CONTRIBUTIONS TO LINEAR QUATERNIONIC ANALYSIS

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PREFACE

This thesis considers and solves two problems in quaternionic Hilbert spaces.

The first problem, whose study constitutes Part I, deals with the description of normal operators. By a simple modification of complex methods their structure is fully clarified, a complete set of unitary invariants is obtained and a functional calculus is worked out. The most interesting theorem here is that every normal operator on a quaternionic Hilbert space is unitarily equivalent to its adjoint.

Part II studies representations in quaternionic Hilbert spaces of a large class of objects including all topological groups. The problem of obtaining all irreducible representations is solved by reducing it to the corresponding problem in the complex case. The methods used are elementary (as the generality of the results perhaps suggests) but the analysis is non-trivial. Of particular methodological interest is the importance accorded to duals of complex Hilbert spaces. Partly depending on the results obtained, interesting theorems in the theory of quaternionic representations of compact groups and locally compact abelian groups are proved. The discussion ends with a brief look at some aspects of quaternionic quantum mechanics.

PART I

NORMAL OPERATORS ON QUATERNIONIC HILBERT SPACES

CONTENTS

INTRODUCTION

Several articles have appeared in recent years which discuss linear transformations on finite-dimensional vector spaces over the quaternions in terms of matrices, but the general infinite-dimensional situation does not seem to have received much attention. In particular, very little is known about linear transformations on quaternionic Hilbert spaces apart from the obvious theory of Hermitian operators. There are hardly any discussions of this subject, apart from the brief treatment of Finkelstein, Jauch, Schiminovitch and Speiser in their fundamental paper on the foundations of quaternionic quantum mechanics (1962), which gives spectral theorems for unitary operators and skew hermitian operators and a form of Stone's theorem which says that weakly continuous one-parameter unitary groups on quaternionic Hilbert spaces have skew-hermitian infinitesimal generators, Varadarajan's book on the geometry of quantum mechanics (1968) in large parts of which Hilbert spaces over the reals, complex numbers and quaternions are discussed simultaneously. and Emch's article (Emch, 1963) on quaternionic quantum mechanics.

We shall now make a thorough study of the central area of problems featuring normal operators and their structure. The first four chapters introduce quaternionic Hilbert spaces and develop a neat and powerful technique of handling normal operators on such spaces and the next four chapters exploit this theory to yield not only the complete structure theory of the individual normal operator but also an understanding of the structure of one-parameter unitary groups and weakly closed abelian algebras of operators. In the treatment of these problems the emphasis is on departures from the complex case and all proofs even remotely resembling proofs of similar theorems on complex Hilbert spaces (almost all of which may be found in the works of either Halmos (1957) or Segal (1951) or Varadarajan (1959)) are omitted.

A brief summary of the contents is as follows:

Chapter I serves to establish notational and other conventions and to collect together the elementary properties of the division ring Q of quaternions. The structure of closed division subrings of Q is then clarified and a classification of equivalence classes of continuous one-parameter groups of unit quaternions in terms of non-negative real numbers is deduced. This is used later in Chapter V where we seek an analogue of Stone's theorem on one-parameter unitary groups on quaternionic Hilbert spaces.

In Chapter II we give a quick sketch of the elementary theory of quaternionic Hilbert spaces. Several examples are discussed (among them a quaternionic analogue of the Fourier-Plancherel transform) to enable the reader to get the feel of quaternionic Hilbert spaces.

If (.,.) is the inner product on a quaternionic Hilbert space $\underline{H}$ then the complex part of (.,.) converts the complex vector space underlying $\underline{H}$ into a complex Hilbert space. This is called the symplectic image of $\underline{H}$. The quickest route to an understanding of normal operators on $\underline{H}$ is via this symplectic image: given a problem of $\underline{H}$ we convert it to a problem on its sympletic image, apply the well-known methods of complex analysis and then get back to $\underline{H}$ again. In Chapter III this method is discussed in detail and the important notion of an imaginary operator (which plays the role of the 'i' in complex Hilbert spaces) is introduced.

Chapter IV poses the eigen-value problem for normal operators, solves it for the finite-dimensional case and explains how this may be generalized to the infinite-dimensional situation. The important point is that spectral measures alone are insufficient to describe the general normal operator on a quaternionic Hilbert space: a description is achievable only in terms of a spectral system which consists of a spectral measure together with an 'admissible' imaginary operator. Given a spectral system it is possible to define the spectral integral for a class of complex valued functions (more specifically, the class of all complex valued essentially bounded measurable functions whose restrictions to a certain fixed subset are real). This definition may appear artificial at first sight but is very well-behaved both algebraically and analytically and on better acquaintance will turn out to be quite natural. Also in this chapter we study the structure of spectral systems and obtain a complete set of unitary invariants for them.

Chapter V motivates and proves an analogue of Stone's theorem on continuous one-parameter unitary groups on quaternionic Hilbert spaces to the effect that every one such can be obtained as an integral of 'simple' groups.

Chapter VI considers the individual normal operator. It is shown that every normal operator A on a quaternionic Hilbert space may be completely described in terms of a spectral system canonically associated to A and that a complete set of unitary invariants for normal operators is given by a multiplicity function based on finite non-negative measures with compact support contained in the upper-half of the complex plane. Two interesting theorems which are not true for normal operators on complex Hilbert spaces are deduced:

i) Every normal operator A on a quaternionic Hilbert space is unitarily equivalent to its adjoint A*.

ii) If A is hermitian and B commutes with every operator commuting with A, then B is hermitian.

In Chapter VII we define the notion of a function of a normal operator on a quaternionic Hilbert space. Based on the definition of spectral integration as it is, it too is artificial at first sight (e.g., according to our definition every function of a hermitian operator is hermitian) but is completely vindicated by the analogues of well-known theorems in the complex case it leads to: On a separable quaternionic Hilbert space if A is normal and B commutes with every operator commuting with A then B is a function of A and the set of all functions of A coincides with the smallest W*-algebra of operators containing A.

The final chapter considers commutative W*-algebras of operators. It is shown that there are two essentially different kinds of such algebras, herein called R-algebras and C-algebras, and that every commutative W*-algebra on a quaternionic Hilbert space may be decomposed into a direct sum of an R-algebra and a C-algebra in a unique fashion. Segal's methods of analysis of operator algebras on complex Hilbert spaces are then applied to R-algebras and C-algebras separately to yield complete sets of unitary invariants for each of them and hence for arbitrary commutative W*-algebras. This theory is then exploited to prove that on a separable quaternionic Hilbert space any commuting family of normal operators may be expressed as functions of a single normal operator - another scoring point for the approach to spectral theory herein developed. Finally we give a proof of the Double Commutant Theorem for commutative W*-algebras which, while having an elementary proof in the complex case, turns out to be a surprisingly deep theorem in the quaternionic case.

I. QUATERNIONS

1.1 Consider the 4-dimensional Euclidean space R^4. Let $1, i, j, k$ denote the vectors of the canonical orthonormal basis for R^4. Every element q of R^4 may be expressed uniquely in the form $q = q_0 1 + q_1 i + q_2 j + q_3 k$ where q_0, q_1, q_2, q_3 are real numbers. R^4 becomes a division ring if we define addition and multiplication of any two elements p and q by

$$p + q = (p_0 + q_0)1 + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3)k$$

and
$$pq = (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3)1 + (p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2)i$$
$$+ (p_0 q_2 + p_2 q_0 + p_3 q_1 - p_1 q_3)j + (p_0 q_3 + p_3 q_0 + p_1 q_2 - p_2 q_1)k$$

respectively. This division ring is called the division ring of (real) quaternions. We shall denote it by Q. The zero of Q is the vector $(0,0,0,0)$ and the identity is the vector 1. For non-zero $q \in Q$, $q^{-1} = q^*/|q|^2$ where $q^* = q_0 1 - q_1 i - q_2 j - q_3 k$ is called the (canonical) conjugate of q and $|q| = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^{1/2}$ is the norm of q. It is easy to verify that for any two quaternions p and q, $(pq)^* = q^* p^*$ and $|pq| = |p| \cdot |q|$.

The set of all quaternions q of the form $q_0 \cdot 1$, q_0 real, is in an obvious isomorphism with R, the field of real numbers. Henceforth we shall identify R with this subfield and drop the '1'. With this understanding Q may be seen to be a Banach division algebra over R. (It is of interest here to note that Arens has shown (Arens, 1947) that any Banach division algebra over R must be isomorphic to one of R, C (the complex field) or Q).

If $q = q_0 + q_1 i + q_2 j + q_3 k$ is a quaternion, q_0 will be called the real part of q and $q - q_0$ the imaginary part of q. A quaternion is real (imaginary) if its imaginary part (real part) is zero. A unit quaternion is a quaternion of norm one. A unit imaginary is a unit imaginary quaternion.

We shall need the following two facts about Q. They are proved in (Varadarajan, 1968). ~~pp. 135-137)~~.

A) A quaternion q commutes with every other quaternion if only if q is real. In other words the reals constitute the centre of Q.

B) If $p, q \in Q$, then p and q are said to be conjugate, in symbols $p \sim q$, if there exists $r \neq 0$ such that $p = r q r^{-1}$. It is easy to check that '$\sim$' is an equivalence relation over Q. The induced equivalence classes are called conjugacy classes. p and q are conjugate if and only if

$Re(p) = Re(q)$ and $|p| = |q|$.

1.2 Our purpose in this chapter is to identify the continuous one-parameter groups of unit quaternions upto equivalence. (Two one-parameter groups (p_t) and (q_t) are said to be equivalent if there exists $r \neq 0$ such that $rp_t r^{-1} = q_t$ for all t). We start by describing the (closed) division subrings of Q. Now if D is one such then D must contain R. It follows that the centre of Q is the only division subring of Q isomorphic to R. What are the other division subrings of Q like? The following theorem answers this question completely.

<u>THEOREM 1.1</u>

i) For every unit imaginary θ, the set $C(\theta) = \{a + b\theta : a, b \text{ real}\}$ is a division subring of Q isomorphic to C, the complex field.

ii) $C(\theta) = C(\varphi)$ if and only if $\varphi = \pm\theta$.

iii) If $\varphi \neq \pm\theta$, then $C(\theta) \cap C(\varphi) = R$.

iv) Every non-real quaternion q belongs to a unique $C(\theta)$.

v) If D is a (closed) division subring of Q different from R and Q then $D = C(\theta)$ for some θ.

vi) Every $C(\theta)$ is a maximal set of commuting elements of Q and conversely every maximal set of commuting elements of Q is some $C(\theta)$.

vii) Given $C(\theta)$, $C(\varphi)$ there exists $r \neq 0$ such that

$$r\, C(\theta)\, r^{-1} = [\, r q r^{-1} : q \in C(\theta)] = C(\varphi).$$

Proof:

i) Observe that $\theta^2 = -1$ and make correspond $a + b\theta$ to the complex number $a + ib$.

ii) That $C(\theta) = C(-\theta)$ is trivial. If $C(\theta) = C(\varphi)$ then $\varphi \in C(\theta)$ is of the form $a + b\theta$. Comparison of real parts and norms gives $a = 0$ and $b = \pm 1$.

iii) Clearly $R \subseteq C(\theta) \cap C(\varphi)$. If there exists $q = a + b\theta = c + d\varphi$, $q \notin R$, then $b \neq 0 \neq d$ and since $C(\theta) \cap C(\varphi)$ is a division subring it follows that $\varphi \in C(\theta)$ and hence that $\varphi = \pm\theta$.

iv) If q is any non-real quaternion then $q \in C(\theta)$ for $\theta = \dfrac{q_1 i + q_2 j + q_3 k}{|q - q_0|}$. The uniqueness is a consequence of iii).

v) Clearly $R \subseteq D$. Since $D \neq R$, there exists a unit imaginary $\theta \in D$ and therefore $C(\theta) \subseteq D$ for that θ. If $C(\theta) \neq D$, then we may find another unit imaginary $\varphi \in D$

such that φ is orthogonal to $C(\theta)$. (Note that $C(\theta)$ and D are linear manifolds in R^4). We would then have the four mutually orthogonal vectors $1, \theta, \varphi$ and $\theta\varphi$ in D forcing us to the conclusion that $D = Q$.

vi) If q is any quaternion commuting with θ, then by equating the corresponding components of $q\theta$ and θq it is easy to see that q must have the form $a + b\theta$ with a, b real. This proves that $C(\theta)$ is a maximal set of commuting quaternions. ...

Conversely, if A is any maximal set of commuting quaternions, then $R \subseteq A$ and if $p, q \in A$, then $-p, p+q, pq \in A$. It follows that we can find a unit imaginary $\theta \in A$ and hence, by the first part of the proof that $A = C(\theta)$.

vii) By B) above we can find $r \neq 0$ such that $r\theta r^{-1} = \varphi$. But then $r\, C(\theta)\, r^{-1} = C(\varphi)$.

The theorem is thus completely proved.

Out of the many $C(\theta)$, we shall choose $C(i)$ and identify it with C. We may then speak meaningfully of quaternions being complex and vice versa. We shall pause here to pick up two useful facts about the relationship between C and Q. Let us denote the set of complex numbers with non-negative imaginary parts by C^+.

1) Every quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ may be written in the form $(q_0 + q_1 i) + (q_3 - q_2 i)k = (q_0 + q_1 i) + k(q_3 + q_2 i)$; i.e. every quaternion q may be written uniquely in the form $\alpha + \beta k = \alpha + k\bar{\beta}$ where α and β are complex. In this representation we shall refer to α as the complex part of q ; in symbols, $\alpha = \text{Com}(q)$.

2) From B) it follows that every quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ is conjugate to $q_0 + |q - q_0|\, i$. i.e. every quaternion is conjugate to an element of C^+. Since distinct elements of C^+ differ either in their real parts or in their norms, we may conclude that the conjugacy classes of Q are indexed by elements of C^+, the conjugacy class corresponding to $\alpha \in C^+$ being the set of all quaternions of the form $p \alpha p^{-1}$, p varying over unit quaternions.

We are now in a position to answer our main question.

THEOREM 1.2

If (q_t) is any continuous one parameter group of unit quaternions then there exist i) a unique non-negative real number λ and ii) a unit quaternion r such that $r q_t r^{-1} = e^{i\lambda t}$ for all t.

Proof: If q_t is real for all t then $q_t \equiv 1$. We may then take $\lambda = 0$ and $r = 1$. If not all q_t are real then from Theorem 1.1 we may conclude that the commuting family (q_t) is contained in a unique (!) maximal family of commuting quaternions and hence in a unique $C(\theta)$. There exists therefore a unit quaternion r such that

$rq_t r^{-1} = \alpha_t$ (say) $\in C$ for all t. (α_t) is then an one-parameter group of complex numbers of modulus one and therefore there exists a real number λ such that $\alpha_t = e^{i\lambda t}$. By replacing r by kr if necessary $((kr)q_t(kr)^{-1} = \bar{\alpha}_t)$ we may assume λ to be non-negative. If μ is any other non-negative real number such that $q_t \sim e^{i\mu t}$ then $e^{i\mu t} \sim e^{i\lambda t}$ implying, by B) again, that $\cos \mu t = \cos \lambda t$ for all t and hence that $\mu = \lambda$.

II. QUATERNIONIC HILBERT SPACES

2.1 Definition: By a Q-space we shall mean a vector space over the quaternions. (All vector spaces we consider will be assumed to be left vector spaces. For the basic theory of vector spaces over division rings the reader is referred to Jacobson (1953)). A norm on a Q-space V is a real-valued function $\|\cdot\|$ on V with the properties

i) $\|x\| \geq 0$, $= 0$ if and only if $x = 0$

ii) $\|qx\| = |q| \cdot \|x\|$

iii) $\|x + y\| \leq \|x\| + \|y\|$

for all $x, y \in V$ and $q \in Q$. A norm on V induces a metric on V in the usual way. A Q-Banach space is a Q-space V with a norm on it such that the resulting metric space is complete.

Examples of Q-Banach spaces are

1) $Q^{(n)}$, the Q-space of all n-tuples $(q_1, \ldots, q_n)$ of quaternions with the supremum norm,

2) $C_Q(X)$, the space of all bounded quaternion valued continuous functions on a topological space X, with the supremum norm,

and 3) the space of all equivalence classes of essentially bounded quaternion valued measurable functions on a measure space with the essential supremum norm.

An inner product on a Q-space V is a quaternion-valued function $(\cdot,\cdot)$ on $V \times V$ with the properties

i) $(x, y) = (y, x)^*$

ii) $(px + qy, z) = p(x, z) + q(y, z)$

and iii) $(x, x) \geq 0$, $= 0$ if and only if $x = 0$

for all $x, y, z \in V$ and $p, q \in Q$. From (i) and (ii) follows

$$(x, py + qz) = (x,y)p^* + (x, z)q^*.$$

It is easy to check that $\|x\| = (x, x)^{1/2}$ is a norm on V (Finkelstein et al, 1962). A Q-space V with an inner product on it which makes V a Q-Banach space with respect to the induced norm is called a quaternionic Hilbert space or a Q-Hilbert space.

The simplest example of a Q-Hilbert space is $Q^{(n)}$, with the inner product given by $(x, y) = \sum_{r=1}^{n} \xi_r \eta_r^*$ where $x = (\xi_1, \ldots, \xi_n)$ and $y = (\eta_1, \ldots, \eta_n)$. To consider more general examples we need a few facts about the theory of integration of quaternion-valued functions (Natarajan and Viswanath, 1967).

Let then (X, Σ) be a measurable space and μ a non-negative (possibly infinite) measure on (X, Σ). A quaternion-valued measurable function f on (X, Σ) is said to be integrable if $\int |f| \, d\mu < \infty$. It is easy to check that $f = f_0 + if_1 + jf_2 + kf_3 = g + kh$ is integrable if and only if the real functions f_0, f_1, f_2, f_3 are integrable or, equivalently, if and only if the complex functions g and h are integrable. For integrable f we define

$$\int f d\mu = \int f_0 \, d\mu + i\int f_1 \, d\mu + j\int f_2 d\mu + k\int f_3 d\mu .$$

The integral has the following properties:

1) $\int (f + g) d\mu = \int f d\mu + \int g d\mu$

2) $\int p f q \, d\mu = p . \int f \, d\mu . \, q$

3) $| \int f \, d\mu | \leq \int |f| \, d\mu$

4) $(\int f \, d\mu)^* = \int f^* d\mu$

where f, g are integrable and $p, q \in Q$.

We define $L_Q^2 (\mu)$ as the space of all equivalence classes of quaternion-valued measurable functions f for which

$|f|^2$ is integrable. For $f, g \in L^2_Q(\mu)$ fg is integrable and if we define $(f, g) = \int fg^* d\mu$ then (f, g) is an inner product and converts $L^2_Q(\mu)$ into a Q-Hilbert space. We point out the important fact that if $f = g + kh$, g, h complex, then $f \in L^2_Q(\mu)$ if and only if $g, h \in L^2_C(\mu)$, the complex $L^2(\mu)$.

We now give a brief sketch of the elementary theory of Q-Hilbert spaces since it does not seem to be available in print.

2.2 Geometry: The geometry of Q-Hilbert spaces is entirely similar to that of complex Hilbert spaces. The norm satisfies the parallelogram identity and the polarization identity takes the form

$$4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$
$$+ j\|x+jy\|^2 - j\|x-jy\|^2 + k\|x+ky\|^2 - k\|x-ky\|^2 .$$

The Jordan-von Neumann characterization of Hilbert spaces among Banach spaces - that a Banach norm is induced by an inner product if and only the norm satisfies the parallelogram identity - is true for the quaternionic case too. The parallelogram identity implies the projection theorem. The concepts of orthogonality, basis etc., are defined in the usual way. The cardinality of a basis of a Q-Hilbert space $\underline{H}$ is defined independently of the basis and is called the dimension of $\underline{H}$. $\underline{H}$ is separable in its metric topology if and only if its dimension is less than or equal to $\aleph_0$. We note in passing that any basis (e_α) of $L^2_C(\mu)$ is a basis of $L^2_Q(\mu)$ too. In particular $[e^{\pm 2\pi inx}]$ is a basis in $L^2_Q[0, 1]$ too.

2.3 Left-orthogonality. Owing to the non-commutative nature of quaternions the structure of Q-spaces is in general richer than that of complex spaces. E.g. in $L^2_C(\mu)$ if $\int f\bar{g}\, d\mu = 0$ then $\int \bar{f} g d\mu = 0$. But in $L^2_Q(\mu)$, $\int fg^* d\mu = 0$ need not imply $\int f^*gd\mu = 0$ in general. We may therefore define $f, g \in L^2_Q(\mu)$ to be left-orthogonal if $\int f^*gd\mu = 0$. For the trivial case when $L^2_Q(\mu) = Q$, $pq^* = 0$ implies that either p or $q = 0$ and hence that $p^*q = 0$, so that the two notions of orthogonality coincide. However in $Q^{(2)}$ (and hence in all $L^2_Q(\mu)$ over measure-spaces with at least two disjoint sets each of finite positive measure) the two concepts are distinct: the vectors $(1 - j, 1 + k)$ and $(i + k, 1 - k)$, for example, are orthogonal but not left orthogonal. Therefore, it is in general a stronger statement when we say that two functions are 'both ways' orthogonal than when we say that they are either orthogonal or left-orthogonal.

We meet with this concept in (Natarajan and Viswanath, 1967) where an analogue of the Peter-Weyl theorem in the quaternionic case is sought.

2.4 Linear Functionals. If V is a Q-space and Λ a linear functional on V then for a non-real quaternion q, $x \to q.\Lambda(x)$ is not a linear functional on V, but $x \to \Lambda(x).q$ is. Consequently the dual of a Q-space V is a right vector space over Q, or, what is the same, a left vector space over Q^{o}, the division ring opposite to Q. But this need not bother us in our study of Q-Hilbert spaces where it is just as true as in the complex case that every bounded linear functional is uniquely of the form $x \to (x,y)$ for some fixed y.

2.5 Bilinear Functionals and Quadratic Forms. A bilinear functional on a Q-Hilbert space $\underline{H}$ is a quaternion-valued function $\phi(.,.)$ on $\underline{H} \times \underline{H}$ with the properties:

i) $\phi(px + qy, z) = p\phi(x, z) + q\phi(y, z)$

and ii) $\phi(x, py + qz) = \phi(x,y)p^* + \phi(x,z)q^*$

where $x, y, z \in \underline{H}$ and $p, q \in Q$. The quadratic form $\hat{\phi}(.)$ induced by ϕ is defined by $\hat{\phi}(x) = \phi(x, x)$ for all $x \in \underline{H}$. A bilinear functional is uniquely determined by its quadratic form, but the polarization identity is not the expdcted one. It is rather

$$4\,\phi(x,y) = \hat{\phi}(x+y) - \hat{\phi}(x-y) + i\,\hat{\phi}(x+iy) - i\,\hat{\phi}(x-iy)$$

$$+\, i\hat{\phi}(x-jy)k - i\,\hat{\phi}(x+jy)k + \hat{\phi}(x+ky)k - \hat{\phi}(x-ky)k.$$

However if ϕ is symmetric (i.e. if $\phi(x, y)^* = \phi(y,x)$) then the above identity reduces to the expected one.

The boundedness of quadratic forms and bilinear functionals is defined in the usual way. A bilinear functional is bounded if and only if its quadratic form is bounded and $\|\hat{\phi}\| \leq \|\phi\| \leq 4\|\hat{\phi}\|$. If ϕ is symmetric $\|\phi\| = \|\hat{\phi}\|$.

2.6 <u>Linear Transformations</u>. We now come to the most important difference between the theory of Q-Hilbert spaces and that of complex Hilbert spaces. It is that, on a Q-Hilbert space $\underline{H}$, the operation of multiplication by a non-real quaternion is not linear: $x \to qx$ is not linear if q is non-real. It is this fact which makes life in Q-Hilbert spaces very much more complicated (or, depending on how one looks at it, very much more interesting) than in complex Hilbert spaces. The best we can do when we want to associate linear transformations with non-real quaternions q is i) to reconcile ourselves to not being able to do this canonically and ii) to choose a basis (e_α) of $\underline{H}$, define A_q by $A_q e_\alpha = qe_\alpha$ for all α and extend A_q by linearity and continuity to the whole of $\underline{H}$. We would then have $A_q x = \sum_{\alpha} (x, e_{\alpha}) q e_{\alpha}$ for all x. As can be expected it is operators such as A_q that we have to handle if we want to penetrate beyond the superficial in our intended study of linear transformations. We shall see in Chapter IV that there is a very neat and almost painless way of doing this.

It is now clear that $\underline{B}(\underline{H})$, the set of all bounded linear transformations of $\underline{H}$ into itself (hereinafter to be called operators) is a Banach algebra only over the reals. But inspite of this fact it is possible to obtain a completely satisfactory description of atleast all the abelian W*-sub-algebras of $B(\underline{H})$ as we show in Chapter VIII.

The existence of adjoints of operators on $\underline{H}$ is proved in the usual way. The definitions of hermitian, unitary and normal operators follow. Every operator A may be expressed uniquely in the form B + C where $B = \frac{A + A^*}{2}$ is hermitian and $C = \frac{A - A^*}{2}$ is skew-hermitian. At this stage we cannot decompose C further, but it will follow as a consequence of our theory of normal operators that C may be written canonically in the form JD where D is hermitian (even positive!) and J is an 'imaginary' operator (c.f. Finkelstein et al, (1962)).

A simple way of defining normal operators on a Q-Hilbert space $\underline{H}$ is to choose a basis (e_α) of $\underline{H}$ and a family of quaternions (q_α) such that $\sup_\alpha |q_\alpha| < \infty$ and define $Ax = \sum_\alpha (x, e_\alpha) q_\alpha e_\alpha$ for all x. This is in fact the only kind of normal operator there is on a finite-dimensional Q-Hilbert space.

On $L_Q^2(\mu)$ we can define a class of normal operators in the following way. If g is an essentially bounded measurable function define R_g on $L_Q^2(\mu)$ by $R_g f = f \cdot g$ ('R' for right multiplication). It is easy to see that R_g is well-defined and that R_g is a normal operator with $\|R_g\| \leq$ ess. sup. $|g|$. If the measure space under consideration has the property that every set of positive measure contains a subset of finite positive measure then we can even show that $\|R_g\| =$ ess. sup $|g|$, $R_g = R_h$ if and only if $g = h\ [\mu]$, R_g is hermitian (unitary) if and only if g is essentially real (of modulus one).

As a particular case of the above let μ be a finite, non-negative measure with compact support on the Borel subsets of C^+, the set of all complex numbers with non-negative imaginary parts. By the canonical operator A_μ associated with μ we shall mean the operator on $L_Q^2(\mu)$ defined by

$(A_\mu f)(\lambda) = f(\lambda).\lambda$ for all $\lambda \in C^+$. One of our basic results will be that every normal operator can be decomposed canonically into a 'direct sum' of operators such as A_μ.

As a final example we consider an analogue of the Fourier-Plancherel transform on $L^2_Q(-\infty, \infty)$. Copying Rudin's exposition (Rudin (1966)) we shall say that one can associate to each $f \in L^2$ an $\hat{f} \in L^2$ such that

i) for $f \in L^1 \cap L^2$, $\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-itx} \, dm(x)$

where m is $\frac{1}{\sqrt{2\pi}}$ times Lebesgue-measure,

ii) $f \to \hat{f}$ is a unitary operator on $L^2_Q(m)$

and iii) If $\phi_A(t) = \int_{-A}^{A} f(x) e^{-ixt} \, dm(x)$

and $\psi_A(x) = \int_{-A}^{A} \hat{f}(t) \, e^{ixt} \, dm(t)$

then $\| \phi_A - \hat{f} \| \to 0$ and $\| \psi_A - f \| \to 0$ on $L^2_Q(m)$ as $A \to +\infty$. As in the complex case one may check that the operator '$\wedge$' intertwines the two operators of right multiplication by $e^{i\alpha x}$ and translation by $-\alpha$, for every real α.

III. THE SYMPLECTIC IMAGE

3.1 If $\underline{H}$ is a Q-Hilbert space, since we have identified the complex field C with a fixed subfield of Q, the additive group of $\underline{H}$ may be considered to be a complex vector space. Let us call this $\underline{H}^S$. For $x, y \in \underline{H}^S$ let us define

$$\langle x, y \rangle = \text{Com}\ [(x,y)]\ (= \text{the complex part of } (x,y)).$$

The complex function $\langle x, y \rangle$ determines the quaternion-valued function (x, y) uniquely. In fact

$$(x, y) = \langle x, y\rangle + \langle x, ky \rangle k$$
$$= \langle x, y \rangle - k \langle k x, y \rangle .$$

It is easy to check that $\langle x, y\rangle$ is a complex inner product on $\underline{H}^S$ and indeed converts $\underline{H}^S$ into a complex Hilbert space since $\langle x, x \rangle = (x, x)$ for all x. We shall call $\underline{H}^S$ the symplectic image of $\underline{H}$ because of the close connection between this concept and that of linear symplectic groups of Chevalley (Chevalley, 1946; see also Finkelstein et al, 1962).

To fix ideas, let us look at the symplectic image of $L_Q^2(\mu)$. For $f = f_1 + kf_2$, $g = g_1 + kg_2$, f_1, f_2, g_1, g_2 complex,

$$(f,g) = \int fg^* d\mu = \int([f_1\bar{g}_1 + \bar{f}_2 g_2] + k[f_2\bar{g}_1 - \bar{f}_1 g_2])d\mu$$

so that

$$\langle f, g \rangle = \int (f_1\bar{g}_1 + \bar{f}_2 g_2)\,d\mu .$$

Identifying $f \in L^2_C(\mu)$ with $f + 0.k$ in $[L^2_Q(\mu)]^s$ we see that for $f, g \in L^2_C(\mu)$, $\langle f, g \rangle = \int f\bar{g}\,d\mu$. Consequently $L^2_C(\mu)$ as a subspace of $[L^2_Q(\mu)]^s$ has the standard Hilbert structure.

Even though $\underline{H}$ and $\underline{H}^s$ are conceptually different, since they have the same underlying set we shall not bother to distinguish between functions on (and subsets of) $\underline{H}$ and $\underline{H}^s$ except in cases where such a distinction helps to make things clearer. A typical example of the sort of statement we will make is: a subspace S of $\underline{H}^s$ is a subspace of $\underline{H}$ if and only if $x \in S$ implies $kx \in S$.

Every operator A on $\underline{H}$ may be considered to be an operator (A^s, say) on $\underline{H}^s$ but not necessarily conversely. In fact the spectrum of every A^s is symmetric about the real axis. For if $(A^s - \lambda I)$ were invertible then so would be $(A^s - \bar{\lambda} I)$: because, given $y \in \underline{H}^s$, there exists $x \in \underline{H}^s$ such that $(A^s - \lambda I)x = -ky$ and then $(A^s - \bar{\lambda} I)kx = y$ so that $(A - \bar{\lambda} I)$ is onto and if $(A^s - \bar{\lambda} I)x = 0$ then $(A^s - \lambda I)(kx) = 0$ implying that $x = 0$ so that $A - \bar{\lambda} I$ is 1-1.

The map $A \Rightarrow A^s$ is clearly a 1-1, norm-preserving homomorphism from $\underline{B}(\underline{H})$ into $\underline{B}(\underline{H}^s)$ considered as a real Banach algebra so that $\underline{B}_o(\underline{H}^s)$, the set of all A^s, is a subring of $\underline{B}(\underline{H}^s)$ and is in fact a real Banach algebra. The map $A \to A^s$ preserves adjoints even: $(A^s)^* = (A^*)^s$. Consequently $A \in \underline{B}(\underline{H})$ is normal (hermitian, unitary) if and only if $A^s \in \underline{B}(\underline{H}^s)$ is normal (hermitian, unitary). We can already draw a non-trivial conclusion from these apparently trivial facts.

We shall write $A \longleftrightarrow B$ to express the fact that A and B commute.

<u>THEOREM 3.1</u> If A is a normal operator on a Q- Hilbert space $\underline{H}$, and B is any operator on $\underline{H}$ such that $B \longleftrightarrow A$, then $B \longleftrightarrow A^*$.

<u>Proof</u>: Since A is normal so is A^s and $B^s \longleftrightarrow A^s$. By Fuglede's theorem $B^s \longleftrightarrow (A^s)^* = (A^*)^s$, so that $B \longleftrightarrow A^*$.

Let K denote the map $x \to kx$ on $\underline{H}^s$. K has the following properties: ($x, y \in \underline{H}^s$ are arbitrary).

1) An operator A on $\underline{H}^s$ is an operator on $\underline{H}$ if and only if $A \longleftrightarrow K$.

2) $K(\alpha x + \beta y) = \bar{\alpha} K(x) + \bar{\beta} K(y)$ for all $\alpha, \beta \in C$.

3) $\langle Kx, Ky \rangle = \langle y, x \rangle$.

4) $\| Kx \| = \| x \|$.

5) $K^2 = -I$, the identity operator.

6) $\langle x, Kx \rangle = 0$. If we agree to write $x \perp^s y$ in place of

$\langle x, y \rangle = 0$ then we may say that $x \perp^{s} Kx$. In particular $L^2_C(\mu) \perp^{s} K[L^2_C(\mu)]$ in $[L^2_Q(\mu)]^s$.

7) If (e_r) is a basis for $\underline{H}$, then (e_r, Ke_r) is a basis for $\underline{H}^s$.

8) If A is an operator on $\underline{H}^s$, then so is $KAK^{-1} = K^{-1}AK$ and $(KAK^{-1})^* = KA^*K^{-1}$.

Proof:

$$\begin{aligned}
\langle KAK^{-1} x, y \rangle &= \text{Com}\,[(KAK^{-1} x, y)] \\
&= \text{Com}\,[k(Ak^{-1} x, y)] \\
&= \text{Com}\,[k \{\langle Ak^{-1} x, y \rangle + \langle Ak^{-1}x, ky\rangle k\}] \\
&= - \langle ky, Ak^{-1} x \rangle \\
&= \langle k^{-1} y, A k^{-1} x \rangle \\
&= \langle A^* k^{-1} y , k^{-1} x \rangle \\
&= \text{Com}\,[(A^*k^{-1} y, k^{-1} x)] \\
&= \text{Com}\,[(A^*k^{-1} y, x)k] \\
&= \text{Com}\,[\{\langle A^*k^{-1} y, x \rangle - k\langle kA^*k^{-1} y, x\rangle\} k] \\
&= \langle x, KA^* K^{-1} y \rangle .
\end{aligned}$$

9) If P is the projection on the subspace S of $\underline{H}^s$, then KPK^{-1} is the projection on the subspace $K[S] = [kx : x \in S]$ of $\underline{H}^s$.

10) If P is the projection on the subspace S of $\underline{H}^S$ and $P \perp^S KPK^{-1}$ then $P + KPK^{-1}$ is the projection (in $\underline{H}$) on the subspace $S \oplus^S K[S]$ of $\underline{H}$.

All these ten results are important to us. In what follows we shall use them uninhibitedly and without explicit mention.

We now introduce and study a class of operators to be called 'imaginary' operators. The proofs involved offer a very good introduction to the way in which we shall exploit the symplectic image later on.

<u>THEOREM 3.2</u> The following conditions on an operator J on $\underline{H}$ are equivalent:

1) J is normal and $-J^2$ is a projection

2) J is normal and $J^2 + J^4 = 0$

3) $J^* = -J$ and $I - JJ^*$ is the projection on the null space of J

4) There exists a basis (e_α) of $\underline{H}$ such that $Je_\alpha = ie_\alpha$ or 0 for all α.

Proof:

<u>1) => 2)</u> Since $-J^2$ is a projection, $(-J^2)^2 = J^4 = -J^2$ so that $J^2 + J^4 = 0$.

2) => 3) J^s is normal on $\underline{H}^s$. Let $E(\cdot)$ be the spectral measure of J^2. Then $E[\lambda : |\lambda^2 + \lambda^4|^2 = 0] = I$ so that $J^s = iE([i]) - iE([-i])$. But then $(J^s)^* = -J^s$ and $I - J^sJ^{s*} = E([0])$, the null space of J.

3) => 4) Notice that $E([-i]) = KE([i])K^{-1}$. Hence if (e_α) is a basis for $E([i])$ in $\underline{H}^s$ then (e_α) is a basis for $E([i]) + E([-i]) = I - E([0])$ in $\underline{H}$. Extending (e_α) to a basis of $\underline{H}$, we have $J e_\alpha = ie_\alpha$ if $e_\alpha \in I - E([0])$ and $Je_\alpha = 0$ if $e_\alpha \in E([0])$.

4) => 1) Trivial.

Note: In the proof of the theorem above we confused a projection with its range. We shall do this more and more as we go along without apology.

DEFINITION 3.1 An operator J on $\underline{H}$ is **imaginary** if it satisfies any one of the 4 (equivalent) conditions of Theorem 3.2. An imaginary operator is **full** if its null space is 0.

We point out the fact (noticed in the proof of the above theorem) that if J is an imaginary operator then J^s is uniquely of the form $iP^+ - iP^-$ where P^+ and P^- are mutually orthogonal projections in $\underline{H}^s$ such that $P^- = KP^+K^{-1}$ and $P^- + P^+ = JJ^*$.

<u>THEOREM 3.3</u> Let J and L be imaginary operators on a Q-Hilbert space $\underline{H}$ such that $Jx = 0$ implies $Lx = 0$. If $J \leftrightarrow L$, then there exist unique mutually orthogonal projections P and Q on $\underline{H}$ such that $L = JP - JQ$ and $P + Q = L^*L$.

<u>Proof:</u> If $J^S = iP^+ - iP^-$ and $L^S = iQ^+ - iQ^-$ then $\{P^+, P^-, Q^+, Q^-\}$ form a commuting family of projections. Since the null space of $J\ (= I - P^+ - P^-)$ is contained in the null space of $L\ (= I - Q^+ - Q^-)$, $Q^+ = Q^+P^+ + Q^+P^-$ and $Q^- = Q^-P^+ + Q^-P^-$. Further $Q^-P^- = KQ^+P^+K^{-1}$ and $Q^+P^- = KQ^-P^+K^{-1}$ so that $P = Q^+P^+ + Q^-P^-$ and $Q = Q^+P^- + Q^-P^+$ are mutually orthogonal projections on $\underline{H}$ and $L = JP - JQ$. Also $P + Q = Q^+(P^+ + P^-) + Q^-(P^+ + P^-) = Q^+ + Q^- = L^*L$. If also $L = JP' - JQ'$ with $P' + Q' = L^*L$ then firstly $P' + Q' = P + Q$ and secondly since $L^*L \subseteq J^*J$, $JP - JQ = JP' - JQ'$ implies, on left multiplication by J^*, that $P - Q = P' - Q'$. Therefore $P = P'$ and $Q = Q'$.

IV. SPECTRAL MEASURES

4.1 Let A be a linear transformation on a vector space V over a division ring $\underline{D}$ which is not a field. Let $q \in \underline{D}$ be an eigen-value of A and $x (\neq 0)$ a corresponding eigen-vector: $Ax = qx$. If q is in the centre of $\underline{D}$ and $y = px$ is any multiple of x then $Ay = qy$; but if q does not belong to the centre of $\underline{D}$, choosing p such that $pq \neq qp$, we see that for $y = px$, $Ay = (pqp^{-1})y \neq qy$, so that in general while a multiple of an eigen-vector is again an eigen-vector it need not correspond to the same eigen-value. This means that there is in general no simple way of describing the action of A on the ray spanned by one of its eigen-vectors. However, for the case $\underline{D} = Q$, a neat description may be achieved in the following manner.

If q is real we have no problem. Suppose q is non-real; by replacing x, if necessary, by px where p is so chosen that $\lambda = pqp^{-1} \in C^+$, we may assume that $Ax = \lambda x$ where $\lambda \in C^+$. If $\alpha + k\beta$ is an arbitrary quaternion with α, β complex, we would then have $A(\alpha x) = \lambda(\alpha x)$ and $A(k\beta x) = \bar{\lambda}(k\beta x)$. Suppose now $\lambda = a + ib$. Define an imaginary operator J by $Jx = ix$ and $Jy = 0$ if $y \perp x$. We then have $A(\alpha x) = (a + Jb)(\alpha x)$ as also $A(k\beta x) = (a + Jb)(k\beta x)$ so that for any quaternion p, $A(px) = (a+Jb)(px)$. We can say therefore that the action of **A** on the ray spanned by x is completely described by the operator $a + Jb$. This observation (due to Finkelstein et al, 1962) is central to our theory. We show in this and ensuing chapters that normal operators can be completely described in terms of real numbers, imaginary operators (these two together replacing the complex numbers in the case of complex Hilbert spaces) and, of course, projections.

4.2 Let us first look at a normal operator A on a finite-dimensional Q-Hilbert space $\underline{H}$. Consider A^s on $\underline{H}^s$. From classical theory we know that there exists $x \neq 0$, $x \in \underline{H}^s$ and $\lambda \in C$ such that $A^s x = \lambda x$. We would then have $A^s(kx) = \bar{\lambda}(kx)$. This means that whenever λ is an eigen-value for A^s, so is $\bar{\lambda}$ and if S is the eigen-subspace corresponding to the eigen-value λ, then $K[S]$ is the eigen-subspace corresponding to $\bar{\lambda}$. If λ is real, it follows that $S = K[S]$ and hence that S is a subspace of $\underline{H}$ even. If λ is non-real then $S \perp^s K(S)$ and $S \oplus^s K(S)$ is a subspace in $\underline{H}$. Let now $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ be the distinct real eigen-values of A^s and $\mu_1, \mu_2, \ldots, \mu_m, \bar{\mu}_1, \ldots, \bar{\mu}_m$ the distinct non-real eigen-values, where the notation is so chosen that $\mu_s \in C^+$ for all s. Let S_r be the eigen-subspace corresponding to λ_r and T_r the eigen-subspace corresponding to μ_r. We may then assert that

$$\underline{H}^s = [(\underset{r}{\overline{+}}) \; S_r] \, (\overline{+}) [(\underset{s}{\overline{+}}) \; T_s] (\overline{+}) [(\underset{s}{\overline{+}}) \; K[T_s]] \; .$$

Observe that for two vectors $x, y \in T_s$, for any s, if $< x, y > = 0$ then $(x, y) = < x, y > + < x, ky > k = 0$, since $T_s \perp^s K [T_s]$, so that a basis for T_s in $\underline{H}^s$ is a basis for $T_s (\overline{+})^s K[T_s]$ in $\underline{H}$ (A similar argument was implicit in the proof of THEOREM 3.2). We now choose a basis for $\underline{H}$ as follows: Choose a basis in $\underline{H}$ for each S_r; choose a basis in $\underline{H}^s$ for each T_s; pool all these together and call the resulting set (e_t). (e_t) is then a basis for $\underline{H}$ such that for each t, $A e_t = \alpha_t e_t$ where $\alpha_t \in C^+$. In other words we have proved that if A is a normal operator on a finite dimensional Q-Hilbert space $\underline{H}$, then there exists a basis (e_t) for $\underline{H}$ and constants $\alpha_t \in C^+$ such that $Ae_t = \alpha_t e_t$ for all t; i.e. every normal operator on a finite dimensional Q-Hilbert space can be 'diagonalized'. (cf. **Finkelstein et al**, 1959).

This representation of normal operators however is not canonical. A canonical form may obtained in the following manner. Let P_r and Q_s be the projections (in $\underline{H}^s$) on S_r and T_s respectively. Then $KQ_s K^{-1}$ is the projection (in $\underline{H}^s$) on $K[T_s]$ and P_r and $R_s = Q_s + KQ_sK^{-1}$ are projections in $\underline{H}$. Define an imaginary operator J by

$$J^s x = 0 \qquad \text{if } x \in (\bar{+}_r)\ S_r$$
$$= ix \qquad \text{if } x \in (\bar{+}_s)^s\ T_s$$
$$= -ix \qquad \text{if } x \in (\bar{+}_s)\ K[T_s].$$

If $\mu_s = a_s + ib_s$, then we have the equation

$$A^s = \Sigma\, \lambda_r P_r + \Sigma(a_s + ib_s)\, Q_s + \Sigma(a_s - ib_s)\, KQ_sK^{-1}$$

or $$A = \Sigma\, \lambda_r P_r + \Sigma(a_s + Jb_s)\, R_s.$$

We have therefore proved the following theorem:

<u>THEOREM 4.1</u> Let A be a normal operator on a finite-dimensional Q-Hilbert space $\underline{H}$. Then there exist

i) a positive integer r

ii) distinct scalars $\lambda_1, \lambda_2, \ldots, \lambda_r \in C^+$

iii) mutually orthogonal projections $P_1, P_2, \ldots, P_r$ with $\sum_m P_m = I$ and

iv) an imaginary operator J such that $J \longleftrightarrow P_r$ for all r and $JJ^* = (\bar{+}_m)\ [P_m : \lambda_m \text{ non-real}]$ such that

$$A = \sum_{m=1}^{r} (a_m + Jb_m)\, P_m$$

where $\lambda_m = a_m + ib_m$, a_m, b_m real.

That this way of decomposing a normal operator is unique will be proved generally in the infinite-dimensional situation.

We shall now examine how Theorem 4.1 may be extended to arbitrary Q-Hilbert spaces.

4.3 Spectral Measures and Integration. The first step is simple. We define a spectral measure on a measurable space (X, Σ) as a countably additive set function E whose values are in $\bar{\underline{L}}(\underline{H})$, the lattice of projections in $\underline{H}$, for which $E(\emptyset) = 0$ and $E(X) = I$. The elementary properties of such spectral measures follow. But the plot thickens when we proceed to spectral integration.

If $E(\cdot)$ is a spectral measure on (X, Σ) with values in $\bar{\underline{L}}(\underline{H})$ then for every $x, y \in \underline{H}$, $(E(\cdot)x, y)$ is a quaternion-valued measure on (X, Σ); and for every real-valued bounded (more generally E-ess. bounded) measurable function f on (X, Σ) we may define $\int f d(E(\cdot)x, y)$ in an obvious way. (We shall not digress to discuss such integrals as we need no more than their definition). Exploiting the fact that reals constitute the centre of Q one may show without difficulty that $\phi(x, y) = \int f d(E(\cdot)x, y)$ is a bounded bilinear functional on $\underline{H}$ (the crucial equation is $\phi(qx, y) = q\,\phi(x, y)$. If f is not essentially real there may exist a q for which this does not hold) and hence that there exists an operator A such that $(Ax, y) = \phi(x, y)$. We write $A = \int f\,dE$. The following lemma is easily proved:

LEMMA 4.1

i) $\|Ax\|^2 = \int |f|^2\, d(E(\cdot)x, x)$.

ii) $\|A\| = N_E(f) = E$ ess. sup $|f|$.

iii) $\int (af + bg)dE = a\int f dE + b\int g dE$, a, b real

iv) (a) $f_n \to f$ uniformly E- a.e.

$\Rightarrow \|A_n - A\| \to 0$

(b) $f_n \to f$ pointwise E- a.e. and

$(N_E(f_n))$ bounded $\Rightarrow \|A_n x - Ax\| \to 0$

where $A_n = \int f_n dE$ and $A = \int f dE$

v) $\int fg dE = \int f dE \int g dE$

and vi) $\int f dE$ is hermitian

where $x, y \in \underline{H}$ and f_n, f, g are E ess.bdd real measurable functions on (X, Σ).

Having obtained these results it is easy to prove the spectral theorem for hermitian operators - that given any hermitian operator A on a Q-Hilbert space $\underline{H}$, there exists a unique compact spectral measure $E(\cdot)$ on the Borel sets of

R such that $A = \int \lambda dE$ where λ is the co-ordinate function on R - either directly, as Finkelstein et al (1962) suggest, or by an appeal to the symplectic image.

For the description of normal, non-hermitian operators however this approach fails completely. As was observed above, the spectral integral is not meaningful unless the integrand is essentially real. We shall now describe a theory which will enable us to handle all normal operators with ease and power. As can be expected, we have to look to the imaginary operators for help.

DEFINITION 4.1 Let $E(.)$ be a spectral measure based on (X, Σ) with values in $\underline{L}(\underline{H})$ and J an imaginary operator on $\underline{H}$. J is admissible w.r.t. to $E(.)$ if

i) $J \longleftrightarrow E(M)$ for all $M \in \Sigma$

and ii) there exists $R_0 \in \Sigma$ such that $JJ^* = E(X - R_0)$.

Condition (ii) says that the null-space of J must be a value of $E(.)$. It is easy to see that R_0 is unique upto E-null sets. It might help the reader to understand the rôle of R_0 in what follows if we mention that when we associate canonical spectral measures with normal operators we shall take $X = C^+$ and $R_0 = R$.

Let now a spectral measure $E(.)$ on (X, Σ) and an imaginary operator J admissible w.r.t. $E(.)$ be given. We shall call the pair (E, J) a spectral system. Choose and fix R_0 such that $I - JJ^* = E(R_0)$. Let $\underline{M}$ denote the real Banach algebra of all (equivalence classes of) complex-valued E-ess. bounded measurable functions on (X, Σ) whose restrictions to R_0 are real, with the norm defined by $N_E(f) = E\text{-ess.} \sup_{\lambda \in X} |f(\lambda)|$.

DEFINITION 4.2 If $f \in \underline{M}$, $f = f_1 + if_2$, f_1, f_2 real then $\int f dE$ w.r.t. J is the operator on $\underline{H}$ defined by

$$\int f dE = \left(\int f_1 dE \right) + J \left(\int f_2 dE \right).$$

THEOREM 4.2 The integral $\int f dE$ w.r.t. J has the following properties: ($x, y \in \underline{H}$ and $f, f_n, g \in \underline{M}$)

i) $\int f dE \longleftrightarrow \int g dE$.

ii) If $A = \int f dE$, then $\| Ax \|^2 = \int |f|^2 d(E(.)x,x)$.

iii) $\| A \| = N_E(f)$.

iv) $\int (af+bg) dE = a \int f dE + b \int g dE$ (a, b real).

v) $\left(\int \bar{f} dE \right) = \left(\int f dE \right)^*$.

vi)(a) $f_n \to f$ uniformly $\Rightarrow \| \int f_n dE - \int f dE \| \to 0$.

(b) $f_n \to f$ pointwise, $(N_E(f_n))$ bounded

$\Rightarrow \| (\int f_n dE) x - (\int f dE) x \| \to 0$.

vii) $\int fg dE = (\int f dE) \cdot (\int g dE)$.

viii) $A = \int f dE$ is normal. A is hermitian if and only if f is E-essentially real and unitary if and only if f is E-essentially of modulus one.

Proof:

i) It is enough to prove that $\int f dE \longleftrightarrow J$ when f is real. When f is a simple function this is true because $J \longleftrightarrow E$. The general case follows.

ii) Let $f = f_1 + if_2$, $B = \int f_1 dE$, $C = \int f_2 dE$. Then

$\|Ax\|^2 = ((B + JC)x, (B + JC)x)$

$$= \|Bx\|^2 + \|Cx\|^2 + (Bx, JCx) + (JCx, Bx)$$

$$(\because f_2 \equiv 0 \text{ on } M_0, JJ^*C = C)$$

$$= \int |f_1|^2 d(E(\cdot)x, x) + \int |f_2|^2 d(E(\cdot)x, x) + (Bx, JCx) + (JCx, Bx)$$

$$= \int |f|^2 d(E(\cdot)x, x) + 2 \operatorname{Re}(Bx, JCx).$$

To complete the proof we have to show that $\operatorname{Re}(Bx, JCx) = 0$. If $x \in E(R_0)$, $Cx = 0$ and hence it is true. Since B, C, J are all reduced by $E(R_0)$ it is now enough to prove it for $x \in E(x - R_0)$. Passing to the symplectic image, it is enough to prove that $\operatorname{Re} \langle Jx, BCx \rangle = 0$ for $x \in P^+$ and P^-

separately where $J^s = iP^+ - iP^-$. But, because $BC = CB$ is hermitian, this is obvious.

iii) $\|Ax\|^2 = \int |f|^2 \, d(E(.)x, x) \leq N_E(f)^2 \|x\|^2$.

The reverse inequality too is obtained in the usual way.

iv) Trivial.

v) By definition $\int \bar{f} dE = B - JC$ and $A^* = B - JC$.

vi) Straightforward.

vii) Let $A = \int f dE = \int f_1 dE + J \int f_2 dE = A_1 + JA_2$

and $B = \int g dE = \int g_1 dE + J \int g_2 dE = B_1 + JB_2$.

Then $\int (fg) dE = \int [(f_1 g_1 - f_2 g_2) + i(f_1 g_2 + f_2 g_1)] dE$

$$= (A_1 B_1 - A_2 B_2) + J(A_1 B_2 + A_2 B_1)$$

$$= (A_1 + JA_2)(B_1 + JB_2)$$

(Because $J^2 A_2 B_2 = - A_2 B_2$).

viii) Straightforward.

We thus see that inspite of its rather odd definition the spectral integral we have introduced behaves exactly like the usual spectral integral in the complex case. Using this integral we shall obtain an understanding of the structure of continuous one-parameter unitary groups in Chapter V and detailed information about normal operators in Chapter VI.

4.4 Structure of spectral systems. We have seen that for the description of a general normal operator a spectral measure alone is not enough. The description is achievable only in the terms of a spectral system (E, J). Consequently, the structure theory of a normal operator on a Q-Hilbert space is not readily deducible from that of the spectral measure, as in the complex case, unless the operator concerned is hermitian. Our purpose in this section is to show that by a refinement of the methods used in the analysis of spectral measures, one may obtain a complete understanding of the structure of spectral systems (considering them to be respectable entities in their own right) the resulting theory being applicable to the study of general normal operators on Q-Hilbert spaces.

In what follows (X, Σ) will denote a measurable space and R_o an arbitrary but fixed set in Σ. All spectral systems (E, J) we consider will satisfy $JJ^* = E(X - R_o)$.

DEFINITION 4.3 Let $\underline{H}$ and $\underline{K}$ be two Q-Hilbert spaces and (E, J) and (F, L) two spectral systems in $\underline{H}$ and $\underline{K}$ respectively with $JJ^* = E(X - R_o)$ and $LL^* = F(X - R_o)$. (E,J) and (F,L) are said to be isomorphic if there exists an isomorphism ϕ between $\underline{H}$ and $\underline{K}$, mapping $\underline{H}$ onto $\underline{K}$ for definiteness, such that

i) $\emptyset E(M)x = F(M)\emptyset(x)$

and ii) $\emptyset Jx = L\emptyset(x)$

for all $x \in \underline{H}$ and $M \in \Sigma$.

Our main aim is to obtain a complete set of unitary invariants for spectral systems. (It is possible to achieve this by considering separately the quaternionic spectral measure E on the null space of J and the complex sepectral measure E^S on $P^+ = [x : Jx = ix]$ and then combining the two theories, but we prefer the direct approach). With this object in view we first analyse an arbitrary but fixed spectral system (E, J) in a Q-Hilbert space $\underline{H}$ with $JJ^* = E(X - R_o)$. $\underline{E}$, the range of E, is then a Boolean σ-algebra and if $\underline{F}$ is the closure of $\underline{E}$ in the set of all operators on $\underline{H}$ of norm less than or equal to one endowed with the weak topology, then $\underline{F}$ is a complete Boolean algebra and is in fact the completion of $\underline{E}$.

Let $\underline{P}$ denote the class of all projections on $\underline{H}$ which commute with $\underline{E}$ (or, equivalently, $\underline{F}$). $\underline{P}$ is a complete lattice. In the standard theory of spectral measures one decomposes $\underline{E}$ in terms of projections in $\underline{P}$. But here we shall go one better and decompose $\underline{E}$ in terms of projections in a sublattice of $\underline{P}$.

This sublattice (let us call it $\underline{Q}$) is the set of all projections in $\underline{P}$ which commute with J. $\underline{Q}$ is a complete lattice by itself. If $J = 0$, then trivially $\underline{Q} = \underline{P}$, but $\underline{Q} \neq \underline{P}$ in general. Anticipating our theory, we can say that a necessary and sufficient condition for $\underline{Q}$ to be equal to $\underline{P}$ is that JJ^* is a row, but for the moment the following simple example suffices to make our point.

<u>EXAMPLE 4.1</u> Let $\underline{H} = Q^{(2)}$, X a singleton set, $E(\emptyset) = 0$, $E(\chi) = I$ and let J be defined by $J(\xi, \eta) = (\xi i, \eta i)$. If now P is the ray generated by the vector (1, k) then $P \in \underline{P}$ but, since $J(1,k) = (i, -ik) \notin P$, $P \notin \underline{Q}$.

Recalling that $S \in \underline{P}$ is called a cycle if there exists an $x \in S$ such that $S = Z(x)$, the subspace generated by all the vectors of the form $E(M)x$, M varying over Σ, the above example shows that there may exist cycles which, while of course belonging to $\underline{P}$, need not belong to $\underline{Q}$. If a cycle does belong to $\underline{Q}$, we shall call it a J-cycle. Lemmas 4.2 to 4.7 below give us all the information we need about cycles and J-cycles.

For x in $\underline{H}$, μ_x will denote the measure on (X, Σ) defined by $\mu_x(M) = (E(M)x, x)$ for all $M \in \Sigma$.

LEMMA 4.2

i) If $x \in S$ and $S \in \underline{P}$, then $Z(x) \subseteq S$.

ii) If $y \perp Z(x)$, then $Z(y) \perp Z(x)$

iii) $x \in E(M_o)$ and $y \perp E(M_o)$ for some M_o in Σ, implies $Z(x+y) = Z(x) \oplus Z(y)$.

iv) If $Z(x)$ is a cycle, then there exists an isomorphism ϕ mapping $L_Q^2(\mu_x)$ onto $Z(x)$ such that $\phi(1)=x$ and $\phi(f\chi_M) = E(M)\phi(f)$ for all $f \in L_Q^2(\mu_x)$, $M \in \Sigma$.

v) The restrictions of the spectral measure E to $Z(x)$ and $Z(y)$ are isomorphic if and only if $\mu_x \equiv \mu_y$.

vi) If $S \subseteq Z(x)$, $S \in \underline{P}$, then there exists $M \in \Sigma$ such that $S = E(M) \cap Z(x) = Z(E(M)x)$.

vii) If $S_1, S_2 \subseteq Z(x)$, $S_1, S_2 \in \underline{P}$, then $S_1 \leftrightarrow S_2$.

viii) If $S_1, S_2 \subseteq Z(x)$, $S_1, S_2 \in \underline{P}$; $S_1 \perp S_2$, then there exists $M \in \Sigma$ such that $S_1 \subseteq E(M)$ and $S_2 \subseteq E(X-M)$.

The proofs of these results are entirely similar to those in the complex case and are omitted.

We now proceed to look at J-cycles. From now on vectors x for which $Jx \equiv JE(X-R_o)x = iE(X-R_o)x$ have a distinguished rôle to play. For convenience, we call such vectors J-vectors.

LEMMA 4.3

i) $Z(x)$ is a J-cycle if and only if $Jx \in Z(x)$.

ii) If x is a J-vector then $Z(x)$ is a J-cycle. Conversely if a subspace S is a J-cycle, then there exists a J-vector $x \in S$ such that $Z(x) = S$.

iii) Every $S \in \underline{Q}$ (in particular $\underline{H}$ itself) is a direct sum of mutually orthogonal J-cycles.

Proof:

i) If $Z(x)$ is a J-cycle then $Z(x) \longleftrightarrow J$, by definition, so that $Jx \in Z(x)$. Suppose, conversely, that $Jx \in Z(x)$. Then $JE(M)x = E(M)(Jx) \in Z(x)$ for all M and since $Z(x)$ is spanned by the $E(M)x$, it follows that J leaves $Z(x)$ invariant. Remembering now that $J^* = -J$, we can conclude that $J \longleftrightarrow Z(x)$, i.e. that $Z(x)$ is a J-cycle.

ii) If x is a J-vector, then $Jx = JE(X - R_o)x = iE(X - R_o)x \in Z(x)$ so that $Z(x)$ is a J-cycle by i) above.

The converse is non-trivial. We are given that S is a J-cycle. Therefore $S = Z(x)$ for some $x \in S$ and $Z(x) \longleftrightarrow J$. Let $y = E(X - R_o)x$ and $z = E(R_o)x$. Then $x = y + z$ and by Lemma 4.2 iii) $Z(x) = Z(y) \oplus Z(z)$. Since $Z(y) = Z(x)\,E(X - R_o)$ and $Z(z) = Z(x)E(R_o)$, $Z(y)$ and $Z(z)$ are J-cycles.

Let $J^S = iP^+ - iP^-$ be the canonical representation of J^S in $\underline{H}^S$. Our result is proved if we prove that there exists

$y_o \in P^+$ such that $Z(y_o) = Z(y)$ - for then $x_o = y_o + z$ is a J-vector and $Z(x_o) = Z(y_o) \;(\mp)\; Z(z)$
$= Z(y)(\mp)\; Z(z) = Z(x)$.

Suppose $y \not\in P^+$ already. We may then write $y = y^+ + ky^-$ with $y^+, y^- \in P^+$. Now, since $Z(y)$ is a J-cycle, $Jy = iy^+ - iky^- \in Z(y)$ and also $iy = iy^+ + iky^- \in Z(y)$. Consequently, $y^+, y^- \in Z(y)$ and $Z(y^+) \subseteq Z(y)$. Let now $y_1 = y^- - Z(y^+)y^-$. Then $y_1 \in P^+$ and $y_1 \perp Z(y^+)$. Hence $Z(y_1) \perp Z(y^+)$. But then $Z(y_1)$ and $Z(y^+)$ are both contained in $Z(y)$ so that an application of (viii) and (iii) of Lemma 4.2 gives, if $y_o = y^+ + y_1$, $Z(y_o) = Z(y^+)(\mp)Z(y_1)$. It is now easy to check that $y_o \in P^+$ and $Z(y_o) = Z(y)$. As observed earlier this completes the proof of (ii).

iii) Let $S \in \underline{Q}$ and let (Z_r) be a maximal family of mutually orthogonal J-cycles contained in S. If $S \neq \underset{r}{(\mp)} Z_r$, consider $S_o = S - \underset{r}{(\mp)} Z_r$. Then $0 \neq S_o \in \underline{Q}$, so that either $S_o E(R_o) \neq 0$ or $S_o E(X - R_o) \neq 0$. In the former case for any non-zero $x \in S_o E(R_o)$, $Z(x)$ is a non-zero J-cycle contained in S and orthogonal to all the Z_r, and in the latter case since $S_o P^+ \neq 0$, for any non-zero $x \in S_o P^+$, $Z(x)$ is again a non-zero J-cycle contained in S and orthogonal to all the Z_r. Since (Z_r) was assumed to be maximal at the outset we have a contradiction in either case. Hence $S = \underset{r}{(\mp)} Z_r$. The lemma is thus completely proved.

DEFINITION 4.4 For any finite, non-negative measure μ on (X, Σ), given $R_o \in \Sigma$, the canonical spectral system (E_μ, J_μ) associated to μ is defined by the equations

i) $E_\mu(M)f = f.\chi_M$

and ii) $J_\mu f = f.i.(1 - \chi_{R_o})$

for all $f \in L^2_Q(\mu)$ and $M \in \Sigma$.

LEMMA 4.4 Let μ and μ' be finite non-negative measures on (X, Σ)

i) The canonical spectral systems (E_μ, J_μ) and $(E_{\mu'}, J_{\mu'})$ are isomorphic if and only if $\mu \equiv \mu'$.

ii) If B is any operator on $L^2_Q(\mu)$ which commutes with $\underline{E}_\mu$ as well as J_μ, then there exists a bounded measurable function h_o on X such that B is coincident with the operator of right multiplication by h_o on $L^2_Q(\mu)$. h_o is essentially complex on $(X - R_o)$ and is unique upto μ-null sets.

iii) $\underline{E}_\mu = \underline{F}_\mu = \underline{P}_\mu = \underline{Q}_\mu$.

The proofs of these results are essentially similar to those of corresponding results in the complex case and are therefore omitted.

DEFINITION 4.5 A subspace S of $\underline{H}$ is said to be of type μ for the spectral system (E, J) if $S \in \underline{Q}$ and the restriction of (E,J) to S is isomorphic to the canonical spectral system (E_μ, J_μ).

<u>LEMMA 4.5</u> For any J-vector $x \in \underline{H}$, $Z(x)$ is of type μ_x for (E, J).

<u>Proof</u>: By (iv) of Lemma 4.2, we can find an isomorphism ϕ from $L_Q^2(\mu_x)$ to $Z(x)$ such that $\phi E_\mu(\cdot)f = E(\cdot)\phi f$. For this same ϕ, since $\phi(1) = x$, for all $M \in \Sigma$

$$\begin{aligned}
\phi J_\mu(\chi_M) &= \phi(\chi_M \cdot i \cdot \chi_{X-R_o}) \\
&= i\, \phi\, E_\mu(M \cap X - R_o) \cdot 1 \\
&= i\, E(M)\, E(X-R_o)\, \phi(1) \\
&= iE(X-R_o)E(M)x \\
&= JE(M)x = J\,\phi(\chi_M)
\end{aligned}$$

because $E(M)x$, together with x, is a J-vector. It now follows, via simple functions, that $\phi J_\mu f = J\phi(f)$ for all $f \in L_Q^2(\mu)$. Consequently $Z(x)$ is of type μ_x for (E, J).

<u>DEFINITION 4.6</u> If $S_1, S_2 \in \underline{Q}$, S_1 and S_2 are said to be equivalent, $S_1 \sim S_2$ in symbols, if the restrictions of the spectral system (E, J) to S_1 and S_2 are isomorphic.

<u>LEMMA 4.6</u> If x, y are any two J-vectors in $\underline{H}$ then $Z(x) \sim Z(y)$ if and only if $\mu_x \equiv \mu_y$.

Proof: Follows from Lemmas 4.4 and 4.5.

Lemma 4.7 below is also easily proved.

LEMMA 4.7. Let x be a J-vector in $\underline{H}$.

i) If $S \subseteq Z(x)$ and $S \in \underline{P}$ then $S \in \underline{Q}$. In fact there exists $M \in \Sigma$ such that $S = E(M) \cap Z(x) = Z(E(M)x)$.

ii) If $S_1, S_2 \subseteq Z(x)$, $S_1, S_2 \in \underline{Q}$, then $S_1 \longleftrightarrow S_2$.

iii) If $S_1, S_2 \subseteq Z(x)$, $S_1, S_2 \in \underline{Q}$, $S_1 \perp S_2$, then there exist disjoint sets M and N such that $S_1 = E(M)Z(x)$ and $S_2 = E(N)Z(x)$.

These six lemmas give us all the crucial results. Once we have these we can just sail through the row-column mechanism effortlessly. We therefore content ourselves with a brief sketch of this part of the theory, displaying only the more important results. The reader may fill in the details for himself by referring to the discussion on complex spectral measures of either Halmos (1957) or Varadarajan (1959).

Every $F \in \underline{F}$ is called a column. For $S \in \underline{Q}$, we define $C(S)$, the column generated by S as the smallest column containing S : $C(S) = \cap [F : S \subseteq F \in \underline{F}]$. Writing $C(x)$ for $C(Z(x))$ for any vector x one may prove that $C(x) = \cap [F : x \in F \in \underline{F}] = \cap [E(M) : x \in E(M), M \in \Sigma]$. Columns of the form $C(x)$ are called primitive columns. Every column is a direct sum of mutually orthogonal primitive columns.

Two subspaces $S_1, S_2 \in \underline{P}$ are called very orthogonal if $S_1 \subseteq F$ and $S_2 \perp F$ for some $F \in \underline{F}$. A row is a direct sum of very orthogonal cycles. Calling a row in $\underline{Q}$ a J-row, it is easy to check that a row R is a J-row if and only if it is a direct sum of very orthogonal J-cycles. If R is a J-row and $S \subseteq R$, $S \in \underline{P}$, then $S \in \underline{Q}$ and is infact a J-row.

A J-row R is said to be full in a column F if R is maximal in F i.e. if $R \subseteq F$ and if R' is any J-row contained in F and containing R, then $R = R'$. If $x \in \underline{H}$ is arbitrary and R is a J-row full in $C(x)$, then there exists a J-vector $y \in R$ such that $R = Z(y)$. Every two J-rows full in a column F are equivalent in the sense of Definition 4.6.

For any column F, every two maximal families of mutually orthogonal J-rows each full in F have the same power and this cardinal, denoted by $u(F)$, is called the multiplicity of F. A column F is said to have uniform multiplicity if $0 \neq G \subseteq F$, $G \in \underline{F}$ implies $u(G) = u(F)$. This happens if and only if there exists a maximal family of mutually orthogonal J-rows full in F which exhausts F. The lemma below may be called the Fundamental Decomposition Theorem for spectral systems.

LEMMA 4.8. Let ϱ be the dimension of $\underline{H}$. Then, for every cardinal number $u \leq \varrho$, there exists a column F_u such that i) F_u is either 0 or has uniform multiplicity u, ii) the (F_u) are mutually orthogonal and iii) $\underline{H} = \bigoplus_{u \leq \varrho} F_u$. Moreover, if $[G_u : u \leq \varrho]$ is any other family of columns with these properties, then $G_u = F_u$ for all u.

From this cardinal valued function u on columns one passes to a multiplicity function u on $\bar{\underline{N}}$, the σ-complete lattice of (equivalence classes of) finite non-negative measures μ on (X, Σ) in the usual way, via primitive columns. To be a little more explicit, we first prove that $S \in \underline{Q}$ is of type μ for (E, J) if and only if there exists a J-vector $x \in S$ such that $S = Z(x)$ and $\mu \equiv \mu_x$. In fact if S is of type μ, there exists a J-vector $y \in S$ such that $S = Z(y)$ and $\mu_y = \mu$. More generally if x is a J-vector and $\mu << \mu_x$, then there exists a J-vector $y \in Z(x)$ such that $\mu = \mu_y$. For $\mu \in \bar{\underline{N}}$, the multiplicity of μ, $u(\mu)$, is defined as the power of any maximal family of mutually orthogonal subspaces of type μ for (E, J). $u(\mu)$ is then a well-defined multiplicity function on $\bar{\underline{N}}$.

From the way we have defined u, it is rather easy to cheek that $u(\mu) \neq 0$ if and only if $\mu \equiv \mu_x$ for some non-zero J-vector $x \in \underline{H}$. But in fact we can say more: $u(\mu) \neq 0$ if and only if $\mu \equiv \mu_x$ for some non-zero vector $x \in \bar{\underline{H}}$, not necessarily a J-vector. This happy refinement is a consequence of the lemma below.

<u>LEMMA 4.9</u> For any $x \in \underline{H}$, there exists a J-vector $y \in \underline{H}$ such that $\mu_y \equiv \mu_x$.

<u>Proof.</u> Let $x_1 = E(X - R_o)\, x$ and $x_2 = E(R_o)x$. Then $\mu_{x_1} \perp \mu_{x_2}$ and $\mu_x = \mu_{x_1} + \mu_{x_2}$. If we now prove that there exists $y_1 \in P^+$ such that $\mu_{y_1} \equiv \mu_{x_1}$, then $y = y_1 + x_2$ is a J-vector and $\mu_y \equiv \mu_x$. In other words, it is enough to prove the lemma when $x \in E(X - R_o)$.

Let then $x \in E(X - R_o)$ and write $x = x^+ + kx^-$ with x^+, $x^- \in P^+$. Then, as is easily verified,

$$\mu_x(M) = \langle E(M)\, x^+, x^+ \rangle + \langle E(M)x^-, x^- \rangle + k \langle E(M)x^-, x^+ \rangle + \langle E(M)x^+, x^- \rangle k^* .$$

The first two terms on the right side of the equality are non-negative and the last two purely imaginary. Consequently $\mu_x(M) = 0$ if and only if $E(M)x^+ = 0 = E(M)x^-$.

If we now define y by $y = x^+ + x^-$ or $x^+ - x^-$ according as Re $(E(M)x^+, x^-)$ is non-negative or negative then $y \in P^+$ and, as may be verified, $\mu_y(M) = 0$ if and only if $E(M)x^+ = 0 = E(M)x^-$. Consequently $\mu_y \equiv \mu_x$. The lemma is therefore proved.

We have shown, over the last few pages, that given a spectral system (E, J) in a Q-Hilbert space $\underline{H}$ based on the measurable space (X, Σ) one may associate with it a multiplicity function u defined on $\underline{N}$, the σ-complete lattice of (equivalence classes of) finite non-negative measures μ. But, even though u is defined is terms of <u>both</u> E and J, in point of fact, it depends only on E. In other words, if E is a spectral measure and (E,J) and (E,L) are any two spectral systems then their multiplicity functions are the same. To prove this, we shall prove that the multiplicity function u' of any (E,J) is the same as that of the spectral system (E,0), or simply of the spectral measure E, say u. Now if μ is such that $\mu(X - R_0) = 0$, then a subspace is of type μ for (E,J) if and only if it is of type μ for E, so that $u'(\mu) = u(\mu)$. It is enough therefore to prove that $u' = u$ when $JJ^* = I$. Let now (μ_α) be a basis for u' and let $(S_{\alpha\beta})$ be a maximal family of mutually orthogonal subspaces of type μ_α for (E,J) such that $(\underset{\alpha}{\dot{+}})(\underset{\beta}{\dot{+}})\, S_{\alpha\beta} = \underline{H}$. (Each $(\underset{\beta}{\dot{+}})\, S_{\alpha\beta}$ is then a primitive column of uniform multiplicity). But then $u'(\mu_\alpha) = u(\mu_\alpha)$ and (μ_α) is a basis for u too. Hence $u' \equiv u$.

After all this trouble, it is now but a few easy steps to our goal.

<u>THEOREM 4.3</u> Let (E, J) and (F, L) be two spectral systems based on (X, Σ) in Q-Hilbert spaces $\underline{H}$ and $\underline{K}$ respectively with $JJ^* = E(X - R_o)$ and $LL^* = F(X - R_o)$ for some $R_o \in \Sigma$. Then (E, J) and (F, L) are isomorphic if and only if they have the same multiplicity function. In particular two spectral measures are isomorphic if and only if they have the same multiplicity function.

<u>COROLLARY 4.1</u> Let E be a spectral measure and J, L two imaginary operators admissible with respect to E such that $JJ^* = LL^*$. Then (E, J) and (E, L) are isomorphic. In particular (E, J) and (E, J*) are always isomorphic.

We bring our discussion of spectral systems to a close with the two theorems below which will be of help to us when we discuss operator algebras. Their proofs parallel those of their analogues in the complex case and are hence omitted.

<u>THEOREM 4.4</u> The following statements regarding $S \in \underline{Q}$ are equivalent:

i) S is a (J-) row

ii) If $S_1, S_2 \subseteq S$, $S_1, S_2 \in \underline{Q}$, then $S_1 \longleftrightarrow S_2$

iii) If Z_1 and Z_2 are J-cycles contained in S then $Z_1 \longleftrightarrow Z_2$.

iv) If Z_1 and Z_2 are J-cycles contained in S and $Z_1 \sim Z_2$, then $Z_1 = Z_2$.

<u>THEOREM 4.5</u> A projection P is a column if and only if $P \longleftrightarrow \underline{Q}$ and $P \longleftrightarrow J$.

V. ONE-PARAMETER UNITARY GROUPS

5.1 In the complex case the one-parameter groups of unit complex numbers are indexed by real numbers λ, the group corresponding to λ being $e^{it\lambda}$, t real. Hence, in a certain obvious sense, the simplest kind of one-parameter unitary group on a complex Hilbert space $\underline{H}$ is of the form $U_t = e^{it\lambda}I$ for some fixed λ. Stone's theorem then says that every weakly continuous one-parameter unitary group on $\underline{H}$ may be obtained as an 'integral' of these simple groups.

Turning now to Q-Hilbert spaces, one may say that a simple kind of one-parameter unitary group on a Q-Hilbert space $\underline{H}$ is obtained by choosing any one-parameter group (p_t) of unit quaternions, any basis (e_α) of $\underline{H}$ and defining (U_t) by $U_t = p_t e_\alpha$ for all α. The obvious question then is whether every well-behaved one-parameter unitary group on $\underline{H}$ is some sort of integral of these simple ones. We show now that this question may be answered in the affirmative.

The crucial observation is that to obtain the class of all 'simple' one-parameter unitary groups as above, it is not necessary for us to consider <u>all</u> the one-parameter groups of unit quaternions - it is enough to restrict ourselves to just one member from each equivalence class of such. This is because, if (p_t) and (q_t) are such that $q_t = r^{-1} p_t r$, $|r| = 1$, then the group defined by $U_t e_\alpha = q_t e_\alpha$ is the same as the group defined by $U_t f_\alpha = p_t f_\alpha$ where $f_\alpha = r e_\alpha$ for all α. Recalling now that we have proved in Theorem 1.2 that every one-parameter group of unit quaternions is equivalent to one of the form $(e^{it\lambda})$ with $\lambda \geq 0$ we see (because of Theorem 3.2) that every one of these simple one-parameter groups may be obtained by choosing a non-negative number λ, a full imaginary operator J and defining (U_t) by $U_t = \cos(\lambda t). I + J \sin(\lambda t). I$ which may be written symbolically as $U_t = e^{J\lambda t}$.

5.2 With this motivation it is clear that Theorem 5.1 below may legitimately be called Stone's theorem for continuous one-parameter unitary groups on Q-Hilbert spaces.

Let R^+ stand for the set of all non-negative real numbers.

<u>THEOREM 5.1</u> Let (U_t) be any weakly continuous one-parameter group of unitary operators on a Q-Hilbert space $\underline{\underline{H}}$. Then there exist a unique spectral measure $E(\cdot)$ on the Borel sets of R^+ and a unique imaginary operator J

admissible with respect to $E(\cdot)$ for which $I - JJ^* = E(\{0\})$ such that

$$U_t = \int_0^\infty e^{it\lambda}\, dE \quad \text{w.r.t. } J.$$

Proof: Consider (U_t^s) on $\underline{H}^s$. Then, by Stone's theorem, there exists a unique spectral measure $E_s(\cdot)$, say, on the real line with values in $\underline{L}(\underline{H}^s)$ such that

$$\langle U_t^s x, y \rangle = \int_{-\infty}^{+\infty} e^{it\lambda}\, d \langle E_s(\cdot) x, y \rangle$$

for all $x, y \in \underline{H}^s$.

We claim that $K E_s(M) K^{-1} = E_s(-M)$ for all Borel sets M. This follows from the string of equalities below and the uniqueness of the Fourier Transform:

$$\int_{-\infty}^{+\infty} e^{-it\lambda}\, d \langle K E_s(\lambda) K^{-1} x, y \rangle$$

$$= \int_{-\infty}^{+\infty} e^{-it\lambda}\, d \langle K E_s(\lambda) K^{-1} x, K K^{-1} y \rangle$$

$$= \int_{-\infty}^{+\infty} e^{-it\lambda}\, d \langle K^{-1} y, E_s(\lambda) K^{-1} x \rangle$$

$$= \int_{-\infty}^{+\infty} e^{-it\lambda}\, d \langle E_s(\lambda) Ky, Kx \rangle$$

$$= \langle U_{-t}^s Ky, Kx \rangle$$

$$= \langle x, U^s_{-t} y \rangle$$

$$= \langle U^s_t x, y \rangle$$

$$= \int_{-\infty}^{+\infty} e^{it\lambda} \, d \langle E_s(\lambda) x, y \rangle$$

$$= \int_{-\infty}^{+\infty} e^{-it\lambda} \, d \langle E_s(-\lambda) x, y \rangle.$$

This result implies that if we now define, for Borel sets M of R^+, $E(M) = E_s(M) \vee E_s(-M)$, then $E(\cdot)$ is a spectral measure with values in $\underline{L}(\underline{H})$, and that if we define J by

$$J = iE_s [\lambda : \lambda > 0] - i E_s [\lambda : \lambda < 0]$$

then J is an imaginary operator on $\underline{H}$ admissible w.r.t. E and $I - JJ^* = E(\{0\})$.

To prove now that $U_t = \int e^{it\lambda} dE$ w.r.t. J, we have to prove that if

$$V_t = \int_0^\infty \cos t\lambda \, dE$$

and
$$W_t = \int_0^\infty \sin t\lambda \, dE$$

then $U_t = V_t + JW_t$ or, what is the same, that

$\langle U_t x, y \rangle = \langle V_t x, y \rangle + \langle W_t x, J^* y \rangle$ for all $x, y \in \underline{H}$.

But

$$\langle U_t x, y \rangle = \int_{-\infty}^{+\infty} e^{it\lambda} d \langle E_s(\cdot) x, y \rangle$$

$$= \langle E_s([0]) x, y \rangle + \int_{0+}^{+\infty} e^{it\lambda} d \langle E_s(\cdot) x, y \rangle$$

$$+ \int_{-\infty}^{0-} e^{it\lambda} d \langle E_s(\cdot) x, y \rangle$$

$$= \langle E([0]) x, y \rangle + \int_{0+}^{+\infty} \cos t\lambda \, d \langle E_s(\cdot) x, y \rangle$$

$$+ i \int_{0+}^{+\infty} \sin t\lambda \, d \langle E_s(\cdot) x, y \rangle$$

$$+ \int_{0+}^{+\infty} \cos t\lambda \, d \langle K E_s(\cdot) K^{-1} x, y \rangle - i \int_{0+}^{+\infty} \sin t\lambda \, d \langle K E_s(\cdot) K^{-1} x, y \rangle$$

$$= \int_{0}^{+\infty} \cos t\lambda \, d(E(\cdot)x, y) + \int_{0}^{+\infty} \sin t\lambda \, d \langle E(\cdot)x, J^* y \rangle$$

$$= \langle V_t x, y \rangle + \langle W_t x, J^* y \rangle,$$

We have now to prove the uniqueness of the spectral representation.

Suppose that $F(\cdot)$ is any spectral measure on R^+ with values in $\underline{L}(\underline{H})$ and L any imaginary operator admissible w.r.t. F such that $I - LL^* = F([0])$ and

$$U_t = \int_0^{+\infty} e^{it\lambda} \, d F \quad \text{w.r.t. } L.$$

Let us define a spectral measure $F_s(\cdot)$ on R with values in $\underline{L}(\underline{H}^s)$ by

$$\begin{aligned} F_s(M) &= F(M)P^+ \quad \text{for all } M \subseteq R^+ - [0] \\ &= F(-M)P^- \quad \text{for all } M \subseteq R - R^+ \\ &= F(M) \quad \text{for } M = [0] \end{aligned}$$

where $L^s = iP^+ - iP^-$ is the canonical decomposition of L^s discussed in Chapter III.

We then have

$$\begin{aligned} \langle U_t x, y \rangle &= \int_0^{+\infty} \cos t\lambda \; d \langle F(\cdot)x, y \rangle \\ &\quad + \int_0^{+\infty} \sin t\lambda \; d \langle F(\cdot)x, L^* y \rangle \\ &= \langle F([0]) x, y \rangle + \int_{0+}^{+\infty} \cos t\lambda \; d \langle F(\cdot)P^+ + F(\cdot)P^-)x, y \rangle \\ &\quad + \int_0^{+\infty} \sin t\lambda \; d \langle (F(\cdot)P^+ + F(\cdot)P^-)x, L^* y \rangle \\ &= \langle F_s([0])x, y \rangle + \int_{0+}^{+\infty} \cos t\lambda \; d \langle F_s(\cdot)x, y \rangle \\ &\quad + \int_{-\infty}^{0-} \cos t\lambda \, d \; \langle F_s(\cdot)x, y \rangle \end{aligned}$$

$$+ \int_{0}^{+\infty} \sin t\lambda \, d < i F_s(\cdot)x, y > - \int_{-\infty}^{0} \sin t\lambda \, d < (-i) F_s(\cdot)x, y >$$

$$= \int_{-\infty}^{+\infty} \cos t\lambda \, d < F_s(\cdot)x, y > + i \int_{-\infty}^{+\infty} \sin t\lambda \, d < F_s(\cdot)x, y >$$

$$= \int_{-\infty}^{+\infty} e^{it\lambda} \, d < F_s(\cdot) x, y >$$

for all $x, y \in \underline{H}^s$ so that the uniqueness of the spectral measure in the complex case helps us to conclude that $F_s(\cdot) = E_s(\cdot)$. But this immediately implies that $E(\cdot) \equiv F(\cdot)$ and $J = L$. This completes the proof of the theorem.

VI. NORMAL OPERATORS

6.1 We are atlast in a position to obtain significant information about normal operators on Q-Hilbert spaces.

We start with the spectral theorem. Let A be a normal operator on a Q-Hilbert space $\underline{H}$. Then A^s is normal on $\underline{H}^s$. Let $E_s(\cdot)$ be the canonical spectral measure of $A^s : A^s = \int \lambda \, dE_s$. We observed earlier that the spectrum of A^s is symmetric about the real axis. This means that the spectrum is known as soon as its intersection with C^+ is known. Much more than this is true. The spectral measure $E_s(\cdot)$ is known as soon as its restriction to C^+ is known.

<u>LEMMA 6.1</u> For every measurable set $M \subseteq C$, $E_s(\bar{M}) = KE_s(M)K^{-1}$, where $\bar{M} = [\lambda : \bar{\lambda} \in M]$.

<u>Proof:</u> If M is a compact set we may prove this using the Stone-Lengyel characterization of spectral subspaces (Halmos, 1957). The regularity of $E_s(\cdot)$ implies the rest.

<u>COROLLARY 6.1</u> If $M = \bar{M}$ then $E_s(M)$ is a projection on $\underline{H}$. Further, if for $M \subseteq C^+$, $E(M) = E_s(M \cup \bar{M})$ then $E(\cdot)$ is a spectral measure on C^+ with values in $\underline{L}(\underline{H})$.

<u>THEOREM 6.1</u> Let A be a normal operator on a Q-Hilbert space $\underline{H}$. Then there exists a unique spectral system (E, J) where E is a spectral measure on the Borel sets of C^+ and J satisfies $JJ^* = I - E(R)$ such that

$$A = \int \lambda \, dE \quad \text{w.r.t. } J.$$

<u>Proof</u>: Define $E(M)$ for measurable $M \subseteq C^+$ by $E(M) = E_s(M \cup \bar{M})$. $E(M)$ is then a spectral measure in $\underline{H}$. Define J on $\underline{H}$ by $J^s = i\, E_s(C^+ - R) - i\, E_s(C - C^+)$. Clearly J is an imaginary operator on $\underline{H}$ and $J^s \longleftrightarrow E_s(\cdot)$ and therefore $J \longleftrightarrow E(\cdot)$. Further $JJ^* = E(C^+ - R)$ so that J is admissible w.r.t. $E(\cdot)$. Consequently (E,J) is a spectral system with the required properties. We have now to prove that $A = \int \lambda \, dE$ w.r.t. J. Let $\lambda = b + ic$ with b, c real, and $B = \int b \, dE$, $C = \int c \, dE$. We have to prove that $A = B + JC$ for which it is sufficient to prove that

$\langle A^s x, y \rangle = \langle (B^s + J^s C^s)x, y \rangle$. But

$$\begin{aligned}
\langle A^s x, y \rangle &= \int \lambda \, d \langle E_s(\cdot) x, y \rangle \quad \text{by definition of } E_s(\cdot) \\
&= \int b\, d \langle E_s(\cdot) x, y \rangle + i \int c\, d \langle E_s(\cdot) x, y \rangle \\
&= \int_R b\, d \langle E_s(\cdot)x, y \rangle + \int_{C^+ - R} b\, d \langle E_s(\cdot)x, y \rangle \\
&\quad + \int_{C - C^+} b\, d \langle E_s(\cdot)x, y \rangle + i \int_{C^+ - R} c\, d \langle E_s(\cdot)x, y \rangle \\
&\quad + i \int_{C - C^+} c\, d \langle E_s(\cdot)x, y \rangle
\end{aligned}$$

$$= \int_R bd \langle E(\cdot)x, y \rangle + \int_{C^+-R} bd \langle E_s(\cdot)x,y \rangle + \int_{C-C^+} bd \langle KE_s(\cdot)K^{-1}x,y \rangle$$

$$+ i \int_{C^+-R} cd \langle E_s(\cdot)x, y \rangle - i \int_{C^+-R} cd \langle KE_s(\cdot)K^{-1}x,y \rangle$$

$$= \int_R bd \langle E(\cdot)x,y \rangle + \int_{C^+-R} bd \langle E(\cdot)x, y \rangle + \int_{C^+-R} cd \langle JE_s(\cdot)x,y \rangle$$

$$+ \int_{C^+-R} cd \langle JKE_s(\cdot)K^{-1} x, y \rangle$$

$$= \int bd \langle E(\cdot)x, y \rangle + \int cd \langle E(\cdot)x, J^*y \rangle$$

$$= \langle B^s x, y \rangle + \langle Cx, J^* y \rangle$$

$$= \langle (B^s + J^s C^s)x, y \rangle.$$

The proof of the uniqueness is equally simple. Let the spectral system (F, L) have the required properties and let $A = \int \lambda dF$ w.r.t. L. Consider L^s on $\underline{H}^s$ and let $L^s = iQ^+ - iQ^-$ where Q^+ is the projection in $\underline{H}^s$ on the eigen subspace of L^s corresponding to the eigen-value i, and $Q^- = KQ^+K^{-1}$. We know that $Q^+ \perp^s KQ^+K^{-1}$, that $Q^+ \longleftrightarrow [F(M)]^s$ ($\because$ $L^s \longleftrightarrow [F(M)]^s$) and $Q^+ + Q^- = L^*L = F(C^+ - R)$. Define now a complex spectral measure $F_s(\cdot)$ with values in $\underline{L}(\underline{H}^s)$ by

$$\begin{aligned} F_s(M) &= [F(M)]^s && \text{if} \quad M \subseteq R \\ &= [F(M)]^s Q^+ && \text{if} \quad M \subseteq C^+ - R \\ &= [F(\bar{M})]^s Q^- && \text{if} \quad M \subseteq C - C^+ \end{aligned}$$

Our theorem will follow from the uniqueness of the spectral representation in the complex case if we prove that $A^s = \int \lambda d\, F_s$, for then $F_s(\cdot) = E_s(\cdot)$ thereby implying $F(\cdot) = E(\cdot)$ as also $J = L$. But we have $A = \int \lambda dF$ w.r.t. L. Therefore,

$$
\begin{aligned}
\langle Ax, y \rangle &= \langle (\int b dF + L \int c dF)x, y \rangle \\
&= \langle (\int b dF)x, y \rangle + \langle (\int c dF)x, L^* y \rangle \\
&= \int bd \langle F(\cdot)x, y \rangle + \int cd \langle F(\cdot)x, L^* y \rangle \\
&= \int_R bd \langle F(\cdot)x, y \rangle + \int_{C^+-R} bd \langle (F(\cdot)Q^+ \\
&\qquad + F(\cdot)Q^-)x, y \rangle \\
&\quad + \int_{C^+-R} cd \langle (F(\cdot)Q^+ + F(\cdot)Q^-)x, L^* y \rangle \\
&= \int_R bd \langle F_s(\cdot)x, y \rangle + \int_{C^+-R} bd \langle F(\cdot)Q^+ x, y \rangle \\
&\quad + \int_{C^+-R} bd \langle F(\cdot)KQ^+K^{-1}x, y \rangle \\
&\quad + \int_{C^+-R} cd \langle F(\cdot)Q^+ x, L^* y \rangle \\
&\quad + \int_{C^+-R} cd \langle F(\cdot)Q^- x, y \rangle \\
&= \int_R bd \langle F_s(\cdot)x, y \rangle + \int_{C^+-R} bd \langle F_s(\cdot)x, y \rangle \\
&\quad + \int_{C-C^+} bd \langle F_s(\cdot)x, y \rangle + i \int_{C^+-R} cd \langle F_s(\cdot)x, y \rangle \\
&\qquad - i \int_{C-C^+} cd \langle F_s(\cdot)x, y \rangle \\
&= \int \lambda d \langle F_s(\cdot)x, y \rangle.
\end{aligned}
$$

The proof of the theorem is complete.

COROLLARY 6.2 Every normal operator A can be written uniquely in the form $B + JC$ where B, J, C are mutually commuting, B is hermitian, C is positive, J is imaginary and $Jx = 0$ if and only if $Cx = 0$.

Proof: If $B = \int \mathrm{Re}\,\lambda\, dE$ and $C = \int \mathrm{Im}\,\lambda\, dE$ then clearly B, C, J satisfy the required conditions and $A = B + JC$. B and C are unique because $B = \frac{A + A^*}{2}$ and $C^2 = -\frac{(A - A^*)^2}{2}$ and C is positive. If now L and J are two imaginary operators whose null space N is the same as that of C, then trivially $Lx = Jx$ for $x \in N$ and if $x \in N^{\perp}$, since there exists $y \in N^{\perp}$ such that $Cy = x$ and $JC = LC = \frac{A - A^*}{2}$, we have $Lx = Jx$ again. This proves the uniqueness of J.

COROLLARY 6.3 An operator B commutes with A if and only if B commutes with $E(\cdot)$ and B commutes with J.

Proof: If $B \longleftrightarrow A$, then $B^S \longleftrightarrow A^S$ and hence $B^S \longleftrightarrow E_S(\cdot)$ implying that $B \longleftrightarrow E(\cdot)$ as well as $B \longleftrightarrow J$. Conversely if $B \leftrightarrow E$ then standard arguments (via simple functions) prove that $B \leftrightarrow \int \mathrm{Re}\,\lambda\, dE$ and $B \leftrightarrow \int \mathrm{Im}\,\lambda\, dE$. If also $B \leftrightarrow J$, then obviously $B \leftrightarrow A$.

We now turn our attention to the problem of obtaining a complete set of unitary invariants for normal operators. We have first the lemma below.

LEMMA 6.2 Two normal operators are unitarily equivalent if and only if their spectral systems are isomorphic.

Proof: Let A, B be normal operators acting on Q-Hilbert spaces $\underline{H}$ and $\underline{K}$ respectively with spectral systems (E, J) and (F, L) respectively. And let E_S and F_S be the spectral measures of A^S and B^S on $\underline{H}^S$ and $\underline{K}^S$. If there exists an isomorphism $\phi : \underline{H} \to \underline{K}$ such that $B = \phi A \phi^{-1}$ then $B^S = \phi^S A^S \phi^{S^{-1}}$ and hence $F_S = \phi^S E_S \phi^{S^{-1}}$ so that

$$F(M)^S = F_S (M \cup \bar{M}) = \phi^S E_S (M \cup \bar{M}) \phi^{S^{-1}} = \phi^S E(M)^S \phi^{S^{-1}} \text{ and}$$

$$L^S = iF_S (C^+ - R) - i F_S(C - C^+) = i\phi^S E_S(C^+ - R)\phi^{S^{-1}}$$

$$- i \phi^S E_S(C - C^+) \phi^{S^{-1}}$$

$$= \phi^S J^S \phi^{S^{-1}}$$

implying that $F = \phi E \phi^{-1}$ as well as $L = \phi J \phi^{-1}$.

Conversely, if (E, J) and (F, L) are isomorphic with $F = \phi E \phi^{-1}$ and $\phi J \phi^{-1} = L$, then it follows rather easily that $B = \phi A \phi^{-1}$.

From this lemma and Theorem 4.3 we may conclude immediately that two normal operators are unitarily equivalent if and only if the multiplicity functions of their associated spectral systems are the same. But we can be more explicit and characterise the multiplicity function attached to a normal operator via its spectral system directly in terms of the operator itself. To see this let $\underline{N}$ be the set of all finite, non-negative measures with compact support in C^+. (We need not consider measures with non-compact support because the multiplicity of every one such will be zero in the case we are interested in). For $\mu \in \underline{N}$ the canonical operator A_μ associated with μ is the normal operator on $L^2_Q(\mu)$ defined by $A_\mu f(\lambda) = f(\lambda).\lambda$. It is easy to check that the spectral system of A_μ is (E_μ, J_μ), the canonical spectral system associated with μ (see Definition 4.4) defined by $E_\mu(M)\ f = f.\ \chi_M$ and

$$J_\mu f = f.i.\ (1 - \chi_R).$$

For $\mu \in \underline{N}$, a subspace S of $\underline{H}$ is said to be of type μ for A if S reduces A and A restricted to S is unitarily equivalent to A_μ; i.e. there is an isomorphism ϕ: $L^2_Q(\mu) \to S$ such that $\phi(A_\mu f) = A\,\phi(f)$. If we observe now that the spectral system of the restriction of A to S is but the restriction of the spectral system of A to S and use Corollary 6.3 and Lemma 6.2 we can say that S is of type μ for A if and only if S is of type μ for (E,J). We therefore have the following theorem:

<u>THEOREM 6.2</u> Let A be a normal operator on a Q-Hilbert space $\underline{H}$. For every finite, non-negative measure μ on the Borel subsets of C^+, let us define $u(\mu)$ as the cardinality of any maximal family of mutually orthogonal subspaces of type μ for A. Then u is a well-defined multiplicity function and two normal operators are unitarily equivalent if and only if their multiplicity functions are the same.

With this theorem the structure theory of normal operators on Q-Hilbert spaces is complete.

We shall now exploit this theory to deduce two interesting properties of normal operators on Q-Hilbert spaces which have no analogues in the complex case.

In the heuristic explanation of the properties of normal operators on complex Hilbert spaces it is often said that normal operators behave like complex numbers. We may make a similar statement with regard to normal operators on Q-Hilbert spaces - They behave like quaternions! The two theorems below reflect the two properties of quaternions that (i) every quaternion q is conjugate to q^* and (ii) if q is real and p commutes with every quaternion commuting with q, then p is real.

<u>THEOREM 6.3</u> Every normal operator A on a Q-Hilbert space is unitarily equivalent to its adjoint A^*.

Proof: Observe that if the spectral system of A is (E,J) then the spectral system of A^* is (E,J^*) and use Corollary 4,1.

THEOREM 6.4. If A is a normal operator on a Q-Hilbert space $\underline{H}$ and B is an operator which commutes with every operator commuting with A, then B is normal. If A is hermitian, so is $B(!)$.

Proof: The first part is trivial. As for the second part, in view of our structure theory,, it is sufficient to prove it when $\underline{H} = L_Q^2(\mu)$, μ being a finite, non-negative, compact measure with support contained in R and $(Af)(\lambda) = f(\lambda).\lambda$. But then $B \leftrightarrow E_\mu(\cdot)$ and therefore by Lemma 4.4 there exists a bounded measurable function h_o such that $B.f = f.h_o$ for all $f \in L_Q^2(\mu)$ and by the hypothesis of the theorem B commutes with every operator of the form $f \rightarrow f.q$ where q is a fixed quaternion. Consequently $q.h_o = h_o q[\mu]$ for all quaternions q and we are forced to the conclusion that h_o is essentially real and therefore that B is hermitian.

VII. FUNCTIONAL CALCULUS

7.1 We shall now give a definition of the concept of a function of a normal operator and develop a functional calculus. This is algebraically (and understandably) more complicated than in the complex case but the geometric ideas underlying it are the same. The vindication of our theory is an analogue of a theorem of von Neumann which we obtain: in the case of a separable Hilbert space, the set of all functions of a normal operator according to our definition coincides with the smallest weakly closed (real) self adjoint algebra of operators containing A.

Throughout this chapter we shall let A be a fixed but arbitrary normal operator on a Q-Hilbert space $\underline{H}$, $E(\cdot)$ ($E_s(\cdot)$) being the spectral measure of A (A^s) and J the imaginary operator of A; $\underline{E}$ will denote the range of E and $\underline{F}$ the completion of $\underline{E}$.

Let $\underline{M}$ denote the class of all complex-valued, measurable, E-ess. bounded functions f defined on C^+ whose restrictions to R are real.

DEFINITION 7.1 For $f \in \underline{M}$, we define $f(A)$ by

$f(A) = \int f\,dE$ w.r.t. J.

Observe that according to this definition every function of a hermitian operator is hermitian !

The following elementary observations and Theorem 7.1 below are immediate consequences of our spectral theory:

i) For a, b real, $f, g \in \underline{M}$, $(af + bg)(A) = af(A)+bg(A)$.

ii) For $f, g \in \underline{M}$, $f(A).g(A) = (f.g)(A)$. In particular $f(A) \leftarrow\rightarrow g(A)$.

iii) For $f_n \in \underline{M}$, $n = 1,2,\ldots$ and $f \in \underline{M}$,

a) if (f_n) converges uniformly to f then $f_n(A)$ converges uniformly to $f(A)$,

b) if f_n converges to f pointwise and boundedly, then $f_n(A)$ converges to $f(A)$ strongly.

iv) $f(A)$ is unitary (hermitian) if and only if f is E-essentially of modulus 1 (real).

v) For any operator B on $\underline{H}$, if $B \leftarrow\rightarrow A$, then $B \leftarrow\rightarrow f(A)$.

<u>THEOREM 7.1</u> If A is a normal operator on a Q-Hilbert space then there exist a positive operator B and a unitary operator U such that $B \leftarrow\rightarrow U$ and $A = BU$.

Our first problem is to give explicitly the spectral system, (F, L) say, of $f(A)$. If the E-essential range of f is a subset of C^+, then one would expect that $F(M) = E(f^{-1}(M))$ and $L = JE(M_+)$ where M_+ is the set of points on which f is non-real. Once this is noticed, Theorem 7.2 below becomes understandable.

Let us introduce the following notation. For $f \in \underline{M}$, let

$$M_+ \ (= M_+^f) = (\lambda : f(\lambda) \in C^+ - R)$$

$$M_- \ (= M_-^f) = (\lambda : f(\lambda) \in C - C^+)$$

$$M_o \ (= M_o^f) = (\lambda : f(\lambda) \in R)$$

and define a function $\hat{f}$ by

$$\hat{f}(\lambda) = f(\lambda) \qquad \text{if} \quad \lambda \notin M_-$$

$$= \overline{f(\lambda)} \qquad \text{if} \quad \lambda \in M_- .$$

<u>THEOREM 7.2</u> If F is the spectral measure and L the imaginary operator of $f(A)$, then $F(M) = E(\hat{f}^{-1}(M))$ and $L = JE(M_+) - JE(M_-)$.

<u>Proof:</u> Define a function f_s on C by

$$f_s(\lambda) = f(\lambda) \qquad \text{if} \quad \lambda \in C^+$$

$$= \overline{f(\bar{\lambda})} \qquad \text{if} \quad \lambda \in C - C^+ .$$

We claim that $[f(A)]^s = f_s(A^s)$. For let $f = f_1 + if_2$, f_1, f_2 real. Then

$$\langle f(A)x, y \rangle = \langle [f_1(A) + J\, f_2(A)]x, y \rangle$$

$$= \langle f_1(A)x, y \rangle + \langle f_2(A)x, J^* y \rangle$$

$$= \int f_1 d \langle E(\cdot)x, y \rangle + \int f_2 d \langle E(\cdot)x, J^* y \rangle$$

$$= \int_R f_1 d \langle E_s(\cdot)x, y \rangle + \int_{C^+-R} f_1 d \langle [E_s(\cdot) + KE_s(\cdot)K^{-1}]x, y \rangle$$

$$+ \int_{C^+-R} f_2\, d \langle [E_s(\cdot) + K\, E_s(\cdot)K^{-1}]x, J^* y \rangle$$

$$= \int_R f_1 d \langle E_s(\cdot)x, y \rangle + \int_{C^+-R} f_1 d \langle E_s(\cdot)x, y \rangle$$

$$+ \int_{C-C^+} f_1 d \langle E_s(\cdot)x, y \rangle + \int_{C^+-R} f_2 d \langle JE_s(\cdot)x, y \rangle$$

$$+ \int_{C-C^+} f_2 d \langle JE_s(\cdot)x, y \rangle$$

$$= \int_R f_1 d \langle E_s(\cdot)x, y \rangle + \int_{C^+-R} (f_1 + if_2)\, d \langle E_s(\cdot)x, y \rangle$$

$$+ \int_{C-C^+} (f_1 - if_2)\, d \langle E_s(\cdot)x, y \rangle$$

$$= \int f_s d \langle E_s(\cdot)x, y \rangle$$

$$= \langle f_s(A^s)x, y \rangle.$$

Therefore, F_s, the spectral measure of $[f(A)]^s$ is but the spectral measure of $f_s(A^s)$. This, we know, is given by $F_s(M) = E_s(f_s^{-1}(M))$ where E_s is the spectral measure of A^s.

Hence for $M \subseteq R$

$$F(M)^s = F_s(M) = E_s(f_s^{-1}(M))$$

$$= E_s[\hat{f}^{-1}(M) \cup (\overline{\hat{f}^{-1}(M)})]$$

$$= E(\hat{f}^{-1}(M))$$

and for $M \subseteq C^+ - R$

$$[F(M)]^s = F_s(M \cup \bar{M}) = E_s(f_s^{-1}(M \cup \bar{M}))$$

$$= E_s(f_s^{-1}(M) \cup f_s^{-1}(\bar{M}))$$

$$= E_s(f_s^{-1}(M)) + E_s(f_s^{-1}(\bar{M}))$$

$$= E_s(f_s^{-1}(M)) + E_s(\overline{[f_s^{-1}(M)]})$$

$$= E_s(f_s^{-1}(M) \cup \overline{[f_s^{-1}(M)]})$$

$$= E_s(\hat{f}^{-1}(M) \cup \overline{[\hat{f}^{-1}(M)]})$$

$$= E(\hat{f}^{-1}(M)).$$

Therefore, for all $M \subseteq C^+$

$$F(M) = E(\hat{f}^{-1}(M)).$$

As for L, since $L = iI$ on

$$F_s(C^+ - R) = E_s(f_s^{-1}(C^+ - R)) = E_s(M_+ \cup \bar{M}_-)$$

$$= E_s(M_+) + E_s(\bar{M}_-)$$

it follows that $L = JE(M_+) - JE(M_-)$. The proof of the theorem is complete.

Next, we ask if a function of a function of A, say $f(g(A))$, is again a function of A. Here again, if the E-essential range of g is contained in C^+ ($f \circ g$ is not defined otherwise), then it is to be expected that $f(g(A)) = (f \circ g)(A)$. The general case is given by Theorem 7.3 below.

We need to introduce some notation again. Fix $f, g \in \underline{M}$ and define

$$N_+ = (\lambda : g(\lambda) \in C^+ - R,\ f(g(\lambda)) \in C^+ - R)$$

$$\cup\ (\lambda : g(\lambda) \in C - C^+,\ f(\bar{g}(\lambda)) \in C - C^+)$$

$$N_- = (\lambda : g(\lambda) \in C^+ - R,\ f(g(\lambda)) \in C - C^+)$$

$$\cup\ (\lambda : g(\lambda) \in C - C^+,\ f(\bar{g}(\lambda)) \in C^+ - R)$$

$$N_o = C^+ - N_+ - N_-.$$

Define the function h on C^+ by

$$h(\lambda) = \hat{f}(\hat{g}(\lambda)) \quad \text{if} \quad \lambda \in N_+$$

$$= \overline{[\hat{f}(\hat{g}(\lambda))]} \quad \text{if} \quad \lambda \in N_-$$

$$= \hat{f}(\hat{g}(\lambda)) \quad \text{if} \quad \lambda \in N_o.$$

THEOREM 7.3 $\quad f(g(A)) = h(A)$.

Proof: For a normal operator B, let us denote the associated spectral measure and imaginary operator by $E_B(\cdot)$ and J_B. We have

$$E_{f(g(A))} = E_{g(A)}(\hat{f}^{-1}(M)) = E(\hat{g}^{-1}(\hat{f}^{-1}(M)))$$
$$= E(\hat{h}^{-1}(M))$$

by Theorem 7.2, since $\hat{h} = \hat{f} \circ \hat{g}$. Also

$$J_{f(g(A))} = J_{g(A)} E_{g(A)}(M_+^f) - J_{g(A)} E_{g(A)}(M_-^f)$$
$$= J_{g(A)} E(\hat{g}^{-1}(M_+^f)) - J_{g(A)} E(\hat{g}^{-1}(M_-^f))$$

and $J_{g(A)} = JE(M_+^g) - JE(M_-^g)$.

Combining these two equations, we deduce that

$$J_{f(g(A))} = J\,E(N_+) - JE(N_-) = J_{h(A)}$$

by Theorem 7.2. Therefore $f(g(A)) = h(A)$.

7.2 These preliminaries concluded we move on to a study of the deeper properties of the functional calculus. Our first result is an analogue of von Neumann's characterization of functions of a normal operator on a separable complex Hilbert space.

For a normal operator B on $\underline{H}$ let $\underline{E}_B$ denote the range of the spectral measure of B, $\underline{F}_B$ its completion and J_B the imaginary operator of B.

<u>THEOREM 7.4</u> If B is any operator on $\underline{H}$ which commutes with every operator commuting with A then B is normal and $\underline{E}_B \subseteq \underline{F}$. If $\underline{E}$ is complete, in particular if $\underline{H}$ is separable, then B is a function of A.

<u>Proof:</u> B is clearly normal. If $P \in \underline{E}_B$ is arbitrary then by Corollary 6.3. $P \leftrightarrow A$ and consequently, by Corollary 6.3 again, $P \leftrightarrow E(\cdot)$ and J. Also if Q is any projection commuting with A, then $B \leftrightarrow Q$ and hence $P \leftrightarrow Q$. By virtue of Theorem 4.5 we may then conclude that $P \in \underline{F}$. Therefore, $\underline{E}_B \subseteq \underline{F}$. If $\underline{\bar{F}} = \underline{E}$, then there exists a complex measurable function g on C^+ with values in C^+ such that $E_B(M) = E(g^{-1}(M))$ (This is proved for the complex case in (Varadarajan, 1959). This proof applies to the quaternionic case too).

Observe first that g must be E-essentially real on R. This follows from the fact that the restriction of B to $E(R)$ is hermitian (a consequence of Theorem 6.4) so that

$$E(R) \subseteq E_B(R) = E(g^{-1}(R))$$

which implies that $E(R - g^{-1}(R)) = 0$. We may therefore assume, without loss of generality, that g is real on R, i.e. that $g \in \underline{M}$. We shall now produce an f such that $\hat{f} = g$ and $B = f(A)$.

For convenience let us write $L = J_B$. Since $B \longleftrightarrow A$, $B \longleftrightarrow J$ and hence $L \longleftrightarrow J$. Also, since $E(R) \subseteq E_B(R)$, $Jx = 0$ implies $Lx = 0$. By Theorem 3.3 $L = JP - JQ$ where P and Q are mutually orthogonal projections commuting with J and $P + Q = L^*L \subseteq J^*J$. But then $J^*L = P - Q$ so that $P = \frac{(L^* + J^*)L}{2}$ and $Q = \frac{(L^* - J^*)L}{2}$. If now R is any arbitrary projection on $\underline{H}$ which commutes with A, then $R \longleftrightarrow J$ and $R \longleftrightarrow B$ so that $R \longleftrightarrow L$. Consequently, $P, Q \longleftrightarrow R$. Hence by Theorem 4.5 $P, Q \in \underline{F} = \underline{E}$. Remembering now that P and Q are orthogonal, we may find two disjoint sets N_+ and N_- such that $P = E(N_+)$ and $Q = E(N_-)$. Define now a function f by

$$\begin{aligned} f(\lambda) &= g(\lambda) \qquad \text{if } \lambda \in N_+ \\ &= \bar{g}(\lambda) \qquad \text{if } \lambda \in N_- \,. \end{aligned}$$

Clearly then $\hat{f} = g$ and one may verify that $B = f(A)$. The proof of the theorem is complete.

We shall now obtain the characterization of the class of all functions $f(A)$ of A promised at the beginning of this chapter. But first, a lemma.

<u>LEMMA 7.1</u> Let M_o be a compact subset of C^+. Let $\underline{P}$ and $\underline{Q}$ be the classes of all real (!) functions of the form $p(\lambda) = \sum_{r=o}^{m} a_r (\frac{\lambda + \bar{\lambda}}{2})^r$ and

$$q(\lambda) = \sum_{s=o}^{n} b_s (\frac{\lambda - \bar{\lambda}}{2})^{2s}$$

respectively, where the real constants a_r, b_s are arbitrary. Let $\underline{C}$ be the class of all finite sums of functions of the form $p \cdot q$ with $p \in \underline{P}$ and $q \in \underline{Q}$. Then $\bar{\underline{C}}$ is uniformly dense in the algebra of real continuous functions on M_o.

<u>Proof:</u> $\underline{C}$ is an algebra separating points and containing constants.

Let now $[A]$ denote the weak closure in $\underline{B}(\bar{\underline{H}})$ of the set of all polynomials $p(A, A^*)$ in A and A^* with real coefficients. $[A]$ is then the smallest W*-algebra containing A(and I). Let $\underline{\Phi}(A)$ denote the set of all functions of A and $[A]''$ the set of all operators which commute with every operator commuting with A.

THEOREM 7.5 $\Phi(A) \subseteq [A] = [A]''$. If $\underline{E}$ is complete, in particular if $\underline{H}$ is separable, $\Phi(A) = [A] = [A]''$.

Proof: Firstly, using the fact that $E(\cdot)$ is concentrated on a compact set and Lemma 7.1 above, it is clear that $f(A) \in [A]$ for all real, E-essentially bounded continuous functions f on C^+. It follows that $E(M) \in [A]$ for all closed sets M and since $E(\cdot)$ is regular, that $\underline{E} \subseteq [A]$. But $[A]$ is weakly closed and every element of $\underline{F}$ is ~~the~~ a weak limit of elements of $\underline{E}$. Therefore $\underline{F} \subseteq [A]$. Also, since $\underline{E} \subseteq [A]$, $f(A) \in [A]$ for all real E-essential bounded f.

Secondly, $J \in [A]$. To prove this let us write $\lambda = b + ic$, b, c real, for all $\lambda \in C^+$. Then $\frac{A - A^*}{2} = \int ic \, dE$ w.r.t. J. Define for every positive integer n, a real-valued function f_n on C^+ by

$$f_n(\lambda) = 1/c \quad \text{if} \quad c \geq 1/n$$
$$= 0 \quad \text{if} \quad c < 1/n .$$

Then $f_n(A) \in [A]$ for all n and

$$J.E[\lambda : c \geq 1/n] = \int_{c \geq 1/n} i\, c \cdot \frac{1}{c}\, dE \quad \text{w.r.t. } J$$

$$= \int i c f_n \, dE \quad \text{w.r.t. } J$$

$$= \frac{A - A^*}{2} f_n(A) \in [A].$$

Taking the limit as $n \to \infty$,

$$J = JE[\lambda : c > 0] \in [A].$$

From these two observations it is obvious that $\Phi(A) \subseteq [A]$.

Coming to the proof of the equality $[A] = [A]''$, we see that it is enough to prove that $[A]'' \subseteq [A]$ as the reverse inequality is trivially true. But if $B \in [A]''$ then by Theorem 7.4. $\underline{E}_B \subseteq \underline{F}$ and also $L \in [A]$ (because $L = JP - JQ$ for $P, Q \in \underline{F}$). Consequently $B \in [A]$ so that $[A]'' \subseteq [A]$.

Finally if $\underline{E}$ is complete $[A]'' \subseteq \Phi(A)$ by Theorem 7.4 again so that $\Phi(A) = [A] = [A]''$. This proves the entire theorem.

VIII. COMMUTATIVE W*-ALGEBRAS

8.1 In this chapter we shall make a thorough study of commutative W*-algebras of operators on a Q-Hilbert space. Inspite of the fact that these are algebras only over the reals it is possible to obtain a complete insight into their structure by applying Segal's methods of analysis of their complex counter parts (Segal, 1951) to the spectral theory developed in the last four chapters.

DEFINITION 8.1 A collection of operators $\underline{A}$ on a Q-Hilbert space $\underline{H}$ is called a W*-algebra if i) $I \in \underline{A}$, ii) If $A, B \in \underline{A}$ then $A + B$, AB, A^* and aA (a real) all belong to $\underline{A}$ and iii) $\underline{A}$ is closed in the weak topology on $\underline{B}(\underline{H})$.

In the case of a W*-algebra of operators on a complex Hilbert space, the algebra is abelian if and only if every operator in the algebra is normal. But not so in the quaternionic case. While every abelian W*-algebra consists only of normal operators, there exist W*-algebras of normal operators which are not abelian. The set of all right multiplication operators on a finite measure space (mentioned in Chapter II) is, for example, one such. The structure of such algebras is very interesting, but we shall restrict ourselves to the abelian case for the present. The non-abelian 'normal' algebras will be discussed elsewhere by the author.

In what follows **A** will denote a commutative W*-algebra (henceforth abbreviated to CW*A). The set of all projections in **A** which is a complete Boolean algebra, will be denoted by F. We note here that one can always find a spectral measure E, based on a suitable measurable space, whose range is exactly F. This is an easy consequence of the well-known Stone-Loomis representation theorem on Boolean σ-algebras (Varadarajan, 1959).

An example of a CW*A on a separable Q-Hilbert space is the set of all functions of a normal operator A (Theorem 7.5). One of the results of this chapter (Theorem 8.7) will be that every CW*A on a separable Q-Hilbert space is of this form for some normal operator A. Observe that we do not say 'for some hermitian operator A'. This is not true in general because, in our functional calculus, every function of a hermitian operator is hermitian.

DEFINITION 8.2 Let **A** and B be two W*A s acting on Q-Hilbert spaces H and K respectively.

i) $\underline{A}$ and $\underline{B}$ are said to be algebraically isomorphic if there exists a 1 - 1 map $\emptyset$ of $\underline{A}$ onto $\underline{B}$ such that for all $A, B \in \underline{A}$ and a real, $\emptyset(a A) = a \emptyset(A)$, $\emptyset(A + B) = \emptyset(A) + \emptyset(B)$ and $\emptyset(AB) = \emptyset(A) \cdot \emptyset(B)$.

ii) $\underline{A}$ and $\underline{B}$ are said to be unitarily equivalent if there exists an isomorphism U of $\underline{H}$ onto $\underline{K}$ such that the canonical map $\emptyset$ which sends A on $\underline{H}$ to $U A U^{-1}$ on $\underline{K}$ is an algebraic isomorphism between $\underline{A}$. and $\underline{B}$.

iii) Let n be any non-zero cardinal number and S a set of cardinality n. For each s in S let $\underline{K}_s$ be a copy of $\underline{K}$. Let $\underline{K}[n]$ be the set of all functions x(s) on S with $x(s) \in \underline{K}_s$ for all s and $\sum_{s \in S} \|x(s)\|^2 < \infty$ converted into a Q-Hilbert space in the usual way. The n-fold copy of $\underline{B}$ on $\underline{K}$, $\underline{B}[n]$ in symbols, is the set of all operators B on $\underline{K}[n]$ which are of the form $(Bx)(s) = (Ax)(s)$ for all $s \in S$ and some $A \in \underline{B}$. $\underline{A}$ is said to be an n-fold copy of $\underline{B}$ if $\underline{A}$ on $\underline{H}$ is unitarily equivalent to $\underline{B}[n]$ on $\underline{K}[n]$.

It is easy to see that if $\emptyset$ is an algebraic isomorphism then (i) $\emptyset(0) = 0$, $\emptyset(I) = I$, (ii) P is a projection if and only if $\emptyset(P)$ is a projection (because a normal operator A is a projection if and only if $A^2 = A$) and (iii) J is imaginary if and only if $\emptyset(J)$ is imaginary (because, by

Theorem 3.2, a normal operator A is imaginary if and only if $A^2 + A^4 = 0$). $\emptyset$ has many more pleasant properties, as will be seen.

DEFINITION 8.3

i) A CW*A **A** is called an R-algebra if every A in **A** is hermitian

ii) A CW*A **A** is called a C-algebra if there exists an (imaginary) operator J in **A** such that $J^2 = - I$.

An example of an R-algebra is the set of all functions of a hermitian operator on a separable Q-Hilbert space and an example of a C-algebra is the set of all functions of a skew hermitian operator with a trivial null-space on a separable Q-Hilbert space. In the notation of Ingelstam (1964) an R-algebra is a R_2 real normed algebra and a C-algebra is a real normed algebra of complex type.

We now prove, in Theorem 8.1 below, that every CW*A is a direct sum of an R-algebra and an C-algebra. It is an amusing coincidence that Theorem 8.1 of Ingelstam says that a commutative real normed algebra satisfying one of two algebraic conditions (neither of which, incidentally, need be satisfied by a CW*A) is a direct sum of a feal normed algebra of complex type and an R_2-algebra.

If $\underline{A}$ is a CW*A and P is a projection which commutes with (every operator in) $\underline{A}$ then $\underline{A}$ may be restricted to a CW*A on P. We denote this by $\underline{A}$ P.

<u>THEOREM 8.1</u> If $\underline{A}$ is a CW*A, then there exists a unique projection F in $\underline{A}$ such that $\underline{A}$ F is a C-algebra and $\underline{A}$.(I - F) is an R-algebra. In other words, every CW*A is a direct sum of an R-algebra and a C-algebra.

<u>Proof:</u> If there exists a non-hermitian operator A in $\underline{A}$, then [A], the smallest W*-algebra containing A is contained in $\underline{A}$ and therefore, by Theorem 7.5, the canonical imaginary operator associated to A, is in $\underline{A}$. Consequently, if $\underline{A}$ were not an R-algebra to start with, $\underline{A}$ contains non-zero imaginary operators J. For every one such J, $-J^2$ is a projection in $\underline{A}$. Consider a maximal family of mutually orthogonal projections F_α in $\underline{A}$ of the form $-J^2$ for some imaginary operator J : $F_\alpha = -J_\alpha^2$, say. Then $F = \sum_\alpha F_\alpha$ is in $\underline{A}$. If we now define $J = \sum_\alpha J_\alpha$ then $J \in \underline{A}$ and $-J^2 = F$ so that $\underline{A}$F is a C-algebra. Also $\underline{A}$ (I - F) is an R-algebra, for, if not, there would exist non-zero imaginary operators J in $\underline{A}$ with $-J^2 \subseteq I - F$, contradicting the maximality of the family (F_α).

To prove uniqueness observe first that the restriction of a C-algebra to every reducing subspace is again a C-algebra. Hence if G were any projection in $\underline{A}$ such that $\underline{A}G$ is a C-algebra then $G \subseteq F$, for if not, $\underline{A}(G - F)$ would be a C-algebra and (F_α) would not be maximal. If G had the further property that $\underline{A}(I - G)$ is an R-algebra, then, for the same reason as above, $F \subseteq G$. Hence the theorem.

We shall refer to $\underline{A}F$ and $\underline{A}(I - F)$ as the complex and real parts of $\underline{A}$ respectively.

If X is a topological space, we denote by $C_R(X)$ $(C_C(X))$ the <u>real</u> Banach algebra of all real-valued (complex-valued) bounded continuous functions on X.

<u>LEMMA 8.1</u>

If $\underline{A}$ is an R-algebra (a C-algebra), then there exists a compact Hausdorff space X, unique upto a homeomorphism, and a map ϕ from $\underline{A}$ to $C_R(X)(C_C(X))$ which is an algebraic isomorphism preserving norms and adjoints.

<u>Proof</u>: Suppose first that $\underline{A}$ is an R-algebra. Let X be the Stone space associated with $\underline{F}$ — $\underline{F}$ is then isomorphio via $P \to P^*$, say, to the Boolean algebra of open-closed subsets of X. For real linear combinations of elements of $\underline{F}$, define ϕ by $\phi(\Sigma a_r P_r) = \Sigma a_r \chi_{P_r^*}$. ϕ is then a

a norm-preserving algebraic isomorphism between uniformly dense subalgebras of **A** and $C_R(X)$ and hence may be extended to a norm-preserving isomorphism between **A** and $C_R(X)$. That X is unique upto a homeomorphism is a consequence of the fact that $C_R(X)$ is determined upto an algebraic isomorphism.

Let now **A** be a C-algebra. We claim that if J is any full imaginary operator in **A**, then every operator in **A** is uniquely of the form B + JC where B and C are hermitian operators in **A** Now every operator A in **A** is normal and hence, by Corollary 6.2, is of the form B' + LC' with B', C' hermitian and L imaginary all belonging to **A** (Theorem 7.5). But then L <--> J and, by Theorem 3.3, is of the form JP - JQ for some mutually orthogonal projections P, Q ∈ **F** (!). Define now B = B' and C = C'P - C'Q. Then B and C are hermitian and one may check that A = B + JC. The uniqueness is clear.

To complete the proof of the lemma we have only to proceed as in the case of the R-algebra adding the extra definition that $\emptyset(J) = i\,1$ for an arbitrarily chosen full imaginary operator J in **A**.

<u>THEOREM 8.2</u> Let **A** and **B** be CW*A s acting on Q-Hilbert spaces H and K respectively. Then any algebraic isomorphism $\emptyset$ between **A** and **B** preserves norms and adjoints.

Proof: Suppose first that $\underline{A}$ and $\underline{B}$ are R-algebras. Then, using Lemma 8.1, and the fact that any algebraic isomorphism between $C_R(X)$ and $C_R(Y)$ where X and Y are topological spaces is norm-preserving we can conclude that $\emptyset$ is norm-preserving. That $\emptyset$ preserves adjoints is trivially true.

If $\underline{A}$ and $\underline{B}$ are C-algebras, then again by Lemma 8.1 and the work of Gelfand, Raĭkov, Šilov (1946),

we can conclude that $\emptyset$ is bicontinuous in the norm topologies. Since every hermitian operator in a CW*A is the uniform limit of a sequence of real linear combinations of projections in the algebra it follows that $\emptyset(A)$ is hermitian if and only if A is hermitian and that $\|\emptyset(A)\| = \|A\|$ for hermitian A. Let now J be any full imaginary operator in $\underline{A}$. Then every $A \in \underline{A}$ is of the form $B + JC$ with B, C in $\underline{A}$ hermitian, by the proof of Lemma 8.1, so that $\emptyset(A^*) = \emptyset(B - JC) = \emptyset(B) - \emptyset(J)\emptyset(C) = \emptyset(A)^*$. Consequently, $\emptyset$ preserves adjoints. That $\emptyset$ preserves norms follows from $\|A\|^2 = \|A^* A\| = \|\emptyset(A^*A)\|$ (because A^*A is hermitian) $= \|\emptyset(A)^* \emptyset(A)\| = \|\emptyset(A)\|^2$.

Suppose, finally, that $\underline{A}$ and $\underline{B}$ are arbitrary CW*A s. Let F, G be the unique projections in $\underline{A}$ and $\underline{B}$ respectively splitting them into their real and complex parts (Theorem 8.1). We claim that ϕ interchanges $\underline{A}$ F with $\underline{B}$ G and $\underline{A}$(I - F) with $\underline{B}$ (I - G). It is sufficient to prove, because of symmetry, that $\phi(F) \subseteq G$, which, in turn, is proved if we prove that $\underline{B}.\phi(F)$ is a C-algebra. But this is true, since, if J is any imaginary operator in $\underline{A}$ which is full in F, then $\phi(J)$ is an imaginary operator in $\underline{B}$ full in $\phi(F)$. Combining this fact with the earlier part of the proof, we have

$$\begin{aligned} \| A \| &= \max (\| AF \| , \| A(I-F) \|) \\ &= \max (\| \phi(A)G \| , \| \phi(A)(I-G) \|) \\ &= \| \phi(A) \| \end{aligned}$$

and $\phi(A^*) = \phi[(AF)^* + (A(I - F))^*] = \phi(A)^*G + \phi(A)^*(I - G)$

$$= \phi(A)^*.$$

This proves the theorem.

We display as a Corollary a result whose proof is implicit in the proof of the above theorem.

<u>COROLLARY 8.1</u> Two CW*A s are algebraically isomorphic if and only if their complex and real parts are (separately) algebraically isomorphic. In particular two CW*A s are unitarily equivalent if and only if their complex and real parts are unitarily equivalent.

Because of this corollary, to obtain unitary invariants for CW*A s it is enough to obtain unitary invariants for C-algebras and R-algebras individually, and this is what we shall do.

DEFINITION 8.4 An R-algebra (C-algebra) is maximal if it is not strictly contained in any R-algebra (C-algebra).

For any W*A $\underline{A}$, let $\underline{S}$ ($\underline{A}$) denote the W*A of all hermitian operators in $\underline{A}$, $\underline{A}'$ the W*A of all operators which commute with $\underline{A}$ and $\underline{C}$ ($\underline{A}$) = $\underline{A} \cap \underline{A}'$, the CW* A of all operators in $\underline{A}$ which commute with $\underline{A}$, i.e. the centre of $\underline{A}$.

THEOREM 8.3

i) An R-algebra $\underline{A}$ is maximal if and only if $\underline{A} = \underline{C}(\underline{A}') = \underline{S}(\underline{A}')$.

ii) A C-algebra $\underline{A}$ is maximal if and only if $\underline{A} = \underline{A}'$.

Proof:

i) We shall first prove that if $\underline{A}$ is any R-algebra then $\underline{C}$ ($\underline{A}'$) $\subseteq$ $\underline{S}(\underline{A}')$. We have only to prove that if $J \neq 0$ is any imaginary operator in $\underline{A}'$, then there exists an imaginary operator L in $\underline{A}'$ such that $JL \neq LJ$. Given such a J, let E be a spectral measure whose range is exactly $\underline{F}$, and consider the spectral system (E, J). Let x be a vector such that $Jx = ix$. Then (E, J) restricted to $Z(x)$ is isomorphic to (E_{μ_x}, J_{μ_x}) on $L^2_Q(\mu_x)$ (Lemma 4.5). Let L be the operator on $\underline{H}$ which is 0 on $Z(x)^{\perp}$ and on $Z(x)$ is isomorphic to the operator $f \to f.j$ on $L^2_Q(\mu_x)$ via the same isomorphism which intertwines (E_{μ_x}, J_{μ_x}) with (E, J) on $Z(x)$. Then $L \not\leftrightarrow J$, but $L \leftrightarrow \underline{F}$ and hence $L \leftrightarrow \underline{A}$. This proves that $\underline{C}(\underline{A}') \subseteq \underline{S}(\underline{A}')$.

Let now $\underline{A}$ be a maximal R-algebra. Obviously $\underline{A} \subseteq \underline{C}(\underline{A}') \subseteq \underline{S}(\underline{A}')$. If $\underline{S}(\underline{A}') \not\subseteq \underline{A}$, then there exists a hermitian operator $A \leftrightarrow \underline{A}$ but not in $\underline{A}$. But then we may construct an R-algebra containing both A and $\underline{A}$, thus violating the maximality of $\underline{A}$. This proves that $\underline{A} = \underline{C}(\underline{A}') = \underline{S}(\underline{A}')$.

Conversely, let $\underline{A}$ be an R-algebra such that $\underline{A} = \underline{C}(\underline{A}') = \underline{S}(\underline{A}')$. Then every hermitian operator which commutes with $\underline{A}$ is by definition in $\underline{S}(\underline{A}')$ and therefore in $\underline{A}$. It follows that $\underline{A}$ is maximal. This concludes the proof of i). ii) If $\underline{A} = \underline{A}'$, then $\underline{A}$ is maximal, as every C-algebra containing $\underline{A}$ must be contained in $\underline{A}'$.

As to the converse, it is enough to prove that every normal operator A in $\underline{A}'$ is in $\underline{A}$. But if $A \notin \underline{A}$, then the smallest W*A generated by A and $\underline{A}$ is a C-algebra containing $\underline{A}$ strictly, a contradiction.

Hence the theorem.

<u>COROLLARY 8.2</u> A CW*A is a maximal abelian self-adjoint algebra if and only if it is a maximal C-algebra.

Let now $\underline{R}$ ($\underline{C}$) denote an arbitrary but fixed R-algebra (C-algebra) acting on a Q-Hilbert space $\underline{H}$. Depending on the context $\underline{F}$ will denote the set of all projections in $\underline{R}$ or $\underline{C}$ and E a spectral measure based on a measurable space (X_o, Σ_o) whose range is exactly $\underline{F}$. $\underline{P}$ will denote the set of all projections on $\underline{H}$ which commute with $\underline{R}$ and $\underline{Q}$ the set of all projections on $\underline{H}$ which commute with $\underline{C}$. Notice that if J is any full imaginary operator in $\underline{C}$, then $\underline{Q}$ is precisely the set of all projections which commute with E and J so that the concepts of J-cycle, J-row etc., with respect to the spectral system (E, J) are defined independently of J.

Let J_o be any one full imaginary operator in $\underline{C}$. The lemma below is a direct consequence of Theorems 4.4 and 8.3.

<u>LEMMA 8.2</u>

i) For the R-algebra $\underline{R}$, if $S \subset \underline{P}$, then S is a row if and only if $\underline{R}S$ is a maximal R-algebra.

ii) For the C-algebra $\underline{C}$, if $S \in \underline{Q}$, then S is a J_0-row if and only if $\underline{C}S$ is a maximal C-algebra.

Consider now $\underline{R}$. If $S_1, S_2 \in \underline{P}$ and are equivalent (in the sense of Definition 4.6) then it is clear that $\underline{R}S_1$ and $\underline{R}S_2$ are unitarily equivalent. Similarly, for $\underline{C}$, if $S_1, S_2 \in \underline{Q}$ are equivalent then $\underline{C}S_1$ and $\underline{C}S_2$ are unitarily equivalent. It follows that for $\underline{R}$ ($\underline{C}$), a projection $F \in \underline{F}$ has uniform multiplicity n if and only if $\underline{R}F$ ($\underline{C}F$) is an n-fold copy of a maximal R-algebra (C-algebra). Theorem 8.4 below is then a direct consequence of Lemma 4.8.

<u>THEOREM 8.4</u> Let $\underline{A}$ be either an R-algebra or a C-algebra on the Q-Hilbert space $\underline{H}$. For each cardinal $n \leq$ the dimension of $\underline{H}$, there exists a projection P_n in $\underline{A}$ such that i) P_n is either 0 or of uniform multiplicity n, ii) the P_n are mutually orthogonal and $\sum_n P_n = I$ and iii) the map $n \to P_n$ of cardinals to projections in $\underline{A}$ with properties i) and ii) is unique.

<u>DEFINITION 8.5</u> Let (Y, T, μ) denote a non-negative measure-space. Let $\underline{A}_\mu$ be the set of all operators R_s on $L^2_Q(\mu)$ of the form $R_s \cdot f = f \cdot s$ for some quaternion-valued, measurable, essentially bounded function s on (Y, T). The sub-collection $\underline{A}^R_\mu$ ($\underline{A}^C_\mu$) of all operators in $\underline{A}_\mu$ with s

essentially real (complex) is called the real (complex) multiplication algebra of the measure-space (Y, T, μ).

If the measure algebra of (Y, T, μ) is a complete measure bearing ring in the sense of Segal (1951) then one may verify that $\underline{A}_{\mu}^{R}$ ($\underline{A}_{\mu}^{C}$) is infact an R-algebra (a C-algebra).

LEMMA 8.3 Let $\underline{A}$ be an R-algebra (a C-algebra). The following conditions are equivalent:

i) There exists a cyclic vector for $\underline{A}$.

ii) $\underline{A}$ is maximal and every projection in $\underline{A}$ satisfies the countable chain condition.

iii) $\underline{A}$ is unitarily equivalent to the real (complex) multiplication algebra of a finite measure space.

The proof of this lemma follows easily from the spectral theory of Chapter IV and is omitted.

LEMMA 8.4 Let $\underline{A}$ and $\underline{B}$ be two maximal R-algebras or C-algebras acting on Q-Hilbert spaces $\underline{H}$ and $\underline{K}$ respectively. Then they are algebraically isomorphic if and only if they are unitarily equivalent. In fact if $\emptyset$ is any algebraic isomorphism between $\underline{A}$ and $\underline{B}$ then $\emptyset$ is induced by an isomorphism between $\underline{H}$ and $\underline{K}$ and is consequently a homeomorphism in the weak, strong and norm topologies. Further for any A in $\underline{A}$ and any function f(A) of A, $\emptyset(f(A)) = f(\emptyset(A))$.

The proof of this result is essentially the same as the proof of its analogue in the complex case and is omitted.

<u>DEFINITION 8.6</u> Let <u>A</u> be either an R-algebra or a C-algebra and let P_n be as in Theorem 8.4. The Boolean algebra B(n) of all projections in <u>A</u> P_n is called the measure algebra of <u>A</u> for the multiplicity n.

The B(n) are complete measure bearing rings in the sense of Segal. Theorem 8.5 below is now proved exactly like Theorem 3 of Segal (1951).

<u>THEOREM 8.5</u> Two R-algebras (C-algebras) are unitarily equivalent if and only if their measure algebras for the same multiplicities are isomorphic.

With this theorem it is clear that a simple set of unitary invariants for either an R-algebra or a C-algebra is given by a Boolean-algebra valued function on cardinals vanishing for sufficiently large cardinals where the Boolean algebras are all complete measure rings and that all such functions may occur. It follows by Corollary 8.1 that for arbitrary CW*A's two such functions serve to characterise them upto unitary equivalence.

Strictly speaking the values of these functions are equivalence classes of Boolean algebras. As Segal suggests these functions may be replaced by functions mapping pairs of cardinals to cardinals with the help of Maharam's classification of measure spaces.

8.2 In this section we deduce some interesting consequences of the theory developed in the previous section. As in the complex case our basic result is as follows.

THEOREM 8.6 Let $\underline{A}$ be an R-algebra (a C-algebra) acting on a Q-Hilbert space $\underline{H}$. Then there exists a maximal R-algebra (C-algebra) $\underline{M}$ acting on a Q-Hilbert space $\underline{K}$ to which $\underline{A}$ is algebraically isomorphic (with preservation of adjoints). Any such isomorphism $\emptyset$ of $\underline{A}$ onto $\underline{M}$ is bicontinuous in the weak topology and has the property that if f is any bounded measurable complex valued function on C^+ which is real on the real line, then for any A in $\underline{A}$, $\emptyset(f(A)) = f(\emptyset(A))$. The algebra $\underline{M}$ is unique within unitarily equivalence and the dimension of $\underline{K}$ is not greater than the dimension of $\underline{H}$.

To prove this theorem we have only to handle the proof of Theorem 5 of Segal (1951) a little more delicately. This is left to the reader. So is the proof of the corollary below.

COROLLARY 8.3 Any algebraic isomorphism between two CW*A s is weakly bicontinuous and preserves the functional calculus for bounded measurable complex functions on C^+ which are real for real arguments.

We elevate another corollary to Theorem 8.6 to the rank of a theorem.

THEOREM 8.7 Let $\underline{H}$ be a separable Q-Hilbert space and $\underline{A}$ any CW*A on $\underline{H}$. Then there exists a normal operator A in $\underline{A}$ such that $\underline{A}$ consists precisely of all functions of A. If $\underline{A}$ is an R-algebra (C-algebra) then A may be chosen to be hermitian (skew-hermitian).

Proof: Suppose first that $\underline{A}$ is an R-algebra (C-algebra). Then, by Theorem 8.6, there exists a separable Q-Hilbert space $\underline{K}$ and a maximal R-algebra (C-algebra) $\underline{B}$ on $\underline{K}$ to which $\underline{A}$ is algebraically isomorphic (via $\emptyset$, say) with preservation of functional calculus. Now, by Lemma 8.3 $\underline{B}$ is unitarily equivalent to the real (complex) multiplication algebra of a finite measure space (Y, T, μ).

Note that $L^2_Q(\mu)$ is then separable. Consequently, we can choose a real measurable function h_o on (Y, T) which lies between 1 and 2 and which has the property that if f is any T-measurable function on Y then there exists a

Borel-measurable function t on $[1, 2]$ such that $f(\lambda) = t(h_o(\lambda))$ for all λ. (Such a h_o always exists. See Varadarajan, 1959).

If now $\underline{A}$ (and hence $\underline{B}$) is an R-algebra let A_o be the operator (in $\underline{B}$) of multiplication by h_o : $A_o f = f.h_o$ for all f. If $\underline{A}$ (and hence $\underline{B}$) is a C-algebra let A_o be the operator (in $\underline{B}$) of right multiplication by ih_o : $A_o f = f.i.h_o$ for all f. In either case one may check, because of the way h_o was chosen, that $\underline{B}$ is precisely the set of all functions of A_o. If now $A = \emptyset^{-1}(A_o)$ then clearly $\underline{A}$ is the set of all functions of A.

If $\underline{A}$ is an arbitrary CW*A, let $F \in \underline{F}$ be such that $\underline{A}F$ and $\underline{A}(I - F)$ are the complex and real parts of $\underline{A}$ respectively. By what we have proved, we may say that there exist B, C in $\underline{A}$ such that BF is skew-hermitian (with spectrum bounded away from 0), C(I - F) is hermitian and $\underline{A}F$ is the set of all functions of BF and $\underline{A}(I - F)$ the set of all functions of C(I - F). Defining now $A = BF + C(I - F)$, it is easy to verify that $\underline{A}$ is the set of all functions of A.

<u>COROLLARY 8.4</u> Any commuting set of normal (hermitian) operators on a separable Q-Hilbert space can be expressed as functions of a single normal (hermitian) operator.

8.3. If $\underline{A}$ is any W*-algebra of operators on a complex Hilbert space, not necessarily abelian, then the Double Commutant Theorem of von Neumann says that $\underline{A} = \underline{A}''$. The proof of this theorem is quite elementary and depends on the observation that for any vector x in the Hilbert space $[Ax : A \in \underline{A}]$ is a subspace.

Even if $\underline{A}$ is a W*-algebra on a Q-Hilbert space one feels that the Double Commutant Theorem ought to be true, but the above proof fails because now $\underline{A}$ is only a real algebra. And no other proof seems available either for the general W*-algebra. However when $\underline{A}$ is a CW*A we can deduce a proof using the theorems of this chapter. But even for this particular case, the proof is non-trivial.

<u>THEOREM 8.8</u> If $\underline{A}$ is a CW*A then $\underline{A} = \underline{A}''$.

<u>Proof:</u> Because of Theorem 8.1 it is enough to prove this theorem separately for R-algebras and C-algebras. We shall prove it now for R-algebras. The case of the C-algebra is handled similarly.

Let then $\underline{A}$ be an R-algebra, $\underline{F}$ the set of all projections in $\underline{A}$. We may assume without loss of generality that $\underline{H}$ is a column of uniform multiplicity, say n. Then there exists a Q-Hilbert space $\underline{K}$ and a maximal R-algebra $\underline{B}$ on $\underline{K}$ such that $\underline{A}$ is the n-fold copy of $\underline{B}$ on $\underline{K}$.

If now $A \in \underline{A}''$, then, in the notation of Definition 8.2, A is reduced by each $\underline{K}_s$. Let A_s be A restricted to $\underline{K}_s$. Since $\underline{A}$ is abelian $A \in \underline{C}(\underline{A}')$ and hence $A_s \in \underline{C}(\underline{B}')$. By Theorem 8.3, $A_s \in \underline{B}$. To prove that $A \in \underline{A}$, we have to prove that $A_{s_1} = A_{s_2}$ for all s_1, s_2. If $s_1 \neq s_2$, consider the operator U on $\underline{K}[n] = \underline{H}$ defined by $Ux(s) = y(s)$ where $y(s_1) = x(s_2)$, $y(s_2) = x(s_1)$ and $y(s) = 0$ if $s \neq s_1, s_2$. Then clearly $U \longleftrightarrow \underline{A}$. Hence $A \longleftrightarrow U$. But then $A_{s_1} = A_{s_2}$.

As remarked earlier this completes the proof of the theorem.

PART II

QUATERNIONIC REPRESENTATIONS

C O N T E N T S

INTRODUCTION

The theory of unitary representations of topological groups in quaternionic Hilbert spaces is of importance in the study of quantum mechanical systems whose logics are assumed to be standard logics associated with quaternionic Hilbert spaces. This is well-known, but to date there have been only three articles on this subject: One by Jauch, Finkelstein and Speiser (1963), one by Emch (1963) and one by Natarajan and Viswanath (1967). The first and the third together give the complete theory for the case of compact groups and the second obtains partial results in the case of the Lorentz group.

We shall now consider quaternionic representations in a very general context, of which quaternionic unitary representations of topological groups constitute a particular case, and show how one may obtain all irreducible quaternionic representations as soon as all irredicible complex representations are known. Our success is due to our ability to define the finite dimensional concepts of conjugates and transposes of matrices for operators on Hilbert spaces in a canonical fashion so that the methods of Frobenius and Schur become applicable to the infinite-dimensional case as well.

A brief chapter-wise summary is as follows:

Chapter I introduces Q-representations and collects together the basic facts regarding them. A theorem which simultaneously characterizes and classifies irreducible Q-representations in terms of the structure of the commuting ring is proved.

Chapter II deals exclusively with the complex case. Duals of Hilbert spaces are introduced and the contragredient of a complex representation is defined canonically as a representation in the dual Hilbert space. Irreducible complex representations are then classified into nonreal, potentially real and pseudoreal ones exactly as in the finite-dimensional case.

In Chapter III the concept of quaternionification of a complex Hilbert space is defined and the relations between the extensions of conjugates and transposes of operators associated with the Hilbert space to its quaternionification are studied. The most important fact here is that extensions of skew-symmetric operators are hermitian.

Chapter IV deals with the relations between complex and quaternionic Hilbert spaces. Necessary and sufficient conditions for the irreducibility of quaternionifications of irreducible complex representations as well as symplectic images of irreducible quaternionic representations are obtained. It is then shown that there is a canonical one-one correspondence between equivalence classes of irreducible quaternionic representations and physical equivalence classes of irreducible complex representations which allows us to compute the former from the latter. This is the main theorem of Part II.

Chapter V deals with compact groups. Orthogonality relations among matrix elements are investigated and a neat analogue of the Peter-Weyl theorem is presented. It is shown that with each irreducible quaternionic representation is associated a real-valued function called its Q-character which determines its equivalence class uniquely. The chapter ends with a brief discussion of the abelian case.

In Chapter V we prove two interesting theorems for the case of a locally compact abelian group G. In one we describe the structure of the space of continuous homomorphisms of G into the unit quaternions and in the other we show how one may express arbitrary quaternionic unitary representations of G as integrals of irreducible ones in an essentially unique manner.

Chapter VI deals with the problem of describing the representations of a locally compact second countable group in the group of automorphisms of the state space of a quantum mechanical system whose logic is assumed to be the lattice of subspaces of a separable quaternionic Hilbert space. This is solved for two important cases:the real line and the Lorentz group. We remark here that Emch too has studied this problem (Emch, 1963) and has reached conclusions similar to ours. But Emch's methods compel him to postulate additional axioms for quaternionic quantum mechanics whereas we do not need to make any assumptions, except of course the basic one about the logic being standard.

I. REPRESENTATIONS

Let S be a set and $s \to s^*$ an involutory bijection on S. Let $\underline{H}$ be a complex or quaternionic Hilbert space and $\underline{B}(\underline{H})$ the set of (bounded linear) operators on $\underline{H}$. A representation of S in $\underline{H}$ is a map A from S to $\underline{B}(\underline{H})$ for which $A(s^*) = A(s)^*$ for all $s \in S$. This definition is clearly general enough to subsume unitary representations of topological groups and representations of Banach algebras as well as spectral measures and systems of imprimitivity. All theorems which we prove for representations of S hold for these special structures too, and the proofs are the same modulo the verification of a few additional topological and algebraic conditions. Since such verifications are invariably trivial we shall not spell out the details.

The concepts of irreducibility, (unitary) equivalence etc., for representations of S are all defined in the usual way (Mackey, 1955). The basic result in the theory of representations is Schur's lemma: If A and B are two representations of S in Hilbert spaces $\underline{H}$ and $\underline{K}$ respectively (either both complex or both quaternionic) and T is any bounded linear map from $\underline{H}$ to $\underline{K}$ intertwining A and B, then the restriction of A to the complement of the null space of T is equivalent to the restriction of B to the closure of the range of T. We state explicitly two consequences of this result which we shall need later on : (i) If A and B are irreducible and $T \neq 0$ intertwines them, then $T/\|T\|$ is unitary, (ii) If $A = \sum_{\alpha} A_{\alpha} = \sum_{\beta} B_{\beta}$ are two decompositions of A into irreducible subrepresentations, then every A_{α} is equivalent to some B_{β}.

Let A be a representation of S in $\underline{H}$. We shall call A a C-representation or a Q-representation according as $\underline{H}$ is a complex or a quaternionic Hilbert space. We shall not consider real Hilbert spaces.

Let $\underline{H}$ be a Q-Hilbert space with inner product $(\cdot,\cdot)$ and $\underline{H}^s$ its symplectic image with inner product $\langle\cdot,\cdot\rangle$. If A is a representation of S in $\underline{H}$, the symplectic image of A is the C-representation A^s of S in $\underline{H}^s$ defined in the obvious way. If K is the map $x \to kx$ on $\underline{H}^s$, then $K \longleftrightarrow A^s$. Further a projection P in $\underline{H}^s$ commutes with A^s if and only if $KPK^{-1} \longleftrightarrow A^s$. (Recall that if P projects onto the subspace P then KPK^{-1} projects onto the subspace $[kx : x \in P]$).

By a <u>half-space</u> of $\underline{H}$ we shall mean a subspace P of $\underline{H}^S$ such that $P \perp^S KPK^{-1}$ and $P \oplus^S KPK^{-1} = \underline{H}^S$. If (e_α) is a basis for $\underline{H}$, then the closure of the set of all finite complex linear combinations of the (e_α) is a half-space of $\underline{H}$ and every half-space may be obtained in this way. If P is a half-space of $\underline{H}$ and (e_α) is a basis for P, then (e_α) is a basis for $\underline{H}$ as well. Every $z \in \underline{H}$ is uniquely of the form $x+ky$ with $x,y \in P$, and $\|z\|^2 = \|x\|^2 + \|y\|^2$. For $x, y \in P$ $(x,y) = \langle x,y \rangle$. All these results are proved in Chapter 3 of Part I.

We shall now prove a theorem characterizing the irreducibility of Q-representations. This is an analogue of the well-known result that a C-representation is irreducible if and only if its commuting ring consists of multiples of the identity.

Let A be a Q-representation of S in $\underline{H}$ and (e_α) a basis for $\underline{H}$. By the matrix elements of A with respect to (e_α) we mean the set of functions on S defined by $(A(s)e_\beta, e_\alpha)$ for every α, β.

<u>DEFINITION 1.1</u>

i) A is of class R if there exists a basis for $\underline{H}$ with respect to which the matrix elements of A are all real.

ii) A is of class C if A is not of class R, but there exists a basis for $\underline{H}$ with respect to which the matrix elements of A are all complex.

iii) A is of class Q if A is neither of class R nor of class C.

In terms of half-spaces A is of class R or C if and only if there exists an invariant half-space for $A(A^s)$.

<u>THEOREM 1.1</u> Let A be a Q-representation of S in $\underline{H}$ and let $\underline{R}$ be the commuting ring of A. Then A is irreducible if and only if $\underline{R}$ is a Banach division ring. Further A is of class R, C or Q according as $\underline{R}$ is isomorphic to the quaternions, the complex numbers or the reals.

Proof: Whatever be A, $\underline{R}$ is clearly a Banach sub-algebra of $\underline{B}(\underline{H})$.

Suppose A is irreducible. Observe first that every M in $\underline{R}$ must be of the form a+Jb, where a, b are real numbers, $b \geq 0$ and if $b > 0$, J is a full imaginary operator on $\underline{H}$ commuting with A (cf. Emch, 1963). Our spectral theory of Part I implies that this is certainly true if M were normal for then the spectral measure of M would be two valued and hence concentrated on a singleton. The general case follows by applying this argument to the real and imaginary parts of M. This observation implies that if $M \in \underline{R}$, then M is invertible and $M^{-1} = (a-Jb)/(a^2+b^2) = M^*/||M||^2$ belongs to $\underline{R}$. Clearly $M \to M^{-1}$ is continuous. Consequently $\underline{R}$ is a Banach division ring. The converse is trivially verified, since the reducibility of A implies the existence of (non-invertible) projections in $\underline{R}$.

If $\underline{R}$ is a division ring it must be necessarily isomorphic to R, C or Q. Suppose $\underline{R}$ is isomorphic to Q. Then there exist full imaginary operators $J, L \in \underline{R}$ such that $JL = -LJ$. We shall prove that there exists a basis (e_α) for $\underline{H}$ such that $Je_\alpha = ie_\alpha$ and $Le_\alpha = ke_\alpha$ for all α.

Let (f_α) be a basis for $\underline{H}$ such that $Jf_\alpha = if_\alpha$ for all α. Such a basis always exists. Since $LJf_\alpha = -JLf_\alpha$ for all α, the matrix of L with respect to (f_α) has the form $(ka_{\alpha\beta})$ with $a_{\alpha\beta}$ complex for all α,β. This implies that if x belongs to P, the half-space spanned by the (f_α), then $kLx \in P$. Therefore, if $0 \neq x \in P$ and $Lx \neq kx$ then x^* defined by $x^* = ix + ikLx \in P$ again and is such that $x^* \neq 0$ and $Lx^* = kx^*$. Consequently there is always an $x \in P$ with $Lx = kx$, $x \neq 0$. Again if $y \in P$ is orthogonal to such an x then so also is $y^* = iy + ikLy \in P$. If now (e_α) is a maximal orthonormal family of vectors in P such that $Le_\alpha = ke_\alpha$ for all α, then (e_α) is easily seen to be a basis for P and hence for $\underline{H}$. Since J and L commute with A and $Je_\alpha = ie_\alpha$, $Le_\alpha = ke_\alpha$ for all α, it is easy to conclude that the matrix elements of A with respect to (e_α) are real.

Suppose conversely that the matrix elements of A with respect to a basis (e_α) are all real. Then $\underline{R}$ contains all operators of the form $Me_\alpha = qe_\alpha$ for all α, $q \in Q$, so that $\underline{R}$ is certainly not isomorphic to R or C. $\underline{R}$ must therefore be isomorphic to Q.

Next let $\underline{R}$ be isomorphic to C. Then there is a full imaginary operator J in $\underline{R}$. Let (e_α) be a basis for $\underline{H}$ such that $Je_\alpha = ie_\alpha$ for all α. Then $A \leftrightarrow J$ implies that the matrix elements of A with respect to (e_α) must be complex, so that A is of class R or C. But A cannot be of class R, since $\underline{R}$ is not isomorphic to Q. Consequently A must be of class C. Conversely, if A is of class C and the matrix elements of A with respect to (e_α) are all complex, then the full imaginary operator J defined by $Je_{\alpha} = ie_{\alpha}$ for all α is in $\underline{\bar{R}}$, so that $\underline{\bar{R}}$ cannot be isomorphic to R. On the other hand $\underline{\bar{R}}$ can not be isomorphic to Q, because A is not of class R. Therefore $\underline{R}$ is isomorphic to $\underline{\bar{C}}$.

The last part now follows easily. The proof of the theorem is complete.

II. CONTRAGREDIENTS AND THE FROBENIUS SCHUR THEORY

In Hilbert space theory the concept of the dual space is generally neglected for the very practical reason that for the majority of problems Riesz's theorem renders its consideration unnecessary. There's however a class of problems, which in the finite-dimensional case are phrased in terms of conjugates and transposes of matrices, in whose study duals of Hilbert spaces figure naturally and prominently, as we shall see in this chapter. Our viewpoint leads us to define the contragredient of a C-representation in a canonical fashion as a representation in the dual Hilbert space. This seems to be the natural way of treating this aspect of representation theory, as it enables us to obtain neat generalizations of some classical theorems due to Frobenius and Schur to the infinite dimensional case. Also, this approach goes well with our handling of the concept of quaternionification which is explained in the next chapter.

Let $\underline{G}$ be a complex Hilbert space and $\underline{G}'$ its dual. We shall denote a typical element of $\underline{G}'$ by f and the element of $\underline{G}'$ related to $x \in G$ via Riesz's theorem by x'. $x \to x'$ is bijective, conjugate linear and norm-preserving. $\underline{G}'$ becomes a complex Hilbert space if we define the inner product on $\underline{G}'$ by $\langle f, g\rangle' = \langle y, x\rangle$ where $f = x'$, $g = y'$ and $\langle \cdot, \cdot \rangle$ is the inner product on $\underline{G}$. The map $x \to x'$ is then anti-unitary (If $\underline{F}$ and $\underline{G}$ are complex Hilbert spaces then $U: \underline{F} \to \underline{G}$ is anti-unitary if it is conjugate linear and satisfies $(x, y) = (Uy, Ux)$ for all $x, y \in \underline{F}$). If (e_α) is a basis for $\underline{G}$, (e'_α) is a basis for $\underline{G}'$ called the basis dual to (e_α). Because a composition of two anti-unitary maps is unitary, we may identify $(\underline{G}')'$ with $\underline{G}$ by identifying x'' with x. We shall do so.

We shall now make a series of definitions, justifying them as we go along. The spaces of operators on $\underline{G}$, on $\underline{G}'$, from $\underline{G}$ to $\underline{G}'$ and from $\underline{G}'$ to $\underline{G}$ will be denoted by $\underline{B}(\underline{G})$, $\underline{B}(\underline{G}')$, $\underline{B}(\underline{G}, \underline{G}')$ and $\underline{B}(\underline{G}', \underline{G})$ respectively.

1) Let A be an operator on $\underline{G}$. Define $\bar{A}$ on $\underline{G}'$ by $\bar{A}x' = (Ax)'$. Clearly $\bar{A} \in \underline{B}(\underline{G}')$. Further if (e_α) is a basis for $\underline{G}$ and $Ae_\beta = \sum_\alpha a_{\alpha\beta} e_\alpha$, then $\bar{A} e'_\beta = \sum_\alpha \bar{a}_{\alpha\beta} e'_\alpha$. $\bar{A}$ is called the <u>conjugate</u> of A. It is easily checked that $(A+B)^- = \bar{A} + \bar{B}$, $(\alpha A)^- = \bar{\alpha}\bar{A}$ (α complex), $(AB)^- = \bar{A}\bar{B}$, $(A^*)^- = (\bar{A})^*$ and $\|A\| = \|\bar{A}\|$.

2) $A' = (A^*)^- = (\bar{A})^*$ is called the transpose of A. Note that $A' \in \underline{B}(\underline{G}')$. If (e_α) is any basis for $\underline{G}$ and $Ae_\beta = \sum_\alpha a_{\alpha\beta} e_\alpha$, then $A'e'_\beta = \sum_\alpha a_{\beta\alpha} e'_\alpha$. Further

$$(A+B)' = A'+B', \ (\alpha A)' = \alpha A', \ (AB)' = B'A',$$

$$(A^*)' = (A')^* = \bar{A}.$$

and $\|A\| = \|A'\|$.

3) Let $C \in \underline{B}(\underline{G}')$ and define $\bar{C}$ on $\underline{G}$ by $\bar{C}x = (Cx')'$. $\bar{C} \in \underline{B}(\underline{G})$ and is the conjugate of C. For $A \in \underline{B}(\underline{G})$, $\bar{\bar{A}}x = (\bar{A}x')' = (Ax)'' = Ax$, so that $C \to \bar{C}$ is the inverse of the map $A \to \bar{A}$. Together with (1) this implies that $A \to \bar{A}$ is a conjugate linear isomorphism between $\underline{B}(\underline{G})$ and $\underline{B}(\underline{G}')$.

4) Let $M \in \underline{B}(\underline{G}, \underline{G}')$. Recall that $M^* \in \underline{B}(\underline{G}', \underline{G})$ is defined by $\langle Mx, f\rangle' = \langle x, M^*f\rangle$ for all $x \in \underline{G}$, $f \in \underline{G}'$. We now define $\bar{M} \in \underline{B}(\underline{G}', \underline{G})$ by $\bar{M}x' = (Mx)'$. $\bar{M}$ is called the conjugate of M. If (e_α) is a basis for $\underline{G}$ and $Me_\beta = \sum_\alpha m_{\alpha\beta} e'_\alpha$, then $\bar{M}e'_\beta = \sum_\alpha \bar{m}_{\alpha\beta} e_\alpha$. Further

$$(M_1+M_2)^- = \bar{M}_1 + \bar{M}_2, \quad (\alpha M)^- = \bar{\alpha}\bar{M} \quad \text{and} \quad \|M\| = \|\bar{M}\|.$$

5) If $N \in \underline{B}(\underline{G}', \underline{G})$, $\bar{N} \in \underline{B}(\underline{G}, \underline{G}')$, the conjugate of N, is given by $\bar{N}x = (Nx')'$ for all $x \in \underline{G}$. Again clearly

$\bar{\bar{M}} = M$ so that $M \to \bar{M}$ is a conjugate linear isomorphism between the Banach spaces $\underline{B}(\underline{G}, \underline{G}')$ and $\underline{B}(\underline{G}', \underline{G})$.

6) It is easy to check that $(\bar{M})^* = (M^*)^- = M'$, say. M' is called the transpose of M. Note that M' maps $\underline{G}$ to $\underline{G}'$. If (e_α) is any basis for $\underline{G}$ and $Me_\beta = \sum_\alpha m_{\alpha\beta} e'_\alpha$ then $M'e_\beta = \sum_\alpha m_{\beta\alpha} e'_\alpha$. $M \to M'$ is a Banach space isomorphism on $\underline{B}(\underline{G}, \underline{G}')$.

7) For $N \in \underline{B}(\underline{G}', \underline{G})$, once again its transpose $N' = (N^*)^- = (\bar{N})^*$ and $N \to N'$ is a Banach space isomorphism on $\underline{B}(\underline{G}', \underline{G})$.

8) The operations $'$, $*$ and $^-$ are mutually commuting and each of these commutes with the operation $^{-1}$ of taking inverses whenever the latter is defined. Further the relations between these various operations are precisely the expected ones.

9) Let $M \in \underline{B}(\underline{G}, \underline{G}')$. M is said to be symmetric or skew-symmetric according as $M = M'$ or $M = -M'$.

These nine observations offer a convenient and canonical way of looking at conjugates and transposes of operators on complex Hilbert spaces. Once we accept it, the generalization of the Frobenius-Schur theory is immediate.

Let A be a representation of S in the complex Hilbert space $\underline{G}$.

<u>DEFINITION 2.1</u> The contragredient of A is the representation $\bar{A} : s \to \overline{A(s)}$ of S in $\underline{G}'$.

Clearly $\bar{A}$ is irreducible, unitary etc., if and only if A is so. Also A is equivalent to B if and only if $\bar{A}$ is equivalent to $\bar{B}$.

<u>THEOREM 2.1</u> Let A be an irreducible representation of S in $\underline{G}$ and let $\bar{A}$ be its contragredient. Then exactly one of the following conditions holds:

i) A is not equivalent to $\bar{A}$.

ii) There is a symmetric unitary operator intertwining A and $\bar{A}$.

iii) There is a skew-symmetric unitary operator intertwining A and $\bar{A}$.

Proof: The proof is the same as that in the finite-dimensional case, but is reproduced here because it is short and elegant.

Suppose A and $\bar{A}$ are equivalent. Then there is a unitary operator M such that $MAM^* = \bar{A}$ (M is unitary if and only if $M^{-1} = M^*$). Taking conjugates we have $\bar{M}\,\bar{A}\,M' = A$.

Hence $M\bar{M}\bar{A}(M\bar{M})^{-1} = \bar{A}$. But $\bar{A}$ is irreducible so that $M\bar{M} = cI$ for some complex number c. On the other hand $MM^* = I$ whence $\bar{M}M' = I$. Now $M\bar{M} = cI$ and $\bar{M}M' = I$ together imply that $cM' = M$, whereas $M^*M' = \bar{c}I$ and $MM^* = I$ together imply that $M' = \bar{c}M$. Consequently $c = \pm 1$ or $M' = \pm M$. The proof is completed if we note that if N is any operator intertwining A and $\bar{A}$ then N is a constant multiple of M.

<u>DEFINITION 2.2</u> (cf. Wigner, 1959) An irreducible representation A is nonreal, potentially real or pseudoreal according as A satisfies condition (i), (ii) or (iii) of Theorem 2.1 above.

We now obtain conditions for A to be potentially real, or pseudoreal in terms of the behaviour of A with respect to a basis. Apart from being very useful to us later on, they establish that each of these three properties is shared by equivalent representations. But first a lemma.

<u>LEMMA 2.1</u> Let $M \in \underline{B}(\underline{G}, \underline{G}')$.

i) M is symmetric unitary if and only if there exists a basis (e_α) for $\underline{G}$ such that $Me_\alpha = \bar{e}_\alpha'$ for all α.

ii) M is skew-symmetric unitary if and only if there exists a basis of the form $(e_\alpha; f_\alpha)$ for $\underline{G}$ such that $Me_\alpha = \bar{f}_\alpha'$ and $Mf_\alpha = -\bar{e}_\alpha'$ for all α.

Proof: i) If M is such that $Me_\alpha = e_\alpha'$ for some basis (e_α) of $\underline{G}$, then clearly M is unitary and $M'e_\alpha = \overline{M}^*e_\alpha = (M^*e_\alpha')' = (M^{-1}e_\alpha')' = e_\alpha' = Me_\alpha$ so that M is symmetric. Conversely suppose M is unitary and symmetric. For $x \in \underline{G}$, let $x^* = ix - iM^*x'$. If $Mx \neq x'$ to start with, then $x^* \neq 0$ and $Mx^* = iMx - ix' = iM'x + (ix)' = i(M^*x')' + (ix)' = x^{*'}$. Therefore, there is always an $x \neq 0$ such that $Mx = x'$. Let (e_α) be a maximal orthonormal family of vectors in $\underline{G}$ such that $Me_\alpha = e_\alpha'$ for all α; (e_α) must then be a basis for $\underline{G}$. For let, if possible, y be such that $\|y\| = 1$ and $y \perp e_\alpha$ for all α. Clearly then $My \neq y'$, so that $y^* = iy - iM^*y' \neq 0$ and $My^* = y^{*'}$, as observed earlier in the proof. But then y^*, as may be easily checked, is orthogonal to all the e_α, contradicting the maximality of the (e_α). (e_α) must therefore be a basis for $\underline{G}$. This proves i).

ii) Let M be such that $Me_\alpha = f_\alpha'$ and $Mf_\alpha = -e_\alpha'$ with respect to a basis $(e_\alpha; f_\alpha)$. Once again M is clearly unitary while $M'e_\alpha = (M^*e_\alpha')' = -f_\alpha' = -Me_\alpha$ and $M'f_\alpha = (M^*f_\alpha')' = e_\alpha' = -Mf_\alpha$ for all α, so that $M = -M'$. Conversely let M be a skew-symmetric unitary operator. If e is any vector in $\underline{G}$, $\|e\| = 1$, and $f = (Me)' (\in \underline{G})$ then $\|f\| = 1$ and $f \perp e$ since

$$\langle e,(Me)' \rangle = \langle Me, e' \rangle' = \langle e, M^*e' \rangle = \langle e,(M'e)' \rangle$$
$$= - \langle e,(Me)' \rangle.$$

Also $Me = f'$ and $Mf = M(Me)' = - M(M'e)' = - M(M^*e')''$ $= - MM^*e' = - e'$. Therefore there always exist orthonormal pairs (e, f) in $\underline{G}$ such that $Me = f'$ and $Mf = - e'$. Let $(e_\alpha; f_\alpha)$ be a maximal family of mutually orthonormal pairs of vectors in $\underline{G}$ with this property. If $x \in \underline{G}$, $\| x \| = 1$, is orthogonal to all these then, as is easy to check, $(x; (Mx)')$ is again such a pair orthogonal to all of them contradicting the maximality of the (e_α). Consequently $(e_\alpha; f_\alpha)$ is a basis for $\underline{G}$. This proves (ii), and hence the lemma.

<u>COROLLARY 2.1</u> If $\underline{G}$ is finite-dimensional and M is skew-symmetric unitary, then $\underline{G}$ is even-dimensional. In particular every pseudoreal finite-dimensional representation is even-dimensional.

<u>THEOREM 2.2</u> Let A be an irreducible representation of S in $\underline{G}$.

i) A is potentially real if and only if there exists a basis (e_α) for $\underline{G}$ such that $\langle Ae_\beta, e_\alpha \rangle$ is real for all α, β.

ii) A is pseudoreal if and only if there exists a basis $(e_\alpha; f_\alpha)$ for $\underline{G}$ such that

$$\langle Ae_\beta, e_\alpha \rangle = \overline{\langle Af_\beta, f_\alpha \rangle} \text{ and } \langle Ae_\beta, f_\alpha \rangle = -\overline{\langle Af_\beta, e_\alpha \rangle}$$

for all α, β.

Proof: i) If (e_α) is a basis for $\underline{G}$ such that $\langle Ae_\beta, e_\alpha \rangle = a_{\alpha\beta}$ is real for all α, β, define M by $Me_\alpha = e_\alpha'$. By Lemma 2.1 M is symmetric unitary and $MAM^{-1} e_\beta' = MAe_\beta = \sum_\alpha a_{\alpha\beta} e_\alpha'$ whereas $\overline{A} e_\beta' = \sum_\alpha \overline{a}_{\alpha\beta} e_\alpha' = \sum_\alpha a_{\alpha\beta} e_\alpha'$ for all β. This implies that $MAM^{-1} = \overline{A}$, so that A is potentially real. Going the other way, if A is potentially real choose (e_α), using Lemma 2.1, such that $Me_\alpha = e_\alpha'$ where M is a symmetric unitary operator intertwining A and $\overline{A}$. Then $\sum_\alpha \overline{a}_{\alpha\beta} e_\alpha' = \overline{A} e_\beta' = MAM^{-1} e_\beta' = \sum_\alpha a_{\alpha\beta} e_\alpha'$ implying thereby that $a_{\alpha\beta} = \overline{a}_{\alpha\beta}$ for all α, β.

ii) To prove the 'if' part define M by $Me_\alpha = f_\alpha'$ and $Mf_\alpha = -e_\alpha'$. Then M is skew-symmetric unitary and the given relations between the matrix elements of A ensure that $MAM^{-1} e_\beta' = \overline{A} e_\beta'$ and $MAM^{-1} f_\beta' = \overline{A} f_\beta'$ for all β. Consequently $MAM^{-1} = \overline{A}$ and A is pseudoreal. For the 'only if' part choose a skew-symmetric unitary M for which $MAM^{-1} = \overline{A}$

and then, by Lemma 2.1, a basis $(e_\alpha; f_\alpha)$ for $\underline{G}$ such that $Me_\alpha = f_\alpha'$ and $Mf_\alpha = -e_\alpha'$. The required identities follow by equating the coefficients of e_α' and f_α' in $MAM^{-1} e_\beta' = \overline{A} e_\beta'$.

The theorem is thus completely proved.

COROLLARY 2.2 If A is nonreal (potentially real, pseudoreal) and B is equivalent to A, then B is nonreal (potentially real, pseudoreal).

The proof is simple and is omitted.

If A and B are two C-representation of S in $\underline{G}$ and $\underline{H}$ respectively, then A and B are said to be physically equivalent if there is an operator intertwining A and B which is either unitary or anti-unitary (Varadarajan, 1968). Clearly physical equivalence preserves irreducibility and every representation is physically equivalent to its contragredient. Consequently A is physically equivalent to B if and only if A is equivalent to B or $\overline{B}$. If A is not nonreal and A is physically equivalent to B then A is equivalent to B. We spell out these details here, but shall make use of them in latter chapters without explicit mention.

We shall end this chapter with a well-known result on compact groups which we shall need later on. The proof is omitted.

<u>THEOREM 2.3</u> Let G be a compact topological group and χ the character of an irreducible unitary C-representation U of G. Then U is nonreal, potentially real or pseudoreal according as

$$\int \chi(g^2)\,dg = 0, +1 \text{ or } -1.$$

III. QUATERNIONIFICATION

Just as a real Hilbert space can be complexified to yield a complex Hilbert space, so a complex Hilbert space can be 'quaternionified' to yield a quaternionic Hilbert space. This process enables us to dispense with the naive point of view that 'every complex matrix is a quaternionic matrix' and helps us to study the inter-relations between C-representations and Q-representations from a sound geometric view-point.

Let $\underline{G}$ be a complex Hilbert space and let $\underline{H}$ be the set of all pairs $[x, y]$ with $x, y \in \underline{G}$. $\underline{H}$ becomes a (left) vector space over the quaternions if the action of $q = \alpha + k\beta \in Q$, α, β complex, on $\underline{H}$ is defined by

$$q \cdot [x, y] = [\alpha x - \bar{\beta} y , \beta x + \bar{\alpha} y].$$

Further $\underline{H}$ becomes a Q-Hilbert space if we define the inner product $(\cdot, \cdot)$ by

$$([x,y], [x^*,y^*]) = \langle x,x^* \rangle + \langle y^*, y\rangle - \langle x,y^* \rangle k + \langle x^*,y \rangle k$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\underline{G}$. $\underline{H}$ is called the quaternionification of $\underline{G}$, in symbols $\underline{H} = \underline{G}_Q$.

Consider now $\underline{H}^s$, the symplectic image of $\underline{H}$. The set of all vectors of $\underline{H}^s$ of the form $[x, 0]$ is a subspace of $\underline{H}^s$ which is in an obvious isomorphism with $\underline{G}$. We shall identify $\underline{G}$ with this subspace. Again, the set of all vectors of the form $[0, x]$ is a subspace of $\underline{H}^s$. This subspace is identifiable with $\underline{G}'$, the dual of $\underline{G}$, with the help of the correspondence $x' \leftrightarrow [0, x]$ since $\alpha\,[0, x] = [0, \bar{\alpha}\, x] = (\bar{\alpha}\, x')' = \alpha\, x'$ for α complex and $([0, x], [0, y]) = \langle y, x \rangle$. Whenever convenient we shall write x for $[x, 0]$ and kx for $k[x, 0] = [0, x]$.

With these two identifications, we may say that $\underline{G}$ and $\underline{G}'$ are half-spaces of $\underline{H}$, mutually orthogonal in $\underline{H}^s$ with $K[\underline{G}] = [kx\colon x \in \underline{G}] \cong \underline{G}'$. Suppose now $\underline{K}$ is an arbitrary Q-Hilbert space and P is a half-space of $\underline{K}$. Then $\underline{K}$ is identifiable with P_Q, the quaternionification of P considered as a complex Hilbert space by itself, and $K[P]$ is clearly isomorphic to P', the dual of P.

In the next chapter we shall have occasion to consider not only the symplectic images of quaternionifications but quaternionifications of symplectic images. In dealing with the latter concept one has to be a little careful to note this point: If $\underline{H}$ is a Q-Hilbert space and $\underline{K} = (\underline{H}^s)_Q$ then for $[x, 0] \in \underline{K}$, $x \in \underline{H}$, $k[x, 0] = [0, x]$ and not $[kx, 0]$.

Our next step is to extend operators associated with $\underline{G}$ to $\underline{G}_Q$. We have the lemma below.

<u>LEMMA 3.1</u> Let $\underline{P}$ and $\underline{Q}$ be half-spaces of Q-Hilbert spaces $\underline{H}$ and $\underline{K}$ respectively. If T is any bounded (complex-) linear map from $\underline{P}$ to $\underline{Q}$, then T may be extended to a unique bounded (quaternion -) linear map T_Q from $\underline{H}$ to $\underline{K}$. The map $T \to T_Q$ is real-linear and preserves norms and adjoints.

<u>Proof</u>. Because $\underline{P}$ is a half-space of $\underline{H}$ every $z \in \underline{H}$ is uniquely of the form $z = x + ky$ with $x, y \in \underline{P}$ and $\| z \|^2 = \| x \|^2 + \| y \|^2$. Define T_Q by $T_Q z = Tx + kTy$. T_Q is easily checked to be quaternion-linear. The uniqueness of the extension is obvious.

$T \to T_Q$ is clearly real-linear. Since

$$\| T_Q \| = \sup_{\| z \| = 1, z \in \underline{H}} \| T_Q z \| \geq \sup_{\| x \| = 1, x \in P} \| T_Q x \| = \| T \| \text{ and}$$

$$\|T_Q z\|^2 = \|Tx + kTy\|^2 = \|Tx\|^2 + \|Ty\|^2 \leq \|T\|^2 \|z\|^2$$

for all $z = x + ky \in \underline{H}$, $x, y \in \underline{P}$, $\|T_Q\| = \|T\|$. Finally, noting that the (complex) inner product on a half-space is but the restriction of the (quaternionic) inner product on the original space, we have, for all

$z = x+ky \in \underline{H}$, $x, y \in \underline{P}$ and $w = u+kv \in \underline{K}$, $u, v \in \underline{Q}$

$$\begin{aligned}(T_Q z, w) &= (Tx+kTy, u+kv) \\ &= (Tx,u) + (Tx,v)k^* + k(Ty,u) + k(Ty,v)k^* \\ &= (x,T^*u) + (x,kT^*v) + (ky,T^*u) + (ky,kT^*v) \\ &= (z,(T^*)_Q w)\end{aligned}$$

implying thereby that $(T_Q)^* = (T^*)_Q$. (Here $(\cdot,\cdot)$ stands for the inner product on $\underline{H}$ as well as $\underline{K}$).

The lemma is proved.

COROLLARY 3.1 If $\underline{G}$ is a complex Hilbert space, $\underline{\bar{H}} = \underline{\bar{G}}_Q$ and T is an operator in $\underline{B}(\underline{G})$ or $\underline{B}(\underline{G}')$ or $\underline{B}(\underline{G}, \underline{G}')$ or $\underline{B}(\underline{G}', \underline{\bar{G}})$ then there is a unique $T_Q \in \underline{B}(\underline{H})$ such that $T_Q x = Tx$ for all x in the domain of T. $T \to T_Q$ is real-linear and $\|T_Q\| = \|T\|$, $(T^*)_Q = (T_Q)^*$.

<u>LEMMA 3.2</u> Let $A, B \in \underline{B}(\underline{G})$, $C, D \in \underline{B}(\underline{G}')$, $M \in \underline{B}(\underline{G}, \underline{G}')$, $N \in \underline{B}(\underline{G}', \underline{G})$.

i) $A_Q [x, y] = [Ax, Ay]$ for all $[x, y] \in \underline{H}$.

ii) $(AB)_Q = A_Q B_Q$, $A_Q = \overline{A}_Q$, $A^*_Q = A'_Q$.

iii) $C_Q = \overline{C}_Q$, $C^*_Q = C'_Q$, $(CD)_Q = C_Q D_Q$.

iv) $M_Q [x, y] = [-(My)', (Mx)']$ for all $[x, y] \in \underline{H}$.

v) $(MA)_Q = M_Q A_Q$, $(CM)_Q = C_Q M_Q$, $M_Q = -\overline{M}_Q$, $M^*_Q = -M'_Q$.

vi) $N_Q = -\overline{N}_Q$, $N^*_Q = -N'_Q$, $(NC)_Q = N_Q C_Q$, $(AN)_Q = A_Q N_Q$,

vii) $(NM)_Q = N_Q M_Q$.

The proof of this lemma is straight forward and is omitted. Note that as a consequence of (v) above M_Q is hermitian if and only if M is skew-symmetric.

IV. THE COMPUTATION OF Q-REPRESENTATIONS

In this chapter we shall show how one may compute all the irreducible Q-representations of S given all the irreducible C-representations of S.

There are two obvious ways of relating C-representations and Q-representations. If A is a C-representation of S in $\underline{G}$, then its quaternionification $A_Q : s \to [A(s)]_Q$ is a Q-representation of S in $\underline{G}_Q$. Again if D is a Q-representation of S in $\underline{H}$ then its symplectic image $D^S : s \to [D(s)]^S$ is a C-representation of S in $\underline{H}^S$. Our first task is to examine when these correspondences preserve irreducibility. We shall find it helpful to reserve the symbols $A, B, \underline{F}, \underline{G}$ for the complex case and $D, E, \underline{H}, \underline{K}$ for the quaternionic case.

<u>THEOREM 4.1</u> Let D be a Q-representation of S in $\underline{H}$ and let D^S be its symplectic image.

i) If D^S is irreducible, then so is D.

ii) If D is irreducible, then D^S is irreducible if and only if D is of class Q.

iii) If D^S is irreducible, then it is pseudoreal.

iv) If D^S is reducible, then $D^S = A + \bar{A}$ for an irreducible C-representation A such that A_Q is equivalent to D.

Proof. i) If D is reducible and $P \longleftrightarrow D$ is a non-trivial projection in $\underline{H}$, then $P^S \longleftrightarrow D^S$ is a non-trivial projection in $\underline{H}^S$, so that D^S is reducible.

ii) Let D be irreducible. If D is not of class Q then there exists a half-space P of $\underline{H}$ reducing D. This is the same as saying that P is a non-trivial subspace of $\underline{H}^S$ reducing D^S, so that D^S must be reducible. Suppose conversely that D^S is reducible. Let P be a non-trivial subspace of $\underline{H}^S$ reducing D^S. Then $K[P] = [kx: x \in P]$ reduces D^S and hence $P - K[P] = P_0$ (say) reduces D^S. $P_0 \neq 0$, for if $P = K[P]$ then P would be a non-trivial subspace of $\underline{H}$ reducing D, contradicting the irreducibility of D. Further $P_0 \perp^S K[P_0]$ because $K[P_0] \subseteq K[P]$. Consequently $P_0 \oplus^S K[P_0] = \underline{H}$ because it is a non-zero subspace of $\underline{H}$ reducing D. This means that P_0 is a half-space of $\underline{H}$ reducing D. Therefore D can not be of class Q. This proves (ii).

iii) If (e_α) is a basis for $\underline{H}$, $(e_\alpha; ke_\alpha)$ is a basis for $\underline{H}^S$ with respect to which the matrix elements of D^S satisfy the identitities of Theorem 2.2 (ii), so that D^S is pseudoreal.

iv) Let D^S be reducible. By the argument in (ii), there exists a half-space P_o of $\underline{H}$ reducing D. Let A be the restriction of D^S to P_o. Then $\bar{A}$ is the restriction of D^S to $K[P_o]$, so that $D^S = A + \bar{A}$. A is irreducible, for if $0 \neq P \subseteq P_o$ reduces A then $P + K[P] \neq 0$ reduces D implying that $P + K[P] = \underline{H}$ and hence that $P = P_o$. To prove that A_Q is equivalent to D, we have only to observe that if (e_α) is any basis for P_o, then (e_α) is simultaneously a basis for P_{oQ} and $\underline{H}$ and that the matrix elements of A_Q and D with respect to (e_α) are the same.

The proof of the theorem is complete.

Theorem 4.2 below was proved for matrix representations of compact groups by Jauch, Schiminovitch and Speiser (1963).

<u>THEOREM 4.2</u> Let A be an irreducible C-representation of S in $\underline{G}$ and let A_Q be its quaternionification.

i) If A_Q is irreducible, then so is A.

ii) If A is irreducible, then A_Q is irreducible if and only if A is not pseudoreal.

iii) If A_Q is irreducible, then it is not of class Q.

iv) If A_Q is reducible, then $A_Q = D + E$ for two equivalent irreducible Q-representations D and E such that D^S is equivalent to A.

Proof: i) If A is reducible and P is a non-trivial subspace of $\underline{G}$ reducing A, then $P + K[P]$ is a non-trivial subspace of $\underline{G}_Q$ reducing A_Q, so that A_Q is reducible.

ii) Let A be irreducible. If A is pseudoreal, then there is a skew-symmetric unitary operator M such that $MA = \bar{A} M$. But then $M_Q A_Q = A_Q M_Q$, because $A_Q = \bar{A}_Q$. Since M is skew-symmetric, M_Q is hermitian on $\underline{H} = \underline{G}_Q$ and is clearly not a (real) multiple of the identity because it maps $\underline{G}$ to $\underline{G}'$. Consequently A_Q is reducible.

To prove the converse, we have to prove that if A_Q is reducible then A is pseudoreal. Now, if A_Q were reducible, then there would exist a hermitian operator T on $\underline{H}$

which is not a multiple of the identity, such that $T \leftrightarrow A_Q$. Consider T^S and $B = [A_Q]^S$ on $\underline{H}^S$. Let P and P' be the projections in $\underline{H}^S$ on $\underline{G}$ and $\underline{G}'$ respectively. Let $T_1 = PT^SP$ and $T_2 = P'T^SP$. Then T_1 and T_2 commute with B. But then T_1 may be considered to be an operator on $\underline{G}$ commuting with A, since $B = A$ on $\underline{G}$. But A is irreducible. Therefore $T_1 = \lambda I$ for some real number λ. This implies that $T_2 \neq 0$, for if $T_2 = 0$, then for all $x \in \underline{G}$, $T^S x = T^S Px = (P + P')T^S Px = T_1 x + T_2 x = \lambda x$ so that $T^S = \lambda I$, contrary to our assumption. Now T_2 may be thought of as an operator from $\underline{G}$ to $\underline{G}'$ which intertwines A and $\bar{A}$, since $B = \bar{A}$ on $\underline{G}'$. Because A (and hence $\bar{A}$) is irreducible and $T_2 \neq 0$, Schur's lemma implies that $T_2/\|T_2\|$ is unitary. To complete the proof that A is pseudoreal we have only to prove that T_2 is skew-symmetric. It is easy to see that $T_2^* : \underline{G}' \to \underline{G}$ is but the restriction of the adjoint of T_2 in $\underline{H}^S$ to $\underline{G}'$. Therefore $T_2^* = PT^SP'$. on $\underline{G}'$. Also, if T_2' is the transpose of T_2, $T_2'x = (T_2^*x')' = KT_2^*Kx$ for $x \in \underline{G}$. Combining these observations and recalling that $P' = KPK^{-1} = -KPK$, we have $T_2' = KPT^SP'K = KPK^{-1}KT^SP'K = P'T^SKP'K = -P'T^SP = -T_2$. T_2 is therefore skew-symmetric. As observed earlier this proves (ii).

iii) is trivial because $\underline{G}$ is a half-space of $\underline{H}$ which reduces A_Q.

iv) Suppose that A_Q is reducible. Then A is pseudoreal by (ii) and hence there is a symmetric unitary operator M such that $MA = \bar{A}M$. Thefefore $M_Q \leftrightarrow A_Q$ and M_Q is a hermitian unitary operator. Consequently there exist two mutually orthogonal projections, P^+ and P^- say, in $\underline{H}$ which reduce A_Q and satisfy $M = P^+ - P^-$, $P^+ + P^- = I$. Let D and E be the restrictions of A_Q to P^+ and P^- respectively.

Because M is skew-symmetric and unitary, by Lemma 2.1, there exists a basis of the form $(e_\alpha; f_\alpha)$ for $\underline{G}$ such that $Me_\alpha = kf_\alpha$ and $Mf_\alpha = -ke_\alpha$ for all α. Define

$$g_\alpha = i\,\frac{e_\alpha + kf_\alpha}{\sqrt{2}} \quad \text{and} \quad h_\alpha = \frac{e_\alpha - kf_\alpha}{\sqrt{2}}.$$

It is not difficult to verify that $(g_\alpha; h_\alpha)$ is a basis for $\underline{H}$ and that $g_\alpha \in P^+$, $h_\alpha \in P^-$ for all α. Consequently (g_α) is a basis for P^+ and (h_α) a basis for P^-. If we now make use of the identities between the matrix elements of A with respect to $(e_\alpha; f_\alpha)$ vouchsafed us by Theorem 2.2 (ii), it is not hard to establish that

$$(Dg_\beta, g_\alpha) = (A_Q g_\beta, g_\alpha)$$

$$\doteq \langle Ae_\beta, e_\alpha \rangle + \langle Ae_\beta, f_\alpha \rangle k$$

$$= (A_Q h_\beta, h_\alpha) = (Eh_\beta, h_\alpha) \quad \text{for all} \quad \alpha, \beta.$$

Clearly then D is equivalent to E. To prove now that D is irreducible, it is enough to prove, by Theorem 4.1, that D^S is irreducible. But this is true since the matrix elements of D^S with respect to the basis $(g_\alpha; kg_\alpha)$ are the same as those of A with respect to $(e_\alpha; f_\alpha)$ so that D^S and A are equivalent and A is irreducible.

The theorem is completely proved.

<u>LEMMA 4.1</u> Let D be an irreducible Q-representation of S in $\underline{H}$ which is of class Q. Then $(D^S)_Q = E_1 + E_2$ for some two irreducible Q-representations E_1 and E_2 equivalent to D.

Proof: By Theorem 4.1 D^S is irreducible and pseudo-real on $\underline{H}^S$. Let $(e_\alpha; f_\alpha)$ be a basis for $\overline{\underline{H}}^S$ so chosen that (e_α) is a basis for $\underline{H}$ and $f_\alpha = ke_\alpha$. Then the matrix elements of D^S with respect to $(e_\alpha; f_\alpha)$ satisfy the identities of

Theorem 2.2. (ii). By the proof of Theorem 4.2. (iv) we may write $(D^S)_Q = E_1 + E_2$ in such a way that E_1 and E_2 are equivalent irreducible Q-representations and the matrix elements of E_1^S with respect to a basis of the form $(g_\alpha; kg_\alpha)$ are the same as those of D^S with respect to $(e_\alpha; f_\alpha)$. But then the matrix elements of D and E_1 with respect to (e_α) and (g_α) respectively are identical, implying thereby that D and E_1 are equivalent. Since E_1 and E_2 are equivalent, the lemma is proved.

<u>LEMMA 4.2</u> i) If A and B are two irreducible C-representations of S acting in $\underline{F}$ and $\underline{G}$ respectively, then A_Q is equivalent to B_Q if and only if A is physically equivalent to B.

ii) If D and E are two irreducible Q-representations of S acting in $\underline{H}$ and $\underline{K}$ respectively, then D^S is physically equivalent to E^S if and only if D is equivalent to E.

Proof: i) If A is physically equivalent to B, then A is equivalent to B or $\overline{B}$. Since $B_Q = \overline{B}_Q$, to prove that A_Q and B_Q are equivalent, we may suppose without loss of generality that A is equivalent to B. But in that case if

$T: \underline{F} \to \underline{G}$ is any unitary operator intertwining A and B then T_Q, the (unique) extension of T to $\underline{F}_Q$ is clearly unitary and intertwines A_Q and B_Q so that A_Q is equivalent to B_Q.

Conversely, let A_Q be equivalent to B_Q. Note then that A_Q^s is equivalent to B_Q^s, since, if U is any unitary operator intertwining A_Q and B_Q then U^s is a unitary operator intertwining A_Q^s and B_Q^s. But now clearly $A_Q^s = A + \overline{A}$ and $B_Q^s = B + \overline{B}$, so that Schur's lemma helps us to conclude that A is equivalent to B or $\overline{B}$ i.e. that A is physically equivalent to B. This proves i).

ii) That D^s is equivalent to E^s if D is equivalent to E was observed in the proof of i) above.

Let D^s be physically equivalent to E^s. We may assume without loss of generality that D^s is equivalent to E^s, for if U intertwines D^s and E^s and is anti-unitary then KU intertwines D^s and E^s and is unitary: $KUD^s = KE^sU = E^sKU$, because $E \leftrightarrow K$. The proof that D is equivalent to E depends on the class of D.

Suppose that D is of class Q. Then, by Theorem 4.1, D^s and E^s are irreducible and pseudoreal. By Lemma 4.1, $(D^s)_Q = D_1 + D_2$ and $(E^s)_Q = E_1 + E_2$ where D_1 and D_2 as also E_1 and E_2 are equivalent irreducible Q-representations such that D_1 is equivalent to D and E_1 is equivalent to E. On the other hand $(D^s)_Q$ is equivalent to $(E^s)_Q$ by i) above, since D^s and E^s are given to be equivalent. Schur's lemma then implies that D_1 is equivalent to E_1. Consequently D is equivalent to E.

Suppose that D is not of class Q. Then D^s and E^s are equivalent reducible C-representations and by Theorem 4.1, $D^s = A + \overline{A}$, $E^s = B + \overline{B}$ where A and B are irreducible C-representations such that A_Q is equivalent to D and B_Q is equivalent to E. By Schur's lemma again A is equivalent to B or $\overline{B}$ so that i) implies that A_Q is equivalent B_Q. Consequently D is equivalent to E.

The lemma is completely proved.

We can now state and prove the main theorem of this chapter. All representations considered are representations of S. The reader is reminded that the adjectives potentially real, nonreal and pseudoreal are applied only to irreducible C-representations of S.

<u>THEOREM 4.3</u> i) If A is a potentially real C-representation, then its quaternionification A_Q is an irreducible Q-representation of class R. Every irreducible

Q-representation of class R is equivalent to the quaternionification of a potentially real C-representation. Two potentially real C-representations are equivalent if and only if their quaternionifications are equivalent.

ii) If A is a nonreal C-representation, then its quaternionification A_Q is an irreducible Q-representation of class C. Every irreducible Q-representation of class C is equivalent to the quaternionification of a nonreal C-representation. Two nonreal C-representations are physically equivalent if and only if their quaternionifications are equivalent.

iii) If D is an irreducible Q-representation of class Q, then its symplectic image D^S is a pseudoreal C-representation. Every pseudoreal C-representation is equivalent to the symplectic image of an irreducible Q-representation of class Q. Two irreducible Q-representations of class Q are equivalent if and only if their symplectic images are equivalent.

Proof: i) If A is potentially real then A_Q is irreducible by Theorem 4.2. If A acts on $\underline{G}$, by Theorem 2.2, there exists a basis (e_α) for $\underline{G}$ with respect to which the matrix elements of A are all real. But then (e_α) is a basis for $\underline{G}_Q$ and the matrix elements of A_Q with respect to (e_α) are all real. Therefore A is of class R.

Conversely, if D is an irreducible Q-representation of class R acting in $\underline{H}$, let (e_α) be a basis of $\underline{H}$ with respect to which the matrix elements of D are all real and let P_o be the half-space of $\underline{H}$ generated by the (e_α). Then the argument of Theorem 4.1 iv) implies that the restriction of D^S to P_o is a potentially real C-representation whose quaternionification is equivalent to D. The last assertion is an immediate consequence of Lemma 4.2, because for potentially real C-representations physical equivalence implies equivalence.

ii) By Theorem 4.2 A_Q is irreducible and is of class R or C. But A_Q cannot be of class R, for if it were, then it would be equivalent to the quaternionification of a potentially real C-representation and Lemma 4.2 would then imply that a nonreal C-representation and a potentially real C-representation are physically equivalent, which is impossible. Therefore A_Q is of class C. Conversely if D is any irreducible Q-representation of class C, then by Theorem 4.1, D^S is reducible and there exists an irreducible C-representation A such that A_Q is equivalent to D. A has to be nonreal, for if A were potentially real D would be of class R and if A were pseudoreal D would be reducible. The last assertion is a consequence of Lemma 4.2.

iii) If D is irreducible and is of class Q, then D^s is pseudoreal by Theorem 4.1. If A is any pseudoreal C-representation then, by Theorem 4.2, A_Q is reducible and there exists an irreducible Q-representation D such that D^s is equivalent to A. Such a D has to be of class Q, for if not, by Theorem 4.1, D^s and hence A would be reducible. The last part once again follows from Lemma 4.2.

The theorem is completely proved.

<u>COROLLARY 4.1</u> Let A be a pseudoreal C-representation of S acting in $\underline{G}$. Let $(e_\alpha; f_\alpha)$ be a basis for $\underline{G}$ with respect to which the matrix elements of A satisfy the identities of Theorem 2.2. (ii). Then the Q-representation D acting in the quaternionification of the (complex) subspace spanned by the (e_α) defined by

$$De_\beta = \sum_\alpha [< Ae_\beta, e_\alpha > + < Ae_\beta, f_\alpha > k] e_\alpha$$

is an irreducible Q-subrepresentation of A_Q which is of class Q.

Theorem 4.3 establishes a one-one correspondence between equivalence classes of irreducible Q-representations and physical equivalence classes of irreducible C-representations and suggests the following procedure for obtaining maximal families of mutually inequivalent Q-representation of S whenever we have available such maximal families of C-representations:

Consider a maximal family of mutually physically inequivalent potentially real (nonreal) C-representations. Then their quaternionifications constitute a maximal family of mutually inequivalent irreducible Q-representations of class R (class C). Consider a maximal family of mutually (physically) inequivalent pseudoreal C-representations and choose an irreducible Q-subrepresentation from each of their quaternionifications. Then they constitute a maximal family of mutually inequivalent irreducible Q-representations of class Q. Pooling all these, we have a maximal family of mutually inequivalent irreducible Q-representations of S.

We end this chapter with a remark on abelian representations of S. Let A be an irreducible C- or Q-representation of S such that the set of operators [A(s)] is a commuting family. Since this family is closed with respect to adjoints, every A(s) is normal. If A is a C-representation then the spectral measure of every A(s) is trivial so that every A(s) is a multiple of the identity and consequently A is one-dimensional. This is true even if A is a Q-representation and may be deduced from the results of this chapter without resorting to the spectral theory of Part I as follows: A can not be of class Q because if it were then A^S would be an irreducible C-representation of dimension > 1. Hence A is of class R or C. By Theorem 4.1 $A^S = B + \overline{B}$ for an irreducible C-representation B such that $B_Q = A$. But then B is one-dimensional so that B_Q, and hence A, is one-dimensional.

In particular we have the result that every irreducible unitary Q-representation of an abelian topological group is one-dimensional.

V. COMPACT GROUPS

As was observed in the beginning of Chapter I, unitary representations of topological groups are a special case of our formulation of the theory of representations. In this chapter we deduce the theory of unitary Q-representations of compact groups as a corollary of our general theory and the classical theory of unitary C-representations of compact groups. Some of the results we obtain are stronger than those of Natarajan and Viswanath (1967).

Let G be a compact topological group. We shall consider only unitary representations of G throughout this chapter and drop the adjective 'unitary'.

<u>THEOREM 5.1</u> Every Q-representation of G is a direct sum of irreducible Q-representations. Every irreducible Q-representation of G is finite-dimensional.

Proof: Let U be a Q-representation of G in $\underline{H}$. Consider U^s on $\underline{H}^s$. Then by the classical theory there exists a subspace P of $\underline{H}^s$ such that the restriction of U^s to P (let us call it A) is irreducible. But then $P_Q = P \oplus^s K[P]$ is a subspace of $\underline{H}$ and the restriction of

U to P_Q is clearly identifiable with A_Q. If A is either nonreal or potentially real then A_Q is an irreducible Q-representation by Theorem 4.3. If A is pseudoreal then the procedure of Corollary 4.1 yields a subspace of P_Q which reduces A_Q (hence U) and on which A_Q is irreducible. In any case there always exists an irreducible subrepresentation of U. A standard application of Zorn's lemma allows us to conclude that U is a direct sum of irreducibles. That every irreducible Q-representation of G is finite-dimensional is a consequence of Theorem 4.3 and the obvious fact that the symplectic images and quaternionifications of finite dimensional spaces are finite-dimensional. The theorem is proved.

Let U be an irreducible Q-representation of G in $\underline{H}$ of dimension n. Then the matrix elements of U with respect to a basis (e_r) of $\underline{H}$ are continuous functions on G and therefore belong to $L^2_Q(G)$, the Q-Hilbert space of quaternion-valued functions on G square-integrable with respect to the Haar measure on G. We shall now investigate the orthogonality relations between these matrix elements. It is convenient to introduce some terminology.

A basis (e_r) of $\bar{H}$ is said to be suited to U if the matrix elements of U with respect to (e_r) are complex if U is of class C and real if $\dot{U}$ is of class R. Let $[u_{rs}] = [(Ue_s, e_r)]$ be the matrix elements of U with respect to a basis (e_r) suited to U. The functions are all real if U is of class R, of the form $a_{rs} + ib_{rs}$ if U is of class C and of the form $a_{rs} + ib_{rs} + jc_{rs} + kd_{rs}$ if U is of class Q, the functions a, b, c, d being all real.

DEFINITION 5.1 The representative functions of U with respect to (e_r) are the n^2 functions $[u_{rs}]$ if U is of class R, the $2n^2$ functions $[a_{rs}, b_{rs}]$ if U is of class C and the $4n^2$ functions $[a_{rs}, b_{rs}, c_{rs}, d_{rs}]$ if U is class Q. 'A set of representative functions of U' is the set of all representative functions of U with respect to some one basis suited to U.

LEMMA 5.1 Let U and V be two inequivalent irreducible Q-representations of G.

i) Every set of representative functions of U consists of mutually orthogonal non-zero real-valued functions in $L^2_Q(G)$.

ii) If U is of class Q then the matrix elements of U with respect to any basis are mutually bothways-orthogonal non-zero functions in $L^2_Q(G)$. If $\dim(U) \geq 2$ and U is such that the matrix elements of U with respect to every basis are mutually orthogonal then U is of class Q.

iii) Every representative function of U is orthogonal to every representative function of V.

Proof: i) Let (e_r) be any basis suited to U and let $[u_{rs}]$ be the matrix elements of U with respect to (e_r). We shall prove that the representative functions of U with respect to (e_r) are non-zero and mutually orthogonal.

If U is of class R then these are the $[u_{rs}]$ themselves. By Theorem 4.3 the $[u_{rs}]$ are the matrix elements of a potentially real C-representation which is in particular irreducible so that by the well-known complex theory

$\int u_{rs} \bar{u}_{r's'} \, dg = \frac{1}{n} \partial_{rr'} \partial_{ss'}$ for all r, s, r', s', where the ∂'s are Kronecker ∂'s and n is the dimension of U.

If U is of class C, then again by Theorem 4.3 the $[u_{rs}]$ are the matrix elements of a nonreal C-representation so that $\int u_{rs} \bar{u}_{r's'} \, dg = \frac{1}{n} \delta_{rr'} \delta_{ss'}$ and

$$\int u_{rs} \bar{\bar{u}}_{r's'} \, dg = 0, \qquad \text{for all } r, s, r', s'.$$

by the complex theory. From these relations it is easy to conclude that, if $u_{rs} = a_{rs} + ib_{rs}$, then the $2n^2$ functions $[a_{rs}, b_{rs}]$ are mutually orthogonal and

$$\int a_{rs}^2 \, dg = \int b_{rs}^2 \, dg = \frac{1}{2n} \qquad \text{for all } r, s$$

so that they are all non-zero.

Finally, if U is of class Q, let

$$\begin{aligned} u_{rs} &= a_{rs} + ib_{rs} + jc_{rs} + kd_{rs} \\ &= a_{rs} + ib_{rs} + (d_{rs} - ic_{rs})k. \end{aligned}$$

Consider U^s on $\underline{H}^s$. By Theorem 4.3 U^s is pseudoreal. By classical results the matrix elements of U^s with respect to the basis (e_r, ke_r) are non-zero and mutually orthogonal. But this latter collection of functions is only $[a_{rs} \pm ib_{rs}, \pm d_{rs} - ic_{rs}]$. It is easy now to conclude that the $4n^2$ functions $[a_{rs}, b_{rs}, c_{rs}, d_{rs}]$ are non-zero and mutually orthogonal.

Thus i) is proved.

ii) The first assertion follows readily from i) since any two distinct matrix elements of U with respect to a basis are linear combinations of disjoint sets of representative functions.

To prove the second assertion we shall prove that if U is of class R or C then there is a basis with respect to which the matrix elements of U are not mutually orthogonal. Let then U be of class R or C. Choose first a basis (e_r) suited to U. If (f_r) is any other basis, let us write $f_s = \sum_r m_{rs} e_r$. Then $e_s = \sum_r m^*_{sr} f_r$ and if the matrix elements of U with respect to (e_r) and (f_r) are respectively $[u_{rs}]$ and $[v_{rs}]$ then $v_{rs} = \sum_{t,w} m^*_{sw} u_{tw} m_{rt}$ for all r, s. Exploiting the orthogonality relations between the representative functions of U with respect to (e_r) obtained in i) we may prove that, for all r, s, r', s',

$$\int v_{rs} v^*_{r's'} \, dg = \frac{1}{(1+\rho)n} \sum_{t,w} m^*_{sw} [m_{rt} m^*_{r't} - \rho i m_{rt} m^*_{r't} i] m_{s'w}$$

where ρ is 0 or 1 according as U is of class R or C.

If we now choose (f_r) such that

$$2f_1 = (1 - j)e_1 + (1 + k)\, e_2$$

$$2f_2 = (i + k)e_1 + (1 - k)e_2$$

$$f_r = e_r \qquad \text{for } r > 2,$$

then as is easily checked, $\int v_{11} v_{22}^* \, dg \neq 0$ so that the $[v_{rs}]$ are not mutually orthogonal. This proves (ii).

iii) is proved by a reasoning similar to that in i).

We can now offer an analogue of the classical Peter-Weyl Theorem. Let $C_Q(G)$ denote the space of quaternion valued continuous functions on G.

<u>THEOREM 5.2</u> Let $[U^\alpha]$ be a maximal family of mutually inequivalent irreducible unitary Q-representations of G. Let $[A^\alpha]$ be a corresponding family of representative functions. Then $[A^\alpha]$ is an orthogonal family of non-zero continuous functions fundamental in $C_Q(G)$ and $L^2_Q(G)$.

The proof this theorem is an immediate consequence of its classical analogue and Lemma 5.1. It is omitted.

The Peter-Weyl Theorem can be formulated in another way in terms of subspaces associated with equivalence classes of representations. To see this let the equivalence classes of irreducible Q-representations of G be indexed by α. n_α, the dimension of any irreducible Q-representation of type α, is called the order of the type α. α is said to be of class R, C or Q according as representations of type α are of class R, C or Q.

<u>DEFINITION 5.2</u> For every type α, $\underline{F}^\alpha$ is the (left) linear manifold spanned by the matrix elements of all possible Q-representations of type α with respect to all possible bases.

<u>LEMMA 5.2</u> i) If $q \in Q$ and $f \in \underline{F}^\alpha$ then $fq \in \underline{F}^\alpha$, so that $\underline{F}^\alpha$ is a right linear manifold.

ii) If $f = f_o + if_1 + jf_2 + kf_3 \in \underline{F}^\alpha$ then $f_o, f_1, f_2, f_3 \in \underline{F}^\alpha$.

iii) If $f \in \underline{F}^\alpha$ then $f^* \in \underline{F}^\alpha$.

iv) $\underline{F}^\alpha$ is of dimension n_α^2, $2n_\alpha^2$ or $4n_\alpha^2$ according as α is of class R, C or Q.

v) $\underline{F}^\alpha$ is bothways orthogonal to $\underline{F}^\beta$ if $\alpha \neq \beta$.

<u>Proof:</u> i) If U is any Q-representation of type α with matrix elements $[u_{rs}]$ with respect to a basis (e_r) then, for any quaternion $q \neq 0$, $|q| = 1$, $[q^* u_{rs} q]$ are

the matrix elements of U with respect to the basis $(q^* e_r)$. implying thereby that $u_{rs}\, q \in \underline{F}^\alpha$ for all r,s. If $q \neq 0$ is arbitrary $u_{rs}\, q = |q| . u_{rs}(q/|q|) \in \underline{F}^\alpha$ for all r,s. Since any $f \in \underline{F}^\alpha$ is a (left) linear combination of matrix elements, the result follows.

ii) We now know that for all $p,q \in Q$, $pfq \in \underline{F}^\alpha$. Therefore $f_0 = \mathrm{Re}\, f = [f - ifi - jfj - kfk] \in \underline{F}^\alpha$. Since f_1, f_2, f_3 are the real parts of $-if_1$, $-jf_2$, $-kf_3$ respectively ii) is proved.

iii) is an immediate consequence of ii).

iv) Let U^α be any Q-representation of type α in $\underline{H}$ and let A^α be the set of representative functions of U^α with respect to a basis (e_r) suited to U^α. By (ii) $A^\alpha \subset \underline{F}^\alpha$. We shall prove that $\underline{F}^\alpha$ is generated by the functions in A^α. Let V be any Q-representation of type α acting in a Q-Hilbert space $\underline{K}$ and let $[v_{rs}]$ be the matrix elements of V with respect to a basis (f_r) in $\underline{K}$. If M is a unitary map from $\underline{H}$ to $\underline{K}$ such that $MUM^{-1} = V$ and $f_s = \sum_r n_{rs} Me_r$, $Me_s = \sum_r n^*_{sr} f_r$ then clearly $v_{rs} = \sum_{t,w} n_{ws}\, u_{tw}\, n^*_{tr}$ and consequently v_{rs} is a (left) linear combination of the representative functions A^α. If we now recall that, because of Lemma 5.1, A^α is a set of non-zero mutually orthogonal functions of cardinality n_α^2, $2n_\alpha^2$ or $4n_\alpha^2$ according as α is of class R, C or Q we find that we have proved iv).

v) now follows from Lemma 5.1 iii) if we remind ourselves that the representative functions are always real and hence commute with all quaternions.

This proves the lemma.

The theorem below is another formulation of Theorem 5.2.

<u>THEOREM 5.2'</u> If $\sum_\alpha \underline{F}^\alpha$ denotes the set of finite sums of elements of $\bigcup_\alpha \underline{F}^\alpha$, where α ranges over all types and $(\sum_\alpha \underline{F}^\alpha)^-$ the uniform closure of $\sum_\alpha \underline{F}^\alpha$, then

$$(\sum_\alpha \underline{F}^\alpha)^- = C_Q(G) \quad \text{and} \quad \oplus_\alpha \underline{F}^\alpha = L^2_Q(G).$$

We now go on to obtain a complete set of equivalence invariants for the irreducible Q-representations of G.

If A is an operator on a finite-dimensional Q-Hilbert space $\underline{H}$, then we define the <u>trace</u> of A by $Tr(A) = 1/2\, Tr(A^S)$ where A^S is the symplectic image of A on $\underline{H}^S$.

It is easy to see that if (e_r) is any basis for $\underline{H}$ then $Tr(A) = Re \sum_{\alpha} (A\, e_r, e_r)$ and that $Tr(A) = Tr(MAM^{-1})$ for all invertible M on $\underline{H}$. The following definition is due to Finkelstein, Jauch and Speiser (1963).

<u>DEFINITION 5.3</u> If U is any irreducible Q-representation of G then its <u>Q-character</u> is the function $X(g) = Tr(U(g))$.

Note that $X(g)$ is always a real valued function. If (e_r) is a basis suited to U, then $X(g) = \sum_{\alpha} Re(U(g)e_r, e_r)$ is a sum of representative functions which are non-zero and mutually orthogonal so that $X(g)$ is itself non-zero. In fact $\int X(g)^2 dg = 1,\ 1/2$ or $1/4$ according as U is of class R, C or Q since, as proved in Lemma 5.1, the square of the L^2-norm of a representative function of U is $\frac{1}{n}, \frac{1}{2n}$ or $\frac{1}{4n}$ according as U is of class R, C or Q, n being the dimension of U. Also, clearly equivalent Q-representations have identical Q-characters and inequivalent Q-representations have orthogonal Q-characters. We have therefore proved the following theorem.

<u>THEOREM 5.3</u> Two irreducible Q-representations of G are equivalent if and only if they have the same Q-character.

Q-characters of inequivalent Q-representations are orthogonal. A Q-representation is of class R, C or Q according as the square of the L^2-norm of its Q-character is 1, 1/2 or 1/4.

The algorithm given in Chapter IV for obtaining maximal families of mutually inequivalent irreducible Q-representations in a general context can now be specialised to provide a rule for computing all the Q-characters of G given all the (irreducible) C-characters of G. Recall Theorem 2.3, which says that an irreducible C-representation with character χ is nonreal, potentially real or pseudoreal according as

$$\int \chi(g^2)\, dg = 0 \qquad (1)$$

$$= +1 \qquad (2)$$

or

$$= -1 \qquad (3)$$

<u>Rule</u>: Every real C-character $\chi(g)$ determines a Q-character $X(g) = \chi(g)$ or $\frac{1}{2}\chi(g)$ according as χ satisfies (2) or (3). Every non-real C-character χ determines a Q-character $X(g) = \mathrm{Re}[\chi(g)]$. All the Q-characters are obtained in this way.

We bring our discussion of compact groups to a close with a few remarks on the abelian case. Let G be a compact abelian group. Then, as was observed in the last chapter, every irreducible representation of G, be it complex or quaternionic, is one-dimensional. Let us look at the converse problem. In the complex case the Peter-Weyl theorem implies that if G is a compact group such that every irreducible C-representation of G is one-dimensional then G is abelian. Further, this is equivalent to saying that if G is a compact group such that every irreducible C-representation of G is abelian then G is abelian. In the quaternionic case however these two statements are not equivalent and in fact, the first is not true while the second is. To prove that the second is true we have only to observe that, under the hypothesis, every irreducible Q-representation is necessarily of dimension 1 and is of class R or C (for if a Q-representation A were of class Q, then A^s would be irreducible, abelian and of dimension > 1 which is impossible) so that every irreducible C-representation is of dimension 1 and consequently, G is abelian. To show that the first statement is not true we give an example of a non-abelian finite group whose irreducible Q-representations are all one-dimensional. We denote by G^o the group opposite to G (i.e., the elements of G^o are those of G and the group operation in G^o is given by $g \cdot h = hg$).

Example: Let G be the quaternion group, i.e., $G = [\pm 1, \pm i, \pm j, \pm k]$. Consider G^o. We show that every irreducible Q-representation of G^o is one dimensional.

If $q \in Q$, let R_q denote the linear transformation of the Q-space Q, given by $R_q(p) = pq$ for all p in Q.

Consider the representations:

(1) $g \to R_g$;

(2) $g \to A_g = R_1$ for all $g \in G^o$;

(3) $g \to A_g = R_1$ if $g = \pm 1, \pm i$,

$= R_{-1}$ otherwise;

(4) $g \to A_g = R_1$ if $g = \pm 1, \pm j$,

$= R_{-1}$ otherwise;

(5) $g \to A_g = R_1$ if $g = \pm 1, \pm k$,

$= R_{-1}$ otherwise.

It is easy to verify that the above five (one-dimensional and hence irreducible) Q-representations are mutually inequivalent. If $\underline{F}^r$ is the subspace in $L^2_Q(G^o) = Q^{(8)}$ associated with the rth-representation above, then $\underline{F}^1$ has dimension four and each of the remaining $\underline{F}^r$ has dimension one. It follows that G^o cannot have any irreducible Q-representation inequivalent to all the five above and in particular that G^o does not have any Q-representation of degree greater than one.

VI. LOCALLY COMPACT ABELIAN GROUPS

In this chapter we shall prove two theorems about unitary Q-representations of a locally compact abelian group G. In one we study the structure of the space X of all continuous homomorphisms of G into the group of unit quaternions endowed with the compact-open topology. We show that X is compact if G is discrete and that if G is compact then certain equivalence classes of X are closed-open in X. In the other we seek to express every unitary Q-representation of G as an integral of irreducible Q-representations of G. It is proved that this can be achieved in a unique manner except for the ambiguity imposed by several forced choices between i and -i. This theorem is a generalization of Theorem 5.1 of Part I.

Let $|Q| = [q: q \in Q, |q| = 1]$ denote the multiplicative group of unit quaternions. The centre of $|Q|$ is the doubleton $[\pm 1]$. For every unit imaginary $\theta \in |Q|$ let $C(\theta) = [a + b\theta : a, b \text{ real}, a^2 + b^2 = 1]$. Then $C(\theta)$ is an abelian subgroup of $|Q|$ and every strict subgroup $|Q|$ different from its centre is of this form. $C(\theta) = C(\phi)$ if and only if $\phi = \pm \theta$; if $\phi \neq \theta$ then $C(\theta) \cap C(\phi) = [\pm 1]$. Given θ and

ϕ there exists $r \in |Q|$ such that $r\phi r^{-1} = \theta$ and hence such that $r\, C(\phi)r^{-1} = C(\theta)$. In particular, writing $C(i) = |C|$ - the multiplicative group of complex numbers of modulus one - given $C(\theta)$ there exists r such that $r\theta r^{-1} = i$ and hence such that $rC(\theta)r^{-1} = |C|$. All these observations are immediate consequences of our results of Chapter 1 of Part I.

We need another important fact about $|Q|$. For every neighbourhood U of $1 \in |Q|$ let $U_n = [x: x, x^2, \ldots, x^n \in U]$. Let us call a neighbourhood U of 1 <u>distinguished</u> if the sets (U_n) constitute a local base at 1. Distinguished neighbourhoods exist. E.g. $U = [q : |q-1| < 1/2]$ is one such. (Proof: Let $q = q_0 + q_1 i + q_2 j + q_3 k$ belong to $|Q|$. $q \in U$ if and only if $q_0 > 7/8$. Therefore if $q \in U$, then $|q^2 - 1|^2 = 4(1 - q_0^2) > 4(1 - q_0) = 2|q-1|^2$ so that $|q^2 - 1| > \sqrt{2}\, |q-1|$. It follows that if $q \in U_{2^n}$, then $|q-1| < 2^{-n/2}$. Consequently as n goes to infinity the diameter of U_{2^n} decreases to 0). It is easy to check that every neighbourhood V of 1 which is contained in U is again a distinguished neighbourhood of 1 so that, we may always choose distinguished neighbourhoods to be symmetric, closed etc.

Consider now a locally compact abelian group G. Let X denote the space of all continuous homomorphisms of G into $|Q|$. For every $x \in X$ and $r \in |Q|$ $r x r^{-1}(g) = r\, x\,(g) r^{-1}$ is again in X. Let us call x and y in X equivalent if there exists r such that $y = r x r^{-1}$. It is easy to check that we have defined a genuine equivalence relation on X. If now U is any irreducible (and hence one-dimensional) unitary Q-representation of G in the Q-Hilbert space $\underline{H}$ then for $u \in \underline{H}$, $\| u \| = 1$, $U(g)u = x(g)u$ for some $x(g) \in |Q|$ for all g in G. $x : g \to x(g)$ clearly belongs to X. If $v \in \underline{H}$ is any other unit vector in $\underline{H}$ then $v = ru$ for some $r \in |Q|$ so that $U(g)v = r\, x(g) r^{-1}\, v$ for all g. This means that U determines an element of X upto equivalence. Conversely it is easy to see that every $x \in X$ determines an irreducible unitary Q-representation of G upto equivalence. Therefore the set of equivalence classes of irreducible unitary Q-representations of G is in an one-to-one correspondence with the set of equivalence classes of X. This is the reason for our interest in the space X.

Observe that every complex character of G belongs to X. X therefore distinguishes between points of G. For $x \in X$ let $E_x = [r x r^{-1} : r \in |Q|]$ be the equivalence class containing x. x is real if and only if E_x is the singleton x. If x is not real then E_x contains exactly two complex characters and these are conjugate to each other. For if x is not real then $x(G)$ is contained in a unique $C(\Theta)$. If r is such that $r\Theta r^{-1} = i$ then $y = r x r^{-1}$ is complex and belongs to E_x. Also $\bar{y} = k x k^{-1} \in E_x$. Suppose now that $z \in E_x$ is complex. Then $z = sys^{-1}$ for some $s = \alpha + k\beta$ (α, β complex) and $sys^{-1} = (\alpha + k\beta) y(\bar{\alpha} - k\beta) = |\alpha|^2 y + |\beta|^2 \bar{y} + k\bar{\alpha}\beta(y - \bar{y})$. But z is complex and $y \neq \bar{y}$. Therefore either α or β must be zero implying thereby that $z = y$ or $\bar{y}$.

For $x, y \in X$ their pointwise product (xy) need not belong to X in general because $|Q|$ is not commutative. In fact $xy \in X$ if and only if either x or y is real or $x(G)$ and $y(G)$ are contained in the same $C(\Theta)$.

Let now X be invested with the compact-open topology. The class of all subsets of X of the form $N_{K,U} = [x: x(K) \subseteq U]$, where K and U are compact and open subsets of G and $|Q|$ respectively, is a subbase for the topology of X. A net (x_α) of elements of X converges to $x \in X$ in this topology if and only (x_α) converges to x uniformly on every compact subset of G. The theorem below describes the structure of this topological space X.

THEOREM 6.1 X equipped with the compact-open topology is a locally compact Hausdorff space which is second countable if G is second countable. The map $[r,x] \rightarrow rxr^{-1}$ is continuous.

Every E_x is compact in X. If X is discrete then every $x \in X$ is real. If G is iscrete then X is compact and if G is compact then every E_x is closed-open in X.

Proof: If $x, y \in X$ and $x(g) \neq y(g)$ for some g in G, let U and V be disjoint neighbourhoods of $x(g)$ and $y(g)$ respectively. Then $N_{[g],U}$ and $N_{[g],V}$ are disjoint neighbourhoods of x and y respectively. X is therefore Hausdorff. To prove that X is locally compact it is now enough to prove that there exists a compact neighbourhood of every x in X. For this it is in turn enough to prove that if K is a compact neighbourhood of the identity $e \in G$ and U is a closed distinguished neighbourhood of $1 \in |Q|$ then $N_{K,U}$ is compact in X. But this may be achieved exactly as in the classical case (Weil,1938) by showing that $N_{K,U}$ is closed in X endowed with the topology of pointwise convergence and that on $N_{K,U}$ the compact-open topology coincides with the topology of pointwise convergence. We omit the details. That X is second countable if G is so is easy to see.

We shall now prove that the map $[r, x] \to r x r^{-1}$ from $|Q| \times X$ to X is continuous. Because $|Q|$ is compact it will follow that E_x is compact for all $x \in X$. Fix x

and r and let $N_{K,U}$ (U open) be a subbasic neighbhourbood of rxr^{-1}. Then $rx(K)r^{-1}$ is compact and is contained in U. There exists therefore an open set V such that $rx(K)r^{-1} \subseteq V \subseteq \bar{V} \subseteq U$. Let $V_1 = r^{-1} V r$, $U_1 = r^{-1} U r$. Then V_1 and U_1 are open and $x(K) \subseteq V_1 \subseteq \bar{V}_1 \subseteq U_1$. Choose a neighbourhood W of $1 \in |Q|$ such that $W \bar{V}_1 W^{-1} \subseteq U_1$. If now $y \in N_{K,V_1}$, $s \in rW$, then $sy(g)s^{-1} \in rWV_1W^{-1} r^{-1} \subseteq rU_1r^{-1} \subseteq U$ for all $g \in G$ so that $sys^{-1} \in W_{K,U}$. Since N_{K,V_1} and rW are neighbourhoods of x and r respectively, we may conclude that $[r, x] \to rxr^{-1}$ is continuous. As remarked earlier E_x is then compact.

Next, we prove that if $x \in X$ is non-real then there exists a net (x_α) in X which converges to x, but is such that $x_\alpha \neq x$ for any α. This will establish that if X is discrete then every $x \in X$ is real. Let $g_0 \in G$ be such that $x(g_0)$ is not real. If V is any neighbourhood of $1 \in |Q|$ then there exists $r_V \in V$ such that $r_V \not\leftrightarrow x(g_0)$, because if it were not true for some V then $x(g_0)$ would commute with every element of that V and hence (since $|Q|$ is connected) with every element of $|Q|$ which would imply that $x(g_0)$ is real, contrary to our assumption. Now (r_V), considered as a net in the obvious way, converges to 1 but

$r_V \neq 1$ for any V. Consequently the net $r_V\, x . r_V^{-1}$ converges to x but $r_V\, x\, r_V^{-1}\,(g_o) \neq x(g_o)$ for any V so that $r_V\, x\, r_V^{-1} \neq x$ for any V.

If G is discrete, taking $K = [e]$ and U to be any closed distinguished neighbourhood of 1, $X = N_{K,U}$ is seen to be compact.

Finally, let G be compact. Fix x in X. Then given any neighbourhood U of 1, using standard methods we may produce a neighbourhood N of x such that if $y \in N$, then $y(g)\, x\,(g)^{-1} \in U$ for all g. Suppose now that x is real and that U is a distinguished neighbourhood of 1. Then for $y \in N$, $z = yx^{-1} \in X$ so that $z(G) \subseteq U$ is a subgroup of $|Q|$. Hence $z(G) \subseteq U_n$ for all n. Since U is distinguished, it follows that $z \equiv 1$, i.e. that $y \equiv x$. This proves that $E_x = [x] = N$ is open, if x is real. If x is nonreal this proof fails because z need not belong to X. We have to approach this case rather carefully.

Let x be nonreal. Since E_x contains complex characters we may assume without loss of generality that x is complex. Again, to prove that E_x is open, it is enough to prove that x is in the interior of E_x, for if $N \subseteq E_x$

is any neighbourhood of x then for any $y = r x r^{-1} \in E_x$, $r N r^{-1}$ is a neighbourhood of y contained in E_x. In other words the theorem is completely proved if we prove that when x is nonreal and complex then there is a neighbourhood N of x such that $y \in N$ implies $y = r x r^{-1}$ for some $r \in |Q|$.

Let U be a distinguished neighbourhood of 1. Choose first a neighbourhood N_1 of x such that $y \in N_1$ implies $y(g)\ x(g)^{-1} \in U$ for all $g \in G$. We may assume without loss of generality that N_1 is of the form $N_1 = \bigcap_{s=1}^{n} N_{K_s, U_s}$ with $x(K_s) \subseteq U_s$, K_s compact, U_s open for all s and $G = \bigcup_{s=1}^{n} K_s$. Choose now open sets V_s such that $x(K_s) \subseteq V_s \subseteq \bar{V}_s \subseteq U_s$ for all s, and a neighbourhood W of 1 such that $W \bar{V}_s W^{-1} \subseteq U_s$ for all s. Let $N_2 = \bigcap_{s=1}^{n} N_{K_s, V_s}$. Then N_2 is a neighbourhood of x such that if $y \in N_2$ and $r \in W$ then $r y r^{-1} \in N_1$. Suppose now we can exhibit another neighbourhood N_3 of x which has the property that for $y \in N_3$, there exists $r \in W$ such that $r y r^{-1}$ is complex. We claim then that the theorem is proved. The reasoning is as follows: Let $N = N_2 \cap N_3$. Then N is a neighbourhood of x. Let $y \in N$. Because $y \in N_3$, there exists $r \in W$ such

such that ryr^{-1} is complex. Because $y \in N_2$ and $r \in W$, $ryr^{-1} \in N_1$. Therefore $ry(g)r^{-1}x(g)^{-1} \in U$ for all $g \in G$. But ryr^{-1} and x, being both complex, commute. Hence $z = ryr^{-1}x^{-1} \in X$. But then the range of z is a subgroup contained in a distinguished neighbourhood so that $z \equiv 1$. Therefore $y = r^{-1}xr$.

We now show how to get hold of N_3.

Since W is a neighbourhood of 1, there exists ∂, $0 < \partial < 1$, such that if $|r-1|^2 < \partial$ then $r \in W$. Since x is nonreal, there exists g_0 in G such that $x(g_0) = c+id$ with $d \neq 0$. By replacing g_0 by g_0^{-1} if necessary we may assume that $d > 0$. Let $\epsilon = \partial d/16$ and let N_3 be a neighbourhood of x such that $y \in N_3$ implies $|y(g)x(g)^{-1}-1| = |y(g) - x(g)| < \epsilon^2$. Such a neighbourhood always exists as we observed earlier in the course of this proof. We shall prove that N_3 has the required property.

Let $y \in N_3$. Write $y(g_0)$ in the form $a+b\theta$ with a, b real and $\theta = \theta_1 i + \theta_2 j + \theta_3 k$ a unit imaginary. By replacing b by $-b$ if necessary we may assume that $\theta_1 \geq 0$. At this stage for all we know b may be 0 and θ arbitrary but we shall soon see that it is not the case.

Now $|y(g_o) - x(g_o)| < \varepsilon^2$

$$\Rightarrow \quad |(a+b\theta) - (c+id)|^2 < \varepsilon^4$$

$$\Rightarrow \quad (a-c)^2 + (b\theta_1 - d)^2 + b^2(\theta_2^2 + \theta_3^2) < \varepsilon^4 \qquad \text{(A)}$$

$$\Rightarrow \quad (a-c)^2 < \varepsilon^4$$

$$\Rightarrow \quad |a-c| < \varepsilon^2$$

$$\Rightarrow \quad |a^2 - c^2| = |a-c|\cdot|a+c| < 2|a-c| < 2\varepsilon^2$$

$$\Rightarrow \quad |b^2 - d^2| = |(1-a^2) - (1-c^2)| = |a^2-c^2| < 2\varepsilon^2$$

$$\Rightarrow \quad d^2 - b^2 < 2\varepsilon^2$$

$$\Rightarrow \quad b^2 > d^2 - 2\varepsilon^2 > d^2 - \frac{d^2\,\partial^2}{128} > d^2/2 \,.$$

Consequently

$$\frac{d^2}{2}(\theta_2^2 + \theta_3^2) < b^2(\theta_2^2 + \theta_3^2) < \varepsilon^4 < \varepsilon^2 \qquad \text{(by (A))}$$

$$\Rightarrow \quad \theta_2^2 + \theta_3^2 < \frac{2\varepsilon^2}{d^2} = \frac{\partial^2}{128} < \partial^2/64 \quad \Rightarrow |\theta_2|,\ |\theta_3| < \partial/8.$$

Again $\theta_1 \geq \theta_1^2 = 1 - \theta_2^2 - \theta_3^2 > 1 - \frac{\partial^2}{128}$ ... (B)

$$\Rightarrow \quad |\theta - i|^2 = 2(1-\theta_1) < \partial^2/64$$

$$\Rightarrow \quad |\theta - i| < \partial/8 \,.$$

Choose now $r = r_o + r_1 i + r_2 j + r_3 k$ such that $r\theta r^{-1} = i$.

By replacing r by $-r$ if necessery we may take r_o to be non-negative. Again, by replacing r by αr where $\alpha = (-r_o + r_1 i)/\sqrt{r_o^2 + r_1^2}$ if necessary, we may assume that $r_1 = 0$. Then

$$|\theta - i| < \partial/8$$

$$\Rightarrow \quad |r^{-1} ir - i|^2 = |ir - ri|^2 = 4(r_2^2 + r_3^2) < \partial^2/64$$

$$\Rightarrow \quad r_2^2 + r_3^2 < \partial^2/256$$

$$\Rightarrow r_o \geq r_o^2 = 1 - r_2^2 - r_3^2 > 1 - \partial^2/256 > 1 - \partial^2/128.$$

This, together with (B) $\Rightarrow |1 - \theta_1 r_o| < \partial/4$.

Therefore $|r-1|^2 = |r\theta r^{-1} - \theta r^{-1}|^2 = |i - \theta r^{-1}|^2$

$$= 2[(1-\theta_1 r_o) + \theta_2 r_3 - \theta_3 r_2]$$

$$< 2[|1-\theta_1 r_o| + |\theta_2||r_3| + |\theta_3||r_2|]$$

$$< \partial/2 + \partial^2/4 < \partial.$$

This implies that $r \in W$, by our choice of ∂.

If we now observe that $y(G) \subseteq C(\theta)'$, then we have proved that there exists $r \in W$ such that ryr^{-1} is complex. As was pointed out earlier, this proves the theorem completely.

We now turn our attention to the problem of expressing every unitary Q-representation of G as an integral of irreducible ones. We consider only the case when G is <u>second countable</u>. The basic idea behind our approach is the same as that explained in Chapter V of Part I for the case of the real line.

Let us first get a few technicalities out of the way. Let Y be the character group of G. The relative topology of Y considered as a subset of X coincides with the usual topology for Y. Let $\underline{B}$ be the smallest σ-algebra of subsets of Y containing the open sets of Y. We shall refer to sets in $\underline{B}$ as Borel sets of Y. Let Y_o be the set of all real characters of G. Y_o is a closed subgroup of Y. A <u>Borel section</u> of Y is a Borel set A of Y such that i) Y_o is contained in A and ii) if $y \notin Y_o$ then exactly one of y and $\bar{y}$ belongs to A. That Borel sections exist may be seen in the following manner:

Introduce an equivalence relation in Y by defining y and z to be equivalent if and only if $y = z$ or $\bar{z}$. This relation is only the restriction to Y of the equivalence relation on X we have been dealing with so far. Let Z denote the resulting space of equivalence classes of Y and let β be the canonical map from Y to Z. If Z is given the quotient topology then β is continuous and open so that Z, along with Y, is a locally compact second countable Hausdorff space. Since Y is σ-compact and β is continuous we may apply the Federer-Morse lemma (see Varadarajan, 1968) to get hold of Borel cross-sections for β. But these are precisely the Borel sections of Y.

Let $\underline{C}$ be the smallest σ-algebra of subsets of Z containing all the open sets of Z. Then $(Y, \underline{B})$ and $(Z, \underline{C})$ are standard Borel spaces in the sense of Mackey (Mackey, 1957). If A is any Borel cross-section for β then β is a one-one Borel map from the standard Borel space A to the separable Borel space Z so that by Lusin's theorem (Kuratowski, 1948) β is a Borel isomorphism between A and Z. (It follows that $\underline{C}$ coincides with the quotient Borel structure on Z). Consequently if A_1 and A_2 are two Borel sections of Y then the natural one-one correspondence between A_1 and A_2 given by $y \rightarrow \beta_2^{-1}\beta_1(y)$, $y \in A_1$, is a Borel isomorphism between A_1 and A_2.

Suppose now that $\underline{H}$ is a Q-Hilbert space. Let (E, J) be a spectral system based on $(Y, \underline{B})$ and acting in $\underline{H}$. If for every $g \in G$ we define the operator $U(g)$ on $\underline{H}$ by $U(g) = \int y(g)\, dE(y)$ with respect to J, then it is easy to prove, using our results of Chapter IV. of Part I, that $g \rightarrow U(g)$ is a unitary Q-representation of G in $\underline{H}$. Conversely if U is any unitary Q-representation of G in $\underline{H}$ we may show without difficulty that it must be an integral with respect to some spectral system based on $(Y, \underline{B})$ (cf. Emch, 1963). The reasoning is as follows:

Since $[U(g)]$ is an abelian family of unitary operators there exists a half-space P of $\underline{H}$ which reduces U^s. This follows readily from our results on CW*A's obtained in Part I. Let V be the restriction of U^s to P. Then V is a unitary C-representation of G in P and therefore there exists a spectral measure F on Y acting in P such that $V(g) = \int y(g)\, dF(y)$. If now we define (E, J) by $E(M) = F(M) + KF(M)K^{-1}$ and $J^s = iF(Y) - iKF(Y)K^{-1}$ then (E, J) is a spectral system based on Y acting in $\underline{H}$ and $U(g) = \int y(g)\, dE(y)$ with respect to J. (The details are easy to check and are omitted).

We have thus shown that every unitary Q-representation of G may be expressed as an integral with respect to some spectral system based on the characters of G. This spectral system need not be unique. In fact, the consideration of a few finite-dimensional examples will show that the spectral system can vary erratically. However if we demand that the spectral systems be concentrated on Borel sections of Y (a demand which is consistent with our approach to the problem) then we can say quite a lot about them as proved by Theorem 6.2 below. Notice that Theorem 5.1 of Part I is a particular case of this theorem.

<u>THEOREM 6.2</u> Let U be a unitary Q-representation of G in $\underline{H}$. Then, given any Borel section A of Y there exists a unique spectral system (E, J) based on A and acting in $\underline{H}$ with $I - JJ^* = E(Y_o)$ such that ~~U(g)~~ $U(g) = \int y(g)\, dE(y)$ with respect to J. The spectral measure E is defined independently of the section A in the sense that given any two sections the natural Borel isomorphism between them preserves E.

Proof: Consider U^s on $\underline{H}^s$. Let E_s be the unique spectral measure on Y such that $U(g)^s = \int y(g) dE_s(y)$. It is easy to check, as in the proof of Theorem 5.1 of Part I, that $E_s(M^{-1}) = KE_s(M)K^{-1}$ for all M in $\underline{B}$, so that, if N is a Borel set in Z, then E: N -> E(N) defined by $E(N) = E_s(\beta^{-1}(N))$ is a spectral measure on Z.

Let now A be any Borel section of Y. Transfer E from Z to A using β. Define J by $J^s = iE_s(A - Y_0) - iE_s(Y - A)$. We may then prove that $U(g) = \int y(g) dE(y)$ with respect to J and that (E, J) is unique exactly as in Theorem 5.1 of Part I. The details are omitted.

The very definition of E shows that it is defined independently of the section A. The theorem is therefore proved.

VII. QUATERNIONIC QUANTUM MECHANICS

In this chapter we study quantum mechanical systems by assuming that the logic of propositions of such a system is represented by the lattice of subspaces of a quaternionic Hilbert space. Our discussion leads us to the conclusion that, roughly speaking, there are exactly as many elementary particles in quaternionic quantum mechanics as in complex quantum mechanics.

The approach we adopt is that of Mackey, as enunciated by Varadarajan (1968).

Let $\underline{H}$ denote a separable infinite-dimensional quaternionic Hilbert space and $\underline{L}$ the lattice of projections of $\underline{H}$. We shall assume that $\underline{L}$ represents the logic of propositions of the physical system we wish to study.

A <u>state</u> of $\underline{L}$ is a real-valued function p on $\underline{L}$ such that (i) $0 \leq p(P) \leq 1$ for all $P \in \underline{L}$ (ii) $p(0) = 0$, $p(I) = 1$ and (iii) $p(\sum_n P_n) = \sum_n p(P_n)$ for any sequence (P_n) of mutually orthogonal projections. $\underline{S}$ is the set of all states of $\underline{L}$. $\underline{S}$ is a σ-convex set. An automorphism of $\underline{S}$ is a bijection on $\underline{S}$ which preserves σ-convexity. The set of all automorphisms, denoted by Aut $(\underline{S})$, of $\underline{S}$ is a group in a natural way.

Let G be a locally compact second countable group. A representation of G in Aut ($\underline{S}$) is a homomorphism $g \to D_g$ of G into Aut ($\underline{S}$) such that for each $P \in \underline{L}$ and $p \in \underline{S}$, the map $g \to (D_g p)(P)$ is Borel on G with respect to the smallest σ-algebra of subsets of G containing all open sets. Representations of G in Aut ($\underline{S}$) are the basic objects of our study.

We shall now describe $\underline{S}$ and Aut ($\underline{S}$) in terms of operators on $\underline{H}$. Let $\underline{W}$ denote the set of all bounded, Hermitian, non-negative operators on $\underline{H}$ of trace one. $\underline{W}$ is a σ-convex set. For every $A \in \underline{W}$ the map $p_A : P \to tr\,(PA)$, $P \in \underline{L}$, is well-defined and is a state on $\underline{L}$. Gleason's theorem asserts that the converse is true: every state of $\underline{L}$ is uniquely of this form. $A \to p_A$ is a σ-convex isomorphism between $\underline{W}$ and and $\underline{S}$.

Let $\underline{U}$ denote the set of all unitary operators of $\underline{H}$. For every $U \in \underline{U}$, $A \to UAU^{-1}$ defines a σ-convex bijection on $\underline{W}$ and hence an automorphism of $\underline{S}$. Conversely, given any $D \in$ Aut ($\underline{S}$) there exists $U \in \underline{U}$, determined uniquely upto sign, such that $Dp_A = p_{UAU^{-1}}$. Consequently Aut ($\underline{S}$) is isomorphic to the group $\underline{P}$ which is the quotient of $\underline{U}$ modulo the doubleton group $[\pm I]$. For the proofs of all these propositions the reader is referred to Varadarajan (1968).

We shall now show that representations of G in Aut ($\underline{S}$) may be replaced by continuous homomorphisms of G into $\underline{P}$. We have first to study $\underline{U}$ and $\underline{P}$ in some detail.

$\underline{U}$, equipped with the strong topology ($\equiv$ the weak topology), becomes a second countable topological group and the associated Borel structure of $\underline{U}$ makes $\underline{U}$ a standard Borel space. Let π denote the canonical hommorphism from $\underline{U}$ to $\underline{P}$. If $\underline{P}$ is endowed with the quotient topology then π is an open continuous map so that $\underline{P}$ too becomes a second-countable topological group. The functions $U \to |(Uf,g)|$ for $f, g \in \underline{H}$ are continuous functions on $\underline{U}$ which are constant on the $\underline{P}$ cosets. They may therefore be lifted to continuous functions on $\underline{P}$. We claim that the topology of $\underline{P}$ is the smallest which makes all these functions continuous.

To see this let $U_o, U_1, U_2, \ldots$ be a sequence in $\underline{U}$ such that $|(U_n f, g)| \to |(U_o f, g)|$ as $n \to +\infty$ for every f, $g \in \underline{H}$. We shall prove that $\pi(U_n) \to \pi(U_o)$. By replacing U_n by $U_n U_o^{-1}$ we may assume that $U_o = I$. Now $\|U_n\| \leq 1$ for all n so that, by the weak-compactness of the set of all operators of norm at most 1, there is an operator U and a subsequence (U_{n_k}) of (U_n) such that, for every $f, g \in \underline{H}$, $(U_{n_k} f, g) \to (Uf, g)$. But then $|(Uf,g)| = |(f,g)|$ for all f, g. In particular Uf

is orthogonal to every element orthogonal to f. Hence for every f, there is associated a quaternion, say q_f such that $Uf = q_f f$. If f and g are orthogonal then $U(f+g) = Uf + Ug$ implies $q_{f+g} = q_f = q_g$. Therefore if (f_n) is an orthonormal basis for $\underline{H}$ then there is a q such that $Uf_n = qf_n$ for all n. Applying this observation to the basis obtained from (f_n) by changing f_1, to pf_1, where p is any unit quaternion, we see that $pqp^{-1} = q$. Consequently q is real. But $|(Uf, g)| = |(f,g)|$ for all f,g. Hence $q = \pm 1$ or $U = \pm I$. Since each U_n is unitary $U_{n_k} \to U$ in $\underline{U}$. We have thus proved that (U_n) has limit points in $\underline{U}$ and also that the only possible limit points are $\pm I$. Choose now a neighbourhood N of I in $\underline{U}$ such that $-N = [-U: U \in N]$ is disjoint with N. Define a sequence ρ_n of numbers by $\rho_n = -1$ if $U_n \in -N$, $\rho_n = +1$ otherwise. Then it is easy to see that $\rho_n U_n \to I$ in $\underline{U}$ and hence that $\pi(U_n) = \pi(\rho_n U_n) \to \pi(I)$ in $\underline{P}$.

Next, we show how to obtain sections for π. For every $f,g \in \underline{H}$, let $E_{f,g} = [U: \mathrm{Re}(Uf, g) > 0]$. $E_{f,g}$ is clearly an open set in $\underline{U}$ (which is nonempty if both f,g are non-zero) meeting each $\underline{P}$ coset of $\underline{U}$ atmost once. If $f = g$, then $E_{f,f}$ is a neighbourhood of the identity $I \in \underline{U}$ and π

restricted to $E_{f,f}$ is a homeomorphism. Choose now a countable dense set D in $\underline{H}$ and look at the collection $E_{f,g}$ with $f, g \in \underline{H}$. Re-enumerate them in the form $E_1, E_2, \ldots$ with $E_1 = E_{f,f}$ for some non-zero $f \in D$. Define a sequence F_n by

$$F_1 = E_1$$

$$F_n = E_n \cap (E_1 \cup (-E_1) \cup E_2 \cup (-E_2) \ldots \cup E_{n-1} \cup (-E_{n-1}))^c$$

$$\text{for } n \geq 2.$$

Let $F = \bigcup_n F_n$. Then F is a Borel set of $\underline{U}$. F meets each π atom exactly once - For if neither U nor $-U$ belongs to F for some $U \in \underline{U}$, then neither U nor $-U$ belongs to $\bigcup_n E_n$. But then $\text{Re}(Uf,g) = 0$ for all $f,g \in D$ implying thereby that $\text{Re}(Uf,g)$ and hence (Uf,g) is 0 for all $f,g \in \underline{H}$ so that $U = 0$ and hence is not unitary.

π is therefore one-one on F and maps F onto $\underline{P}$. Since F, with the relativised Borel structure is standard and the Borel structure of $\underline{P}$ is separable π is a Borel isomorphism between F and $\underline{P}$. In particular $\underline{P}$ is a standard Borel space.

Let c be the inverse of π on F. Then c maps $\underline{P}$ onto $\underline{F}$ and has the following properties:

i) $c: u \to c(u)$ is Borel

ii) $\pi(c(u)) = u$ for all $u \in \underline{P}$.

iii) $c(\pi(I)) = I$.

iv) There is a neighbourhood N of $\pi(I)$ such that for $u, v \in N$ $c(uv) = c(u)\, c(v)$ and such that c is continuous on N.

In what follows by a section for π we shall mean a map c from $\underline{P}$ into $\underline{U}$ satisfying (i) - (iv) above.

THEOREM 7.1 If G is any locally compact second countable group then representations of G in Aut ($\underline{S}$) correspond in a one-one fashion to Borel (and therefore continuous) homomorphisms of G into $\underline{P}$.

Proof: First, let $g \to D_g$ be a representation of G in Aut ($\underline{S}$) and let $g \to u_g$ be the induced homomorphism of G into $\underline{P}$. For g choose $U_g \in \underline{U}$ such that $\pi(U_g) = u_g$. To prove that $g \to u_g$ is Borel, it is enough to prove that for every two unit vectors $x, y \in \underline{H}$; $g \to |(U_g x, y)|$ is Borel. Write A and P for the projections in $\underline{H}$ onto the rays spanned by x and y respectively. We may take A to belong to $\underline{W}$ and P to $\underline{L}$. Then

$$D_g p_A(P) = p_{U_g A U_g^{-1}}(P) = |(U_g x, y)|^2 .$$

Since $g \to D_g$ is a representation of G in Aut ($\underline{S}$) it follows that $|(U_g x, y)|^2$ and hence $|(U_g x, y)|$ is Borel on G. As observed earlier this proves that $g \to u_g$ is Borel.

Conversely, let $g \to u_g$ be a Borel homomorphism of G into $\underline{P}$. Choose a section c for π and **define** $U_g = c(u_g)$ for all g. $g \to U_g$ is then a Borel map of G into $\underline{U}$. Define, for every $g \in G$, D_g in Aut ($\underline{S}$) by $D_g p_A = p_{U_g A U_g^{-1}}$. $g \to D_g$ is clearly a homomorphism of G in Aut ($\underline{S}$) which induces $g \to u_g$ and is in fact a representation of G in Aut ($\underline{S}$) since for any $p = p_A \in \underline{S}$ and $P \in \underline{L}$,

$$D_g p_A(P) = p_{U_g A U_g^{-1}}(P) = \mathrm{tr}\,(P U_g A U_g^{-1})$$

is Borel because $g \to U_g$ is Borel. This proves the theorem.

Because of this theorem the objects of our study are Borel (continuous) homomorphisms of G in $\underline{P}$. For convenience we shall call them representations of G in $\underline{P}$. Every unitary representation $g \to U_g$ of G induces a representation $g \to \pi(U_g)$ of G in $\underline{P}$. If G is connected then distinct unitary representations induce distinct representations in $\underline{P}$. This is because if (U_g) and (V_g) are two unitary representations such that $\pi(U_g) = \pi(V_g)$ for all g then $V_g = a(g)U_g$ for a function $a(g)$ taking values ± 1. Since (U_g) and (V_g) are continuous, so is $a(g)$. But $a(e) = 1$ and G is connected so that $a(g) \equiv 1$.

<u>THEOREM 7.2</u> If G is simply connected, then every representation of G in $\underline{P}$ is induced by a unique unitary representation of G.

<u>Proof</u>: Let $g \to u_g$ be a representation of G in $\underline{P}$. Choose a section c for π and let N be a neighbourhood of $\pi(I)$ on which c is continuous and has the property that for $u, v \in N$, $c(uv) = c(u)c(v)$. Define U_g, for all g for which $u_g \in N$, by $U_g = c(u_g)$. Then $g \to U_g$ is a continuous local homomorphism. Because G is simply connected, this may be extended to a unitary representation say (U_g) again. (U_g) induces u_g since $\pi(U_g) = u_g$ in a neighbourhood of the identity in G and hence (because G is connected) everywhere. The uniqueness of (U_g) is a consequence of the connectedness of G, as observed earlier.

The theorem is therefore proved.

From the two theorems above it follows immediately that the dynamical group of a physical system (which is a representation of the real line R in Aut ($\underline{S}$)) whose logic of propositions is identified with $\underline{L}$ is of the form

$$D_t\, p_A \to p_{U_t A U_t^{-1}}$$

for a unique unitary representation (U_t) of R. To (U_t) we may apply Stone's theorem which we have obtained in Chapter 5 of Part I. According to this theorem there is a unique spectral system (E, J) on the non-negative real numbers such that $I - JJ^* = E[0]$ and $U_t = \int e^{i\lambda t}\, dE(\lambda)$ with respect to J. If we write H for the unbounded self-adjoint operator on the Q-Hilbert space $\underline{H}$ defined by the spectral measure then we may write the above equation in the form $U_t = e^{JHt}$. The following theorem is immediate.

<u>THEOREM 7.3</u> Let $\underline{H}$ be an infinite-dimensional separable Q-Hilbert space. If (H, J) is any pair of operators on $\underline{H}$ such that H is self-adjoint and non-negative (possibly unbounded), J is imaginary and $Jx = 0$ if and only if $Hx = 0$, then there is a unique dynamical group D_t of $\underline{S}$ such that for each t, D_t transforms the state corresponding to $A \in \underline{W}$ to the state corresponding to $e^{JHt} A e^{-JHt}$. Conversely every dynamical group of $\underline{S}$ is induced in the above fashion for a unique pair of operators (H, J).

We now go on to a discussion of representations in $\underline{P}$ of the relativity group G - the connected component of the complete inhomogeneous Lorentz group. Since G is not simply connected we can not describe these in terms of unitary representations of G, but in the particular case which interests us we may show that they are induced atleast by the unitary representations of G^*, the covering group of G.

We shall call a representation $g \to u_g$ of G in $\underline{P}$ <u>elementary</u> if the collection of operators $\pi^{-1}[u_g]$ is irreducible. Two elementary representations (u_g) and (v_g) are <u>equivalent</u> if there exists a unitary operator M on $\underline{H}$ such that for any choice of (U_g) and (V_g) such that $\pi(U_g) = u_g$ and $\pi(V_g) = v_g$, $\pi(MU_gM^{-1}) = \pi(V_g)$. <u>An elementary particle in its free state</u> or a <u>free particle</u> is an equivalence class of elementary representations of G.

Let G^* denote the covering group of G and ρ the covering homomorphism. The kernel of ρ is a doubleton. We shall write it as $[\pm e^*]$ where e^* is the identity of G^*.

<u>THEOREM 7.4</u> Let G denote the connected component of the inhomogeneous Lorentz group and G^* its covering group. Then there is a one-one correspondence between elementary representations of G and irreducible unitary representations

of G^* which preserves equivalence.

Proof: Let $g^* \to U_{g^*}$ be an irreducible unitary representation of G^*. Then $U^2_{-e^*} = I$ so that $U_{-e^*} = P - Q$ for two mutually orthogonal projections P and Q whose sum is I. But U_{-e^*} commutes with every U_{g^*} and (U_{g^*}) is irreducible. Consequently either P or Q is 0 and $U_{-e^*} = \pm I$. This implies that $g^* \to u_{g^*} = \pi(U_{g^*})$ is constant on the atoms of ϱ so that there exists a homomorphism $g \to u_g$ of G in $\underline{P}$ such that $u_{g^*} = u_{\varrho(g^*)}$ $\cdot(u_g)$ is even continuous, because ϱ is open and continuous, so that it is a representation of G in $\underline{P}$. Since (U_{g^*}) is irreducible (u_{g^*}) and hence (u_g) is elementary.

Conversely let $g \to u_g$ be an elementary representation of G. Then $g^* \to u_{\varrho(g^*)}$ is an elementary representation of G^*. Since G^* is simply connected, by Theorem 7.2, there is a unitary representation (U_{g^*}) of G^* which induces (u_{g^*}) and hence (u_g), as in the previous paragraph. (U_{g^*}) is clearly irreducible.

Coming now to the problem of equivalence, it is easy to see that if (U_{g^*}) and (V_{g^*}) are equivalent irreducible unitary representations of G^*, then the induced elementary representations of G are equivalent. Conversely, let (u_g) and (v_g)

be two equivalent elementary representations of G. Then $g^* \to u_{\varrho(g^*)}$ and $g^* \to v_{\varrho(g^*)}$ are equivalent elementary representations of G^*. Let (U_{g^*}) and (V_{g^*}) be their corresponding unitary representations. Then there is a unitary operator M such that $\pi(V_{g^*}) = \pi(MU_{g^*} M^{-1})$ for all g^* so that there is a function $a(g^*)$ taking only the values ± 1 such that $V_{g^*} = a(g^*) MU_{g^*} M^{-1}$. But then $a(g^*)$ is continuous and $a(e^*) = 1$. Since G^* is connected, $a(g^*) \equiv 1$ or $V_{g^*} = MU_{g^*}M^{-1}$ for all g^* so that (V_{g^*}) and (U_{g^*}) are equivalent.

The theorem is therefore proved.

Because of this theorem the free particles in quaternionic quantum mechanics may be identified with equivalence classes of irreducible unitary Q-representations of G^*. To obtain the free particles, therefore, we have to write down a maximal family of mutually inequivalent irreducible unitary Q-representations of G^*. Since it so happens that the free particles having physical significance all correspond to Q-representations of class $\mathcal{C}$, we shall in fact write down only these latter. By our results of Chapter IV, to achieve this we have only to get hold of a maximal family of mutually physically inequivalent (irreducible) nonreal C-representations of G^*. But this has been done by Wigner (Theorem 12.4 of Varadarajan, 1968). So there is nothing else for us to do.

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