc (u_1, x_1) and adding the arc (u_1, x_k). Evidently G^* is strong and (u_1, x_k) is a cut arc of G^*. If some cut arc (x, y) of G becomes a noncut arc in G^*, then $(x, y) = (x_i, x_{i+1})$ for some $i \leq k-1$. Also then there is a path from x_i to u_1, hence by Lemma 2.3.1, $(x_i, u_p) \in G$, a contradiction. Thus no cut arc of G becomes a noncut arc in G^*. Since $s(G^*) \geq s(G)$ and $m(G^*) = m(G)$, G^* is extremal. Next let a noncut arc (x, y) of G become a cut arc in G^*. Then since $[u_1, x_k, x_1]$ is a path from u_1 to x_1 in $G + (u_1, x_k)$, it follows that $(x, y) = (x_k, x_1)$. Now construct the graph G^{**} from G^* by adding the new arc (u_p, x_k). It can be seen that this does not convert any cut arc of G^* belonging to μ to a noncut arc in G^{**}. Also (u_1, x_k) is a noncut arc in G^{**}. Thus

$$s(G) = s(G^*) - 1 = s(G^{**}).$$

This is a contradiction since $m(G^{**}) = m(G) + 1$. Thus $s(G^*) = s(G)$. In the graph G^* we have the path $[u_1, x_k, u_p]$ from u_1 to u_p. Thus we may choose the path η in G with length 2. Let then $\eta = [u_1, x_1, u_p]$.

If now (u_1, x_1) is a noncut arc of G, then $G + (u_p, x_1)$ has the same cut arcs as G, a contradiction. Thus we may assume that (u_1, x_1) and (x_1, u_p) are cut arcs of G. If $p - 1$ = the length of $\mu \geq 2$, add the new arc (u_p, x_1) and delete the arc (u_1, x_1). This does not change the number of cut arcs in the graph since $(u_1, u_p) \in G$. Thus the size of the tree can be increased when $p \geq 3$. So let $p = 2$.

If there is a chain, other than (u_1, x_1) and (x_1, u_2) connecting x_1 and a vertex of T and with all intermediate vertices outside T, then as before it can be shown that there is a path with the same properties. This contradicts the assumption that (u_1, x_1) and (x_1, u_2) are cut arcs. Thus either $d^+(x_1) = d^-(x_1) = 1$ or x_1 is a cut vertex of the undirected graph H associated with G. In the latter case, a piece of H with respect to x_1 which does not include any vertex of T contains a symmetric arc and this piece can be transferred (in G) to a vertex of T in such a way that the size of the tree increases. Thus we may take $d^+(x_1) = d^-(x_1) = 1$. If there is any path connecting two vertices of T with all intermediate vertices outside T, we first reduce its length to 2 and then make the indegree and the outdegree of the middle vertex unity. Thus we take that if x is any vertex outside T, then there are exactly two arcs (u_1, x) and (x, u_2) incident with x and u_1, u_2 are adjacent in T.

Let now q be the number of vertices outside T. Let J be the subgraph of G generated by the vertices of T. Then

$$n(J) = n - q, \quad m(J) = m - 2q.$$

Also let $s(J) = s(G) - 2q + \beta$. Clearly $\beta \leq q$. Now consider an extremal graph J_1 on $q + 1$ vertices with $2q - \beta$ cut arcs. Evidently such a graph exists and has at least $2q$ arcs unless $q = \beta = 1$. Now attaching J_1 suitably to J, we get a graph with the same number of cut arcs as G and the size of the tree is increased.

If $q = \beta = 1$ then let x be the vertex outside T and let $(u_1, x), (x, u_2) \in G$. Since $\beta = 1$, (u_1, u_2) is a cut arc of J. If (u_2, u_1) is a noncut arc of J, then replace the arc (u_1, x) by (x, u_1) and add the new arc (u_2, x). The resulting graph has the same number of cut arcs as G, thus G is not extremal. Thus (u_2, u_1) is a cut arc of J, then each of the vertices, u_1, u_2 is either a pendant vertex of T or a cut vertex of the undirected graph H associated with T. If J has an arc (u, v) not belonging to T, we may assume that u, v belong to that piece of H with respect to u_1 which does not contain u_2. Then this piece can be turned around at u_1 (in G) such that the arc (u, v) becomes incident into u_1. Then (u, u_2) is an arc of, and $\beta = 0$ for, the new graph. Thus we can increase the size of T unless $q = 1$ and $T = J$, i.e., $s = 2n-3$. This completes the proof of the lemma.

It is easy to deduce the following

Corollary. The maximum number of cut arcs in a strong graph on n vertices is 2n-2.

This result was proved earlier by Gupta [6].

Theorem 2.3.4. The maximum number of arcs in a strong digraph with n vertices and with s cut arcs is

$$B(n, s) = \begin{cases} \binom{n-s-1}{2} + \binom{n}{2} + n - 1 & \text{if } 0 \leq s \leq n-1 \\ \binom{2n-s-1}{2} + s & \text{if } n \leq s \leq 2n-2. \end{cases}$$

Proof. If $s = 2n-3$, there is no graph with n vertices, s cut arcs and with a spanning symmetric tree. Now by Lemma 2.3.3, there is an extremal graph, with n-1 vertices generating a symmetric tree and with the indegree and outdegree of the n-th vertex being unity. Thus the theorem is proved when $s = 2n - 3$.

So let $s \neq 2n - 3$. Then we prove the theorem by induction on n. The theorem is trivial for $n = 2$ and assume the result for $n - 1$. *By Lemma 2.3.3, there is an* extremal graph G with n vertices, s cut arcs and with a spanning symmetric tree T. Let (x, y) be a pendant edge of T and x a pendant vertex. We have four cases.

<u>Case (i).</u> Both (x, y) and (y, x) are cut arcs of G. Then $d^+(x) = d^-(x) = 1$. Also by induction hypothesis, $G - x$ has at most $B(n-1, s-2)$ arcs and so G has at most $B(n-1, s-2) + 2 \leq B(n, s)$ arcs.

<u>Case (ii).</u> (x, y) is a cut arc of G and (y, x) is not a cut arc of G. Then $d^+(x) = 1$. Also $G - x$ has at least $s-1$ cut arcs and so by induction hypothesis has at most $B(n-1, s-1)$ arcs. If $s \leq n-1$, then there are at most n arcs incident with x in G. If $s \geq n$, then there are at least $s-n+1$ edges (u, v) of $T-x$ such that both (u, v) and (v, u) are cut arcs of G. If now $[x, \ldots, u, v]$ is a chain in T, then $(v, x) \notin G$. Thus we see that if $s \geq n$, there are at most $2n-s-1$ arcs of G incident with x. Thus G has at most $B(n, s)$ arcs.

Case (iii). (x, y) is a noncut arc of G and (y, x) is a cut arc of G. This is similar to case (ii).

Case (iv). Both (x, y) and (y, x) are noncut arcs of G. Then $G - x$ has at least s cut arcs and so has at most $B(n-1, s)$ arcs. Now let $[x, \ldots, u, v]$ be a chain of T. If (u, v) is a cut arc of G, then $(x, v) \notin G$. If (v, u) is a cut arc of G, then $(v, x) \notin G$. Thus there are at most $2n-s-2$ arcs of G incident with x. So G has at most $B(n-1, s) + 2n - s-2 \leq B(n, s)$ arcs.

To show that the bound $B(n, s)$ is attained, consider the following graphs $H_1(n, s)$ defined on vertices $1, 2, \ldots, n$.

In $H_1(n, s)$, (i, j) is an arc if either $i < j$ or $i = j+1$ or $k \leq j < i \leq k + n - s - 1$ where k is a fixed integer such that $1 \leq k \leq s + 1$. Here $s < n - 2$.

In $H_2(n, s)$, (i, j) is an arc if either $i < j$ or $i = j + 1$ and $i \neq k + 1, k + 2$ or $i = k + 2$ and $j = k$ or $i = \lambda$ and $j = \lambda - 1$ where k, λ are fixed integers such that $1 \leq k \leq n - 2$ and $k + 1 \leq \lambda \leq k + 2$. Here $s = n - 2$.

$H_3(n, s)$ is the graph obtained by attaching symmetric trees with a total of $2k$ arcs at some of the vertices of the following graph H. H has vertices $1, 2, \ldots, n-k$ where $0 \leq k \leq s + 1 - n$ and $k \leq n - 3$ if $s = 2n - 3$. In H, (i, j) is an arc if either $1 \leq i < j \leq 2n - s - 1$ or $i = j + 1 \leq 2n - s - 1$. Further for each i with $2n - s \leq i \leq n - k$, $d^+_H(i) = d^-_H(i) = 1$ and if (i_1, i), $(i, i_2) \in H$, then $1 \leq i_1 < i_2 \leq 2n - s - 1$. Here $s \geq n - 1$.

This completes the proof of the theorem.

It may be remarked that Theorem 2.3.4 can be proved without using Lemma 2.3.3 if it can be proved that there is an extremal graph on n vertices with s cut arcs and with a noncut vertex. Also if every extremal graph has a noncut vertex then it can be proved that $H_1(n, s)$, $H_2(n, s)$, $H_3(n, s)$ are the only extremal graphs.

<u>Corollary.</u> Any strong graph G on n vertices with $2n - 2$ cut arcs is a symmetric tree.

<u>Proof.</u> By Theorem 2.3.4, G has at most $2n - 2$ arcs, thus G is extremal. By Lemma 2.3.2, G has no circuit of length greater than 2 and so G is a symmetric tree.

This was proved by Gupta [6].

<u>Theorem 2.3.5.</u> The maximum number of cut arcs in a strong digraph with n vertices with m arcs is $s = s(n, m)$ where

$$s(n, m) = \max \{q: q \leq m \leq B(n, q)\}.$$

<u>Proof.</u> By Theorem 2.3.4, the number of cut arcs in any graph with n vertices with m arcs is at most $s(n, m)$. To show that the bound is the best possible, consider the following graph. If $m < 2n - 2$, consider a circuit on $2n - m$ vertices with an attached symmetric path of length $m - n$. If $m \geq 2n - 2$, take a graph with a spanning symmetric tree with n vertices, with $s = s(n, m)$ cut arcs and with $B(n, s)$ arcs and delete $B(n, s) - m$ arcs not belonging to the spanning symmetric tree. This completes the proof of the theorem.

Now it is not difficult to prove that the range of s, the number of cut arcs in a digraph with n vertices with m arcs, is

$$0 \leq s \leq s(n, m) \qquad \text{if } m \geq 2n,$$

$$2n - m + 1 \leq s \leq s(n, m) \qquad \text{if } n+1 \leq m \leq 2n-1,$$

$$s = n \qquad \text{if } m = n.$$

2.4 Cut vertices in undirected graphs

A vertex x of an undirected graph is called a cut vertex if $G-x$ has more components than G. A connected graph is said to have the property $P(n, r, d)$ if it has (exactly) n vertices, r cut vertices and degree of each vertex $\geq d$. Let $f(n, r, d)$ be the maximum number of edges in a graph with property $P(n, r, d)$, if one such exists and zero otherwise. An extremal graph is a graph with property $P(n, r, d)$ and with $f(n, r, d)$ edges. In this section we determine the value of $f(n, r, d)$. If $r = 0$, then $f(n, r, d)$ equals $\binom{n}{2}$ whenever $n > d$ and zero otherwise. So we assume that $r > 0$. If x is a cut vertex of G, by removing a piece of G with respect to x we mean removing the edges and vertices of that piece excepting the vertex x itself.

We observe the following simple facts. Let G be a graph with property $P(n, r, d)$. By adding a new edge to a block of G we again get a graph with property $P(n, r, d)$. If there are more than two blocks containing a cut vertex x of G then, by combining any two blocks containing x and completing it, we have another graph with property $P(n,r,d)$ which has more edges than G. Thus in an extremal graph every block is complete and there are exactly two blocks containing any cut vertex x. Thus the block graph G^* of an extremal graph G is a tree, where the block graph G^* of a graph G is a graph whose vertex set is the set of all blocks of G, two blocks are joined in G^* if and only if they have non-empty intersection in G [11].

Lemma 2.4.1. In an extremal graph G, there is no block of size p, with $3 \leq p \leq d-1$.

Proof. The lemma is vacuously true if $d \leq 3$. So let $d > 3$ and, if possible, let A generate a block of size p in G, $3 \leq p \leq d-1$. Clearly, every vertex of A is a cut vertex of G. Suppose now, if possible, the degree of every vertex of A is $\geq d+1$ in G and x a vertex of A. Then, remove the edges incident at x in this block and join x to every vertex of a pendant block of G in the piece of G with respect to x not containing A. Since, a pendant block has at least d+1 vertices and $p \geq 3$, the resulting graph has the property P(n, r, d) and has more edges than G, a contradiction, which proves that some vertex x_1 (say) of A has degree d in G. Since, x_1 is a cut vertex of G, there are exactly $d-p+2$ (≥ 3) vertices in the other block C_1 (say) containing x_1. Now, if the degree of every vertex in C_1, other than x_1, is $\geq d+1$ in G then, removing the edges incident at x_1 in C_1 and, as above, joining x_1 to every vertex of a suitable pendant block, results in a graph with property $P(n, r, d)$, since $d-p+2 \geq 3$, having more edges than G. Thus at least one of the vertices x_2 (say) of C_1, other than x_1, has degree d in G, x_2 is a cut vertex of G, and the size of the other block C_2 (say) containing x_2 is p. Proceeding as above, since the graph is finite, we finally get a cut vertex x_k (say) of G such that one of the two blocks C_{k-1}, C_k containing x_k has size p and the other $d-p+2$, the degree of every vertex of C_k, other than x_k, is $\geq d+1$ in G. Modification of the graph described above yields a graph with property $P(n, r, d)$ having more edges than G, a contrad diction. This completes the proof of the lemma.

Lemma 2.4.2. Let G be an extremal graph. Then the following statements are true.

(1) There is no chain $[D_1, D_2, D_3]$ in the block graph G^* of G with $|V(D_1)|, V(D_3)| \geq \max\{2, d\}$ and $|D_2| \geq 3$.

(2) There is at most one pair of intersecting blocks each of size provided $d \geq 3$.

Proof. To prove (1), let if possible, $[D_1, D_2, D_3]$ be a chain satisfying (1). Further let $\{x_1\} = V(D_1) \cap V(D_2)$ and $\{x_2\} = V(D_2) \cap V(D_3)$. Clearly then x_1, x_2 are cut vertices of G. Then remove all edges in D_2 adjacent to x_1, x_2 except the edge (x_1, x_2) and join each vertex of $V(D_2) - \{x_1, x_2\}$ to every vertex of a suitable pendant block. Since $|D_2| \geq 3$, x_1, x_2 are cut vertices of the new graph as well. Further $x \neq x_1, x_2$ is a cut vertex of the new graph if and only if it is a cut vertex of G. Further, since $|V(D_1)|, |V(D_3)| \geq \max\{2, d\}$ the new graph has the property $P(n, r, d)$ and has more edges than G, a contradiction. If $d = 1$, this proves that all but one pendant block are of size 2.

To prove (2), let if possible, (C_1, C_2), (C_3, C_4) be two distinct pairs of intersecting blocks such that $|V(C_i)| \geq \max\{2, d\}$, $1 \leq i \leq 4$. If two of these four blocks coincide then, by (1), we get a contradiction. Thus these four blocks are distinct. Since G^*, the block graph of G, is a tree there is a unique chain μ, without loss of generality we can assume that it is of the form $[C_1, C_2, D_1, \ldots, D_k, C_3, C_4]$ in G^*, where $k \geq 0$. If $k = 0$ or $|V(D_1)| \geq d$ (≥ 3, by hypothesis), by (1), we get a contradiction. So $k \geq 1$ and

$|V(D_1)| \leq d-1$. Then, by Lemma 2.4.1, $|V(D_1)| = 2$. Since, minimum degree in $G \geq d$, $|V(D_2)| \geq d-1$ so, by Lemma 2.4.1, $|V(D_2)| \geq d$. Now suppress the edge of D_1, amalgamate the two end vertices of this edge and using this edge separate C_1 and C_2. Again we get an extremal graph H. Since, $|V(C_2)| \geq d$ and $|V(D_2)| \geq d$ in H, as above, $|V(D_3)| = 2$ and $|V(D_4)| \geq d$. Thus proceeding, as above, we finally get an extremal graph with property $P(n, r, d)$ and a chain of three blocks each of size $\geq d$. This is a contradiction by (1), and this completes the proof of the lemma.

Now we start with an extremal graph G' and describe a procedure by which we can get an extremal graph H_0 having the following structure: H_0 consists of λ complete graphs each of size d separated by $\lambda - 1$ edges with $\lambda(d-2)+1$ of the $\lambda(d-2)+2$ terminal chains consisting of a cut edge and a complete graph on $d+1$ vertices at its end where λ is given by the following relation $r = 2(d-1).\lambda + 1 + \partial$, $0 \leq \partial \leq 2d-3$, and the exceptional chain has $\epsilon(\partial)$ vertices (not counting the vertex a (say) by which this chain is attached to the rest of the graph), ∂ cut vertices, where $n = (d^2-2).\lambda + d + 1 + \epsilon(\partial)$, and is of the structure described below provided $d \geq 3$. If $\partial = 0$, the chain is the complete graph on $\geq d$ vertices. Otherwise, the structure is as shown in the Figs. 1, 2 in which each A_i, $1 \leq i \leq [\frac{\partial}{2}]$ is of size $d+1$ and $A_{[\frac{\partial}{2}]+1}$ has size $\geq d+1$.

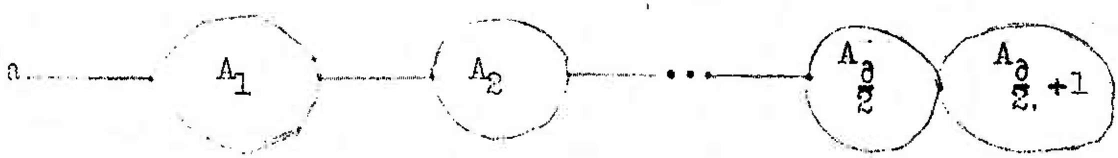


Fig. 1. ∂ is even.

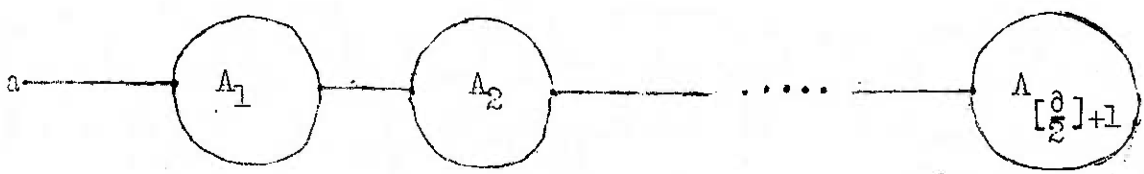
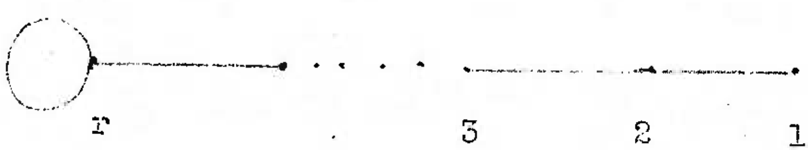
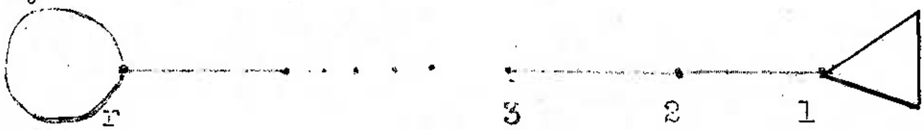


Fig. 2. ∂ is odd.

Let G be an extremal graph with property $P(n, r, d)$. If $d=1$ then, by Lemma 2.4.2, every block of G except possibly one, has size 2 and hence its structure looks like



If $d=2$, again by Lemma 2.4.2, there are exactly two pendant blocks of G. Further, one pendant block has size 3 and the second pendant block has size ≥ 3 (since G is extremal) and every other block has size 2. Thus its structure is given by



So start with an extremal graph G with property $P(n, r, d)$, where $d \geq 3$. Let D_o be a block of maximum size in G. Clearly, $|V(D_o)| \geq d+1$ and the size of any other block of $G \leq d+1$. If D_o is not a pendant block of G then we get an extremal graph G_1 having D_o as a pendant block as follows: Choose and fix a piece of G not containing D_o with respect to a cut vertex x_1 of G belonging to D_o. Then, remove all the pieces of G not containing D_o with respect to other cut vertices of G belonging to D_o and join them chainwise at a pendant block of the fixed piece of G with respect to x_1. The new graph G_1 is an extremal graph with property $P(n, r, d)$ having D_o as a pendant block. Further, in G_1 every other block has size $\leq d+1$.

In G_1, if the number of nonpendant blocks of size $d+1$ is greater than one, we further modify G_1 to get an extremal graph G_2 which has exactly one nonpendant block of size $\geq d+1$. So, suppose that E_1, E_2 are two nonpendant blocks each of size $\geq d+1$ in G_1. Let $[E_1, F_1, F_2, \ldots, F_k, E_2]$ be the unique chain in G_1^* connecting E_1 to E_2. We consider only the case $k \geq 1$, for the other case is similar. Let then

$\{x_1\} = V(E_1) \cap V(F_1)$ and $\{y_1\} = V(F_k) \cap V(E_2)$ and $x_1, x_2, \ldots, x_p$ be the cut vertices of G_1 belonging to E_1. Further let D_{i-1} be the piece of G_1 not containing E_1 with respect to x_i, $2 \leq i \leq p$. Now at the first stage remove the piece D_1 and attach it at some noncut vertex of G_1 belonging to E_2. At the j-th stage, $j \leq p - 1$, remove the piece D_j and attach it at some noncut vertex of G_1 belonging to E_2 at which D_k's, $1 \leq k \leq j-1$, are not attached. If there is no noncut vertex available at some stage i_o, $1 \leq i_o \leq p-1$, we get a graph with property $P(n, r, d)$ having more edges than G_1 as follows: Choose a cut vertex $y \neq y_1$ belonging to E_2 of the graph at the $(i_o - 1)$-th stage such that the other block containing y does not belong to the same piece of this graph with respect to y as D_o. Then suppress the edges incident at y in the other block containing y and join y to every vertex of D_o. Since $|V(E_2)| = d+1$ and $|V(D_o)|$ is maximum, the new graph has the property $P(n, r, d)$ and has more edges than G_1, a contradiction. This proves (repeating if necessary) that G_1 can be modified to get an extremal graph G_2 which has at most one nonpendant block of size $d+1$ and which has D_o as a pendant block. Further, every pendant block $\neq D_o$ of G_2 has size equal to $d+1$. Thus we have two cases.

Case (i). In G_2, there is no nonpendant block of size $d+1$. Then G_2, by Lemma 2.4.2, has at most one pair of intersecting blocks each of size $\geq d$. If there is no such pair take $H_o = G_2$. H_o has the structure described at the beginning of this reduction procedure by Lemmas 2.4.1 and 2.4.2 with the exceptional chain consisting of a cut edge and a complete graph on $\geq d+1$ vertices at its end. So assume that G_2 has a pair of intersecting blocks, C_1, C_2 (say), each of size $\geq d$. Assume r (the number of cut vertices) is > 1. Now we consider two subcases, first in which one of them C_1 (say) is a pendant block of G_2, and the second in which both of them are nonpendant blocks of

G_2. In the first subcase, C_1 and D_0 (the pendant block of maximum size) can be interchanged in an obvious way and, by Lemmas 2.4.1, 2.4.2 and by hypothesis, the new graph has the required structure with the exceptional chain consisting of the complete graph D_0. In the second subcase, by Lemma 2.4.2, $|V(C_1)| = |V(C_2)| = d$. Then these two blocks can be separated as shown in Fig. 3 and repeating this suitably, we get a graph having the required structure with the exceptional chain consisting of the complete graph D_0. This completes the proof in case (i).

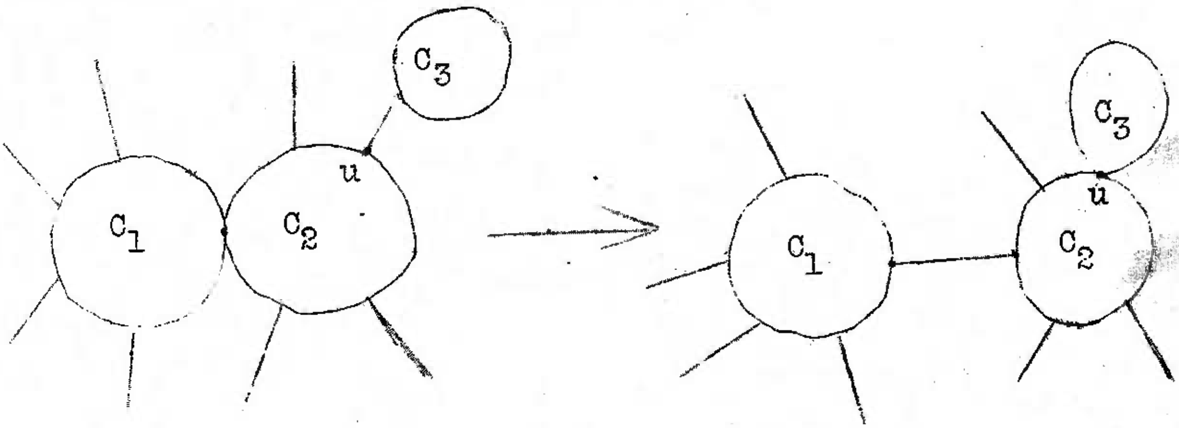


Fig. 3.

Case (ii). There is exactly one nonpendant block C (say) of size $d+1$ in G_2. Then, by Lemma 2.4.2, there is at most one pair of intersecting blocks each of size $\geq d$. First suppose that there is no such pair of intersecting blocks. Let then $x_1, x_2, \ldots, x_p$ be the cut vertices of G_2 belonging to C. Since C is a nonpendant block $p \geq 2$. Further $p \leq d-1$, for otherwise removing all the edges incident in C at some suitable x_i and joining x_i to every vertex of D_o we have a graph with property P(n, r, d) having more edges than G_2. Since there is no pair of intersecting blocks each of size $\geq d$, the other block containing x_i contains only one other vertex y_i (say), $1 \leq i \leq p$. Without loss of generality we can assume that the piece of G_2 not containing C with respect to x_i is not the graph consisting of a cut edge and a complete graph at its end for $1 \leq i \leq p_1$ namely

Fig. 4.

Now we consider the case $p_1 \geq 2$ (the other case is similar). Let then z_i be the nearest end vertex from x_i of a farthest cut edge from x_i in the piece of G_2 containing y_i with respect to x_i, $1 \leq i \leq p_1$. Further let D_i be the piece of G_2 not containing y_i with respect to z_i, $1 \leq i \leq p_1$. Obviously D_i is isomorphic to the graph of Fig. 4. Then remove the edges (x_i, y_i), $1 \leq i \leq p_1$ and the pieces D_i, $2 \leq i \leq p_1$, from the graph G_2; add the edges (y_1, y_2),

(z_i, y_{i+1}), $2 \le i \le p_1 - 1$, (z_{p_1}, x_{p_1}), then arrange the pieces D_i, $2 \le i \le p_1$, and the pieces of G_2 not containing C with respect to the other cut vertices x_i, $p_1+1 \le i \le p$, in the form of a chain as shown in Fig. 5 and attach it at x_1.

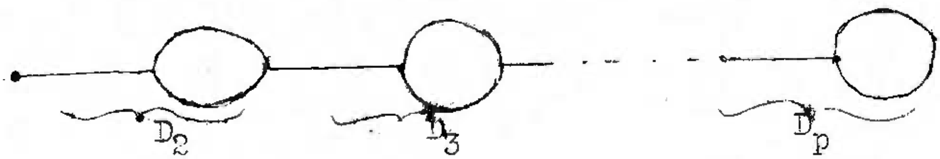


Fig. 5.

We can without loss of generality assume that $D_p = D_o$. Let H_o be the new graph thus obtained. Then H_o, by Lemmas 2.4.1, 2.4.2, has the required structure with the exceptional chain consisting of block C together with the chain described in Fig. 5 in which the number of cut vertices $\le 2d-4$ (since $p \le d-1$).

Now assume that G_2 has a pair of intersecting blocks, C_1, C_2 (say), each of size $\ge d$. Then, making the modifications as shown in Fig. 4, if necessary, we can take that one of them C_1 (say) is the block C. Now, as above, it can be proved that we can get an extremal graph with the required structure in which the exceptional chain consists of C together with the chain described in Fig. 6.

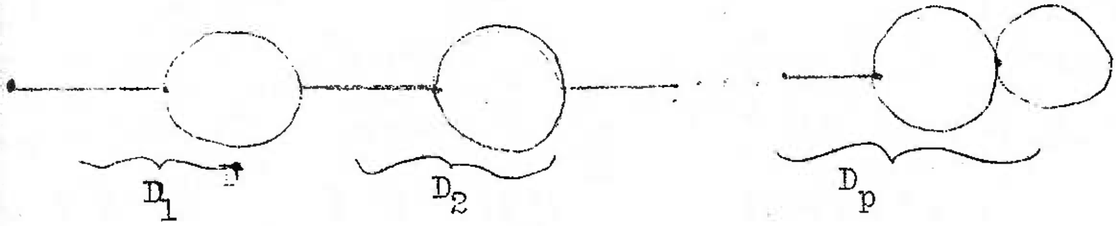


Fig. 6.

Here the exceptional chain has $\leq 2d-3$ cut vertices. Thus always we can get an extremal graph with structure described at the beginning of the reduction procedure.

Define

$$\epsilon_1(\theta) = (\tfrac{\theta}{2} + 1)d + \tfrac{\theta}{2} \cdot$$

$$m_1(\theta) = \tfrac{\theta}{2} + \tfrac{\theta}{2}\binom{d+1}{2} + \binom{1+n-\lambda d-(2\lambda+2d-7+\frac{\theta}{2})(d+1)}{2}.$$

Now counting the number of vertices, cut vertices and edges in H_o we have the following set of relations.

$$r = 2(d-1)\cdot\lambda + 1 + \partial, \qquad 0 \leq \partial \leq 2d-3 \qquad (1)$$

$$n = (d^2-2)\cdot\lambda + d + 1 + \epsilon(\partial) \qquad (2)$$

$$\epsilon(\partial) \geq \begin{cases} \epsilon_1(\partial) & \text{if } \partial \text{ is even} \\ \epsilon_1(\partial-1)+1 & \text{if } \partial \text{ is odd} \end{cases} \qquad (3)$$

Further by constructing the graph H_o we have the following

<u>Theorem 2.4.3.</u>

$$f(n,r,d) = \binom{d}{2}\cdot\lambda + \lambda - 1 + 2\lambda + 2d - 7$$

$$+ \binom{(2\lambda + 2d - 7)(d+1)}{2} + m(\partial)$$

provided $n, r, d(\partial)$ satisfy the relations (1) to (3) above and zero otherwise, where

$$m(\partial) = \begin{cases} m_1(\partial) & \text{if } \partial \text{ is even} \\ m_1(\partial - 1) + 1 & \text{if } \partial \text{ is odd.} \end{cases}$$

Now we are able to give a partial solution to a problem of Ramachandra Rao [19]. Let <u>$A(n, m, d)$</u> be the maximum number of cut vertices in a connected graph on n vertices with m edges and with minimum degree $\geq d$. Then we have the following

<u>Theorem 2.4.4.</u> $A(n, m, d) \leq r_o$

where $r_o = \max \{r : r \leq n,\ m \leq f(n, r, d)\}$. Further the bound is the best possible whenever $f(n, r_o + 1, d) \neq 0$.

<u>Proof</u>. Clearly $A(n, m, d) \leq r_o$. To show that the bound is attained whenever $f(n, r_o + 1, d) \neq 0$, consider <u>the graph H_o</u> with property $P(n, r_o, d)$. Now it is easy to

check that we can remove edges, one by one, from the exceptional chain and the block of size $\geq d + 2$ (exists since $f(n, r_o + 1, d) \neq 0$) of H_o to get a graph with $f(n, r_o + 1, d) - 1$ edges and with property $P(n, r_o, d)$ such that at each stage we have a graph with property $P(n, r_o, d)$. Thus the bound is the best possible and this completes the proof of the theorem.

2.5 Cut edges in undirected graphs

A connected graph is said to have the property $Q(n, s, d)$ if it has exactly n vertices, s cut edges and minimum degree $\geq d$. Let $g(n, s, d)$ be the maximum number of edges in a graph with property $Q(n, s, d)$ if such a graph exists and zero otherwise. An extremal graph is a graph with property $Q(n, s, d)$ and with $g(n, s, d)$ edges. Clearly, if $s = 0$ then $g(n, s, d) = \binom{n}{2}$ whenever $n \geq d$ and zero otherwise. So we assume that $s > 0$. If x is a cut vertex of G, by removing a piece of G with respect to x we mean removing the edges and vertices of that piece excepting the vertex x itself.

The following facts can be easily proved. In an extremal graph G every **block** is complete and there is at most one block of size ≥ 3 containing a fixed cut vertex.

Now we prove the non-existence of blocks of certain sizes in an extremal graph by proving the following

Lemma 2.5.1. In an extremal graph G there is no block of size p, $3 \leq p \leq d$.

Proof. Let, if possible, $A = \{x_1, x_2, \ldots, x_p\}$, $3 \leq p \leq d$, generate a block in G. Then construct a new graph H from G by shrinking A to a vertex, x_1 (say), and completing a pendant block of G by adding the vertices of $A - \{x_1\}$ to that block. Clearly, H has no multiple edges and has n vertices. Since $p \geq 3$, no edge of the block A is a cut edge of G so H has the same number of cut edges as G. Further, since $p(d - p + 1) \geq d$ and a pendant block has at least $d+1$ vertices, it follows that the minimum degree in H is also $\geq d$. Thus H has the property $Q(n, s, d)$ and $m(H) > m(G)$, a contradiction. This completes the proof of the lemma.

Lemma 2.5.2. Let $1 \leq s \leq d-1$.

Then

$$g(n, s, d) = \begin{cases} s + s\binom{d+1}{2} + \binom{n-s(d+1)}{2} & \text{if } n \geq (s+1)(d+1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let G be an extremal graph in which $s \leq d-1$. Since $s \leq d-1$, there is exactly one block A_x (say) of size ≥ 3 containing a cut vertex x, by the observation made at the beginning of this section. Now, by Lemma 2.5.1, $|V(A_x)| \geq d+1$. Then arrange the blocks of G in the form a chain in which blocks of size $\geq d+1$ are separated by cut edges of G. Now in the new extremal graph every block of size ≥ 3, except possibly one, has size equal to $d+1$. Counting the number of vertices and edges in this graph we have $n \geq (s+1)(d+1)$ and $g(n,s,d) \leq s + s\binom{d+1}{2} + \binom{n-s(d+1)}{2}$.

To show that the bound is attained, construct a graph G on n vertices as follows. Partition the n vertices into s+1 sets A_i, $1 \leq i \leq s+1$ such that $|A_i| = d+1$, $1 \leq i \leq s$ and $|A_{s+1}| \geq d+1$. This is possible whenever $n \geq (s+1)(d+1)$. Take complete graphs on A_i's and arrange them in the form of a chain where A_i's are separated by cut edges. This completes the proof of the lemma.

<u>Lemma 2.5.3.</u> If $s \geq d$, then in any extremal graph G there is a cut vertex x such that every block containing x is of size equal to 2.

<u>Proof</u>. Suppose in some extremal G there is no cut vertex with the property stated in the lemma. Then, by Lemma 2.5.1, there is exactly one block of size $\geq d+1$ containing any fixed cut vertex. Choose and fix a cut vertex x_0 of G and collect all the cut edges of G to the vertex x_0 as follows. Let (u, v) be a cut edge of G with

$u, v \neq x_0$. Without loss of generality assume that the piece of G with respect to u containing v does not contain x_0. Remove this piece and attach it at x_0. The new graph H thus obtained is also an extremal graph. Thus, repeating if necessary, collect all the cut edges at x_0. Since $s \geq d$, the degree of x_0 in the new extremal graph is $\geq 2d+1$. Now remove the piece of G with respect to x_0 which contains the block C (say) of size $\geq d+1$ containing x_0, and attach it at a pendant block B by joining every vertex of $C - \{x_0\}$ to those of B. We thus get another graph with property $Q(n, s, d)$. Further, since a pendant block has at least $d+1$ vertices, this graph has more edges than G, a contradiction and this completes the proof of the lemma.

Now we proceed to find the value of $g(n, s, d)$. We start with an extremal graph G and describe a procedure by which we can get another extremal graph H in which all the cut edges form a tree T on $s+1$ vertices, the degree of every nonpendant vertex of T, except possibly one, is equal to $\max\{2, d\}$ and the degree of the exceptional vertex lies between d and $2d-2$. Further H consists of complete graphs on $d+1$ vertices attached at all but one pendant vertex of the tree T and a complete graph on $\geq d+1$ vertices attached at the exceptional pendant vertex of T, whenever $d \geq 2$.

So we start with an extremal graph G in which $s \geq d$. First we modify G to get an extremal graph in which every block of size ≥ 3 is a pendant block. Suppose, in G there is a nonpendant block C (say) of size ≥ 3. Since $s \geq d$, by Lemma 2.5.3, there is a cut vertex x_0 of G such that every block of G containing x_0 is of size 2. Then, remove all the pieces of G with respect to cut vertices of G in C, except the piece containing x_0, and attach them at x_0. The new extremal graph thus obtained has less number of nonpendant blocks of size ≥ 3 than G. Thus, repeating if necessary, we can get an extremal graph in which every block of size ≥ 3 is a pendant block of G.

Thus we may and do assume that in G itself every block of size ≥ 3 is a pendant block. Clearly then the s cut edges of G form a tree T (say) on $s+1$ vertices. Choose and fix a nonpendant vertex x_0 of the tree T. If some vertex $y \neq x_0$ of the tree T has degree $\geq d+1$ in T, remove all but d pieces of G with respect to y not containing x_o and attach them at x_o. Let H be the new extremal graph thus obtained. In H also every block of size ≥ 3 is a pendant block. If now the degree of x_o in H is greater than $2d-2$, remove all but d pieces of H with respect to x_o and attach them at a pendant vertex y_o (say) of the tree formed by the cut edges of H. Then remove the piece of H with respect to y_o containing the block of size ≥ 3 at y_o, add it to a pendant block of H and complete it. The new graph has the property $Q(n, s, d)$ and has more edges than G, a contradiction. Thus, in H the degree of $x_o \leq 2d-2$. Further, since H is extremal every pendant block of H, except possibly one, has size $d+1$ and thus H has the structure described above.

Let now p be the number of pendant vertices in the tree T formed by the s cut edges of H. Then, if $d > 1$,

$$p = (s+1) - \left[\frac{s-1}{d-1} \right] \quad \text{and}$$

$$n \geq dp + s+1 = (d+1)(s+1) - \left[\frac{s-1}{d-1} \right] d.$$

If $d = 1$, $p = 2$ and $n \geq s+1$. Now counting the number of edges in H we have

$$m(H) = g(n,s,d) \leq s + (p-1)\binom{d+1}{2} + \binom{n-s-d(p-1)}{2},$$

whenever $n \geq (d+1)(s+1) - [\frac{s-1}{d-1}]d$ and $d > 1$,

$g(n, s, d) \leq s + \binom{n-s}{2}$ whenever $n \geq s+1$ and $d = 1$

$g(n, s, d) = 0$ otherwise.

To show that the bound is attained, construct a graph G on n vertices as follows. If $d=1$, take a chain on $s+1$ vertices and attach at one of its ends a complete graph on $n - s - 1$ vertices. So let $d > 1$ and $n \geq (d+1)(s+1) - [\frac{s-1}{d-1}]d$. A nonpendant vertex x of a tree T is called a <u>next vertex</u> if every vertex, except possibly one, adjacent to it is a pendant vertex in T. Let $V = \{1, 2, \ldots, n\}$. Construct a tree T on $A = \{1, 2, \ldots, n\}$ in which degree of every nonpendant vertex, except possibly one next vertex, is d and the degree of the exceptional vertex is between d and $2d-2$. Let $x_1, x_2, \ldots, x_p$ be the pendant vertices of T where

$p = (s+1) - [\frac{s-1}{d-1}]$. Partition $V - A$ into p sets A_i, $1 \leq i \leq p$, with $|A_i| = d$ for $1 \leq i \leq p-1$, and $|A_p| \geq d$. Since $n \geq (d+1)(s+1) - [\frac{s-1}{d-1}]d$, such a partition is possible. Now take complete graphs on A_i and join x_i to every vertex of A_i, $1 \leq i \leq p$. Let G be the resulting graph. Then G has n vertices, s cut edges, minimum degree $\geq d$ and

$s + (p-1)\binom{d+1}{2} + \binom{n-s-d(p-1)}{2}$ edges. We modify the structure of G to get another extremal graph G_1 (which will be used in the proof of Theorem 2.5.4.). Let x_0 be the next vertex of the tree T. Suppose degree of $x_0 > d$ in G. Remove all but d pieces of G with respect to x_0 not containing the vertex of degree > 1 adjacent to x_0 (if any) and arrange them in the form of a chain where blocks of size ≥ 3 are separated by cut edges and attach this chain at a pendant block of G.

Thus we have the following

Theorem 2.5.4. If $s \geq d$, then

$$g(n,s,d) = \begin{cases} s + \left(s - \left[\frac{s-1}{d-1}\right]\right)\binom{d+1}{2} + \binom{n-s-d\left(s - \left[\frac{s-1}{d-1}\right]\right)}{2} \\ \quad \text{whenever } n \geq (d+1)(s+1) - \left[\frac{s-1}{d-1}\right]d \text{ and } d > 1 \\ s + \binom{n-s}{2} \quad \text{if } n \geq s+1 \text{ and } d = 1 \\ 0 \quad \text{otherwise.} \end{cases}$$

Now we give a partial solution to a problem of Ramachandra Rao [19]. Let $B(n, m, d)$ be the maximum number of cut edges in a graph on n vertices, m edges having minimum degree $\geq d$.

<u>Theorem 2.5.5.</u> $B(n, m, d) \leq s_o$,

where $s_o = \max\{s : s \leq n-1, \ m \leq g(n, s, d)\}$

Further the bound is the best possible whenever $g(n, s_o + 1, d) \neq 0$, or $s_o \leq d - 1$ and $m \geq \frac{nd}{2} + 1$ if d is even, or $s_o \leq d-1$ and $n \geq (s_o+1)(d+1) + 2$.

<u>Proof.</u> It is evident that $B(n, m, d) \leq s_o$. We prove the equality in the case $g(n, s_o+1, d) \neq 0$ and $s_o \geq d$. Let G_1 be the extremal graph with property $Q(n, s_o, d)$ described above. Then it is easy to check that we can remove edges, one by one, from the exceptional chain (if any) and the block of size $\geq d+2$ (exists since $g(n, s_o+1, d) \neq 0$) of G_1 until we get a graph with $g(n, s_o+1, d) - 1$ edges such that at each stage we have a graph with property $Q(n, s_o, d)$. Thus the equality holds in this case. Similarly the equality can be proved in the other cases and this completes the proof of the theorem.

CHAPTER 3

ON THE POWERS OF A TREE

The concept of the square of a graph is due to Ross and Harary who introduced it in [23] and obtained a criterion for a graph to be the square of a tree. In [13] Mukhopadhyay obtained necessary and sufficient conditions for a graph to be the square of some graph. The n-th power of a graph was defined by Harary, Karp and Tutte in [10] and they obtained a criterion for the planarity of the n-th power of a graph. In this chapter we obtain necessary and sufficient conditions for a graph to be the cube of a tree, the fourth power of a tree. In Section 1 we obtain necessary conditions for a graph to the cube of a tree and prove that T^3 determines the tree T uniquely, up to isomorphism. In Section 2 we give a criterion for a graph to be the cube of a tree and an algorithm to determine the tree cube root of a graph. In Section 3 necessary conditions for a graph to be the 2k-th power of a tree are obtained which generalise some of the results of Ross and Harary [23]. Using these results, in Section 4, we obtain necessary and sufficient conditions for a graph to be the fourth power of a tree. We give an algorithm to determine all the tree fourth roots of a graph. Finally graphs with a unique tree fourth root are characterised. The algorithms afore mentioned utilise a result of Harary and Ross [8] for determining all the cliques in a given graph.

Let k be a positive integer. The k-th power of a graph, denoted by G^k, has the same set of vertices as G, and the vertices x and y $(x \neq y)$ are adjacent in G^k if and only if they are joined in G by a chain of length less than or equal to k.

If A is a subset of V(G), then G[A] denotes the subgraph of G spanned by A. A set A of vertices is complete if G[A] is the complete graph. A clique of G is a maximal complete set of vertices of G. Let $C_k(G)$ be the set of all cliques of G with at least 2k+1 vertices. Let L(G) be the set of all cliques with at least four vertices of G. Let $\eta(G)$ be the class of all complete sets of vertices A of G with at least three vertices such that $A = A_1 \cap A_2$, where $A_i \in L(G)$ and $G[A_1 \cup A_2 - A]$ is disconnected. Let $M_1(G)$ be the set of all vertices of G which belong to exactly one member of $\eta(G)$ and $M_2(G)$ the set of all vertices of G which belong to at least two distinct members of $\eta(G)$.

Let T be a tree and $S(k, T)$ be the set of all vertices x of T such that there are two vertices y, z at distance k from x in T and belonging to different pieces of T with respect to x. Let $N(x, k, T) = \{y: d_T(x,y) \leq k\}$, $d_G(x, y)$ denotes the distance from x to y in the graph G.

3.1 The cube of a tree

Throughout this section we assume that $G = T^3$ where T is a tree with at least four vertices. Also let $N(x, T) = N(x, 1, T)$, and $S(T) = S(2, T)$. In this section we find necessary conditions for G to be the cube of a tree. Further we prove that T^3 determines T uniquely up to isomorphism.

Lemma 3.1.1. G is complete if and only if $S(T)$ is empty.

Proof. We observe that G is complete if and only if diameter of $T \leq 3$, that is, if and only if $S(T)$ is empty.

In what follows we assume that G is not the complete graph.

Lemma 3.1.2. $A \in L(G)$ if and only if there is a vertex x in $S(T)$ such that

i) $N(x, T)$ is a proper subset of A,

ii) $A - N(x, T)$ is the set of all vertices at distance two from x and belonging to a fixed piece of T with respect to x.

Proof. Let A be a set of vertices and x, a vertex in $S(T)$ satisfying conditions (i) and (ii) of the lemma. Then by (i), $|A| \geq 4$ and by (ii), A is a clique of G. Thus $A \in L(G)$.

Conversely, let $A \in L(G)$. Then if y, z are vertices in A, the maximality of A implies that every vertex in the unique chain joining y and z in T is also in A. Therefore $T[A]$ is a tree. Since G is not complete, diameter of $T[A]$ is three. Let $[x_0, x_1, x_2, x_3]$ be a diameteral chain of $T[A]$. It is evident that one of the vertices x_1, x_2, say x_1, belongs to $S(T)$. Since $G[A]$ is complete, every vertex of A is adjacent to x_1 or x_2 in T. Now the maximality of A implies that $N(x_i, T) \subseteq A$, $i = 1,2$. Thus taking $x = x_1$, conditions (i) and (ii) of the lemma are satisfied and this completes the proof of the theorem.

Corollary 3.1.3. If $A_1 \in L(G)$ then there exists $A_2 \in L(G)$ such that $A_1 \cap A_2 \in \eta(G)$.

Proof. Let x_1 be a vertex corresponding to A_1 given by Lemma 3.1.2. Since $x_1 \in S(T)$, there is a chain $[x_1, y, u]$ in T such that $u \notin A_1$. To complete the proof take $A_2 = N(x_1, T) \cup N(y, T)$.

Theorem 3.1.4. $A \in \eta(G)$ if and only if there is a vertex x in $S(T)$ such that $A = N(x, T)$. Also then A determines x uniquely.

Proof. Let $A \in \eta(G)$. By definition, $|A| \geq 3$ and there are sets of vertices A_1, A_2 such that $A = A_1 \cap A_2$ where $A_1 \neq A_2$ and $A_i \in L(G)$, $i = 1,2$. Let x_i be a vertex of $S(T)$ corresponding to A_i given by Lemma 3.1.2. If $x_1 = x_2$, then $A = N(x_1, T)$. If $d_T(x_1, x_2) = 1$, since $|A| \geq 3$, A is either $N(x_1, T)$ or $N(x_2, T)$. If $d_T(x_1, x_2) = 2$ and if x is the middle vertex in the unique chain in T joining x_1, x_2, then $A = N(x, T)$. If $d_T(x_1, x_2) \geq 3$, then $|A| \leq 2$, a contradiction. Thus there exists a vertex x in $S(T)$ such that $A = N(x, T)$. The uniqueness of x is trivial.

Conversely, let x be in $S(T)$ and $A = N(x, T)$. Let x_1, x_2 be vertices at distance two from x in T and belonging to different pieces of T with respect to x. Define D_i to be the set of all vertices y such that $d_T(x, y) = 2$ and y belongs to the same piece of T with respect to x as x_i, $i = 1,2$. Let $A_i = N(x, T) \cup D_i$. Then $A = A_1 \cap A_2$ and, by Lemma 3.1.2, $A_i \in L(G)$, $i = 1,2$. Thus $A \in \eta(G)$ and this completes the proof of the theorem.

Lemma 3.1.5. If $|\eta(G)| \geq 2$, then $M_2(G) = S(T)$.

Proof. If x_1, x_k are vertices in $S(T)$ and $[x_1, x_2, \ldots, x_k]$ is the unique chain in T connecting x_1 to x_k, then there exist vertices $y \neq x_2$ and $z \neq z_{k-1}$ such that (x_1, y) and (x_k, z) are edges in T, hence $x_i \in S(T)$ for every i, $2 \leq i \leq k-1$. Thus $T[S(T)]$ is a tree. Now the result follows easily from Lemma 3.1.4.

Lemma 3.1.6. Every pair of adjacent vertices of G is contained in a member of $L(G)$ and G is connected.

Proof. If x, y are any two vertices connected by a chain of length at most three in T, then two other vertices, z, u exist such that x, y, z, u form a complete set of vertices in G. Now the lemma follows from the definitions of $L(G)$ and T^3.

Before proving the uniqueness of the tree T, we prove the following

Lemma 3.1.7. Let T be a tree such that $|S(T)| \geq 3$. Let $S_*(T) = \cup \{ N(b, T): b \in S(T) \}$.

(1) Two vertices a, b in S(T) are adjacent in T if and only if there exist A, B in $\eta(G)$ such that $A \cap B = \{ a, b \}$.

(2) Let $a \in S_*(T) - S(T)$ and $b \in S(T)$ then a, b are adjacent in T if and only if the following conditions (i) and (ii) are satisfied.

i) There exists c in $\eta(G)$ containing both a and b, and there is no B in $\eta(G)$ such that $a, b \notin B$ and both a and b are joined in T^3 to all vertices of B,

ii) If a set $B \in \eta(G)$ contains b then $a \in B$ or a is joined in G to every vertex of B.

(3) No two vertices of $S_*(T) - S(T)$ are adjacent in T.

Proof. Statements (1) and (3) and the only if part of (2) are easily proved. To prove the if part of (2) let conditions 2(i) and 2(ii) be satisfied. Then by condition 2(i), $d_T(a, b) \leq 2$. Now if possible let [a, c, b] be a path of length two in T. Since $a \in S_*(T) - S(T)$, it follows that

$c \in S(T)$. By 2(i), b is the only vertex of $S(T)$ adjacent to c in T. Further, by 2(ii), c is the only vertex of $S(T)$ adjacent to b in T. Thus $|S(T)| = 2$, a contradiction which shows that a, b are adjacent in T. This completes the proof of the lemma.

The following theorem proves the uniqueness of the tree cube root of G.

Theorem 3.1.8. Let T be a tree such that $|S(T)| \geq 1$. Then T^3 determines T uniquely, up to isomorphism.

Proof. The theorem can be easily proved when $|S(T)|$ is one or two. So we assume that $|S(T)| \geq 3$. Let $S_*(T) = M_1(T^3) \cup M_2(T^3)$. By Lemma 3.1.7, $T[S_*(T)]$ is uniquely determined by T^3. Let now $b \in S(T)$, $y_1, \ldots, y_k$ be the vertices of $S_*(T) - S(T)$ adjacent to b in T and d_i be the number of pendant vertices of T adjacent to y_i in T, $1 \leq i \leq k$. Then the nonzero numbers among $d_1, \ldots, d_k$ are the sizes of the components of $T^3[D]$ which are disjoint with $S_*(T)$, where D is the set of all vertices outside $N(b, T)$ which are adjacent in T^3 to every vertex of $N(b, T)$. Thus T^3 determines T uniquely up to isomorphism. This completes the proof of the theorem.

3.2 Characterisation and algorithm

In this section we give a characterisation of the cube of a tree and an algorithm for determining the tree cube root of a graph when it exists.

Let G be the complete graph on n vertices. Then G is the cube of any tree whose set of vertices is V(G) and whose diameter is ≤ 3. The number of non-isomorphic tree cube roots of G is equal to

$$\left[\frac{|V(G)| - 4}{2}\right] + 2.$$

Theorem 3.2.1. Let G be a graph such that $\eta(G)$ has exactly one element A(say). Then G is the cube of a tree if and only if the following conditions are satisfied.

(1) Every vertex of V(G) - A is joined to all vertices of A.

(2) Every connected component of G[V(G) - A] is complete and their number is $\leq |A| - 1$.

Proof. The proof is easy and is omitted. Further it can be proved that G determines T uniquely up to isomorphism.

The following theorem gives a criterion for a graph to be the cube of a tree.

Theorem 3.2.2. Let G be such that $|\eta(G)| \geq 2$. Then G is the cube of a tree if and only if G satisfies the following conditions.

(1) G is connected and every pair of adjacent vertices of G belongs to a member of $L(G)$.

(2) If $L_1 \in L(G)$, there exists $L_2 \in L(G)$ such that $L_1 \cap L_2 \in \eta(G)$.

(3) a) Any two distinct members of $\eta(G)$ intersect in at most two vertices.

b) If G^* is the graph whose vertices are the members of $\eta(G)$, two vertices A_1, A_2 of G^* being joined by an edge if $|A_1 \cap A_2| = 2$, then G^* is a tree.

c) If $A_1, A_2 \in \eta(G)$ and $|A_1 \cap A_2| = 1$, then $d_{G^*}(A_1, A_2) = 2$.

d) If $A_1, \ldots, A_k$ are all the vertices of G^* adjacent to some vertex A of G^*, then $A_i \cap A_j$ contains exactly one vertex b and this vertex is independent of i and j, $i \neq j$, $i \leq i < j \leq k$. We will write then $A = A_b$. If G^* is not an edge and if A is a terminal vertex of G^* adjacent to A_b, then we will write $A = A_c$ where

$A_b \cap A = \{b, c\}$. If G^* has exactly two vertices A_1

and A_2 and $A_1 \cap A_2 = \{a, b\}$, then write $A_1 = A_a$ and $A_2 = A_b$.

(4) Let D_b denote the set of all vertices of G outside A_b which are joined in G to every vertex of A_b. Then

(a) If $d \in A_b \cap M_2(G)$ and $d \neq b$, then $A_d - \{b, d\}$ is a component of $G[D_b]$. If a component F of $G[D_b]$ intersects $M_1(G) \cup M_2(G)$ then there is a vertex d in $M_2(G) \cap A_b$ such that $d \neq b$ and $F = A_d - \{b, d\}$.

(b) If p_b denotes the number of components of $G[D_b]$, then $p(b) \leq |A_b| -1$.

(c) If a vertex belongs to D_b and D_c with $b \neq c$ then it belongs to some member of $\eta(G)$.

Proof. First let G be the cube of a tree T. Then condition (1) follows from Lemma 3.1.6, condition (2) from Corollary 3.1.3, and condition 3(a) from Theorem 3.1.4. Conditions 3(b), 3(c) and 3(d) follow from Theorem 3.1.4 and the fact that G^* is isomorphic to $T[S(T)]$ and $T[S(T)]$ is a tree. To prove conditions 4(a) and 4(b) observe that D_b is the set of all vertices which are at distance two from b in T and belonging to the same piece of T with respect to b. To prove 4(c), let $x \in D_b \cap D_c$. Then since $T[S(T)]$ is a tree, some vertex y of $S(T)$ is adjacent in T to x and $x \in A_y$.

To prove sufficiency, let, G be a graph satisfying conditions (1) to (4) of the theorem. Then we construct a tree T with $V(T) = V(G)$ and prove that $T^3 = G$.

First we observe that A_b <---> b is a one-one correspondance between members of $\eta(G)$ and elements of $M_2(G)$ such that

i) $b \in A_b$,

ii) if A_b and A_c intersect in two vertices, then $A_b \cap A_c = \{b, c\}$,

iii) if $b, c \in M_2(G)$ and $b \in A_c$ then $c \in A_b$.

(i) and (ii) follow from condition 3(d). (iii) follows from conditions 3(c) and 3(d).

Step 1. Find all the cliques of G, a method for which is given in [8]. Then determine the class $\eta(G)$.

Step 2. Construct a graph on the vertices of $M_2(G)$ as follows. Join two vertices b and c if and only if A_b and A_c intersect in two vertices. The resulting graph T_1 is a tree by condition 3(b).

Step 3. Join each of the vertices of $M_1(G) \cap A_b$ to b. Let the resulting tree be T_2. The vertices of T_2 are the elements of $M_1(G) \cup M_2(G)$.

Step 4. Let $D_1, D_2, \ldots,$ be the connected components of $G[D_b]$ which are disjoint with the vertices of T_2. Then join all vertices of D_j to some vertex y in $M_1(G) \cap A_b$, the vertices y being different for different components D_j. This is possible by conditions 4(a) and 4(b). If a vertex belongs to D_b and D_c, then it belongs to T_2 by condition 4(c); thus the resulting graph T is a tree. By (1) and (2), every vertex of G belongs either to $M_1(G) \cup M_2(G)$ or to D_b for some b. Hence $V(T) = V(G)$.

Suppose now $b \in M_2(G)$ and E is a component of D_b. If $[x_1, x_2, x_3]$ is a chain in E and (x_1, x_3) is not an edge in G, then $A_b \cup \{x_1, x_2\}$ and $A_b \cup \{x_2, x_3\}$ are contained in two members of $L(G)$ whose intersection contains $A_b \cup \{x_2\}$. This is a contradiction to 3(a). Thus E is complete.

Let b be an end vertex of T_1. Since $A_b \in \eta(G)$, $G[D_b]$ has at least two components E_1, E_2. By 4(a), at least one of E_1, E_2 is disjoint with $M_1(G) \cup M_2(G)$. Thus by our construction there exists a vertex of $V(T)-V(T_2)$ at distance two from b in T. Hence every end vertex of T_1 and so every vertex of T_1 belongs to $S(T)$. By construction there is no other vertex in $S(T)$, thus $S(T) = M_2(G)$.

It can be easily verified that $A_b = N(b, T)$ for all $b \in S(T)$.

Let now $L_1 \in L(G)$. Then by (2) there exists $L_2 \in L(G)$ such that $L_1 \cap L_2 \in \eta(G)$. Let $A_b = L_1 \cap L_2$. Evidently $L_1 - A_b \subseteq D_b$. Let E be a component of $G[D_b]$ containing $L_1 - A_b$. Then E is complete in G. If possible, let $x \in V(E) - (L_1 - A_b)$. Then $L_1 \cup \{x\}$ is complete in G, a contradiction since $L_1 \in L(G)$. Thus $V(E) = L_1 - A_b$. If $V(E)$ is disjoint with the vertices of T_2, then by construction and by Lemma 3.1.2, $L_1 = (A_b \cup V(E)) \in L(T^3)$. If $V(E)$ intersects $M_1(G) \cup M_2(G)$, then by 4(a) and Lemma 3.1.2, $L_1 = (A_b \cup V(E)) \in L(T^3)$. Conversely, let $L_1 \in L(T^3)$. Then by Lemma 3.1.2 and condition 4(a) it follows that $L_1 \in L(G)$. Thus $L(G) = L(T^3)$. By condition (1) and Lemma 3.1.7, it follows now that $G = T^3$. This completes the proof of the theorem.

3.3 The even powers of a tree

In this section we obtain some necessary conditions for a graph G to be the $2k$-th power of a tree where k is a positive integer. So let k be a fixed positive integer and let $G = T^{2k}$ where T is a tree.

Lemma 3.3.1. G is complete if and only if $|S(k, T)| \leq 1$.

Proof. G is complete if and only if diameter of $T \leq 2k$, that is, if and only if $|S(k, T)| \leq 1$.

In what follows we assume that G is not a complete graph.

Lemma 3.3.2. $T[S(k, T)]$ is a tree.

Proof. If two vertices belong to $S(k, T)$ then every vertex in the unique chain joining them in T also belongs to $S(k, T)$. Hence $T[S(k, T)]$ is a tree.

Now we proceed to prove our main theorem of this section.

Theorem 3.3.3. If $A \in C_k(G)$, then there is a unique x in $S(k, T)$ such that $A = N(x, k, T)$. Conversely, if x is in $S(k, T)$, then $N(x, k, T)$ belongs to $C_k(G)$.

Proof. Let A be in $C_k(G)$. Since A is maximal, every vertex in the unique chain in T joining any two vertices of A also belongs to A. Hence $T[A]$ is a tree. Clearly, diameter of $T[A] \leq 2k$. If possible, let the diameter of $T[A]$ be less than $2k$. Since G is not the complete graph there exists a vertex x in $V(G) - A$ adjacent in T to some vertex of A. Now $A \cup \{x\}$ is complete in G, a contradiction. Thus diameter of $T[A]$ equals $2k$. Let then x be the unique centre of $T[A]$. Clearly x belongs to $S(k, T)$ and $A = N(x, k, T)$. The uniqueness of x is evident.

Conversely, let x be in $S(k, T)$ and $A = N(x,k,T)$. Clearly, $|A| \geq 2k + 1$, and distance in T between any two vertices of $A \leq 2k$. Hence A is complete in G. Let x_1, x_2 be vertices at distance k from x in T and belonging to different pieces of T with respect to x. Now if y is a vertex outside A then $d_T(x_i, y) > 2k$ for at least one $i = 1$ or 2. So in G no vertex outside A is joined to all vertices of A. Hence $G[A]$ is a maximal complete subgraph of G. Thus $A \in C_k(G)$ and this completes the proof.

Lemma 3.3.4. Two vertices x_1, x_2 of $S(k, T)$ are adjacent in T if and only if the only members of $C_k(G)$ containing $N(x_1, k, T) \cap N(x_2, k, T)$ are $N(x_i, k, T)$, $i = 1,2$ and $|N(x_1, k, T) \cap N(x_2, k, T)| \geq 2k$.

Proof. This lemma follows from Lemma 3.3.2 and Theorem 3.3.3.

Remark. If $k = 1$, then two vertices x_1, x_2 of $S(k, T)$ are adjacent in T if and only if $|N(x_1, 1, T) \cap N(x_2, 1, T)| = 2$. This was obtained earlier by Ross and Harary [23].

Lemma 3.3.5. Every pair of adjacent vertices of G belongs to a member of $C_k(G)$.

Proof. Let (x_1, x_2) be an edge of G. By definition of $G = T^{2k}$, $d_T(x_1, x_2) \leq 2k$. Let μ be the unique chain in T joining x_1 to x_2. If there is a vertex y of $S(k, T)$ on μ then there also exists z of $S(k, T)$ on μ such that $d_T(x_i, z) \leq k$, $i = 1,2$ and x_1, x_2 belong to $N(z, k, T)$ which by Theorem 3.3.3 is a member of $C_k(G)$. Thus the theorem will be proved in this case so assume that no vertex of μ belongs to $S(k, T)$. Define

$$\partial_i = \min \{ d_T(x_i, z) : z \in S(k, T) \}, \ i = 1,2.$$

By Lemma 3.3.1, $S(k, T)$ is nonempty so ∂_i is well defined and $\partial_i \leq k$. Choose $z_i \in S(k, T)$ such that $d_T(x_i, z_i) = \partial_i$, $i = 1, 2$. We now prove that $z_1 = z_2$. Let ξ be the first vertex at which the chain in T joining z_1 to x_1 meets μ. Since $\xi \notin S(k, T)$, by Lemma 3.3.2, z_1, z_2 belong to the same piece of T with respect to ξ. But then, by definition of ∂_i, $z_1 = z_2$. Since $\partial_i \leq k$, x_1, x_2 belong to $N(z_1, k, T)$ which belongs to $C_k(G)$ by Theorem 3.3.3 and this completes the proof of the lemma.

Remark. It can be easily seen that the results proved in this section are generalisations of the lemmas and theorem of [23], by taking $k = 1$.

3.4 The fourth power of a tree

In this section we give a criterion for a graph G to be the fourth power of a tree and an algorithm for determining a tree fourth root of G when it exists. Further we characterise all graphs with a unique tree fourth root.

Clearly, if G is the complete graph then G is the fourth power of any tree T with $V(T) = V(G)$ and whose diameter ≤ 4.

Theorem 3.4.1. If $|C_2(G)| = 2$, let A_1, A_2 be the members of $C_2(G)$. Then G is the fourth power of a tree if and only if

(1) every pair of adjacent vertices of G is contained in A_1 or A_2 and G is connected,

(2) $\partial = |A_1 \cap A_2| \geq 4$.

Further, the tree fourth root of G is unique, up to isomorphism, if and only if $\partial = 4$, or $\partial = 5$ and $|V(G)| = 7$.

Proof. The necessity is easy to prove. To prove sufficiency, let G be a graph satisfying conditions (1) and (2) of the theorem. Construct a tree with $V(T) = V(G)$ as follows. Join two vertices v_1, v_2 belonging to $A_1 \cap A_2$ by an edge. Partition the remaining set of vertices of $A_1 \cap A_2$ into two sets B_1, B_2 and join v_i to each vertex of A_i, $i = 1,2$. Then join each vertex of $A_i - A_1 \cap A_2$ to some vertex of B_i, $i = 1,2$. Now it can be easily shown that $T^4 = G$ and the above construction gives all the tree fourth roots of G. Now the second part follows easily. This completes the proof.

Let now $|C_2(G)| \geq 3$ and A_i's, $1 \leq i \leq p$, be the members of $C_2(G)$. Construct a graph G^* with $V(G^*) = \{ A_i, 1 \leq i \leq p \}$ as follows. (A_i, A_j), $i \neq j$, is an edge of G^* if and only if there is no A_k, $k \neq i, j$, containing $A_i \cap A_j$. Let $\underline{U_i}$ be the set of all vertices of G which belong to A_i and to no other member of $C_2(G)$, and $\underline{\alpha_i} = |U_i|$. Further let $\underline{\rho_i}$ be the degree of A_i in G^*. Then we prove the following

<u>Theorem 3.4.2.</u> G is the fourth power of a tree if and only if G satisfies the following conditions.

(1) G has no isolated vertices and every pair of adjacent vertices of G is contained in a member of $C_2(G)$.

(2) The number of edges in G^* is $|C_2(G)| - 1$.

(3) If $[A_{i_0}, A_{i_1}, \ldots, A_{i_{s-1}}, A_{i_s}]$ is a chain in G^* then each member of this chain contains $A_{i_0} \cap A_{i_s}$ and

$$|A_{i_0} \cap A_{i_s}| \begin{cases} = 0 & \text{if } s \geq 5 \\ = 5 - s & \text{if } s = 3, 4 \\ \leq \rho_{i_1} + 1 & \text{if } s = 2 \\ \geq \rho_{i_1} + 2 & \text{if } s = 2 \text{ and } \alpha_{i_1} > 0 \\ \geq 4 & \text{if } s = 1 . \end{cases}$$

(4) If A_i, A_j belong to $C_2(G)$, let

$B(A_i)$ = the set of all chains in G^* of length four whose middle vertex is A_i.

$C(A_i)$ = the set of all chains in G^* of length two whose middle vertex is A_i.

$D(A_i, A_j)$ = the set of all chains in G^* of length three whose second and third vertices are A_i, A_j respectively.

Then the intersection of the first and last members of every element of $B(A_i)$ is the same. Similarly for $C(A_i)$ and $D(A_i, A_j)$.

<u>Proof</u>. To prove the necessity of conditions (1) to (4) of the theorem, let G be the fourth power of a tree T. Then (1) follows from Lemma 3.3.5. By Lemma 3.3.4, G^* is isomorphic to $T[S(2, T)]$ and condition (2) follows from Lemma 3.3.2. To prove (3) and (4) we observe, by Theorem 3.3.3, that the members of $C_2(G)$ are $N(x, 2, T)$'s with x in $S(2, T)$.

To prove sufficiency, let G satisfy the conditions (1) to (4) of the theorem. Then we give an algorithm to construct a tree T such that $T^4 = G$.

<u>Step 1</u>. Find all the cliques of G by the method described in [8]. Then, by condition (1), they are members of $C_2(G)$. Let A_i, $1 \leq i \leq p$, be the members of $C_2(G)$.

Construct the graph G^* with the vertex set $\{A_i, 1 \leq i \leq p\}$. By conditions (2) and (3), G^* has no cycles and has $p-1$ edges, so G^* is a tree.

Step 2. Labeling of G^*. We label each vertex A_i of G^* with a label v_i where v_i is a vertex belonging to A_i as follows. We consider three cases.

Case (i). Diameter of G, $d(G^*) = 2$. Let $[A_i, A_j, A_k]$ be a chain of length 2 in G^*. Label all A_i's, $1 \leq i \leq p$, with any p vertices in $A_i \cap A_k$. This is possible because by (3), $|A_i \cap A_k| \geq \rho_j + 1$.

Case (ii). $d(G^*) = 3$. Let $[A_i, A_j, A_k, A_\ell]$ be a chain of length 3 in G^*. Label A_j and A_k with distinct vertices of $A_i \cap A_\ell$. This is possible since $|A_i \cap A_\ell| = 2$ by condition (2). Label all vertices of G^* adjacent to A_j in $G^*(\neq A_k)$ with distinct vertices in $(A_i \cap A_k)-(A_i \cap A_\ell)$ and all vertices adjacent to A_k in G^* $(\neq A_j)$ with distinct vertices in $(A_j \cap A_\ell) - (A_i \cap A_\ell)$.

Case (iii). $d(G^*) \geq 4$. If $B(A_i)$ is nonempty for some i then, by (4) and (3) the intersection of the first and last members of any element of $B(A_i)$ contains exactly one vertex. Label A_i with this vertex. Repeat this for all A_i's for which $B(A_i)$ is nonempty. If now $\rho_i \geq 2$ for

some unlabeled vertex then there is a chain of length 3 with A_i as the second vertex, $[A_j, A_i, A_k, A_\lambda]$ (say), such that $\rho_j = 1$ and A_k is labeled, for otherwise $d(G^*) \leq 3$. Label A_i with the vertex in $(A_j \cap A_\lambda) - \{v_k\}$. Repeat this procedure for all unlabeled A_i's for which $\rho_i \geq 2$. If now $\rho_i = 1$, then there is a chain of length 2 with A_i as the first vertex, $[A_i, A_j, A_k]$ (say), such that A_j is labeled. Then label all unlabeled vertices of G^* adjacent to A_j with distinct vertices in $A_i \cap A_k$ which are not used previously for labeling. Repeat this procedure as long as there are unlabeled vertices A_i for which $\rho_i = 1$. This is possible by (3) and this completes the labeling of G^*. Let H be the resulting graph thus obtained. Then
$V(H) = \{v_1, v_2, \ldots, v_p\}$.

By conditions (4), (3) and by the labeling procedure it can be shown that if $A_i \neq A_j$ then $v_i \neq v_j$. Let now
$E_i = N(v_i, 1, H)$.

<u>Step 3.</u> If $\rho_i = 1$, consider any vertex v_k at distance 2 from v_i in H, then by (3) and (4) $A_i - A_k$ is independent of such a selection of v_k and
$|A_i - A_k| \geq \alpha_i + 1$. Join now v_i to every vertex of $(A_i - A_k) - U_i$. If $\rho_i \geq 2$, consider any chain of length 2

with middle vertex v_i in H, $[v_j, v_i, v_k]$ (say). By (3) and (4) $A_j \cap A_k$ is independent of such a selection of the chain in H and $|A_j \cap A_k| > \rho_i + 1$ if $\alpha_i > 0$. Join now v_i to every vertex of $(A_j \cap A_k) - E_i$.

<u>Step 4.</u> Let $U_i = \{ v_{i,1}, v_{i,2}, \ldots, v_{i,\alpha_i} \}$. Join $v_{i,j}$ to a vertex to which v_i is joined in step 3, $1 \leq j \leq \alpha_i$ and $1 \leq i \leq p$. Observe that if $\alpha_i > 0$, then v_i will be joined to a new vertex in step 3. This completes the construction.

Let T be a resulting graph. T is a tree with $V(T) = V(G)$. Using condition (4) it can be easily verified that $S(2, T) = \{ v_i, 1 \leq i \leq p \}$. Further using conditions (4), (3) and the labeling procedure it can be proved that $N(v_i, 2, H)$ is a subset of A_i and no vertex in $A_i - N(v_i, 2, H)$ is used as a label in G*. Then we can verify using steps 3 and 4 of the construction that $N(v_i, 2, T)$ is a subset of A_i. Next it can be shown that $A_i = N(v_i, 2, T)$. Hence, by Theorem 3.3.3, it follows that $C_2(T^4) = C_2(G)$. Then, by condition (1) and Lemma 3.3.5, it further follows that $T^4 = G$. This completes the proof of the theorem.

Further the above construction gives all the tree fourth roots of G by Lemmas 3.3.2, 3.3.4 and steps 3, 4 of the construction. Now it is easy to prove the following theorem which characterises graphs with a unique tree fourth root.

Theorem 3.4.3. Let G be a graph with a tree fourth root T. Let $S(2, T) = \{ v_i, 1 \leq i \leq p \}$. Define β_i to be the number of vertices adjacent to v_i in T and not in S(2, T). Then T is unique, up to isomorphism, if and only if $\beta_i = 1$ whenever $\alpha_i \geq 2$.

CHAPTER 4

ON A PROBLEM OF ORE ON MAXIMAL TREES

A connected graph G is said to have the property P_o (after Ore who posed the problem) if for every maximal tree (i.e., spanning tree) T of G there exists $a_T \in V(G)$ such that $d_T(a_T, x) = d_G(a_T, x)$ for every $x \in V(G)$, where $d_G(x_1, x_2)$ denotes the distance in G from x_1 to x_2. The following problem was posed by Ore (see [15], page 103, Problem 4): Determine all graphs with property P_o. In Section 1 we present a solution to the above mentioned problem in the finite case, i.e., if $|V(G)| < \infty$.

4.1 Two theorems

The following theorem characterises all finite biconnected graphs with property P_o.

Theorem 4.1.1. A finite biconnected graph G has the property P_o if and only if it is a cycle (Type I) or a complete bipartite graph $K(S, V(G) - S)$ with $|S| = 2$

and $|V(G) - S| \geq 2$ (Tupe II).

<u>Proof.</u> It is easy to check that the biconnected graphs mentioned in the statement of the theorem have the property P_o.

Conversely, let G be a finite biconnected graph with property P_o and $d(G)$ its diameter. If $d(G) = 1$, then G is a triangle hence of type I. So assume that $d(G) \geq 2$. We note the following facts.

(1) If T is a maximal tree of G then $d(T) \leq 2d(G)$ and further if $d(T) = 2d(G)$ then a_T (given by the property P_o) is the unique centre of T.

(2) Every partial subgraph of G which is a tree can be extended to a maximal tree of G.

Let x_o, y_o be vertices of G such that $d_G(x_o, y_o) = d(G)$. Since G is biconnected, by Menger's theorem, there is a elementary circuit μ containing x_o, y_o. Without loss of generality assume that $\mu = [x_o, x_1, \ldots, x_t, x_o]$. Clearly length of μ, hitherto denoted by $L(\mu)$, is greater than or equal to $2d(G)$. We show that $L(\mu) \leq 2d(G) + 1$. Suppose not, then consider the partial subgraph $\mu[x_o, x_t]$ whose length $\geq 2d(G) + 1$ and which is a tree. By (2) and

(1) we get a contradiction. Thus $L(\mu) \leq 2d(G) + 1$. We consider two cases.

<u>Case (i). $L(\mu) = 2d(G) + 1$.</u>

Let $A = \{x_o, x_1, \ldots, x_t\}$, then we show that $A = V(G)$. For otherwise, let y be a vertex of $V(G) - A$ adjacent to some vertex of A, say x_i. Consider the partial subgraph

$\xi = (y, x_i) + \mu[x_i, x_o] + \mu[x_o, x_{i-1}]$ which is a tree and whose diameter $\geq 2d(G) + 1$. By (2) and (1) we get a contradiction. Now we claim that $G = \mu$. Otherwise, let (x_i, x_j) be an edge of G, with j different from $i-1$, $i+1$. Consider the partial graph

$T = \mu[x_{i+1}, x_j] + (x_j, x_i) + \mu[x_o, x_{i-1}]$. T is a maximal tree of G and $d(T) = 2d(G)$. Since G has the property P_o, a_T is the unique centre of T by (2). But it can be easily shown that there is a vertex x in $V(G)$ such that $d_T(a_T, x) > d_G(a_T, x)$, a contradiction. Thus $G(= \mu)$, is a cycle (Type I).

<u>Case (ii). $L(\mu) = 2d(G)$.</u>

Let as above $A = \{x_o, x_1, \ldots, x_t\}$. Define

$B_i = \{y\colon y \in V(G) - A$ and y is adjacent to x_i in $G\}$,

for every i, $0 \leq i \leq t$.

Further let $B = \bigcup_{i=0}^{t} B_i$. If B is empty it can be shown as in the proof of case (i) that $G = \mu$. So assume that B is nonempty. We show that B is an independent set in G. Let if possible x, y be vertices in B and (x, y) be an edge of G with y(say) in B_{i_0}. Then consider the following partial subgraph ξ of G,

$\xi = (x, y) + \mu [x_{i_0}, x_0] + \mu [x_0, x_{i_0 - 1}]$, which is a tree and whose length is $2d(G) + 1$. Hence, by (2) and (1), this yields a contradiction. Thus B is an independent set of G. Further, if z is in B_i, (z, x_{i+1}) and (z, x_{i-1}) are not edges of G. Since B is an independent set and G is biconnected, z is joined in G to x_j for some j, $0 \leq j \leq t$, and $j \notin \{i-1, i+1\}$. Now we prove that $d(G) = 2$. If $d(G) > 2$, consider the partial subgraph ξ,

$$\xi = [x_{j+1}, x_{j+2}, \ldots, x_i, z, x_j, x_{j-1}, \ldots, x_{i+1}].$$

Clearly length of $\xi = 2d(G)$. So by (2) and (1), ξ can be extended to a maximal tree T and a_T is the unique centre of T. But it can be easily proved that there is a vertex $x \in V(G)$ such that $d_T(a_T, x) > d_G(a_T, x)$, a contradiction. Thus this proves $d(G) = 2$, so $\mu = [x_0, x_1, x_2, x_3, x_0]$.

Since B is nonempty, at least one of B_i, $0 \leq i \leq 3$, is nonempty. Assume that B_o is nonempty. Now if x is in $V(G) - A$ it belongs to B_o and B_2. Let $S = \{x_o, x_2\}$, then $G = K(S, V(G) - S)$, the complete bipartite graph with $|V(G) - S| \geq 2$, hence of type II and this completes the proof of the theorem.

Theorem 4.1.2. A finite connected graph with property P_o on n vertices is a tree or consists of a subgraph H on n_o $(3 \leq n_o \leq n)$ vertices of type I or type II to which trees with a total of $n - n_o$ edges are attached at some vertices of H.

Proof. If G is a finite biconnected graph with property P_o the present theorem follows from Theorem 4.1.1. So we can assume that G has a cut vertex x. Now it can be easily shown that at most one piece of G with respect to x is not a tree and each piece of G with respect to x has the property P_o. Now again Theorem 4.1.2 follows from Theorem 4.1.1.

Remark. Perhaps it is true that $G = K(S, V(G) - S)$, the complete bipartite graph with $|S| = 2$, is the only biconnected graph with property P_o if $|V(G)|$ is infinite.

The results of this section appeared in [28].

CHAPTER 5

SOME EXTREMAL PROBLEMS CONCERNING RADIUS AND DIAMETER IN DIGRAPHS

In [25], Vizing found the maximum number of edges in an undirected graph on n vertices with radius r. In Section 5.1, we obtain the maximum number of arcs in a directed graph (not necessarily strong) on n vertices with radius r and characterise the extremal graphs. Further, we obtain an expression for the maximum number of arcs in a strong digraph on n vertices with radius r for $r \leq 3$ and state a conjecture in the general case. In Section 5.2, we consider diameter critical digraphs, that is digraphs whose diameter decreases by adding any new arc to them, and characterise all diameter critical k-connected digraphs and obtain an expression for the maximum number of arcs in a k-connected digraph on n vertices with diameter d. The analogous results in the undirected case were obtained by Ore [16]. In Section 5.3, we give a partial solution to the following problem of Murty [14]. For what values of n is it possible to orient the complete graph in such a way that the resulting tournament has diameter ≤ 2 and the tournament obtained by removing any k or fewer vertices has diameter ≤ 2 ?

5.1 Maximum number of arcs in a diagraph with given radius

In [25], Vizing, V. G. found the maximum number of edges in an undirected graph with n vertices and radius r. In this section we determine the maximum number of arcs in a directed graph with n vertices and radius r. Further we obtain an expression for the maximum number of arcs in a strong directed graph with n vertices and radius ≤ 3 and state a conjecture for general r.

Lemma 5.1.1. Let G be a directed graph with n vertices, radius r and with maximum number of arcs. If there is a vertex x_0 of G with $d_G^-(x_0) = 0$, then G has exactly $n(n-r) + \frac{(r+1)(r-2)}{2}$ arcs.

Proof. Let $X = \{x_0\}$, $X_i = \{x: d(x_0, x) = i\}$, and $n_i = |X_i|, 1 \leq i \leq r$. Since $d_G^-(x_0) = 0$, x_0 is the unique centre of G and X_i's, $1 \leq i \leq r$, are nonempty and $\bigcup_0^r X_i = X$. Let now $x \in X_i$ and $y \in X_j$. Then since G is extremal (has maximum number of arcs), $(x, y) \in G$ if $i \geq j - 1 \geq 0$. Since G is extremal and $d_G^-(x_0) = 0$ it follows that $r \neq 1$, and the lemma is trivial for $r = 2$, so let $r \geq 3$. If now X_r contains two vertices x, y, then the graph obtained from G by adding an arc (z, y) joining a vertex z of X_{r-2} to y has radius r, a contradiction. Thus $n_r = 1$. If possible let $n_i \geq 2$, $n_j \geq 2$ with $|i-j| \geq 2$. Shifting a vertex from X_j to X_i increases the number of arcs by $\partial = 2(n_{i-1} + n_i + n_{i+1}) - 2(n_{j-1} + n_j - 1 + n_{j+1})$ and the radius r is unaltered. Since G is extremal $\partial \leq 0$. Now shifting a vertex from X_i to X_j increases the number of arcs by $4 - \partial > 0$ and does not alter the radius, a contradiction which proves that $n_i = 1$ for all i except possibly for i_o and i_o+1 for some i_o with $1 \leq i_o \leq r-2$. Now counting the number of arcs in the graph, the lemma is proved.

It may be noted that in the proof of the above lemma we have actually determined all the extremal graphs.

Lemma 5.1.2. An extremal graph G with n vertices and with radius $r \geq 3$, contains a vertex with indegree in G being equal to zero.

Proof. First we observe that

$m(G) \geq n(n-r) + \frac{(r+1)(r-2)}{2}$ as is seen from the proof of Lemma 5.1.1. Now let x_o be a vertex of G with maximum outward degree in G and S be the set of all vertices not accessible from x_o. Evidently $d_G^+(x_o) \geq n-r+1$. Since radius of G is r, S is nonempty. Let θ be the number of vertices in S. Now evidently there is no arc from X - S to S. Let x_1 be a centre of G, then clearly $x_1 \in S$. Now remove from G all the vertices of $S-x_1$ and join x_1 to $y \in X-S$ if it is not already joined and if there is an arc in G from S to y. If G* is the new graph thus obtained, the number of arcs in G* is at least $m(G) - (\theta-1) - (\theta-1)(n-\theta-1)$, since the outdegree of x_1 is reduced by at most $\theta - 1$ and the outdegree in G of any vertex of $S-x_1$ is at most $n-\theta-1$. Thus $m(G) \leq m(G^*) + (\theta-1)(n-\theta)$. Now let the radius of G* be $r - \theta + \partial$, $\partial \geq 1$. Since $d^+(x_1) = 0$ in G*, it follows from Lemma 5.1.1 that

$$m(G^*) \leq (n-\theta+1)(n-\theta+1-r+\theta-\partial) + \frac{(r-\theta+\partial+1)(r-\theta+\partial-2)}{2}.$$

Thus if $\theta > 1$, we will get a contradiction by showing that

$$n(n-r) + \frac{(r+1)(r-2)}{2} > (n-\theta+1)(n-r-\partial+1) + (\theta-1)(n-\theta) + \frac{(r-\theta+\partial+1)(r-\theta+\partial-2)}{2}.$$

Simplifying, this reduces to

$$2n(\partial-1) + r(r+1) > (r+\partial-\theta-1)(r+\partial-\theta-2).$$

Since $n \geq r - \theta + \partial + 1 + (\theta - 1) = r + \partial$, it is enough to show that

$$\partial(\partial + 1) > (2 - \theta)(\theta + 1).$$

This is true if $\theta \geq 2$, since $\partial \geq 1$. Thus we conclude that $\theta = 1$ and the lemma is proved.

From Lemmas 5.1.1 and 5.1.2 we get the following

Theorem 5.1.3. The maximum number of arcs in a directed graph with n vertices with radius r is $n(n-r) + \frac{(r+1)(r-2)}{2}$. Further any extremal graph has a vertex with indegree zero and is of the form described in the proof of Lemma 5.1.1.

This theorem also solves the problem of determining the maximum radius of a directed graph with n vertices with m arcs, the answer being

$$n-1 \quad \text{if} \quad m \leq \binom{n}{2}$$

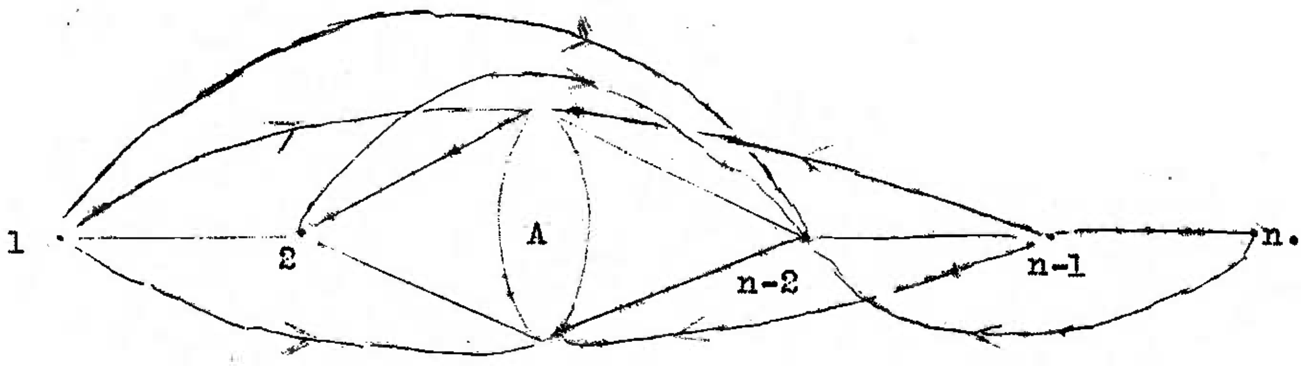
$$\left[\frac{2n+1 - \sqrt{8m+9-4n(n-1)}}{2}\right] \quad \text{if} \quad m > \binom{n}{2}.$$

Now we consider the problem of determining the maximum number of arcs in a strong directed graph with n vertices and with radius r.

The problem is trivial for $r = 1$, for $r = 2$ the answer is $n(n-2)$. For $r = 3$ we have the following

Theorem 5.1.4. The maximum number of arcs in a strongly connected directed graph with n vertices with radius 3 is $n(n-4) + 4$.

Proof. Evidently the maximum number of arcs $\geq n(n-4) + 4$ as the following graph shows:



In the above figure $A \cup \{2\}$ and $A \cup \{n-2\}$ are complete symmetric.

Let now G be an extremal graph with n vertices and with radius 3. Let x_o be a vertex with the maximum out-degree. Evidently $d_G^+(x_o) = n-3$. Let B be the set of all vertices b such that (x_o, b) and (b, x_o) are arcs, C be the set of all vertices c not in B such that (x_o, c) is an arc, and let $k = |C|$. Let y, z be the vertices not in $\{x_o\} \cup B \cup C$.

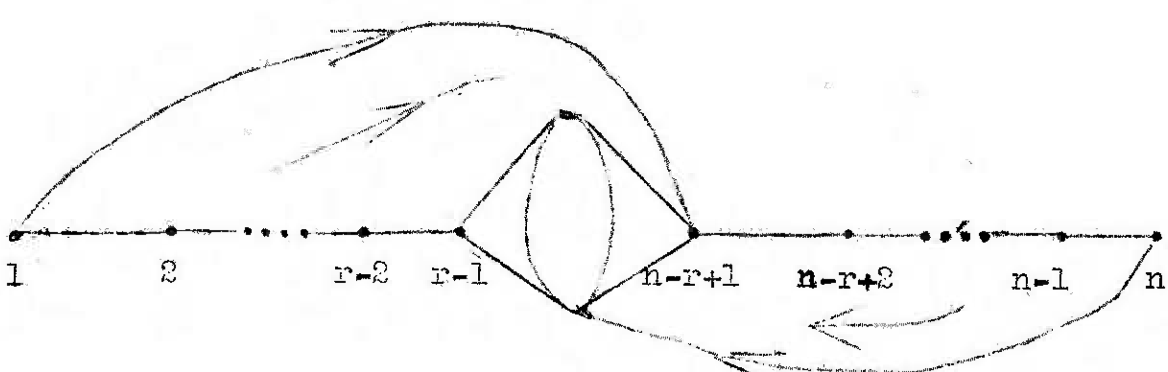
If (b, y) is an arc for some $b \in B$, then every path from x_o to z must pass through y (for otherwise the radius will be 2, x_o being the centre). Hence (y, z) is an arc and the radius of G is 2, b being the centre. This contradiction shows that (b, y), (b, z) are not arcs for all $b \in B$. So let without loss of generality (c_o, y), (y, z) be arcs for some $c_o \in C$. Now we show that out of the possible $n(n-1)$ arcs, at least $3n-4$ are missing in G.

Firstly there are no arcs from $C \cup \{y\}$ to x_o, $X - \{y, z\} = \{x_o\} \cup B \cup C$ to z, and from $B \cup \{x_o\}$ to y. Thus $(k+1) + (n-2) + (n-k-2) = 2n-3$ arcs are missing in G. Further either (y, b) or an arc from b to some vertex of $B \cup C - \{b\}$ is missing in G for every $b \in B$. Thus $n-k-3$ more arcs are missing in G. Next either c to y or an arc from c to some vertex of $B \cup C - \{c\}$

is missing in G for every $c \in C$. Thus k more arcs are missing in G. Finally two arcs from z to $X - \{z\}$ are missing in G since $d^+(z) \leq n-3$. Thus $3n-4$ arcs are missing in G, hence $n(G) \leq n^2 - 4n + 4$. This completes the proof of the theorem.

For $r \geq 4$, we conjecture that

$f(n,r) = n(n-2r+2) + r(r-1) - 2$ if $n \geq 2r$, and an extremal graph looks like



5.2. Diameter critical digraphs

Call a strongly connected digraph k-connected if the graph obtained by removing any $k-1$ or fewer vertices is strongly connected. Call a k-connected digraph G diameter critical if the addition of any new arc decreases its diameter. Let $h(n, k, d)$ be the maximum number of arcs in a k-connected digraph on n vertices and with diameter d. A k-connected digraph with diameter d and with $h(n,k,d)$ arcs is called an extremal graph. Clearly an extremal graph G is a diameter critical digraph. In this section we determine the diameter critical digraphs and the value of $h(n,k,d)$. The analogous results for undirected graphs were obtained by Ore [16].

Let x, y be vertices of a k-connected diameter critical digraph G with $d_G(x, y)$ = diameter of $G = d$. Let A_i be the set of all vertices at distance i from x and $n_i = |A_i|$, $0 \leq i \leq d$. Since G is k-connected, $n_i \geq k$ for every i, $1 \leq i \leq d-1$. Further, since G is diameter critical, $A_i \cup A_{i+1}$ is complete symmetric in G.

As in the proof of Lemma 5.1.1, it can be proved that $n_d = 1$ and if $x \in A_i$, $y \in A_j$ then (x, y) is an arc of G

if $i > j$ and (x, y) is not an arc if $i < j-1$. Now if G is an extremal graph with diameter d as in the proof of Lemma 5.1.1 it can be shown that all $n_i = k$ except possibly for $i = i_o, i_o+1$ where i_o is a fixed integer between 1 and $d-2$. Now the following two theorems can be proved easily.

Theorem 5.2.1. A k-connected digraph G on n vertices and with diameter $d > 1$ exists if and only if $n \geq 2 + k(d-1)$ and is diameter critical if and only if the vertices of G can be partitioned into $d+1$ sets A_i, $0 \leq i \leq d$, such that $|A_o| = |A_d| = 1$ and $|A_i| \geq k$, $1 \leq i \leq d-1$, and if $x \in A_i$, $y \in A_j$ then (x, y) is an arc of G if and only if $i \geq j-1$.

Theorem 5.2.2. A k-connected digraph G on n $\geq 2+k(d-1)$ vertices with diameter d is extremal if and only if the vertices of G can be partitioned into $d+1$ sets A_i, $0 \leq i \leq d$, such that $|A_o| = |A_d| = 1$ and $|A_i| = k$ for i except possibly i_o, i_o+1 where i_o is a fixed positive integer such that $1 \leq i_o \leq d-2$ and if $x \in A_i$, $y \in A_j$ then (x, y) is an arc of G if and only if $i \geq j-1$. Further

$$h(n,k,d) = \begin{cases} \binom{n}{2} + (d-2)\binom{k}{2} + \binom{n-(d-2)k-1}{2} + k(n-k-1) & \text{if } d > 1 \\ 2\binom{n}{2} & \text{if } d = 1 . \end{cases}$$

5.3. Orientation of the complete graph

In his thesis [14], U. S. R. Murty asked the following question. For what values of n does there exist a D_V (n, 2, 2, s) where D_V (n, k, λ, s) denotes a tournament (an oriented complete graph) on n vertices with diameter $\leq$ k such that the diameter of the tournament obtained by removing any s or fewer vertices is $\leq \lambda$? We obtain a partial solution to this problem.

First we make some observations. With any tournament G we can associate its adjacency matrix N of order n whose (i, j)-th element is 1 or 0 according as there is an arc in G from vertex i to vertex j or not. A tournament G on n vertices is a D_V (n, 2, 2, s) if and only if

$$N_{i*} N_{i*}' - N_{i*} N_{j*}' \geq s+1 \qquad \text{if} \quad n_{ij} = 0$$

where N_{i*} denotes the i-th row of N. Call a (0, 1)-matrix $A = ((a_{ij}))$ of order n skew symmetric if $a_{ij} = 0$ if and only if $a_{ji} = 1$ whenever $i \neq j$, $1 \leq i, j \leq n$.

Lemma 5.3.1. N is the adjacency matrix of a regular D_V (4s+ 3, 2, 2, s) if and only if it is skew-symmetric and is the incidence matrix of a (4s+ 3, 2s+1, s) - configuration.

Proof. Suppose first that N is the adjacency matrix of G which is a regular D_V (4s+ 3, 2, 2, s). Then N is skew symmetric since G is a tournament. Also G is regular of degree 2s+ 1. Now consider $\sum_{j\neq i}^{4s+3} N_{i*} N'_{j*}$. This is equal to 2s (2s+ 1) since there are 2s+ 1 1's in the i^{th} row, and each column sum is 2s + 1. Now since there are exactly 2s + 1 1's in any row, we get by hypothesis, $N_{i*} N'_{j*} \leq s$ for all $j \neq i$. Hence we have $N_{i*} N'_{j*} = s$ for all i and j with $i \neq j$ and N is the incidence matrix of a (4s+ 3, 2s+ 1, s) configuration.

Let now N be skew-symmetric and let N be the incidence matrix of a (4s+ 3, 2s+ 1, s) - configuration. Evidently then N is the adjacency matrix of a regular tournament G on 4s+ 3 vertices. The diameter of G $\leq$ 2, since the number of paths of length 2 from vertex i to vertex j is $N_{i*} N'_{i*} - N_{i*} N'_{j*} = s+1 > 0$ whenever $n_{ij} = 0$. It also follows that the tournament obtained by removing any s or fewer vertices from G has diameter $\leq$ 2, completing the proof of the lemma.

Lemma 5.3.2. Let G be a tournament with diameter 2. Let $e_{ij} = N_{i*} N'_{i*} - N_{i*} N'_{j*}$,

$s(G) = \min \{ e_{ij} : n_{ij} = 0 \} - 1$ and

$\varepsilon(G) = \min \{ e_{ij} - 1 : n_{ij} = 1 \}$ where

N is the adjacency matrix of G. Further let

$$N_1 = \begin{pmatrix} N & I + N \\ N & N' \end{pmatrix}, \quad N_2 = \begin{pmatrix} N & I + N & \underline{0} \\ \underline{N} & \underline{N'} & \underline{1} \\ \underline{1} & \underline{0} & 0 \end{pmatrix}$$

and

$$N_3 = \begin{pmatrix} N & I + N & \underline{0} & \underline{1} \\ N & N' & \underline{1} & \underline{0} \\ \underline{1} & \underline{0} & 0 & 0 \\ \underline{0} & \underline{1} & 1 & 0 \end{pmatrix}$$

Then N_1, N_2 and N_3 are the adjacency matrices of tournaments G_1, G_2 and G_3 respectively, where

(1) $s(G_1) \geq \min \{ s(G) + \varepsilon(G) + 1,\ 2s(G) + 1,\ d^+(G),\ d^-(G) - 1 \}$,

(2) $\varepsilon(G_1) \geq \min \{ s(G) + \varepsilon(G) + 1,\ 2\varepsilon(G) - 1,\ d^+(G), d^-(G) - 1 \}$,

(3) $s(G_2) \geq \min \{ s(G) + \varepsilon(G) + 1,\ 2s(G) + 1,\ d^+(G),\ d^-(G) \}$,

(4) $\varepsilon(G_2) \geq \min \{ s(G) + \varepsilon(G) + 1,\ 2\varepsilon(G) - 1,\ d^+(G),\ d^-(G) \}$,

(5) $s(G_3) \geq \min \{ s(G) + \varepsilon(G) + 1,\ 2s(G) + 1,\ d^+(G) - 1,\ d^-(G) \}$,

(6) $\varepsilon(G_3) \geq \min \{ s(G) + \varepsilon(G) + 1,\ 2\varepsilon(G) - 1,\ d^+(G),\ d^-(G) \}$,

Proof. It is evident that N_1, N_2 and N_3 are the adjacency matrices of tournaments G_1, G_2 and G_3. We will now prove (2), the rest are proved similarly.

Let us number the rows and the columns of G_2 as $1, 2, \ldots, n; 1', 2', \ldots, n'$ in that order. If $n_{ij} = 1$, then $\epsilon_{ij}(G_1) = 2\epsilon_{ij}(G)$. Also $\epsilon_{ii'}(G_1) = d_G^+(i) + 1$. If $n_{ij} = 1$, then $\epsilon_{ij'}(G_1) = d_G^+(i) + 1$. If $n_{ij} = 1$, then $\epsilon_{i'j}(G_1) = d_G^-(j)$. If $n_{ij} = 1$, then $\epsilon_{j'i'}(G_1) = \epsilon_{ji}(G) + \epsilon_{ij}(G) \geq s(G) + 1 + \epsilon(G) + 1 = s(G) + \epsilon(G) + 2$. This completes the proof of the lemma.

Corollary. If $D_V(n, 2, 2, s)$ exists then $D_V(2n, 2, 2, s)$, $D_V(2n+1, 2, 2, s+1)$ and $D_V(2n+2, 2, 2, s)$ exist. If the minimum indegree in a $D_V(n, 2, 2, s) \geq s+2$ then $D_V(2n, 2, 2, s+1)$ exists. If the minimum outdegree in a $D_V(n, 2, 2, s) \geq s+2$ then $D_V(2n+2, 2, 2, s+1)$ exists.

The following lemma can be easily proved.

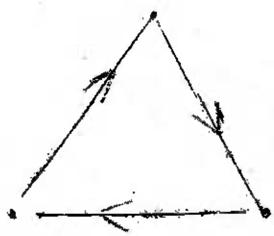
Lemma 5.3.3. If $D_V(n, 2, 2, s)$ exists, then $D_V(n-1, 2, 2, s-1)$ exists.

Let $S_i = \{ n : D_V(n, 2, 2, i) \text{ exists} \}$, $i \geq 0$.

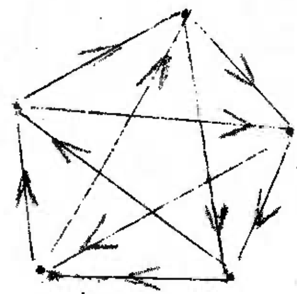
Now we are able to prove the following

<u>Theorem 5.3.4.</u> $D_V(n, 2, 2, 0)$ exists if and only if $n = 3$ or $n \geq 5$. $D_V(n, 2, 2, 1)$ exists if and only if $n = 7$ or $n \geq 9$.

<u>Proof</u>: Evidently $3, 5 \in S_o$ as the following graphs show:



$s = 0, \epsilon = 0$



$s = 0, \epsilon = 0$

Hence by Corollary to Lemma 5.3.2, we see that $10 \in S_1$, so $9 \in S_o$. Now applying the same Corollary we can prove that $5, 6, 7, 8, 9 \in S_o$, hence $\{n : n = 3, \text{ or } n \geq 5\} \subseteq S_o$. To prove equality it is enough to observe that $D_V(4,2,2,0)$ does not exist.

It may be remarked that the set S_o was determined earlier by Murty [14] by an entirely different method.

Now we show that $9 \in S_1$. For this consider the graph G defined as follows: G has vertices $1, 2, \ldots, 9$ and (i,j) is an arc if and only if $i - j = 1, 2, 4$ or $6 \pmod 9$.

G is a tournament since (i, j) is an arc if and only if $j-i = 8, 7, 5$ or $3 \pmod 9$. Since G is regular of degree 4, it is enough to show that for any arc (i,j), $\Delta(i,j) \leq 2$, where $\Delta(i,j)$ is the number of vertices k such that both (i, k) and (j,k) are arcs of G. For this we may take without loss of generality that $i = 1$ and a simple verification gives the result.

By using Corollary to Lemma 5.3.2, we can now show that

$\{n: 9 \leq n \leq 17\} \subseteq S_1$, hence

$\{n: n = 7$ or $n \geq 9\} \subseteq S_1$. To show equality we now prove that $8 \notin S_1$. Take any tournament G on 8 vertices, say $1, 2, \ldots, 8$. Then it has a transitive subtournament on 4 vertices say $1, 2, 3, 4$ where $1 \to 2 \to 3 \to 4$.

Let now $G = D_V(8, 2, 2, 1)$. Then there exist two paths of length 2 from 4 to 3, say $4 \to 7 \to 3$, $4 \to 8 \to 3$. Since there are two paths of length 2 from 3 to 2, we must have $3 \to 5 \to 2$ and $3 \to 6 \to 2$. Considering paths from 2 to 1 we get $2 \to 7 \to 1$, $2 \to 8 \to 1$. Considering paths from 3 to 1 we get $3 \to 5 \to 1$, $3 \to 6 \to 1$. Now it is evident that at most one path of length two from any vertex among 5, 6, 7, 8 to any other can use 1, 2, 3, or 4, hence the subtournament on 5, 6, 7, 8 must be a D_V (4, 2, 2, 0), a contradiction. Thus $8 \notin S_1$. A similar argument proves that 2, 3, 4, 5, 6 $\notin S_1$. This completes the proof.

Now let us consider the set S_2. Since there exists a perfect difference set namely the quadratic residues in the field of residues modulo 11, it follows that $11 \in S_2$. To show that $13 \in S_2$, consider the graph with vertex set $\{1, 2, \ldots, 13\}$, an arc going from i to j if and only if $i - j \in \{1, 2, 3, 5, 6, 9\}$ modulo 13. This is a tournament since (i, j) is an arc if and only if $j - i \in \{12, 11, 10, 8, 7, 4\}$ modulo 13. Also the graph is regular of degree 6 and it can be easily verified that $\Delta(i, j) \leq 3$ for all $i \neq j$, hence the graph is a D_V (13, 2, 2, 2) and $13 \in S_2$. Since $7 \in S_1$, by Corollary to Lemma 5.3.2, 14, 15, $16 \in S_2$. Since the D_V (9, 2, 2, 1) given in the proof of Theorem 5.3.4 is regular of degree 4 and has $\epsilon = 1$, it follows by Lemma 5.3.2, that $18 \in S_3$, hence 17, 18, 19, $20 \in S_2$. Since 10, 11, $12 \in S_1$, it follows that 21, 22, 23, 24, $25 \in S_2$. Hence we have

$\{n: n = 11$ or $n \geq 13\} \subseteq S_2$. We now show that if $G = D_V$ (12, 2, 2, 2) exists then G can not contain a transitive set of 5 vertices. For suppose that the vertices of G are numbered $1, 2, \ldots, 12$ such that $1, 2, \ldots, 5$ forms a transitive set and $1 \to 2 \to 3 \to 4 \to 5$. Clearly, if $x \to y$ in G then there are at least three vertex disjoint paths of length two from y to x. Thus without loss of generality we may take $5 \to 12 \to 4$, $5 \to 11 \to 4$, $5 \to 10 \to 4$ and $4 \to 9 \to 3$, $4 \to 8 \to 3$, $4 \to 7 \to 3$. Since $2 \to 3$ in G, at least three vertices of $\{6, 7, 8, 9\}$ are joined to 2 in G. Also, since $2 \to 4$ in G, at least two vertices of $\{10, 11, 12\}$ are joined to 2 in G. Thus in all at least five vertices of $\{6, 7, \ldots, 12\}$ are joined to 2 in G. But $1 \to 2$ in G and there do not exist three vertex disjoint paths of length two from 2 to 1, a contradiction. Thus G can not contain a transitive set on five vertices.

We think that $D_V(12, 2, 2, 2)$ does not exist. We conclude this thesis with a more general conjecture that

$$S_k = \{n : n = 4k + 3 \text{ or } n \geq 4k + 5\}.$$

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