CONTRIBUTIONS TO MEASURE THEORY

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A thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirements for the degree of Doctor of Philosophy

Calcutta

1971

ACKNOWLEDGEMENTS

I am grateful to Professor C. R. Rao, F. R. S., Directo of the Research and Training School of the Indian Statistical Institute for extending to me all the facilities to carry out the research work.

I am grateful to Professor Ashok Maitra for supervising this work. I sincerely thank Dr. Ashok Maitra for going thro the thesis meticulously and offering valuable suggestions for its improvement. The present form of the thesis owes a lot t him.

I am grateful to Professors Jayanta Kumar Ghosh and Mahendra Ganapathy Nadkarni for lending their patient ear to me through my various expeditions and for offering valuable clues at critical junctures.

No scale on this earth can gauge my indebtedness to Mr. K. P. S. Bhaskara Rao. I cannot forget his geniality in drawing me out of my shell during moments of helplessness and actively engaging me in stimulating discussions. I truly learned the meaning of the word 'tenacity' from him. I am thankful to him for giving permission to me to include some results we jointly obtained in this thesis.

I take this opportunity to thank all my colleagues with whom I had useful discussions.

Finally, it is a pleasure to thank Mr. Gour Mohon Das for his neat, careful and quick typing of this thesis.

CONTENTS

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INTRODUCTION

This thesis is devoted to a study of measures with emphasis on nonatomic measures. We briefly describe here the work carried out in various chapters.

In Chapter 1, we study various aspects of nonatomic measures based on some characterisations obtained early in the Chapter.

In Chapter 2, the problem - when a mixture of nonatomic measures is nonatomic - is examined. An example and five sufficient conditions are given.

In Chapter 3, we examine when a mixture of invariant non-ergodic measures is non-ergodic. An example and three sufficient conditions are given.

In Chapter 4, we study certain borel structures which do not admit nonatomic measures.

In Chapter 5, we give necessary and sufficient conditions for an algebra to admit a nonatomic charge.

In Chapter 6, we characterise extreme points of the set of all probability measures on a product borel structure with prescribed marginal probabilities.

Some of the results of the Thesis are submitted to various journals for publication. See [3], [4], [5], [6] and [7].

CHAPTER 1

NONATOMIC MEASURES - SOME CHARACTERISATIONS

1. Introduction: The following are some of the definitions and notation used in this thesis.

i) A borel structure is a pair $(X, \underline{\underline{A}})$ where X is a set and $\underline{\underline{A}}$ is a σ-algebra of subsets of X, i.e., $\underline{\underline{A}}$ is closed under complementation, countable unions and contains empty set $\emptyset$.

ii) A σ-algebra $\underline{\underline{A}}$ is said to separable if it has a countable generator, i.e., there exists a sequence of sets $A_1, A_2, \ldots$ in $\underline{\underline{A}}$ such that the smallest σ-algebra containing $A_1, A_2, \ldots$ is $\underline{\underline{A}}$.

iii) A set A in a σ-algebra $\underline{\underline{A}}$ is said to be an atom of $\underline{\underline{A}}$ if (a) $A \neq \emptyset$ and (b) $B \in \underline{\underline{A}}$, $B \subset A$ implies $B = \emptyset$ or $B = A$.

iv) A σ-algebra $\underline{\underline{A}}$ on X is said to be atomic if X is the union of all atoms of $\underline{\underline{A}}$.

(v) A function μ defined on a σ-algebra $\underline{\underline{A}}$ is said to be a measure on $\underline{\underline{A}}$ if (a) $\mu(\emptyset) = 0$, (b) μ is non-negative, (c) $A_1, A_2, \ldots \in \underline{\underline{A}}$, $A_i \cap A_j = \emptyset$ for $i \neq j$ implies $\mu(\bigcup_{i \geq 1} A_i) = \sum_{i \geq 1} \mu(A_i)$ and $\underline{\mu(X) < \infty}$.

(vi) A measure μ on a σ-algebra $\underline{\underline{A}}$ is said to be two-valued if there exists a real number $\alpha > 0$ such that $\mu(A) = 0$ or α for every A in $\underline{\underline{A}}$ and $\mu(X) = \alpha$.

(vii) A measure μ on an atomic σ-algebra $\underline{\underline{A}}$ is said to be continuous if $\mu(A) = 0$ for every atom A of $\underline{\underline{A}}$.

(viii) Let μ be a measure defined on a σ-algebra $\underline{\underline{A}}$. A set A in $\underline{\underline{A}}$ is said to be a μ-atom if (a) $\mu(A) > 0$ and (b) $B \in \underline{\underline{A}}$, $B \subset A$ implies $\mu(B) = 0$ or $\mu(A)$.

(ix) A measure μ on a σ-algebra $\underline{\underline{A}}$ is said to be nonatomic if there are no μ-atoms in $\underline{\underline{A}}$. Equivalently, $A \in \underline{\underline{A}}$, $\mu(A) > 0$ implies there exists $B \in \underline{\underline{A}}$, $B \subset A$ such that $0 < \mu(B) < \mu(A)$.

(x) A measure μ on a σ-algebra $\underline{\underline{A}}$ is said to have Darboux property if the range of μ is an interval, i.e., $\{\mu(A) : A \in \underline{\underline{A}}\} = [0, \mu(X)]$.

(xi) Let $(X, \underline{\underline{A}})$ and $(Y, \underline{\underline{B}})$ be two borel structures. The product borel structure $(X \times Y, \underline{\underline{A}} \times \underline{\underline{B}})$ is defined to be the product space $X \times Y$ and the product σ-algebra $\underline{\underline{A}} \times \underline{\underline{B}}$ generated by the family $\{A \times B: A \in \underline{\underline{A}}, B \in \underline{\underline{B}}\}$. If μ and η are two measures defined on $\underline{\underline{A}}$ and $\underline{\underline{B}}$ respectively, the product measure on $\underline{\underline{A}} \times \underline{\underline{B}}$ is denoted by $\mu \times \eta$.

(xii) If λ is a measure on a product σ-algebra $\underline{\underline{A}} \times \underline{\underline{B}}$, the marginals of λ on $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are denoted by λ_1 and λ_2 respectively and are defined by the following equations

$$\lambda_1 (A) = \lambda (A \times Y) \quad \text{for } A \text{ in } \underline{\underline{A}}$$
$$\lambda_2 (B) = \lambda (X \times B) \quad \text{for } B \text{ in } \underline{\underline{B}}.$$

(xiii) Let μ and η be two measures on a σ-algebra $\underline{\underline{A}}$. μ is said to be absolutely continuous with respect to η (in notation, $\mu << \eta$) if $A \in \underline{\underline{A}}$, $\eta(A) = 0$ implies $\mu(A) = 0$.

xiv) Let μ and η be two measures on a σ-algebra. μ and η are said to be equivalent (in notation, $\mu \equiv \eta$) if $\mu << \eta$ and $\eta << \mu$.

xv) If $\underline{\underline{D}}$ is any collection of subsets of a set X, $\sigma(\underline{\underline{D}})$ denotes the smallest σ-algebra on X containing $\underline{\underline{D}}$.

FOR ANY OTHER UNEXPLAINED TERMINOLOGY USED IN THE SEQUEL, REFER [11, Halmos] or [1, Berberian] or [22, Neveu].

Some characterisations of nonatomic measures are given in Section 3 of this chapter.

2. Some known results: Here, we record some well known results which we use in the sequel. If the proofs of these results are simple, we write them down, otherwise we give references.

Proposition 1.2.1: Every separable σ-algebra is atomic.

Proof: Let $A_1, A_2, \ldots$ be a generator for the σ-algebra $\underline{\underline{A}}$ on X. For a subset $A \subset X$, we denote $A = A^0$, $A^c = X - A = A^1$. The non-empty sets among the collection

$\left\{ A_1^{i_1} \cap A_2^{i_2} \cap A_3^{i_3} \cap \ldots : \; i_1, i_2, \ldots \right.$ is a sequence of 0's and 1's $\left.\right\}$ are the atoms of $\underline{\underline{A}}$ and their union is X.

Proposition 1.2.2: There is no two-valued continuous measure on a separable σ-algebra.

Proof: Suppose μ is a $0-\alpha$ $(\alpha > 0)$ valued continuous measure on a separable σ-algebra $\underline{\underline{A}}$. Let $A_1, A_2, \ldots$ be a generator for $\underline{\underline{A}}$. For any natural number n, there exists $i_n = 0$ or 1 such that $\mu(A^{i_n}) = \alpha$. $A_1^{i_1} \cap A_2^{i_2} \cap \ldots$ is an atom of $\underline{\underline{A}}$ and $\mu(A_1^{i_1} \cap A_2^{i_2} \cap \ldots) = \alpha$, giving a contradiction to the continuity of the measure μ.

Corollary 1.2.3: A measure μ on a separable σ-algebra $\underline{\underline{A}}$ is nonatomic if and only if μ is continuous.

Proof: If μ is nonatomic, then it is obvious that μ is continuous. Suppose μ is continuous and not nonatomic. Let A be a μ-atom of $\underline{\underline{A}}$. Consider the borel structure $(A, A \cap \underline{\underline{A}})$, where $A \cap \underline{\underline{A}}$ is the trace of the σ-algebra $\underline{\underline{A}}$ on A, i.e. $A \cap \underline{\underline{A}} = \{A \cap B : B \in \underline{\underline{A}}\}$, with the measure η defined by $\eta = \mu / \Lambda \cap \underline{\underline{A}}$, the restriction of μ to $\Lambda \cap \underline{\underline{A}}$. But $\Lambda \cap \underline{\underline{A}}$ is a separable σ-algebra and η is a two-valued measure on $\Lambda \cap \underline{\underline{A}}$. *This is a contradiction* to proposition 1.2.2.

<u>Proposition 1.2.4:</u> (Liapounoff): Every nonatomic measure has the Darboux property.

For proofs, see [20, Lindenstrauss, pp. 971] or [18, Koshi, pp. 29].

<u>Corollary 1.2.5:</u> Let μ be a nonatomic measure on a σ-algebra $\underline{\underline{A}}$. Given $A \in \underline{\underline{A}}$, $\mu(A) \geq 0$, $0 \leq \alpha \leq \mu(A)$, we can find $B \in \underline{\underline{A}}$, $B \subset A$ such that $\mu(B) = \alpha$.

<u>Proof:</u> Restrict the measure μ to $\Lambda \cap \underline{\underline{A}}$ and observe that the restriction is nonatomic on $\Lambda \cap \underline{\underline{A}}$. Then apply proposition 1.2.4.

<u>Remark 1:</u> It is not true that if a measure has the Darboux property it is nonatomic. Here is an example.

Let X be the set of natural numbers, $\underline{\underline{A}}$ the class of all subsets of X and the measure μ on $\underline{\underline{A}}$ is determined by the conditions $\mu(\{n\}) = \frac{1}{2^n}$ for every natural number n.

Range of $\mu = [0, 1]$, but μ is not nonatomic.

3. Characterisations of nonatomic measures: We begin with a

Lemma 1.3.1: Let μ be a nonatomic measure on a σ-algebra $\underline{\underline{A}}$ and $A \in \underline{\underline{A}}$. Then there exists a separable sub σ-algebra $\underline{\underline{B}}$ of $\underline{\underline{A}}$ containing A such that μ is nonatomic on $\underline{\underline{B}}$.

Proof: Case (i) $\mu(A) > 0$ and $\mu(A^c) > 0$. Let $A_0 = A$ and $A_1 = A^c$. By Corollary 1.2.5, we can find sets A_{00}, A_{01}, A_{10} and A_{11} in $\underline{\underline{A}}$ such that (a) $A_{00} \cup A_{01} = A_0$, (b) $A_{00} \cap A_{01} = \emptyset$, (c) $\mu(A_{00}) = \mu(A_{01}) = \frac{1}{2}\mu(A_0)$, (d) $A_{10} \cup A_{11} = A_1$, (e) $A_{10} \cap A_{11} = \emptyset$, and (f) $\mu(A_{10}) = \mu(A_{11}) = \frac{1}{2}\mu(A_1)$. More generally, we can define, for every finite sequence $i_1, i_2, \ldots, i_k$ of 0's and 1's, sets $A_{i_1, i_2, \ldots, i_k}$ in $\underline{\underline{A}}$, by induction on k, satisfying

(a) $A_{i_1, i_2, \ldots, i_k} \subset A_{i_1, i_2, \ldots, i_{k-1}}$,

(b) $A_{i_1, i_2, \ldots, i_{k-1}, 0} \cup A_{i_1, i_2, \ldots, i_{k-1}, 1}$

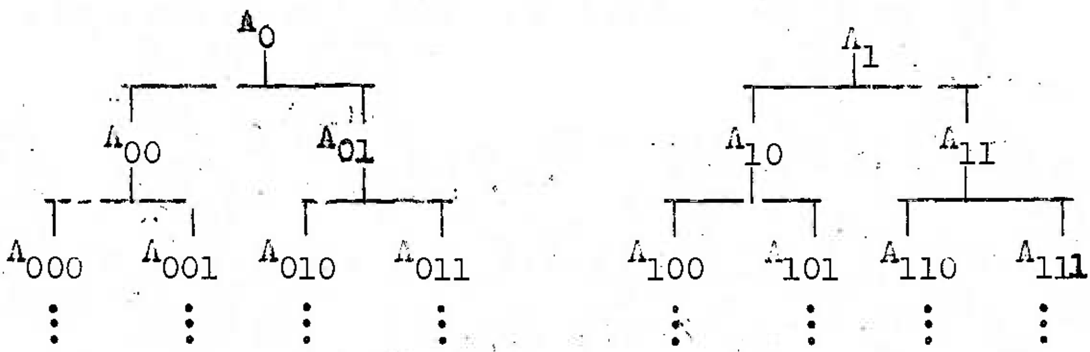
$$= A_{i_1, i_2, \ldots, i_{k-1}}$$

(c) $A_{i_1, i_2, \ldots, i_{k-1}, 0} \cap A_{i_1, i_2, \ldots, i_{k-1}, 1} = \emptyset$,

and

(d) $\mu(A_{i_1, i_2, \ldots, i_k}) = \frac{1}{2}\mu(A_{i_1, i_2, \ldots, i_{k-1}})$.

This scheme can be represented pictorially as follows:



Let $\underline{\underline{B}} = \sigma\{A_{i_1, i_2, \ldots, i_k}$: $i_1, i_2, \ldots$ is any sequence 0's and 1's$\}$.

$\underline{\underline{B}}$ is separable, and the atoms of $\underline{\underline{B}}$ are the nonempty sets among $\bigcap_{k\geq 1} A_{i_1, i_2, \ldots, i_k}$, where $i_1, i_2, \ldots$, is any sequence of 0's and 1's. From the construction of these sets, it is clear that μ-measure of every atom of $\underline{\underline{B}}$ is zero. Hence, μ is nonatomic on $\underline{\underline{B}}$. Obviously, $A \in \underline{\underline{B}}$.

Case (ii) $\mu(A) = \mu(X)$.

Let B be a set in $\underline{\underline{A}}$ such that $B \subset A$, $\mu(B) > 0$

and $\mu(A - B) > 0$. Let $A_0 = B$ and $A_1 = A - B$. Repeat the procedure outlined in case (i) to get a separable sub σ-algebra containing A on which μ is nonatomic.

Case (iii) $\mu(A) = 0$.

As in case (ii), operate with A^c.

<u>Theorem 1.3.2:</u> Let μ be a measure on a σ-algebra $\underline{\underline{A}}$. The following statements are equivalent.

i) μ is nonatomic on $\underline{\underline{A}}$.

ii) μ is nonatomic on some sub σ-algebra of $\underline{\underline{A}}$.

iii) μ is nonatomic on some separable sub σ-algebra of $\underline{\underline{A}}$.

<u>Proof:</u> i) => (ii) is trivial.

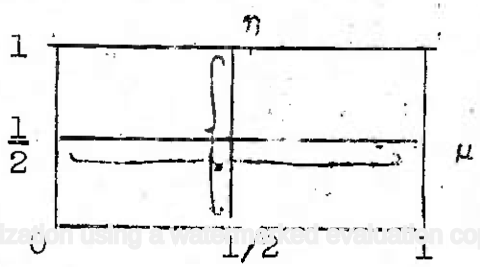
ii) => (iii) follows from Lemma 1.3.1.

iii) => (i) Let $\underline{\underline{B}}$ be a separable sub σ-algebra of $\underline{\underline{A}}$ such that μ is nonatomic on $\underline{\underline{B}}$. Let $A \in \underline{\underline{A}}$. Let $\underline{\underline{C}} = \sigma\{\underline{\underline{B}}, A\}$. Clearly $\underline{\underline{C}}$ is separable. If $B_1, B_2, \ldots,$ is a generator for $\underline{\underline{B}}$, then $A, B_1, B_2, \ldots$ is a generator for $\underline{\underline{C}}$. Consequently, every atom of $\underline{\underline{C}}$ is contained in some atom of $\underline{\underline{B}}$. Since μ is continuous on $\underline{\underline{B}}$, μ is continuous on $\underline{\underline{C}}$. Hence μ is nonatomic on $\underline{\underline{C}}$. This implies μ is nonatomic on $\underline{\underline{A}}$.

<u>Proposition 1.3.3:</u> Let λ be a measure on a product σ-algebra $\underline{\underline{A}} \times \underline{\underline{B}}$ of $X \times Y$, and let λ_1 and λ_2 be marginals of λ on $\underline{\underline{A}}$ and $\underline{\underline{B}}$ respectively. If one of the marginals is nonatomic, then λ is nonatomic.

<u>Proof</u>: λ_1 is nonatomic on $\underline{\underline{A}}$ is equivalent to saying that λ is nonatomic on the sub σ-algebra $\underline{\underline{A}} \times Y$ of $\underline{\underline{A}} \times \underline{\underline{B}}$. By Theorem 1.3.2, λ is nonatomic on $\underline{\underline{A}} \times \underline{\underline{B}}$.

<u>Remark</u> 1): The converse of the above proposition is not true. Here is an example. Let I^2 be the unit square $\left\{(x, y): 0 \leq x \leq 1, 0 \leq y \leq 1\right\}$ equipped with the usual borel σ-algebra. Let μ and η be two continuous measures on this borel σ-algebra with total mass $\frac{1}{2}$ each concentrated on the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$ respectively.



$\mu + \eta$ is nonatomic, but none of the marginals is nonatomic. In fact, for both the marginals $\left\{\frac{1}{2}\right\}$ is a measure atom.

However, in the case of product measures, we have a different picture.

<u>Theorem 1.3.4</u>: Let $(X, \underline{\underline{A}})$ and $(Y, \underline{\underline{B}})$ be two borel structures with measures μ and η on $\underline{\underline{A}}$ and $\underline{\underline{B}}$ respectively. Let $(X \times Y, \underline{\underline{A}} \times \underline{\underline{B}})$ be the product borel structure with the product measure $\mu \times \eta$ on $\underline{\underline{A}} \times \underline{\underline{B}}$. $\mu \times \eta$ is nonatomic if and only if atleast one of μ and η is nonatomic.

<u>Proof</u>: Suppose μ is nonatomic on $\underline{\underline{A}}$.

Then $\mu \times \eta = [\eta(Y)]\, \mu$, a scalar multiple of μ, on $\underline{\underline{A}} \times Y$. Consequently $\mu \times \eta$ is nonatomic on $\underline{\underline{A}} \times Y$. By Theorem 1.3.2. $\mu \times \eta$ is nonatomic on $\underline{\underline{A}} \times \underline{\underline{B}}$.

Conversely, suppose $\mu \times \eta$ is nonatomic on $\underline{\underline{A}} \times \underline{\underline{B}}$. By Theorem 1.3.2., there exists a separable sub σ-algebra $\underline{\underline{C}}$ of $\underline{\underline{A}} \times \underline{\underline{B}}$ on which $\mu \times \eta$ is nonatomic. We can find separable sub σ-algebras $\underline{\underline{A}}_1 \subset \underline{\underline{A}}$ and $\underline{\underline{B}}_1 \subset \underline{\underline{B}}$ such that $\underline{\underline{C}} \subset \underline{\underline{A}}_1 \times \underline{\underline{B}}_1$. $\mu \times \eta$ is nonatomic on $\underline{\underline{A}}_1 \times \underline{\underline{B}}_1$. If μ and η are not nonatomic on $\underline{\underline{A}}_1$ and $\underline{\underline{B}}_1$ respectively, then they are not continuous on $\underline{\underline{A}}_1$ and $\underline{\underline{B}}_1$ respectively. Consequently, we can find A_1 atom of $\underline{\underline{A}}_1$ and B_1 atom of $\underline{\underline{B}}_1$ such that $\mu(A_1) > 0$ and $\eta(B_1) > 0$. Observe that $A_1 \times B_1$ is an atom of $\underline{\underline{A}}_1 \times \underline{\underline{B}}_1$ and $(\mu \times \eta)(A_1 \times B_1) = \mu(A_1).\eta(B_1) > 0$. But this is a contradiction to the nonatomicity and hence continuity of $\mu \times \eta$ on $\underline{\underline{A}}_1 \times \underline{\underline{B}}_1$. So, either μ or η is nonatomic.

<u>Alternative proof of Theorem 1.3.4 using Fubini's Theorem.</u>

'if part'. Suppose, μ is nonatomic. Let $E \in \underline{\underline{A}} \times \underline{\underline{B}}$ satisfying $\mu \times \eta\,(E) > 0$. Then $\mu \times \eta\,(E) = \int \eta(E_x)\,\mu(dx)$, where E_x = x-section of $E = \{y \in Y : (x, y) \in E\}$. See Theorem B of [11, Halmos, pp. 144]. Let

$A = \{x \in X;\ \ \eta(E_x) > 0\}$. Since $\mu \times \eta\,(E) > 0$, we have $\mu(A) > 0$. There exists $A_1 \subset A$, A_1 in $\underline{\underline{A}}$ such that $0 < \mu(A_1) < \mu(A)$. Let $F = (A_1 \times Y) \cap E$.

$$\mu \times \eta(F) = \int \eta(F_x)\ \mu(dx)$$

$$= \int_{A_1} \eta(E_x)\ \mu(dx) > 0 .$$

$$\mu \times \eta(E) = \int_{A} \eta(E_x)\ \mu(dx) .$$

Since $\mu(A_1) < \mu(A)$, we have

$$\mu \times \eta(F) < \mu \times \eta(E).$$

This proves the nonatomicity of $\mu \times \eta$.

Conversely, let $\mu \times \eta$ be nonatomic. We shall prove that if A in $\underline{\underline{A}}$ is a μ-atom and B in $\underline{\underline{B}}$ is a η-atom, then $A \times B$ is a $\mu \times \eta$ - atom.

Let C be in $\underline{\underline{A}} \times \underline{\underline{B}}$ contained in $A \times B$.

$$\mu \times \eta\,(C) = \int \eta(C_x)\,\mu(dx)$$

$$= \int_A \eta(C_x)\,\mu(dx).$$

Since $C_x \subset B$, $\eta(C_x) = 0$ or $\eta(B)$.

Let $A_1 = \left\{x \in X:\ \eta(C_x) > 0\right\}$. Then $A_1 \subset A$ and $\mu(A_1) = 0$ or $\mu(A)$. Consequently, $\mu \times \eta\,(C) = 0$ or $\mu(A).\mu(B)$. This shows that either μ is nonatomic or η is nonatomic.

<u>Theorem 1.3.5:</u> Let μ and η be two measures defined on a σ-algebra $\underline{\underline{A}}$. Let μ be nonatomic and $\eta << \mu$. Then η is nonatomic.

Proof: By repeated applications of Theorem 1.3.2 and Corollary 1.2.3, we have the following deductions. μ is non-atomic on some separable sub σ-algebra $\underline{\underline{B}}$ of $\underline{\underline{A}}$ $\Rightarrow$ μ is continuous on $\underline{\underline{B}}$ $\Rightarrow$ η is continuous on $\underline{\underline{B}}$ $\Rightarrow$ η is non-atomic on $\underline{\underline{B}}$ $\Rightarrow$ η is nonatomic on $\underline{\underline{A}}$.

Alternate proof of Theorem 1.3.5 using Radon-Nikodym theorem.

Let $\eta \ll \mu$. Let f be a version of the Radon-Nikodym derivative of η with respect to μ. Let A in $\underline{\underline{A}}$ be such that $\eta(A) > 0$. Let $B = \{x : f(x) > 0\}$. We have $\eta(A) = \int_A f \, d\mu = \int_{B \cap A} f \, d\mu$. Since μ is nonatomic, there exists a C in $\underline{\underline{A}}$ contained $B \cap A$ such that $0 < \mu(C) < \mu(B \cap A)$. Thus, we have $0 < \eta(C) < \eta(A)$. This proves the nonatomicity of η.

Theorem 1.3.6: Let μ be a measure on a σ-algebra $\underline{\underline{A}}$. μ is nonatomic if and only if the range of every probability measure $\eta \ll \mu$ is $[0, 1]$.

Proof: The proof of the 'only if' part is a consequence of Theorem 1.3.5 and Liapounoff's theorem. Suppose A in $\underline{\underline{A}}$

is a μ-atom. Define a set function η on $\underline{\underline{A}}$ as follows.

$$\eta(C) = 1 \quad \text{if} \quad \mu(A \cap C) = \mu(A)$$

$$= 0 \quad \text{if} \quad \mu(A \cap C) = 0 .$$

η is a well defined probability measure on $\underline{\underline{A}}$ and $\eta << \mu$. But Range of $\eta = \{0, 1\}$. This contradiction shows that μ is nonatomic.

4. A Generalisation

Let $\mu_1, \mu_2, \ldots, \mu_n$ be n measures defined on a Borel structure $(X, \underline{\underline{A}})$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$. μ is said to be nonatomic if each μ_i, $1 \leq i \leq n$, is nonatomic.

Theorem 1.4.1. μ is nonatomic if and only if there exists a separable sub σ-algebra $\underline{\underline{B}}$ of $\underline{\underline{A}}$ such that μ is non-atomic on $\underline{\underline{B}}$ and Range of μ on $\underline{\underline{B}}$ is same as the Range of μ on $\underline{\underline{A}}$.

Proof: If μ is nonatomic on $\underline{\underline{B}}$, then, by Theorem 1.3.2., μ is nonatomic on $\underline{\underline{A}}$. If μ is nonatomic on $\underline{\underline{A}}$, then, by repeated application of Theorem 1.3.2, we can find a separable sub σ-algebra $\underline{\underline{C}}$ of $\underline{\underline{A}}$ such that μ is nonatomic on $\underline{\underline{C}}$.

The range of μ on $\underline{\underline{A}} = \left\{ (\mu_1(A), \ldots, \mu_n(A)) : A \in \underline{\underline{A}} \right\}$ is a compact subset E of R^n. For a proof of this result, see [12, Halmos, pp. 421]. Let $y_1, y_2, \ldots$ be a dense subset of E. Choose one A_i from each of $f^{-1}(\{y_i\})$, $i \geq 1$, where the map $f: \underline{\underline{A}} \longrightarrow E$ is given by $f(A) = (\mu_1(A), \mu_2(A), \ldots, \mu_n(A))$. Let $\underline{\underline{D}} = \sigma\left\{A_i: \ i \geq 1\right\}$, and $\underline{\underline{B}} = \sigma\left\{\underline{\underline{C}}, \underline{\underline{D}}\right\}$. $\underline{\underline{B}}$ is separable and μ is nonatomic on $\underline{\underline{B}}$. Further, the range of μ on $\underline{\underline{B}}$ contains the sequence y_i, $i \geq 1$. Hence range of μ on $\underline{\underline{B}}$ = range of μ on $\underline{\underline{A}}$.

5. A property of nonatomic measures

Theorem 1.5.1. Let μ and η be two nonatomic probability measures defined on a σ-algebra $\underline{\underline{A}}$, satisfying $\mu(A) = r$ if and only if $\eta(A) = r$ for some $0 < r < 1$, or equivalently $\left\{A \in \underline{\underline{A}}: \mu(A) = r\right\} = \left\{A \in \underline{\underline{A}} : \eta(A) = r\right\}$. Then $\mu = \eta$.

Proof: Step 1. Suppose $r = \frac{1}{2}$.

We prove, by induction on n, $\mu(A) = \frac{1}{2^n}$ if and only if $\eta(A) = \frac{1}{2^n}$, where n is any natural number. Suppose, $\mu(A) = \frac{1}{2^k}$ if and only if $\eta(A) = \frac{1}{2^k}$, where k is some natural number. Let $B \in \underline{\underline{A}}$ be such that $\mu(B) = \frac{1}{2^{k+1}}$. By Liapounoff's theorem, we can find A in $\underline{\underline{A}}$ such that $\mu(A) = \frac{1}{2^k}$ and $B \subset A$. We can also find B_1 and B_2 in $\underline{\underline{A}}$ satisfying

$$B_2 \subset B_1 \subset A^c \quad \text{and}$$

$$\mu(B_1) = \frac{1}{2^k}, \quad \mu(B_2) = \frac{1}{2^{k+1}}$$

Observe that

$$\mu(B) + \mu(B_2) = \frac{1}{2^k} = \mu(B \cup B_2)$$

$$\Rightarrow \quad \eta(B \cup B_2) = \frac{1}{2^k} = \eta(B) + \eta(B_2),$$

and

$$\mu(B) + \mu(B_1 - B_2) = \frac{1}{2^k} = \mu[B \cup (B_1 - B_2)]$$

$$\Rightarrow \eta[B \cup (B_1 - B_2)] = \frac{1}{2^k} = \eta(B) + \eta(B_1 - B_2),$$

by induction hypothesis.

By addition, we obtain

$$2\eta(B) + \eta(B_1) = 2(\frac{1}{2^k}).$$

Consequently, $\eta(B) = \frac{1}{2^{k+1}}$.

A similar argument shows that if $B \in \underline{\underline{A}}$ and $\eta(B) = \frac{1}{2^{k+1}}$, then $\mu(B) = \frac{1}{2^{k+1}}$. Any real number r, in $(0, 1]$ admits a dyadic expansion

$$r = \frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + \ldots, \text{ where } n_1, n_2, \ldots \text{ is}$$

an (finite or infinite) increasing sequence of positive natural numbers. Consequently, $\mu(A) = r$ if and only if $\eta(A) = r$ for every $0 < r \leq 1$. $\mu(A) = 0$ if and only if $\eta(A) = 0$ follows from the fact that $\mu(B) = 1$ if and only if $\eta(B) = 1$. Hence $\mu = \eta$.

<u>Step 2.</u> $r = \frac{1}{2^n}$ for some natural number n. We prove that $\mu(A) = \frac{1}{2^{n-1}}$ if and only if $\eta(A) = \frac{1}{2^{n-1}}$. Let A in $\underline{\underline{A}}$ be such that $\mu(A) = \frac{1}{2^{n-1}}$. Let B in $\underline{\underline{A}}$ be such that $B \subset A$ and $\mu(B) = \frac{1}{2^n}$.

$$\mu(B) = \frac{1}{2^n} \Rightarrow \eta(B) = \frac{1}{2^n}$$

$$\mu(A - B) = \frac{1}{2^n} \Rightarrow \eta(A - B) = \frac{1}{2^n}.$$

Consequently, $\eta(A) = 2(\frac{1}{2^n}) = \frac{1}{2^{n-1}}$.

Repeating this argument, we note that

$$\mu(A) = \frac{1}{2} \text{ if and only if } \eta(A) = \frac{1}{2}.$$

Step 1 shows that $\mu = \eta$.

Step 3. Let r be any real number in $(0, 1)$. Assume, without loss of generality, $r < \frac{1}{2}$. We prove, by induction on n, $\mu(A) = r/2^n$ if and only if $\eta(A) = r/2^n$.

Suppose $\eta(A) = r/2^n$ if and only if $\eta(A) = r/2^n$ is true for some natural number n. Let B in $\underline{\underline{A}}$ be such that $\mu(B) = r/2^{n+1}$. Let A in $\underline{\underline{A}}$ be such that $B \subset A$ and $\mu(A) = r/2^n$. Let B_1 and B_2 in $\underline{\underline{A}}$ be such that

$$B_2 \subset B_1 \subset A^c \quad \text{and}$$

$$\mu(B_1) = r/2^n, \quad \mu(B_2) = r/2^{n+1}.$$

By induction hypothesis, it follows that

$$\mu(B) + \mu(B_2) = r/2^n = \mu(B \cup B_2)$$

$$\Rightarrow \eta(B \cup B_2) = r/2^n = \eta(B) + \eta(B_2),$$

and

$$\mu(B) + \mu(B_1 - B_2) = r/2^n = \mu[B \cup (B_1 - B_2)]$$

$$\Rightarrow \eta[B \cup (B_1 - B_2)] = r/2^n = \eta(B) + \eta(B_1 - B_2).$$

By addition, we obtain,

$$2\eta(B) + \eta(B_1) = 2(r/2^n).$$

Therefore, $\eta(B) = r/2^{n+1}$. Thus, we find that $\mu(A) = r/2^n$ if and only if $\eta(A) = r/2^n$ for any natural number n.

Let $0 < s < r$. There exists $0 < s' < 1$ such that $s = rs'$. But s' has dyadic expansion $s' = \frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + \ldots$ for some increasing (finite or infinite) sequence $n_1, n_2, \ldots$ of positive natural numbers. So, $s = \frac{r}{2^{n_1}} + \frac{r}{2^{n_2}} + \ldots$.

Consequently, $\mu(A) = s$ if and only if $\eta(A) = s$. Let k be any natural number such that $\frac{1}{2^k} < r$. So, we have in particular,

$$\mu(A) = \frac{1}{2^k} \quad \text{if and only if} \quad \eta(A) = \frac{1}{2^k}$$

Step 2 shows that $\mu = \eta$.

<u>Alternate proof of Theorem 1.5.1</u>.

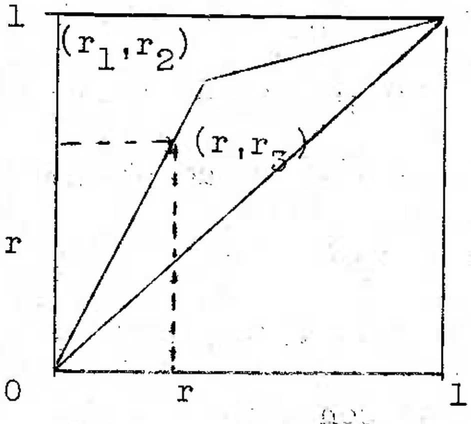
The proof of Theorem 1.5.1 can be very much simplified if we apply the following theorem of Liapounoff.

'Let $\mu_1, \mu_2, \ldots, \mu_n$ be n nonatomic measures defined on a borel structure $(X, \underline{\underline{A}})$. Then the range of

$$\mu = (\mu_1, \mu_2, \ldots, \mu_n) = \{(\mu_1(A), \mu_2(A), \ldots, \mu_n(A)) : A \in \underline{\underline{A}}\}$$

is a compact and convex subset of the n-dimensional Euclidean Space, R^n. For proofs see [18] and [20].

In the proof of the Theorem, it is enough if we show that the range of (μ, η) is the diagonal, **D**, of the unit square, $I \times I$.



If the range, R of $(\mu, \eta) \neq D$, let $(r_1, r_2) \in R$ with the property $r_1 \neq r_2$. Since R is convex, then the lines L_1 and L_2 joining (0, 0) and (r_1, r_2), and (r_1, r_2) and (1, 1) respectively lie entirely in R. The line $x = r$ either cuts L_2 or L_1 at (r, r_3). Clearly $r_3 \neq r$. This gives a contradiction to the hypothesis of the Theorem.

6. The case of infinite measures

In this section, we shall see how far the results of Section 3 of this Chapter are valid in the case of infinite measures. In this section, we allow measures to take values in the nonnegative extended real line $[0, \infty]$. Before embarking on this problem, let us be clear about the definitions of nonatomicity in the case of infinite measures vogue in the literature. According to Johnson [13, pp. 650], a measure μ defined on a σ-algebra $\underline{\underline{A}}$ is nonatomic if $A \in \underline{\underline{A}}$, $\mu(A) > 0$ implies there exists $B \in \underline{\underline{A}}$, $B \subset A$ such that $\mu(B) > 0$ and $\mu(A - B) > 0$. The definition we adopt here is the following. $A \in \underline{\underline{A}}$, $\mu(A) > 0$ implies there exists $B \in \underline{\underline{A}}$, $B \subset A$ such that $0 < \mu(B) < \mu(A)$. These two definitions are not same. A measure nonatomic in our sense is nonatomic according to the definition of Johnson also. Take any atomless σ-algebra $\underline{\underline{A}}$ on X, i.e., $A \in \underline{\underline{A}}$, $A \neq \emptyset$ implies there exists a $B \in \underline{\underline{A}}$ such that $B \neq \emptyset$, $B \subset A$ and $B \neq A$. Define μ on $\underline{\underline{A}}$ as follows. $\mu(A) = 0$ if $A = \emptyset$, and $\mu(A) = \infty$ if $A \neq \emptyset$ for $A \in \underline{\underline{A}}$. μ is nonatomic according to the definition of Johnson but is not nonatomic in our sense.

However, these two definitions of nonatomicity are equivalent in the case of semi-finite measures. A measure μ on a σ-algebra $\underline{\underline{A}}$ is said to be <u>semi-finite</u> if $A \in \underline{\underline{A}}$, $\mu(A) = \infty$ implies there exists a $B \in \underline{\underline{A}}$, $B \subset A$ such that $0 < \mu(B) < \mu(A)$. The equivalence of the two definitions of nonatomicity is easy to verify in the case of semi-finite measures. Note that every σ-finite measure is semi-finite.

<u>Proposition 1.6.1.</u> Let μ be a semi-finite measure on a separable σ-algebra $\underline{\underline{A}}$. μ is nonatomic on $\underline{\underline{A}}$ if and only if μ is continuous on $\underline{\underline{A}}$.

<u>Proof:</u> If μ is nonatomic, then it is continuous. Conversely, let μ be continuous and not nonatomic. Let $A \in \underline{\underline{A}}$ be a μ-atom. Clearly $\mu(A) > 0$. Since μ is semi-finite, $\mu(A) < \infty$. Now, consider the borel structure $(A, A \cap \underline{\underline{A}})$ and the measure $\mu/A \cap \underline{\underline{A}}$ on $A \cap \underline{\underline{A}}$. $A \cap \underline{\underline{A}}$ is separable and $\mu/A \cap \underline{\underline{A}}$ is two-valued and continuous measure on $A \cap \underline{\underline{A}}$. But this is a contradiction to Proposition 1.2.2.

Lemma 1.6.2. Let μ be a σ-finite nonatomic measure on a σ-algebra $\underline{\underline{A}}$. Let $A \in \underline{\underline{A}}$. Then there exists a separable sub σ-algebra $\underline{\underline{B}}$ of $\underline{\underline{A}}$ such that $A \in \underline{\underline{B}}$ and μ is nonatomic on $\underline{\underline{B}}$.

Proof: Since μ is σ-finite, we can write $\mu = \sum_{i \geq 1} \mu_i$, where each μ_i is a finite measure on $\underline{\underline{A}}$ and supports of μ_i's are disjoint. By Lemma 1.3.1, for each $i \geq 1$, we can find a separable sub σ-algebra $\underline{\underline{B}}_i$ of $\underline{\underline{A}}$ such that $A \in \underline{\underline{B}}_i$ and μ_i is nonatomic on $\underline{\underline{B}}_i$. Let $\underline{\underline{B}} = \sigma\left\{ \underline{\underline{B}}_i : i \geq 1 \right\}$. Clearly, $\underline{\underline{B}}$ is separable, $A \in \underline{\underline{B}}$ and each μ_i is nonatomic on $\underline{\underline{B}}$. Hence $\sum_{i \geq 1} \mu_i = \mu$ is nonatomic on $\underline{\underline{B}}$.

Remark 1.6.3: The above Lemma is not true for any nonatomic measure. The following is a counter example. Let for every $x \in I$, the unit interval, $(Y_x, \underline{\underline{A}}_x, \mu_x)$ be a nonatomic probability space such that $Y_x \cap Y_z = \emptyset$ if $x \neq z$. Let $Y = \bigcup_{x \in I} Y_x$ and $\underline{\underline{A}} = \left\{ \bigcup_{x \in I} A_x : A_x \in \underline{\underline{A}}_x \text{ and } A_x = \emptyset \text{ for all but a countable number of } x\text{'s or } A_x = Y_x \text{ for all but a countable of } x\text{'s} \right\}$. $\underline{\underline{A}}$ is a σ-algebra on Y. Let $\mu = \sum_{x \in I} \mu_x$. μ is a semi-finite, non σ-finite, nonatomic measure on $\underline{\underline{A}}$. Now, we claim that μ is not nonatomic on any separable sub σ-algebra of $\underline{\underline{A}}$. Let $\underline{\underline{B}}$ be a separable sub σ-algebra of $\underline{\underline{A}}$. Let $B_1, B_2, \ldots$ be a generator for $\underline{\underline{B}}$. Since the collection of sets of the form $\bigcup_{x \in I} A_x$ with $A_x = \emptyset$ for all but a countable number of x's is a generator for $\underline{\underline{A}}$, we can assume each B_i is of the above form. $\bigcap_{i \geq 1} B_i^c$ is an atom of $\underline{\underline{B}}$ and is of the form $\bigcup_{x \in X} A_x$ with $A_x = Y_x$ for all but a countable number of x's. Consequently, this atom of $\underline{\underline{B}}$ has μ-measure ∞. Hence μ is not nonatomic on $\underline{\underline{B}}$.

Theorem 1.6.4: Let μ be a σ-finite measure defined on a σ-algebra $\underline{\underline{A}}$. The following statements are equivalent

i) μ is nonatomic on $\underline{\underline{A}}$.

ii) μ is nonatomic on some sub σ-algebra of $\underline{\underline{A}}$.

iii) μ is nonatomic on some separable sub σ-algebra of $\underline{\underline{A}}$.

Proof: (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii) follows from Lemma 1.6.2.

(iii) $\Rightarrow$ (i) Let $\underline{\underline{B}}$ be a separable sub

σ-algebra of $\underline{\underline{A}}$ such that μ is nonatomic on $\underline{\underline{B}}$. Since μ is σ-finite, there exists a sequence $A_1, A_2, \ldots$ of sets in $\underline{\underline{A}}$ such that $A_i \cap A_j = \emptyset$ if $i \neq j$, $\bigcup_{i \geq 1} A_i = X$ and $\mu(A_i) < \infty$ for each i. Since μ is continuous on $\underline{\underline{B}}$, $\mu / A_i \cap \underline{\underline{B}}$ is continuous on $A_i \cap \underline{\underline{B}}$ for every i. Hence $\mu / A_i \cap \underline{\underline{B}}$ is nonatomic on $A_i \cap \underline{\underline{B}}$. This implies that $\mu / A_i \cap \underline{\underline{A}}$ is nonatomic on $A_i \cap \underline{\underline{A}}$. Thus, we observe that μ is nonatomic on $\underline{\underline{A}}$.

<u>Remark 1.6.5:</u> Theorem 1.6.4. is not true if we relax the condition of σ-finiteness of the measure μ. The example given in the Remark 1.6.3 will serve also as a counter example in this situation.

<u>Theorem 1.6.6.</u> Let μ be any σ-finite measure and let λ be any semi-finite measure on a σ-algebra $\underline{\underline{A}}$ such that $\lambda << \mu$. If μ is nonatomic, then λ is nonatomic.

<u>Proof:</u> If λ is finite, then, by Theorems 1.6.4 and 1.3.2, λ is nonatomic. In the general case, suppose λ is not nonatomic. Let $A \in \underline{\underline{A}}$ be λ-atom of $\underline{\underline{A}}$. Since λ is semi-

semi-finite, $\lambda(\Lambda) < \infty$. Define $\eta(B) = \lambda(B \cap \Lambda)/\lambda(\Lambda)$, for $B \in \underline{\underline{A}}$. η is 0-1 valued measure on $\underline{\underline{A}}$ and $\eta \ll \mu$. By what we have remarked above, η is nonatomic. But this is a contradiction. Hence λ is nonatomic.

Remark 1.6.7. Johnson [13, Theorem 2.4, pp. 653] proved the following result. 'Let μ be a σ-finite measure and λ be any arbitrary measure defined on a σ-algebra $\underline{\underline{A}}$ and $\lambda \ll \mu$. If μ is nonatomic, then so is λ.' Note that his definition of nonatomicity is different from ours. We do not have such a general theorem in our case. In our Theorem 1.6.6, neither the semi-finiteness condition on λ nor the σ-finiteness condition on μ can be relaxed. The following examples make this point clear.

Let μ be any nonatomic measure on $\underline{\underline{A}}$. Define $\lambda(A) = 0$ if $\mu(A) = 0$, $= \infty$ if $\mu(A) > 0$, for A in $\underline{\underline{A}}$. λ is a measure on $\underline{\underline{A}}$ and $\lambda \ll \mu$. Certainly, λ is not nonatomic. Note that λ is not semi-finite. This shows that Theorem 1.6.6. is not true for any general measure λ.

Let μ be the measure on $\underline{\underline{A}}$ described in Remark 1.6.3. We note that μ is nonatomic, semi-finite, but not σ-finite, measure on $\underline{\underline{A}}$. Define $\lambda(\bigcup_{x \in I} B_x) = 0$ if $B_x \in \underline{\underline{A}}_x$ and $B_x = \emptyset$ for all but a countable number of x's and $= 1$, otherwise. λ is a 0-1 valued measure on $\underline{\underline{A}}$ and $\lambda \leq \mu$. Certainly, λ is not nonatomic. This shows that the σ-finiteness condition on μ of Theorem 1.6.6 cannot be relaxed.

CHAPTER 2

MIXTURES OF NONATOMIC MEASURES

1. Introduction. The problem tackled in this Chapter is the following. Let $(X, \underline{\underline{A}})$ and $(Y, \underline{\underline{B}})$ be two borel structures. Let P be a nonatomic transition probability defined on $X \times \underline{\underline{B}}$, i.e., P is a function defined $X \times \underline{\underline{B}}$ taking values in $[0, 1]$ satisfying

(i) $P(x, \cdot)$ is a nonatomic probability measure on $\underline{\underline{B}}$ for every x in X, and

(ii) $P(\cdot, B)$ is $\underline{\underline{A}}$ - measurable function for every B in $\underline{\underline{B}}$.

Let λ be a probability measure on $\underline{\underline{A}}$. It is easy to verify that the set function μ defined by the formula

$$\mu(B) = \int P(x, B)\, \lambda(dx) \quad \text{for } B \text{ in } \underline{\underline{B}}$$

is a probability measure on $\underline{\underline{B}}$. Question: Is μ nonatomic?

Let us call μ the mixture of P with respect to λ.

The answer turns out to be no, in general. Section 2 gives five sufficient conditions under which every mixture becomes nonatomic. Section 3 gives an example of a nonatomic transition probability P and a probability measure λ such that the mixture μ of P with respect to λ is not nonatomic.

THROUGHOUT THIS CHAPTER P STANDS FOR A NONATOMIC TRANSITION PROBABILITY, UNLESS OTHERWISE SPECIFIED.

The following proposition gives a sufficient condition on λ to ensure the mixture μ to be nonatomic.

Proposition 2.1.1. If λ is a discrete probability measure, i.e., concentrated on a countable subset of X, then the mixture μ of P with respect to λ is nonatomic.

Proof: Let $\sum_{n\geq 1} \alpha_n \partial_{x_n}$ be a representation of λ, where α_n's are positive, $\sum_{n\geq 1} \alpha_n = 1$ and $x_1, x_2, \ldots$ is a sequence of points in X. Then

$$\mu(\cdot) = \sum_{n\geq 1} \alpha_n \; P(x_n, \cdot) .$$

It is a straightforward verification to show that μ is nonatomic.

2. Sufficient conditions

Theorem 2.2.1 If $\underline{\underline{B}}$ is separable, then the mixture μ of any P with respect to any probability measure λ is non-atomic.

Proof: By Corollary 1.2.3., we know that continuity and nonatomicity of measure on $\underline{\underline{B}}$ are equivalent. Since $P(x, \cdot)$ is a continuous measure on $\underline{\underline{B}}$ for every x in X, μ is continuous and hence nonatomic on $\underline{\underline{B}}$.

Theorem 2.2.2. If $\left\{P(x, \cdot) : x \text{ in } X\right\}$ is a dominated family of nonatomic probability measures, i.e., there exists a σ-finite measure ρ on $\underline{\underline{B}}$ such that $P(x, \cdot) \ll \rho$ for every x in X, then any mixture μ of P is nonatomic.

Proof: By a theorem of Halmos and Savage (see, for example, [22, Neveu, pp. 122]), the family $\left\{P(x, \cdot): x \text{ in } X\right\}$ is equivalent to a countable sub-family $\left\{P(x_n, \cdot): n \geq 1\right\}$, i.e., $P(x, B) = 0$ for every x in X if and only if $P(x_n, B) = 0$ for every $n \geq 1$. If $\eta(\cdot) = \sum_{n \geq 1} (\frac{1}{2^n}) P(x_n, \cdot)$, then the family $\left\{P(x, \cdot) : x \text{ in } X\right\}$ is equivelent to η.

By direct argument, one can see that η is nonatomic on $\underline{\underline{B}}$. If μ is a mixture of P, then $\mu << \eta$. By Theorem 1.3.5., μ is nonatomic on $\underline{\underline{B}}$.

<u>Theorem 2.2.3</u>. Let X be a topological space having a countable dense subset, and $\underline{\underline{A}}$ be the σ-algebra generated by open subsets of X. Further, assume that $P(\cdot, B)$ is a continuous function on X for every B in $\underline{\underline{B}}$. Then any mixture of P is nonatomic.

In order to prove this Theorem, we need the following Lemmas.

<u>Lemma 2.2.4.</u> Let φ be a nonatomic measure on $\underline{\underline{B}}$ and η be a two-valued measure on $\underline{\underline{B}}$. Then φ and η are mutually singular.

<u>Proof</u>: By Lebesgue decomposition theorem (see, for example, [11, Halmos, Theorem C, pp. 134]), we can write $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 << \eta$, and φ_2 and η are mutually singular. Since η is two-valued, φ_1 can take on at most 2 values. Since $\varphi_1 \leq \varphi$, by Theorem 1.3.5, φ_1 is nonatomic. Hence, $\varphi_1 = 0$,

$\varphi_2 = \varphi$ and so φ and η are mutually singular.

<u>Lemma 2.2.5.</u> If $\varphi_1, \varphi_2, \ldots$ is a sequence of nonatomic measures and η is a two-valued measure all defined on $\underline{\underline{B}}$, then there exists B in $\underline{\underline{B}}$ such that $\eta(B) = \eta(Y)$ and $\varphi_i(B) = 0$ for every $i \geq 1$.

<u>Proof</u>: Assume, without loss of generality, $\varphi_i(Y) > 0$ for every $i \geq 1$. Let

$$\varphi(\cdot) = \sum_{n \geq 1} \left(\frac{1}{2^n}\right)\left(\frac{1}{\varphi_n(Y)}\right) \varphi_n(\cdot) .$$

Then φ is a nonatomic measure, and by Lemma 2.2.4., φ and η are singular. There exists a B in $\underline{\underline{B}}$ such that $\eta(B) = \eta(Y)$ and $\varphi(B) = 0$. This implies $\varphi_i(B) = 0$ for every $i \geq 1$.

<u>Proof of the Theorem 2.2.3.</u>

Let μ be a mixture of P with respect to λ. Suppose μ is not nonatomic. Let B_o be a μ-atom. Since $\mu(B_o) > 0$, the open set $U = \left\{x \text{ in } X: P(x, B_o) > 0\right\}$ has positive λ-measure. As X contains a dense denumerable set, U contains a dense denumerable set, say, $x_1, x_2, \ldots$. The measure μ on $B_o \cap \underline{\underline{B}}$ is two-valued. $P_1(x_n, \cdot) = P(x_n, \cdot)/B_o \cap \underline{\underline{B}}$, the restriction of $P(x_n, \cdot)$ to $B_o \cap \underline{\underline{B}}$ is a sequence of non-zero nonatomic measures on $B_o \cap \underline{\underline{B}}$. By Lemma 2.2.5., there exists a B in $\underline{\underline{B}}$, $B \subset B_o$ such that $\mu(B) = \mu(B_o)$ and $P(x_n, B) = 0$ for every $n \geq 1$. Let $C = \{x$ in X: $P(x, B) = 0\}$. C is a closed set and each x_n is a member of C. So, $U \subset$ closure of $\{x_n : n \geq 1\} \subset C$. Now,

$$0 \neq \mu(B_o) = \mu(B) = \int P(x, B)\, \lambda(dx)$$

$$= \int_U P(x, B)\, \lambda(dx) + \int_{U^c} P(x, B)\, \lambda(dx)$$

$$= 0 + 0 = 0, \quad \text{a contradiction.}$$

The second term is zero follows from the fact

$$P(x, B) \leq P(x, B_o) = 0 \text{ if } x \in U^c.$$

This shows that μ is nonatomic.

<u>Theorem 2.2.6.</u> Let X be a Lindelof topological space and $\underline{\underline{A}}$ be the σ-algebra on X generated by the open subset of X. Further, assume that $P(\cdot, B)$ is a continuous function on X for every B in $\underline{\underline{B}}$. Then any mixture of $\underline{\underline{P}}$ is nonatomic.

<u>Proof</u>: Let μ be a mixture of P with respect to λ. Let $B_o \in \underline{\underline{B}}$ be such that $\mu(B_o)$ is positive. Then $V = \{x \text{ in } X: P(x, B_o) > 0\}$ is an open set of positive λ-measure. Let $V_n = \{x \text{ in } X: P(x, B_o) \geq \frac{1}{n}\}$. Then $V = \bigcup_{n \geq 1} V_n$. Since $\lambda(V)$ is positive, there exists a natural number N such that $\lambda(V_N)$ is positive. Since V_N is closed, it is Lindelof. For every $B \subset B_o$, B in $\underline{\underline{B}}$, define $V_B = \{x \text{ in } X: P(x, B_o) > P(x, B) > \frac{1}{N+1}\}$. V_B is an open set for every $B \subset B_o$, B in $\underline{\underline{B}}$. Now, $\{V_B : B \subset B_o \text{ and } B \text{ in } \underline{\underline{B}}\}$ is an open cover for V_N. For, let $x \in V_N$. Then $P(x, B_o) \geq \frac{1}{N} > \frac{1}{N+1}$. By the nonatomicity of $P(x, \cdot)$, there exists C in $\underline{\underline{B}}$, $C \subset B_o$ such that

$$P(x, B_o) > P(x, C) > \frac{1}{N+1}.$$

Consequently, x is in V_C. By Lindelof property of V_N, we can find a countable sub cover for V_N from $\{V_B : B \subset B_o \text{ and } B \text{ in } \underline{\underline{B}}\}$. Hence, there exists B in $\underline{\underline{B}}$, $B \subset B_o$ such that $\lambda(V_B)$ is positive. On V_B, $\frac{1}{N+1} < P(x, B) < P(x, B_o)$. So, $0 < \mu(B) < \mu(B_o)$. This proves the nonatomicity of μ.

For the next result, we need the following definitions and results.

A cardinal number m is said to have measure zero if X is any set of cardinality m, and μ is a finite continuous measure on the power set P(X) of X implies μ is identically zero.

Sikorski and Marczewski [25, Theorem 3, pp. 137] proved the following theorem.

Theorem. Let X be a metric space such that it contains a dense subset whose cardinality has measure zero. Let μ be any finite measure on the σ-algebra $\underline{\underline{A}}$ on X generated by all open subsets of X. Let $V_\alpha : \alpha \in D$ be any family of open subsets of X such that $\mu(V_\alpha) = 0$ for every α in D. Then $\mu(\bigcup_{\alpha \in D} V_\alpha) = 0$.

Theorem 2.2.7: Let X be a metric space such that it contains a dense subset whose cardinality has measure zero. Let $\underline{\underline{A}}$ be the σ-algebra on X generated by the open subsets of X. Further, assume that P(•, B) is a continuous function on X for every B in $\underline{\underline{B}}$. Then any mixture of P is nonatomic.

Proof: Let μ be a mixture of P with respect to λ. Let $B_o \in \underline{\underline{B}}$ be such that $\mu(B_o) > 0$. Then the set $V = \{x \text{ in } X: P(x, B_o) > 0\}$ has λ-measure positive. For every set B in $\underline{\underline{B}}$, $B \subset B_o$, define $V_B = \{x \text{ in } X: P(x, B_o) > P(x, B) > 0\}$. V_B is an open set and the family

$$\{V_B : B \text{ in } \underline{\underline{B}},\ B \subset B_o\}$$

covers V. For, if $x \in V$, $P(x, B_o) > 0$. Since $P(x, \cdot)$ is nonatomic on $\underline{\underline{B}}$, there exists C in $\underline{\underline{B}}$, $C \subset B_o$ such that $0 < P(x, C) < P(x, B_o)$. Consequently, $x \in V_C$. By the Theorem quoted in the paragraph preceding the statement of Theorem 2.2.7, there exists a E in $\underline{\underline{B}}$, $E \subset B_o$ such that $\lambda(V_E) > 0$ since $\lambda(V) > 0$. Consequently, $0 < \mu(E) < \mu(B_o)$. This proves the nonatomicity of μ.

Remark. For the nonatomicity of all mixtures of P, Theorem 2.2.1 lays conditions on the borel structure $(Y, \underline{\underline{B}})$, Theorem 2.2.2 on the family $\{P(x, \cdot) : x \text{ in } X\}$ of measures, and Theorems 2.2.3, 2.2.6 and 2.2.7 on $(X, \underline{\underline{A}})$

and the family $\{P(\cdot, B) : B \text{ in } \underline{\underline{B}}\}$ of functions.

3. Example. An examination of the theorems proved in Section 2 makes it clear that the desired counter-example will be pathological. The proof of the following proposition is easy.

Proposition 2.3.1. Let $(X, \underline{\underline{A}})$ be a borel structure, where X is any uncountable set and $\underline{\underline{A}}$ the countable co-countable σ-algebra on X. Then any real valued function defined on X is $\underline{\underline{A}}$-measurable if and only if it is constant on a co-countable subset of X.

Example. Let X be any uncountable set and $\underline{\underline{A}}$, the countable co-countable σ-algebra on X. For each x in X, let $(Y_x, \underline{\underline{B}}_x, \mu_x)$ be a nonatomic probability space. Let $Y = \prod_{x \in X} Y_x$, the product space and $\underline{\underline{B}} = \prod_{x \in X} \underline{\underline{B}}_x$, the product σ-algebra. Fix f_o in Y. Let $f_o^x = f_o / X - \{x\}$, the restriction of f_o to $X - \{x\}$. For every B in $\underline{\underline{B}}$, define $B_x = f_o^x$- th section of $B = \{g(x) \in Y_x : g \in B \text{ and } g = f_o \text{ on } X - \{x\}\}$. If we identify $Y = Y_x \times \prod_{z \in X - \{x\}} Y_z$, as a product space of two spaces and $\underline{\underline{B}} = \underline{\underline{B}}_x \times \prod_{z \in X-\{x\}}$, as the corresponding product of the σ-algebras, we can easily verify that $\{B_x : B \in \underline{\underline{B}}\} = \underline{\underline{B}}_x$. (See the general discussion in [11, Halmos, Section 34, pp. 141-142].) $P : X \times \underline{\underline{B}} \longrightarrow [0, 1]$ is defined as follows $P(x, B) = \mu_x(B_x)$. For every x in X, $P(x, \cdot)$ is a nonatomic probability measure on $\underline{\underline{B}}$. For, suppose $P(x, B) > 0$. There exists C in $\underline{\underline{B}}_x$ such that $C \subset B_x$ and $0 < \mu_x(C) < \mu_x(B_x)$. Let $D = (C \times \prod_{z \in X-\{x\}} Y_x) \cap B$. Note that D is in $\underline{\underline{B}}$, $D \subset B$ and $D_x = C \cap B_x = C$. Consequently, $0 < P(x, D) < P(x, B)$. Since every B in $\underline{\underline{B}}$ is a countable dimensional cylinder, it follows that $B_x = \emptyset$ for all but a countable number of x's, or, $= Y_x$ for all but a countable number of x's. Consequently, $P(x, B) = 0$ for all but a countable number of x's, or, $= 1$ for all but a countable number of x's. By Proposition 2.3.1, $P(\cdot, B)$ is $\underline{\underline{A}}$ - measurable for every B in $\underline{\underline{B}}$. Let λ be the $0 - 1$ valued measure on $\underline{\underline{A}}$ defined by $\lambda(A) = 0$ or 1, according as A is countable or cocountable. Let μ be the mixture of P with respect to λ. Then μ is $0 - 1$ valued and hence cannot be nonatomic. In fact, μ is a degenerate measure at f_0.

Remark. If one wants to construct a borel structure $(Z, \underline{\underline{C}})$, a transition probability Q on $Z \times \underline{\underline{C}}$, and a probability measure η on $\underline{\underline{C}}$ such that the mixture of Q with respect to η is not nonatomic, one can proceed as follows.

Let $(X, \underline{\underline{A}})$ and $(Y, \underline{\underline{B}})$ be two borel structures, P a transition probability on $X \times \underline{\underline{B}}$. At this stage, do not assume anything additional on P. Let λ be a probability measure on $\underline{\underline{A}}$, and μ the mixture of P with respect to λ. Define a transition probability Q on $(X \times Y) \times (\underline{\underline{A}} \times \underline{\underline{B}})$ as follows.

$$Q[(x, y), C] = [\lambda \times P(x, \cdot)](C) \quad \text{for } C \text{ in } \underline{\underline{A}} \times \underline{\underline{B}}.$$

$Q[(x, y), \cdot]$ is a probability measure on $\underline{\underline{A}} \times \underline{\underline{B}}$. Further, $Q[\cdot, C]$ is $\underline{\underline{A}} \times \underline{\underline{B}}$-measurable for every C in $\underline{\underline{A}} \times \underline{\underline{B}}$. For, let $\underline{\underline{D}} = \{C \text{ in } \underline{\underline{A}} \times \underline{\underline{B}} : Q[\cdot, C] \text{ is } \underline{\underline{A}} \times \underline{\underline{B}}\text{-measurable}\}$. Then $\underline{\underline{D}}$ contains $\{A \times B: A \text{ in } \underline{\underline{A}} \text{ and } B \text{ in } \underline{\underline{B}}\}$, closed under complementation and countable disjoint unions. Further,

$\underline{\underline{D}}$ is a monotone class. Hence $\underline{\underline{D}} = \underline{\underline{A}} \times \underline{\underline{B}}$. Note also that Q is nonatomic if P is nonatomic. Now, $\lambda \times \mu$ is an invariant measure for Q, i.e.,

$$\lambda \times \mu\,[C] = \int Q[(x, y), C]\, d(\lambda \times \mu)(x, y), \quad \text{for } C \text{ in } \underline{\underline{A}} \times \underline{\underline{B}}.$$

For, if

$$\eta[C] = \int Q[(x, y), C]\, d(\lambda \times \mu)(x, y),$$

then

$$\begin{aligned} \eta(A \times B) &= \int Q[(x, y), A \times B]\, d(\lambda \times \mu)(x, y) \\ &= \int [\lambda \times P(x, \cdot)](A \times B)\, d(\lambda \times \mu)(x, y) \\ &= \int \lambda(A)\, P(x, B)\, d(\lambda \times \mu)(x, y) \\ &= \lambda(A) \int P(x, B)\, \lambda(dx) = \lambda(A)\mu(B). \end{aligned}$$

Thus the probability measures η and $\lambda \times \mu$ agree on $\{A \times B : A \text{ in } \underline{\underline{A}} \text{ and } B \text{ in } \underline{\underline{B}}\}$. Hence $\eta = \lambda \times \mu$.

Regarding the counter-example in the setup of a single borel structure, take $Z = X \times Y$, $\underline{\underline{C}} = \underline{\underline{A}} \times \underline{\underline{B}}$, where the borel structures $(X, \underline{\underline{A}})$ and $(Y, \underline{\underline{B}})$ are the ones constructed at the beginning of this section. Since μ and λ are 0-1 valued, $\lambda \times \mu$ is also 0-1 valued, and hence $\lambda \times \mu$ is not nonatomic on $\underline{\underline{C}}$. The mixture of Q with respect to $\lambda \times \mu$ is itself, where Q is the transition probability on $Z \times \underline{\underline{C}}$ constructed as above using P and λ. Further, note that Q is nonatomic.

<u>Further remark.</u> The continuity conditions quoted in Theorems 2.2.3, 2.2.6 and 2.2.7 cannot be relaxed to measurability. If one wishes to construct a counter example, one can take the example given in Section 3 and proceed as follows. Take X to be any separable, or Lindelöf, or metric space with a dense subset of cardinality measure zero and λ any continuous measure on the borel σ-algebra of X. The rest of details proceed exactly as they are carried out in Section 3.

4. <u>Notes.</u> The following Theorem 1.2 which appears in [13, Johnson, pp. 651] bears some resemblance to the problem considered in this Chapter.

Suppose μ_α is a family of nonatomic measures and $\mu = \Sigma \mu_\alpha$. Then μ is nonatomic also.

It should be noted here that his definition of nonatomicity of a measure is different from ours. His Theorem 1.2 is not true with our definition of nonatomicity. For, let μ be a nonatomic probability measure on a σ-algebra $\underline{\underline{A}}$, and let $\mu_n = \mu$ for every natural number n. Then $\lambda = \sum_{n \geq 1} \mu_n$ is nonatomic according to the definition of Johnson, but not according to our definition of nonatomicity.

CHAPTER 3

MIXTURES OF INVARIANT NON-ERGODIC PROBABILITIES

1. Introduction. Let $(X, \underline{\underline{A}})$ and $(Y, \underline{\underline{B}})$ be two borel structures, T a measurable transformation from Y to Y, i.e., $T^{-1}B$ is in $\underline{\underline{B}}$ for every B in $\underline{\underline{B}}$. A measure μ on $\underline{\underline{B}}$ is said to be invariant if $\mu(B) = \mu(T^{-1}B)$ for every B in $\underline{\underline{B}}$. A measure μ on $\underline{\underline{B}}$ is said to be ergodic if B in $\underline{\underline{B}}$, $B = T^{-1}B$ implies $\mu(B) = 0$ or $\mu(B^c) = 0$. A measure μ on $\underline{\underline{B}}$ is said to be non-ergodic if it is not ergodic. Let P be a transition probability on $X \times \underline{\underline{B}}$ with the following property.

$P(x, \cdot)$ is an invariant non-ergodic probability measure on $\underline{\underline{B}}$ for every x in X.

Let μ be a mixture of P with respect to some probability measure λ on $\underline{\underline{A}}$. It is obvious that μ is an invariant probability measure on $\underline{\underline{B}}$.

Question: Is μ non-ergodic?

What is the relevance of this problem? What can you say about mixtures of ergodic probability measures? The representation theorem developed in [7, Blum and Hanson, pp. 1125-1129], as we will find out, is a problem of mixtures of invariant ergodic probability measures, and see how we are led to the formulation of the above problem. The following is a brief description of the representation theorem of [8].

Let $\underline{\underline{P}}$ denote the collection of all invariant probability measures on $\underline{\underline{B}}$, and $\underline{\underline{P}}_e$ the collection of all ergodic measures in $\underline{\underline{P}}$. The following assumptions are made.

i) $\underline{\underline{P}} \neq \emptyset$

ii) $\underline{\underline{P}} \equiv \underline{\underline{P}}_e$, i.e., $\lambda(B) = 0$ for every λ in $\underline{\underline{P}}$ if and only if $\eta(B) = 0$ for every η in $\underline{\underline{P}}_e$. A suitable σ-algebra $\underline{\underline{E}}$ on $\underline{\underline{P}}_e$ is defined insuring the measurability of the following real valued maps on $\underline{\underline{P}}_e$.

$$\eta \longrightarrow \eta(B), \quad \eta \text{ in } \underline{\underline{P}}_e \text{ and } B \text{ in } \underline{\underline{B}} .$$

The theorem is : given any λ in $\underline{\underline{P}}$, there exists a probability measure μ_λ on $\underline{\underline{E}}$ such that

$$\lambda(B) = \int \eta(B) \, d\mu_\lambda(\eta) .$$

A close scrutiny of this equation suggests that we can reformulate the representation theorem as follows. Define $P : \underline{\underline{P}}_e \times \underline{\underline{B}} \longrightarrow [0, 1]$ by the formula

$$P(\eta, B) = \eta(B).$$

P satisfies the following properties.

i) $P(\eta, \cdot)$ is an invariant ergodic probability measure on $\underline{\underline{B}}$ for every η in $\underline{\underline{P}}_e$, and

ii) $P(\cdot, B)$ is $\underline{\underline{E}}$-measurable for every B in $\underline{\underline{B}}$.

The representation theorem says that every invariant probability measure λ, ergodic or non-ergodic, is a mixture of the invariant ergodic transition probability P with respect to some probability measure μ_λ on the borel structure $(\underline{\underline{P}}_e, \underline{\underline{E}})$.

These results led us to the question whether mixtures of invariant non-ergodic transition probabilities retain the flavour of non-ergodicity or not.

A solution to this problem is the core of this Chapter. Section 2 gives a counter example and Section 3 gives three sufficient conditions for every mixture of P to be non-ergodic.

2. Example. The example given here is modelled along the lines of the example given in Chapter 2.

Let X be any uncountable set, $\underline{\underline{A}}$ the countable - co-countable σ-algebra on X, and λ the probability measure on $\underline{\underline{A}}$ given by $\lambda(A) = 0$ or 1 according as A is countable or co-countable subset of X.

For every x in X, a quintuplet $(Y_x, \underline{\underline{B}}_x, \mu_x, T_x, y_x)$ is associated with the following properties.

i) $(Y_x, \underline{\underline{B}}_x)$ is a borel structure.

ii) T_x is a measurable transformation from Y_x into Y_x.

iii) $y_x \in Y_x$ and is a fixed point of T_x, i.e., $T_x y_x = y_x$.

iv) μ_x is an invariant, non-ergodic probability measure on $\underline{\underline{B}}_x$.

Let $Y = \prod_{x \in X} Y_x$, the product space, and $\underline{\underline{B}} = \prod_{x \in X} \underline{\underline{B}}_x$, the product σ-algebra on Y. Define a transformation $T : Y \longrightarrow Y$ by $(Tf)(x) = T_x[f(x)]$, f in Y. It is a routine piece of work to check that T is measurable. Let f_o be the element in Y defined by $f_o(x) = y_x$, for x in X. Clearly, $T f_o = f_o$. Let $f_o^x = f_o / X - \{x\}$, the restriction of f_o to $X - \{x\}$. For every B in $\underline{\underline{B}}$, define $B_x = f_o^x$-th section of $B = \{ g(x) \text{ in } Y_x : g \in B \text{ and } g = f_o \text{ on } X - \{x\} \}$. Note that $\underline{\underline{B}}_x = \{ B_x : B \text{ in } \underline{\underline{B}} \}$. Now, $P : X \times \underline{\underline{B}} \longrightarrow [0, 1]$ is defined by the formula

$$P(x, B) = \mu_x(B_x), \quad \text{for } x \text{ in } X \text{ and } B \text{ in } \underline{\underline{B}}.$$

The properties of P are listed below.

i) $P(x, \cdot)$ is a probability measure on $\underline{\underline{B}}$ for every x in X.

ii) $P(x, \cdot)$ is an invariant measure for every x in X.

This follows from the fact that for every B in $\underline{\underline{B}}$, $(T^{-1}B)_x = T_x^{-1}B_x$. The fact that is instrumental in establishing the above identity is $Tf_o = f_o$.

iii) $P(x, \cdot)$ is a non-ergodic measure for every x in X.

Let B_x be an invariant set under T_x with the property $0 < \mu_x(B_x) < 1$.

Let $B = \coprod_{z \in X} C_z$, where $C_x = B_x$ and $C_z = Y_z$ for $z \neq x$. B is available in $\underline{\underline{B}}$ and invariant under T. Clearly $0 < P(x, B) < 1$.

iv) $P(\cdot, B)$ is $\underline{\underline{A}}$-measurable for every B in $\underline{\underline{B}}$.

Since every B in $\underline{\underline{B}}$ is a countable dimensional cylinder, it follows that $B_x = \emptyset$ for all but a countable number of x's, or $= Y_x$ for all but a countable number of x's. Consequently, $P(x, B) = 0$ for all but a countable number of x's or $= 1$ for all but a countable number of x's.

Now, let us take the mixture μ of P with respect to λ. μ is an invariant, $0 - 1$ valued measure on $\underline{\underline{B}}$. Hence μ is ergodic.

Remark. In the convex set $\underline{\underline{P}}$ of all invariant measures, an invariant measure is an extreme point of $\underline{\underline{P}}$ if and only if it is ergodic. Our example shows that a certain generalised convex combination of non-extreme points gives an extreme point!

3. Sufficient conditions. If $\mu_1, \mu_2, \ldots$ is a sequence of invariant non-ergodic probability measures on $\underline{\underline{B}}$ and $\alpha_1, \alpha_2, \ldots$ is a sequence of nonnegative numbers with the property $\sum_{i \geq 1} \alpha_i = 1$, then the probability measure $\sum_{i \geq 1} \alpha_i \mu_i$ is invariant and non-ergodic.

Theorem 3.3.1. Let X be a Lindelof topological space, $\underline{\underline{A}}$ the σ-algebra on X generated by open subsets of X. Let P be an invariant non-ergodic transition probability defined on $X \times \underline{\underline{B}}$ with the additional property that $P(\cdot, B)$ is a continuous function on X for every B in $\underline{\underline{B}}$. Then any mixture of P is non-ergodic.

<u>Proof</u>. Let μ be a mixture of P with respect to some probability measure λ on $\underline{\underline{A}}$. Suppose μ is ergodic. For every B in $\underline{\underline{B}}$, define $U_B = \{x \text{ in } X: P(x, B) > 0\}$, and $V_B = \{x \text{ in } X: P(x, B) < 1\}$. Because of continuity, U_B and V_B are open subsets of X. Consider the family $\underline{\underline{F}} = \{U_B : B \text{ in } \underline{\underline{B}}, B \text{ invariant and } \mu(B) = 0\} \cup \{V_C : C \text{ in } \underline{\underline{B}}, C \text{ invariant and } \mu(C) = 1\}$. Note that $\lambda(U_B) = 0$ if B is in $\underline{\underline{B}}$, B invariant and $\mu(B) = 0$. Also $\lambda(V_C) = 0$ if C is in $\underline{\underline{B}}$, C invariant and $\mu(C) = 1$. Further, the family $\underline{\underline{F}}$ is an open cover for X. For, if x is in X, there exists an invariant set D in $\underline{\underline{B}}$ such that $0 < P(x, D) < 1$, since $P(x, \cdot)$ is non-ergodic. Since μ is ergodic, $\mu(D) = 0$ or $= 1$. In any case, $x \in U_D$ or $x \in V_D$. From $\underline{\underline{F}}$ we can extract a countable sub-cover for X. Since every set in $\underline{\underline{F}}$ has λ measure zero, $\lambda(X) = 0$, a contradiction. This proves the theorem.

The basic idea in the proofs of the following theorems is essentially the same as that used in Theorem 3.3.1.

<u>Theorem 3.3.2.</u> Let X be a metric space containing a dense subset whose cardinality has measure zero. Let $\underline{\underline{A}}$ be the

σ-algebra on X generated by open subsets of X. Let P be an invariant non-ergodic transition probability on $X \times \underline{\underline{B}}$ with the additional property that $P(\cdot, B)$ is a continuous function on X for every B in $\underline{\underline{B}}$. Then any mixture of P is non-ergodic.

Proof. Let μ be a mixture of P with respect to λ. Suppose μ is ergodic. For every B in $\underline{\underline{B}}$, define $U_B = \{x \text{ in } X: P(x, B) > 0\}$ and $V_B = \{x \in X : P(x, B) < 1\}$. As in the proof of Theorem 3.3.1, we note the following.

i) U_B and V_B are open subsets of X.

ii) The family $\underline{\underline{F}} = \{U_B : B \text{ in } \underline{\underline{B}},\ B \text{ invariant and } \mu(B) = 0\} \cup \{V_C : C \text{ in } \underline{\underline{B}},\ C \text{ invariant and } \mu(C) = 1\}$ is an open cover for X.

iii) For every set U_B in $\underline{\underline{F}}$ and V_C in $\underline{\underline{F}}$, we have $\lambda(U_B) = 0 = \lambda(V_C)$.

Consequently, by the Theorem quoted in the paragraph preceding the statement of Theorem 2.2.7, λ-measure of the union of all sets in $\underline{\underline{F}}$ is zero. This implies $\lambda(X) = 0$. But this is a contradiction. Hence μ is non-ergodic.

Theorem 3.3.3. Let X be a Hausdorff topological space, $\underline{\underline{A}}$ the σ-algebra on X generated by open subsets of X, and λ a regular probability measure on $\underline{\underline{A}}$, i.e., $\lambda(A) = \operatorname{Sup}\left\{\lambda(C):\ C \text{ compact subset of } X \text{ and } C \subset A\right\}$, for every A in $\underline{\underline{A}}$. Let P be an invariant non-ergodic transition probability on $X \times \underline{\underline{B}}$ with the additional property that $P(\cdot, B)$ is a continuous function on X for every B in $\underline{\underline{B}}$. Then the mixture μ of P with respect to λ is non-ergodic.

Proof. First, observe the following. If $V_\alpha : \alpha \in D$ is a family of open subsets of X and $\lambda(V_\alpha) = 0$ for every $\alpha \in D$, then $\lambda\left(\bigcup_{\alpha \in D} V_\alpha\right) = 0$. This follows from the regularity of λ. More precisely, for every compact set $C \subset \bigcup_{\alpha \in D} V_\alpha$, $\lambda(C) = 0$.

As in the case of the previous theorems, define $U_B = \left\{x \in X : P(x, B) > 0\right\}$ and $V_B = \left\{x \in X : P(x, B) < 1\right\}$ for every B in $\underline{\underline{B}}$.

Note:

i) U_B, V_B are open subsets of X for every B in $\underline{\underline{B}}$,

ii) the family $\underline{\underline{F}} = \{U_B : B$ in $\underline{\underline{B}}$, B invariant and $\mu(B) = 0\} \cup \{V_C : C$ in $\underline{\underline{B}}$, C invariant and $\mu(C) = 1\}$ is an open cover for X, and

iii) every U_B in $\underline{\underline{F}}$, and V_C in $\underline{\underline{F}}$ has λ-measure zero.

Consequently, λ-measure of the union of all sets in $\underline{\underline{F}}$ is zero. This implies $\lambda(X) = 0$. This contradiction shows that μ is non-ergodic.

CHAPTER 4

NON-EXISTENCE OF NONATOMIC MEASURES

1. Introduction. The following interesting result appears in [23a, Rudin, Theorem 5; pp. 41]. 'There are no regular nonatomic measures on the Borel σ-algebra of any compact Hausdorff space X, no subset of which is perfect'. Question: Are there any nonatomic measures on the Borel σ-algebra? When we say measures, the measure identically equal to zero is excluded in our arguments.

The purpose of this Chapter is two-fold. One is to examine the question posed in the first paragraph, in detail, in certain ordinals space and the other is studying the borel structure itself of this space.

The space, we have in mind, is $\bar{X} = [0, \Omega]$, the collection of all ordinals less than or equal to the first uncountable ordinal Ω . Let $X = [0, \Omega)$. Equip $\bar{X}$ and X with order topologies. $\bar{X}$ is a compact Hansdorff space containing no perfect subsets. A nonempty closed subset of a topological space is said to be perfect if it does not contain any isolated point. For the topological aspects of the spaces $\bar{X}$ and X are concerned, refer [14, Kelley, pp. 29, 30, 57, 59, 76, 131, 132, 162, 164, 165, 167, 172, 173]. These spaces are used, mostly, in constructing counter-examples in Topology and so their topological properties are investigated, in detail, in the literature. But their borel structure does not seem to have received proper attention. In [11, Halmos, pp. 231], a property of the borel subsets of $\bar{X}$ is observed.

A measure λ defined on the Borel σ-algebra, i.e., the σ-algebra generated by open subsets of a Hausdorff topological space is said to be regular, if for every Borel set B,

$$\mu(B) = \operatorname{Sup}\left\{\mu(C) : C \text{ compact and } C \subset B\right\}.$$

We were unable to obtain a complete solution to the problem posed in the first paragraph of this Section. However, towards the end of this Chapter, it is observed that there are no nonatomic measures on the borel σ-algebra of some special spaces of interest.

2. <u>The borel structure of $\bar{X}$.</u> A subset A of X is said to be unbounded if given any α in X there exists a β in A such that $\beta > \alpha$. In [11, Halmos, pp. 231], it is observed that the class of all unbounded closed subsets of X is closed under the formation of countable intersections. Further, every borel subset A of $\bar{X}$ has the property: either A or A^c contains an unbounded closed subset of X. Let $\underline{\underline{B}}$ denote the borel σ-algebra on $\bar{X}$. We define a measure μ on $\underline{\underline{B}}$ as follows.

$$\mu(B) = 1 \quad \text{if } B \text{ contains an unbounded closed subset of } X,$$
$$= 0 \quad \text{otherwise, for } B \text{ in } \underline{\underline{B}}.$$

In view of the above remarks, it is easy to verify that μ is a measure, and further it is continuous, i.e., $\mu(\{x\}) = 0$ for x in $\bar{X}$. Further, note that μ is not regular. The following theorem gives a complete characterisation of the borel subsets of $\bar{X}$.

Theorem 4.2.1. Let $\underline{\underline{B}}_1 = \{A \subset \bar{X} : A$ or A^c contains an unbounded closed subset of $X\}$. Then $\underline{\underline{B}} = \underline{\underline{B}}_1$.

Proof. We need only to prove $\underline{\underline{B}}_1 \subset \underline{\underline{B}}$. Let A be any unbounded closed subset of X. It suffices to prove that every subset B of A^c is Borel. Assume, without loss of generality, $\Omega \notin B$ and $0 \in A$.

1°. For every α in A, α' stands for the first succeeding ordinal of α in A. α' exists for any α in A since A is unbounded.

We shall express B as a countable union of sets A_n where each A_n is a set having atmost one element in common with (α, α') for every α in A and we show that any such set is Borel.

2°. Let f be any function defined on a subset C of A such that $\alpha < f(\alpha) < \alpha'$ for every α in C. We shall prove that $f(C)$ ($\subset A^c$) is Borel. Since $\alpha < f(\alpha) < \alpha'$, we can write $f(\alpha) = \bigcap_{i \geq 1} A_{i\alpha}$ where each $A_{i\alpha}$ is open and $A_{i\alpha} \subset (\alpha, \alpha')$. Now, $f(C) = \bigcap_{i \geq 1} B_i$ where $B_i = \bigcup_{\alpha \in C} A_{i\alpha}$. Each B_i is open and hence $f(C)$ is Borel.

3°. Since A is closed, for any β in A^c, $\beta \neq \Omega$, there exists an α in A such that $\alpha < \beta < \alpha'$. For, if $\bar{\alpha}$ is the first element in A such that $\bar{\alpha} > \beta$ and $\alpha = \operatorname{Sup}\{\partial : \partial \in A$ and $\partial < \beta\}$, then $\bar{\alpha} = \alpha'$.

4°. Define a set valued function g on A by

$$g(\alpha) = \{\beta \in B ; \ \alpha < \beta < \alpha'\}, \text{ for } \alpha \text{ in } A.$$

It is easily seen that $\bigcup_{\alpha \in A} g(\alpha) = B$. For each α in A, we shall fix an enumeration for the elements of $g(\alpha)$ whenever $g(\alpha) \neq \emptyset$. The sets $A_n = \bigcup_{\alpha \in A} \{n^{th}$ element in the enumeration of $g(\alpha)$ whenever there is n^{th} element in the enumeration of $g(\alpha)\}$, $n \geq 1$, are Borel by 2°. So, $B = \bigcup_{n \geq 1} A_n$. Hence B is Borel.

3. <u>Measures on $\bar{X}$</u>. The following result of Ulam is well known. See, for example, Lemma 2 of [12, Halmos, pp. 111]. We give here an alternate proof.

<u>Lemma 4.3.1.</u> Let Y be any set of cardinality $\leq \aleph_1$. There is no continuous probability measure on the power set $P(Y)$ of Y.

<u>Proof</u>. It is enough if we treat the case: cardinality of Y $= \aleph_1$. Suppose there is such a measure λ on $P(Y)$. By theorem 1 of [23, B. V. Rao, pp. 614], product σ-algebra $P(Y) \times P(Y) = P(Y \times Y)$. Identify Y with X. Let $D_1 = \{(\alpha, \beta): \alpha \leq \beta, \alpha, \beta \text{ in } X\}$ and $D_2 = \{(\alpha, \beta): \alpha > \beta, \alpha, \beta \text{ in } X\}$. By Fubini's theorem, $\lambda \times \lambda(D_1) = 0 = \lambda \times \lambda(D_2)$. Hence $\lambda \times \lambda(Y \times Y) = 0$. Hence such a λ does not exist.

<u>Lemma 4.3.2.</u> Let λ be any continuous measure on $\underline{\underline{B}}$. Then $\lambda = c\mu$ for some $c \geq 0$.

<u>Proof</u>. By Lebesgue decomposition theorem [11, Halmos, pp. 134], we can write $\lambda = \lambda_1 + \lambda_2$, where $\lambda_1 << \mu$ and $\lambda_2 \perp \mu$. So,

$\lambda_1 = c\mu$ for some $c \geq 0$ and there exists a Borel set Y such that $\lambda_2(Y) = \lambda_2(\bar{X})$ and $\mu(Y) = 0$. By Theorem 4.2.1., $Y \cap \underline{\underline{B}} = P(Y)$, and $\lambda_2 / Y \cap \underline{\underline{B}}$ is continuous. Hence by Lemma 4.3.1., $\lambda_2 \equiv 0$.

Corollary 4.3.3. There is no nonatomic measure on $\underline{\underline{B}}$.

Theorem 4.3.4. Any measure λ on $\underline{\underline{B}}$ can be written as $c\mu + \varphi$, where $c \geq 0$ and φ is concentrated on a countable subset of $\bar{X}$.

Proof. λ can be written as $\lambda_1 + \lambda_2$ where λ_1 is continuous and λ_2 is concentrated on a countable subset of $\bar{X}$. Lemma 4.3.2. completes the proof.

Corollary 4.3.5. Every regular measure on $\underline{\underline{B}}$ is concentrated on a countable number of points of $\bar{X}$.

Proof. Let λ be a regular measure on $\underline{\underline{B}}$. By Theorem 4.3.4., we can write $\lambda = c\mu + \varphi$, where φ is concentrated on a countable number of points of $\bar{X}$. Since μ is not regular, $c = 0$.

4. Measurable functions on $\bar{X}$.

Theorem 4.4.1. A real valued function on $\bar{X}$ is $\underline{\underline{B}}$-measurable if and only if it is constant on an unbounded closed subset of X.

Proof. Let f be any real valued function defined on $\bar{X}$. Suppose f is constant on an unbounded closed subset of X, i.e., $f(x) = c$, for every x in A, where c is a real number and A is an unbounded closed subset of X. Let B be any borel subset of the real line. If $c \in B$, then $f^{-1}(B) \supset A$, and hence, $f^{-1}(B) \in \underline{\underline{B}}$. If $c \notin B$, then $[f^{-1}(B)]^c \supset A$, and hence, $f^{-1}(B) \in \underline{\underline{B}}$. f is $\underline{\underline{B}}$-measurable. Conversely, let f be $\underline{\underline{B}}$-measurable. If $\underline{\underline{C}}$ denotes the borel σ-algebra of the real line, then $f^{-1}(\underline{\underline{C}})$ is a separable sub σ-algebra of $\underline{\underline{B}}$. Restrict the measure μ to $f^{-1}(\underline{\underline{C}})$. By proposition 1.2.2., μ is concentrated on an atom of $f^{-1}(\underline{\underline{C}})$, i.e., there exists an atom A of $f^{-1}(\underline{\underline{C}})$ such that $\mu(A) = 1$. Evidently A must contain an unbounded closed subset of X, and observe that f is constant on the atoms of $f^{-1}(\underline{\underline{C}})$.

5. Some general cases. The following Lemma is in the folklore.

Lemma 4.5.1. Let X be a compact T_2 topological space and $\underline{\underline{B}}$ its borel σ-algebra. For regular measures on $\underline{\underline{B}}$, continuity and nonatomicity are equivalent.

Proof. If μ is nonatomic, then it is, obviously, continuous. Suppose μ is continuous and not nonatomic. Let B in $\underline{\underline{B}}$ be a μ-atom. Since μ is regular, we can choose B to be compact. Further, μ is two-valued on $B \cap \underline{\underline{B}}$. In view of these, it suffices to show that there are no two-valued regular continuous measures on $\underline{\underline{B}}$. For any measure μ on $\underline{\underline{B}}$, there exists a point x_o in X such that $\mu(U) > 0$ for every open subset U of X containing x_o. If not, for every x in X, there exists an open set U_x containing x such that $\mu(U_x) = 0$. $\{U_x : x \text{ in } X\}$ is an open cover for X. Extracting a finite subcover, we note that $\mu(X) = 0$. If μ is two-valued, regular and continuous, then $\mu(\{x_o\}) = \text{Inf}\{\mu(U) : U$ open subset of X containing $x_o\}$. This gives us

$\mu(\{x_o\}) = \mu(X)$, a contradiction to the continuity of the measure μ.

<u>Theorem 4.5.2:</u> Let X be a compact Hausdorff space, no subset of which is perfect and every singleton is a G_δ. Then there are no nonatomic measures on the borel σ-algebra $\underline{\underline{B}}$ of X.

<u>Proof</u>. Let $\underline{\underline{C}}$ denote the Baire σ-algebra of X, i.e., the σ-algebra generated by compact G_δ subsets of X. Every measure on $\underline{\underline{C}}$ can be extended, uniquely, to $\underline{\underline{B}}$ as a regular measure. By the hypothesis of the theorem, all singletons are available in $\underline{\underline{C}}$. Let μ be a nonatomic measure on $\underline{\underline{B}}$. Let μ_o be the regularisation of μ. This μ_o can be obtained in two ways. Let $C(X)$ be the collection of all continuous functions on X. The linear functional $L : C(X) \longrightarrow R$ defined by

$$L(f) = \int f d\mu \qquad \text{for } f \text{ in } C(X),$$

is nonnegative and continuous. By Riesz representation theorem, there exists a regular measure μ_o on $\underline{\underline{B}}$ such that

$$\int f d\mu = \int f d\mu_o \quad \text{for every } f \text{ in } C(X).$$

The other way is to restrict μ to $\underline{\underline{C}}$ and then find the regular extension μ_o to $\underline{\underline{B}}$.

Since μ is nonatomic, it is continuous. μ_o is also continuous. Reason: $\mu / \underline{\underline{C}}$ is continuous. By the preceding Lemma, any regular continuous measure is nonatomic. Hence μ_o is nonatomic. The result of Rudin, stated at the beginning of the Introduction, is contradicted.

Remark. The problem of the existence of nonatomic measures on the power set $P(X)$ of a set X comes under the purview of the central problem stated at the beginning of this Chapter. For, it is possible to give a compact Hausdorff topology $\mathcal{T}$ on X such that (i) no subset of X is perfect under $\mathcal{T}$ and (ii) the σ-algebra generated by $\mathcal{T}$ is the class of all subsets of X, $P(X)$. More precisely, this topology $\mathcal{T}$ can be obtained as follows. Let x_o be a fixed point of X. Let $\mathcal{T}_o$ be the discrete topology on $X - \{x_o\}$. Do one point compactification to $X - \{x_o\}$ by adjoining the point x_o to $X - \{x_o\}$. The resultant topology $\mathcal{T}$ fulfils all the requirements. So, the following question remains open. Does there exist a nonatomic measure on the power set of any set? Under Continuum Hypothesis, Ulam [26, Satz 3, pp. 149] proved that there is no nonatomic measure on the power set of any set.

CHAPTER 5

EXISTENCE OF NONATOMIC CHARGES

1. Introduction. There are no satisfactory necessary and sufficient conditions for a σ-algebra to have a nonatomic measure defined on it. However, in certain topological measure spaces, such characterisations were obtained. See, for example, Theorem 1 of [17, Knowles, pp. 64] and Corollary 3.5 of [21, Luther, pp. 458]. The problem tackled here is to give characterisations of algebras admitting nonatomic charges. The characterisations given here turn out to be very simple. One characterisation gives conditions on the algebra and another on its Stone space. These results provide a characterisation of superatomic Boolean algebras.

The following are the pertinent definitions used in this Chapter. There may be a repetetion of one or two definitions. These are included to make this Chapter complete in all details.

i) A <u>field</u> is a collection of subsets of a set - closed under complementation, finite unions and contains empty set, $\emptyset$.

ii) For the definition of Boolean algebra, we refer to [12, Halmos, pp. 5] or [24, Sikorski, pp. 1]. The zero element and the unit element of a Boolean algebra are denoted by 0 and 1 respectively. Further, the join and meet of two elements are denoted by $\vee$ and $\wedge$ respectively, and the complement of an element of Boolean algebra by prime, $'$.

Boolean algebras are indicated by Roman capital letters A, B, C ... and the elements by a, b, c, ... etc., with or without suffixes. Fields of sets are denoted by $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$ ... and the elements by A, B, C ... etc., with or without suffixes. Whether we are tackling with a Boolean algebra or a field of sets will be made clear in every situation so as to give no room for confusion in the notation.

The join, meet and complementation in the case of field of sets are denoted by $\cup$, $\cap$ and ^{c} respectively.

A <u>σ-field</u> of subsets of a set X is a field on X closed under countable unions.

Here, we wish to point out that these conventions are used only in this Chapter. In the other Chapters, we, usually, deal with σ-field of sets and revert to the nomenclature of calling such entities by σ-algebras.

There is nothing to distinguish between field of sets and Boolean algebras in view of the Stone representation Theorem. This statement is amplified at the end of this Section. Whatever notion we introduced in Boolean algebras, they can be defined, verbatim, for field of sets.

iii) <u>A charge</u> is a <u>non-zero</u>, real valued, nonnegative and finitely additive function whose domain of definition is either a Boolean algebra or a field of sets, vanishing at the zero element or empty set, as the case may be.

iv) A charge μ on a Boolean algebra A is said to be <u>nonatomic</u> if for every a in A with $\mu(a)$ positive, there exists a b in A such that $b < a$ and $0 < \mu(b) < \mu(a)$.

v) Two Boolean algebras A and B are said to be isomorphic if there exists a one-one and onto map T from A to B preserving the operations - join and complementation.

vi) A collection of non-zero elements $a_{i_1, i_2, \ldots, i_k}$: $k \geq 1$ and $i_1, i_2, \ldots, i_k$ is any sequence of 0's and 1's in a Boolean algebra A is said to be a tree in A if

(1) $a_{i_1, i_2, \ldots, i_{k-1}, 0} \vee a_{i_1, i_2, \ldots, i_{k-1}, 1} =$

$$a_{i_1, i_2, \ldots, i_{k-1}},$$

(2) $a_{i_1, i_2, \ldots, i_{k-1}, 0} \wedge a_{i_1, i_2, \ldots, i_{k-1}, 1} = 0,$

and (3) $a_0 \vee a_1 = 1.$

vii) In a compact Hausdorff space, X, Baire (borel) σ-field is the σ-field generated by compact G_δ (compact) subsets of X.

viii) A <u>measure</u> is a charge whose domain of definition is either field of sets or a σ-field of sets and is countably additive.

ix) A measure on the borel σ-field of a compact Hausdorff space is said to be <u>regular</u> if the measure of any borel set can be approximated from below by the measure of compact subsets.

x) A Boolean algebra is said to be super-atomic if its Stone space is scattered, i.e., contains no perfect subsets. See [24, Sikorski, pp. 35].

xi) A Boolean algebra B is said to be <u>atomless</u> if $b \in B$, $b \neq 0$ implies there exists $a \in B$ such that $a \neq 0$, $a \neq b$ and $a < b$.

The following results are used in the sequel without explicit mention.

Any measure on the Baire σ-field of a compact Hausdorff space can be extended uniquely as a regular measure to its borel σ-field. See [11, Halmos, pp. 239].

Any Boolean algebra A is isomorphic to the field of all clopen subsets of some compact Hausdorff totally disconnected space X. X is called the Stone space of A. See [24, Sikorski, pp. 24] or [12, Halmos, pp. 79].

2. Preliminary results.

Proposition 5.2.1. Let μ be a measure on a σ-field $\underline{\underline{A}}$ and $\underline{\underline{B}}$ a field which generates $\underline{\underline{A}}$. If μ is nonatomic on $\underline{\underline{A}}$, then μ is nonatomic on $\underline{\underline{B}}$.

Proof. Suppose μ is not nonatomic on $\underline{\underline{B}}$. Then there exists a set B in $\underline{\underline{B}}$ such that $\mu(B)$ is positive and C in $\underline{\underline{B}}$, $C \subset B$ implies $\mu(C) = 0$ or $\mu(B)$. In other words, μ is two-valued on $B \cap \underline{\underline{B}}$. $B \cap \underline{\underline{B}}$ is a generator for $B \cap \underline{\underline{A}}$. Hence μ is two-valued on $B \cap \underline{\underline{A}}$. This is a contradiction.

Remark. The converse of this Proposition is not true. Let $X = [0, 1]$, $\underline{\underline{B}}$ the field generated by the intervals

$(a, b] \subset [3/8, 5/8)$. Restriction of Lebesgue measure λ to this field is nonatomic and [Range of $\lambda / \underline{\underline{B}}$] $\cap (\frac{1}{4}, \frac{3}{4}) = \emptyset$. Hence $\lambda / \sigma(\underline{\underline{B}})$ does not have Darboux property. For, [Range of $\lambda / \underline{\underline{B}}$] is a dense subset of [Range of $\lambda / \sigma(\underline{\underline{B}})$]. λ can not be nonatomic on $\sigma(\underline{\underline{B}})$.

The following result is proved in [21, Luther, Lemma 3.4, pp. 457]. Here, we give an alternate proof.

<u>Proposition 5.2.2.</u> Let X be a compact Hausdorff space, $\underline{\underline{B}}$ and $\underline{\underline{A}}$ Baire and borel σ-fields of X respectively.
(i) If μ is a nonatomic measure on $\underline{\underline{B}}$, then any extension of this measure to $\underline{\underline{A}}$ is nonatomic. (ii) If μ is a regular nonatomic measure on $\underline{\underline{A}}$, then its restriction to $\underline{\underline{B}}$ is nonatomic.

<u>Proof</u>. (i) follows from Theorem 1.3.2. For (ii), suppose μ is not nonatomic on $\underline{\underline{B}}$. Let B be a μ-atom of $\underline{\underline{B}}$. Since every measure on $\underline{\underline{B}}$ is regular, we can choose B to be a compact G_δ. μ is two valued measure on $B \cap \underline{\underline{B}}$. Further,

$B \cap \underline{\underline{B}}$ and $B \cap \underline{\underline{A}}$ are the Baire and borel σ-fields of the compact Hausdorff space B. In view of these observations, it is sufficient to prove that if μ is 0-1 valued on $\underline{\underline{B}}$, then its regular extension to $\underline{\underline{A}}$ is also 0-1 valued. If μ is 0-1 valued on $\underline{\underline{B}}$, there exists a point x in X such that every open Baire set (hence every Baire set) containing x has measure 1. The degenerate measure ∂_x is the regular extension of μ to $\underline{\underline{A}}$ and hence 0-1 valued.

The following theorem is taken from [17, Knowles, Theorem 1, pp. 64].

<u>Theorem 5.2.3.</u> There exists a regular nonatomic measure on the borel σ-field of a compact Hausdorff space if and only if X contains a perfect set.

Combining Proposition 5.2.2 and Theorem 5.2.3, we obtain the following result.

<u>Theorem 5.2.4.</u> There exists a nonatomic measure on the Baire σ- field of a compact Hausdorff space X if and only if X

contains a perfect set.

Proposition 5.2.5. Let X be a compact Hausdorff space and $\underline{\underline{A}}$ the field of all clopen subsets of X. Then every charge on $\underline{\underline{A}}$ is a measure on $\underline{\underline{A}}$.

Proof. Let $A_1, A_2, \ldots$ be a disjoint sequence of sets in $\underline{\underline{A}}$ such that $\bigcup_{i \geq 1} A_i$ is also in $\underline{\underline{A}}$. Since $\bigcup_{i \geq 1} A_i$ is compact, all but a finite number of elements of this sequence are empty.

Proposition 5.2.6. Let X be a compact Hausdorff totally disconnected space, i.e., a Stone space. The collection of clopen subsets of X is a generator for the Baire σ-field of X.

Proof. Finite linear combinations of indicator functions of clopen sets, by Stone-Weierstrass' theorem, is dense in the space of all continuous functions on X.

3. Main Results.

Theorem 5.3.1. Let A be a Boolean algebra. The following are equivalent.

i) There is a nonatomic charge on A.

ii) A contains a tree.

iii) A has a countable atomless sub-algebra.

iv) The Stone space of A contains a perfect subset.

Proof. (i) => (ii) is obvious.

(ii) => (iii). Let $\{a_{i_1, i_2, \ldots, i_k} : i_1, i_2, \ldots, i_k$ is any finite sequence of 0's and 1's, $k \geq 1\}$ be a tree in A. Let B be the algebra generated by this tree. Then B is countable and atmosless sub-algebra of A. (Finite disjoint joins of elements of a tree is the Boolean algebra B generated by the tree.)

(iii) => (iv). Observe that any atomless countable Boolean algebra is isomorphic to the field of all clopen subsets $\underline{C}$ of the Cantor space $\{0, 1\}^{\aleph_0}$.

See [24, Sikorski, C), pp. 28]. Consequently, there is a natural homomorphism from $\underline{C}$ into A, which is also one-one. Let X be the Stone space of A. There exists a continuous function f from X onto $\{0, 1\}^{\aleph_0}$. For this result, see the discussion in [24, Sikorski, First three paragraphs of page 34]. Now, we claim that there exists a minimal closed set $P \subset X$ such that the range of $f/P = \{0, 1\}^{\aleph_0}$. For this, we proceed as follows. The collection $\underline{\underline{Z}} = \{C \subset X : C$ closed in X and Range of $f/C = \{0, 1\}^{\aleph_0}\}$ is nonempty and is partially ordered by set inclusion. Let $C_\alpha : \alpha \in D$ be a chain in $\underline{\underline{Z}}$. Then $\bigcap_{\alpha \in D} C_\alpha \in \underline{\underline{Z}}$. For, let $y \in \{0, 1\}^{\aleph_0}$. There exists $x_\alpha \in C_\alpha$ such that $f(x_\alpha) = y$ for every $\alpha \in D$. Since $C_\alpha : \alpha \in D$ is a chain in $\underline{\underline{Z}}$, $x_\beta \in C_\alpha$ if $\beta \geq \alpha$. Since X is compact, there exists a subnet of $x_\alpha : \alpha \in D$ converging to some element x_0 of X. It is plain that $x_0 \in C_\alpha$ for every $\alpha \in D$ and $f(x_0) = y$. By Zorn's lemma, there exists a minimal closed set $P \subset X$ with the property that the range of $f/P = \{0, 1\}^{\aleph_0}$. Now, we observe that P is perfect.

For, if not, let x be an isolated point of P. Since $P - \{x\}$ is compact, $f(P - \{x\}) = f(P) - \{f(x)\}$ is closed in $\{0, 1\}^{\aleph_0}$. This implies that $\{f(x)\}$ is open and hence clopen. But $\{0, 1\}^{\aleph_0}$ is perfect. This contradiction shows that P is perfect. This completes the proof.

(iv) $\Longrightarrow$ (i). Let X be the Stone space of A, $\underline{\underline{D}}$ the collection of all clopen subsets of X and $\underline{\underline{B}}$ the Baire σ-field on X. Since X contains a perfect subset, there exists a nonatomic measure μ on $\underline{\underline{B}}$, by Theorem 5.2.4. By Proposition 5.2.1., $\mu / \underline{\underline{D}}$ is a nonatomic charge on $\underline{\underline{D}}$. Transfer $\mu / \underline{\underline{D}}$ to A. Thus we have a nonatomic charge on A.

<u>Remark</u>. If a Boolean algebra A contains a tree, then we can define a nice nonatomic charge on the algebra B generated by the tree. This charge on the sub-algebra B can be extended to A as a charge. See [24, Sikorski, (b_7), pp. 211]. But there is no guarantee that some extension of a given nonatomic charge will be nonatomic. An inkling of such a situation is provided in the Remark following Proposition 5.2.1.

Corollary 5.3.2. On any infinite σ-field of sets, there is a nonatomic charge.

Proof. Let $N = \{1, 2, 3, \ldots\}$. We can find N_o, N_1 such that $N_o \cap N_1 = \emptyset$, $N_o \cup N_1 = N$ and N_o, N_1 are both infinite. Thus, by induction on k, we can find sets $N_{i_1, i_2, \ldots, i_k}$ for every finite sequence $i_1, i_2, \ldots, i_k$ of 0's and 1's such that

(a) $N_{i_1, i_2, \ldots, i_{k-1}, 0} \cap N_{i_1, i_2, \ldots, i_{k-1}, 1} = \emptyset$,

(b) $N_{i_1, i_2, \ldots, i_{k-1}, 0} \cup N_{i_1, i_2, \ldots, i_{k-1}, 1} =$

$N_{i_1, i_2, \ldots, i_{k-1}}$, and

(c) $N_{i_1, i_2, \ldots, i_{k-1}, 0}$ and $N_{i_1, i_2, \ldots, i_{k-1}, 1}$

are infinite.

If $\underline{\underline{B}}$ is an infinite σ-field on X, we can find an infinite sequence of disjoint, nonempty sets $B_1, B_2, \ldots$ in $\underline{\underline{B}}$ such that $\bigcup_{i\geq 1} B_i = X$. Apply the above technique to $B_1, B_2, \ldots,$ to get a tree in $\underline{\underline{B}}$. Theorem 5.3.1 completes the proof.

Corollary 5.3.3. A Boolean algebra B is superatomic if and only if there is no nonatomic charge on B.

Proof. Trivially follows from Theorem 5.3.1.

4. Position of nonatomic probability charges in the space of all probability charges. A charge μ on a Boolean algebra A is said to be probability charge if $\mu(1) = 1$. Let $\underline{\underline{P}}$ denote the collection of all probability charges on A. Equip $\underline{\underline{P}}$ with a topology by defining convergence as follows. A net μ_α in $\underline{\underline{P}}$ converges to a μ in $\underline{\underline{P}}$ if $\mu_\alpha(a)$ converges to $\mu(a)$ for every a in A.

Theorem 5.4.1. The collection of all nonatomic probability charges on a Boolean algebra A is a dense subset of $\underline{\underline{P}}$ if and only if A is atomless, or, equivalently, if and only if the Stone space, X, of A is perfect.

Proof. Identify A with the field $\underset{=}{C}$ of all clopen subsets of X, and assume, without loss of generality, that every μ in $\underset{=}{P}$ is defined on $\underset{=}{C}$. Let $\underset{=}{B}$ be the Baire σ-field on X. Every μ on $\underset{=}{C}$ is countably additive and consequently, we can extend μ from $\underset{=}{C}$ to $\underset{=}{B}$ as a measure, uniquely. Denote this extension by $\bar{\mu}$. Let $\underset{=}{Q}$ denote the collection of all probability measures on $\underset{=}{B}$. The weak-star topology is defined as follows. A net λ_α in $\underset{=}{Q}$ converges to a λ in $\underset{=}{Q}$ if $\int f\,d\lambda_\alpha$ converges to $\int f\,d\lambda$ for every continuous function f on X. Define a map φ from $\underset{=}{P}$ to $\underset{=}{Q}$ as follows. $\varphi(\mu) = \bar{\mu}$, for μ in $\underset{=}{P}$. φ is one-one and onto. Further, it is a homeomorphism. For this, use the fact that finite linear combinations of indicator functions of sets in $\underset{=}{C}$ is dense in the space of all continuous functions on X, with supremum norm.

Let A be atomless, i.e., X is perfect. Knowles [17, pp. 65] proved that if X is perfect, the collection of all nonatomic probability measures $\underset{=}{N}$ on $\underset{=}{B}$ is a dense

G_δ subset of $\underline{\underline{Q}}$. By proposition 5.2.1, the collection of all nonatomic probability charges on $\underline{\underline{C}}$ contains $\psi^{-1}(\underline{\underline{N}})$, and, consequently is a dense subset of $\underline{\underline{P}}$.

Conversely, suppose X is not perfect. Let **x** be an isolated point of X. Note $\{x\} \in \underline{\underline{C}}$. The degenerate charge $\partial_x \in \underline{\underline{P}}$. Since the collection of all nonatomic charges is a dense subset of $\underline{\underline{P}}$, there exists a net μ_α of probability charges in $\underline{\underline{P}}$ converging to ∂_x. But $\mu_\alpha(\{x\}) = 0$ for every α and $\partial_x(\{x\}) = 1$. This contradiction shows that X is perfect.

CHAPTER 6

MEASURES WITH PRESCRIBED MARGINALS

1. Introduction. Let $(X, \underline{\underline{B}})$ and $(Y, \underline{\underline{C}})$ be two borel structures, and $(X \times Y, \underline{\underline{B}} \times \underline{\underline{C}})$ the product borel structure. Let μ and λ be two probability measures on $\underline{\underline{B}}$ and $\underline{\underline{C}}$ respectively. Let $\underline{\underline{M}}(\mu, \lambda)$ be the collection of all probability measures ξ on $\underline{\underline{B}} \times \underline{\underline{C}}$ with marginals μ and λ on $\underline{\underline{B}}$ and $\underline{\underline{C}}$ respectively, i.e., $\xi(B \times Y) = \mu(B)$ for every B in $\underline{\underline{B}}$ and $\xi(X \times C) = \lambda(C)$ for every C in $\underline{\underline{C}}$. The product measure $\mu \times \lambda$, obviously, belongs to $\underline{\underline{M}}(\mu, \lambda)$. It is easy to check that $\underline{\underline{M}}(\mu, \lambda)$ is a convex set.

Problem. What are the extreme points of $\underline{\underline{M}}(\mu, \lambda)$? A motivation for this problem came from the following theorem of [15, Kemp, pp. 1356]. '$\mu \times \lambda$ is an extreme point of $\underline{\underline{M}}(\mu \times \lambda)$ if and only if either μ is 0-1 valued or λ is 0-1 valued'.

A complete solution to the problem posed above is obtained in the case when X and Y are finite sets. More precisely, let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m$ be $m + n$ non-negative real numbers. Let $\underline{\underline{A}}$ be the collection of all non-negative matrices $A = (a_{ij})$ of order $m \times n$ with marginal sums $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$, i.e.,

$$\sum_{j=1}^{n} a_{ij} = b_i, \quad \text{for } i = 1,2,\ldots, m,$$

$$\sum_{i=1}^{m} a_{ij} = a_j, \quad \text{for } j = 1,2,\ldots, n, \quad \text{and}$$

$$a_{ij} \geq 0 \text{ for every } i \text{ and } j.$$

It is easy to see that $\underline{\underline{A}}$ is a compact convex set. ($\underline{\underline{A}}$ may be empty. In fact, $\underline{\underline{A}}$ is nonempty if and only if $\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j$.) In Section 2, some characterisations are given for a matrix A in $\underline{\underline{A}}$ to be an extreme point of $\underline{\underline{A}}$. As a simple consequence, Birkhoff- von Neumann's theorem on doubly stochastic matrices is obtained. In Section 4, supports of extreme points are analysed. In Section 5, a characterisation of extreme points is obtained in the general case. An explicit construction of extreme points is made using measure preserving transformations.

Let $A = (a_{ij})$ be any matrix. An ordered sequence $[a_{i_1j_1}, a_{i_2j_2}, \ldots a_{i_kj_k}]$ of elements in A is said to be a <u>loop</u> in A if (i) k is even, (ii) $i_1 = i_2, i_3 = i_4, \ldots, i_{k-1} = i_k$, (iii) $j_2 = j_3, j_4 = j_5, \ldots, j_k = j_1$, and (iv) the pairs (i_p, j_p), where $p = 1,2,\ldots, k$, are all distinct. A loop in A is said to be <u>positive</u> if every element in the loop is positive. <u>Support of A</u> is defined to be the collection of all pairs (i, j) such that a_{ij} is positive.

2. <u>Characterisations</u>. In this Section, we obtain several characterisations of extreme points of $\underline{\underline{A}}$.

<u>Lemma 6.2.1</u>. Every row (column) of a matrix A in $\underline{\underline{A}}$ contains an even number of elements of any loop in A.

<u>Proof.</u> Trivial.

<u>Lemma 6.2.2.</u> If $a_{i_1j_1}, a_{i_1j_2}; a_{i_2j_2}, a_{i_2j_3}; \ldots;$ $a_{i_pj_p}, a_{i_pj_{p+1}}; \ldots$ is an infinite sequence of elements from a given matrix A with the property that any two consecutive suffixes are distinct, then we can find a loop in A consisting of elements from the given sequence.

<u>Proof.</u> Let b' be the first element in the sequence whose suffix agrees with the suffix of one of the elements a' preceding b'.

Case (i). $a' = a_{i_kj_k}$ and $b' = a_{i_pj_p}$.

Then $[a_{i_kj_k}, a_{i_kj_{k+1}}; \ldots; a_{i_{p-1}j_{p-1}}, a_{i_{p-1}j_p}]$ is a loop in A.

Case (ii). $a' = a_{i_kj_{k+1}}$ and $b' = a_{i_pj_{p+1}}$.

Then $[a_{i_{k+1}j_{k+1}}, a_{i_{k+1}j_{k+2}}; \ldots; a_{i_pj_p}, a_{i_pj_{p+1}}]$ is a loop in A.

Case (iii). $a' = a_{i_kj_k}$ and $b' = a_{i_pj_{p+1}}$.

Then $[a_{i_{k+1}j_{k+1}}, a_{i_{k+1}j_{k+2}}; \ldots; a_{i_pj_p}, a_{i_pj_{k+1}}]$ is a loop in A.

Case (iv). $a' = a_{i_kj_{k+1}}$ and $b' = a_{i_pj_p}$.

This case can be disposed of in the same way as Case (iii).

In the proofs of many of the theorems, the basic idea is to exhibit a sequence satisfying the hypothesis of the Lemma 6.2.2.

Theorem 6.2.3. A matrix A in $\underline{\underline{A}}$ is an extreme point of $\underline{\underline{A}}$ if and only if there is no positive loop in A.

Proof. Suppose $[a_{i_1j_1}, a_{i_2j_2}, \ldots, a_{i_kj_k}]$ is a positive loop in A. Let $0 < \epsilon <$ minimum of $a_{i_1j_1}, \ldots, a_{i_kj_k}$. Define two matrices $B = (b_{ij})$ and $C = (c_{ij})$ as follows.

$$b_{i_pj_p} = a_{i_pj_p} + \epsilon, \quad \text{if } p \text{ is odd},$$

$$= a_{i_pj_p} - \epsilon, \quad \text{if } p \text{ is even},$$

and $\quad b_{ij} = a_{ij}$ if $(i, j) \neq (i_p, j_p)$ for any $p = 1, 2, \ldots, k$.

$$c_{i_pj_p} = a_{i_pj_p} + \epsilon, \quad \text{if } p \text{ is even},$$

$$= a_{i_pj_p} - \epsilon, \quad \text{if } p \text{ is odd},$$

and $$c_{ij} = a_{ij} \quad \text{if} \quad (i, j) \neq (i_p, j_p) \quad \text{for any}$$

$p = 1, 2, \ldots, k$.

Note that B and C are in $\underline{\underline{A}}$, distinct and

$$A = \frac{1}{2}(B + C).$$

Conversely, suppose A is not an extreme point of $\underline{\underline{A}}$. We can write $A = \alpha B + (1 - \alpha)C$ for some distinct $B = (b_{ij})$ and $C = (c_{ij})$ in $\underline{\underline{A}}$ and $0 < \alpha < 1$. If $a_{ij} = 0$, then $b_{ij} = c_{ij} = 0$. Since A and B are distinct, there exists a pair (i_1, j_1) such that $a_{i_1j_1} \neq b_{i_1j_1}$. Clearly $a_{i_1j_1}$ is positive. Assume, without loss of generality, $a_{i_1j_1} > b_{i_1j_1}$. There exists $j_2 \neq j_1$ such that $a_{i_1j_2} < b_{i_1j_2}$. Note that $a_{i_1j_2}$ is positive. There exists $i_2 \neq i_1$ such that $a_{i_2j_2} > b_{i_2j_2}$, $j_3 \neq j_2$ such that $a_{i_2j_3} < b_{i_2j_3}$, and so on. Thus, we can construct a sequence of positive elements, $a_{i_1j_1}$, $a_{i_1j_2}$; $a_{i_2j_2}$, $a_{i_2j_3}$; ...... with the property that any two consecutive suffixes distinct. By Lemma 6.2.2, we can find a positive loop in A. This proves the theorem.

Lemma 6.2.4. If every row and column of A contains atleast two positive elements, then A contains a positive loop.

Proof. It is easy to construct an infinite sequence of positive elements in A satisfying the hypothesis of Lemma 6.2.2.

Theorem 6.2.5. The following are equivalent.

(i) There is a positive loop in A.

(ii) There exists a square submatrix D of A of order k such that the number of positive elements in D is greater than or equal to 2k.

Proof. Suppose A contains a positive loop $[a_{i_1j_1}, a_{i_2j_2}, \ldots, a_{i_rj_r}]$. Let p_1 = the number of distinct elements among $i_1, i_2, \ldots, i_r$ and p_2 = the number of distinct elements among $j_1, j_2, \ldots, j_r$. Assume, without loss of generality, $p_1 \leq p_2$. Let E be the submatrix of A of order $p_1 \times p_2$ determined by $i_1, i_2, \ldots, i_r$ th rows and $j_1, j_2, \ldots, j_r$th columns of A. Let D be a submatrix of E obtained by deleting $p_2 - p_1$ columns from E. Since every column of D contains atleast two positive elements, the number of positive elements in $D \geq 2p_1$, and the order of the matrix D is $p_1 \times p_1$.

Conversely, let D be a square submatrix of order k such that the number of positive elements of D is $\geq 2k$ and let D satisfy, furthermore, the minimality property that for every square submatrix G of order $q \leq k-1$, G has less than $2q$ positive elements. We prove that every row and column of D contains atleast two positive elements. Suppose this is not true. Let some row, say i^{th}, contain atmost one positive element. Let $\alpha = \min_{\text{columns}} \{$the number of positive elements in a column of D$\}$. Case (i). $\alpha = 0$ or 1. Choose a column, say j^{th}, containing α positive elements. Let C be the matrix obtained from D by deleting the i^{th} row and the j^{th} column in D. By the minimal property of D, the number of positive elements in $C < 2k - 2$. On the other hand, by direct computation, the number of positive elements in C is $\geq 2k - 2$. Thus, we have a contradiction. Case (ii). $\alpha \geq 2$. If the i^{th} row contains only zeros, the number of positive elements in C, as constructed above, is $\geq \alpha(k - 1)$. This leads to a contradiction. If the i^{th} row contains one positive element, let F be the matrix obtained from D by deleting the i^{th} row and the column which contains the positive element of the i^{th} row. By direct argument, the number of positive elements in F is $\geq \alpha(k - 1)$. By the minimal property of D, the number of positive elements in $F < 2k - 2$. This is a contradiction. Lemma 6.2.4 completes the proof.

From what we have done till now, we have the following comprehensive "Theorem".

<u>Theorem 6.2.6</u> Let A be any matrix in $\underline{\underline{A}}$. The following are equivalent.

i) A is not an extreme point of $\underline{\underline{A}}$.

ii) There is a positive loop in A.

iii) There exists a submatrix E of A satisfying the property that every row and column of E has atleast two positive elements.

iv) There exists a square submatrix D of A satisfying the property that every row and column of D contains atleast two positive elements.

v) There exists a square submatrix F of A, say, of order k, satisfying the property that the number of positive elements in F is greater than or equal to 2k.

The equivalence of (i) and (v) is proved by Lindenstrauss [19, pp. 382]. After obtaining the results of the present Section, the author came to know that the equivalence of (i) and (ii) is also proved in [16, Theorem 4, pp. 265].

3. Some consequences.

Theorem 6.3.1. (Birkhoff - von Neumann). Let $\underline{\underline{A}}$ be the collection of all doubly stochastic matrices of order n. A matrix B in $\underline{\underline{A}}$ is an extreme point of $\underline{\underline{A}}$ if and only if B is a permutation matrix.

<u>Proof.</u> It is obvious that if B is a permutation matrix, then B is an extreme point of $\underline{\underline{A}}$. Suppose B is not a permutation matrix. There exists (i_1, j_1) such that $0 < b_{i_1 j_1} < 1$. There exist $j_2 \neq j_1$ such that $0 < b_{i_1 j_2} < 1$, $i_2 \neq i_1$ such that $0 < b_{i_2 j_2} < 1$, and so on. Thus, we have an infinite sequence $b_{i_1 j_1}, b_{i_1 j_2}$; $b_{i_2 j_2}, b_{i_2 j_3}$; ...... of positive elements in B with the property that any two consecutive suffixes are distinct. By Lemma 6.2.2, we can find a loop in B. Theorem 6.2.3 completes the proof.

<u>Remark.</u> For an alternate proof of Birkhoff-von Neumann's theorem, see [2, Berge, pp. 105-106].

Assume a is positive, where $a = a_1 + a_2 + \dots + a_n = b_1 + b_2 + \dots + b_m$. Define $a_{ij} = (b_i \cdot a_j)/a$, for $i = 1$ to m and $j = 1$ to n. Then $A = (a_{ij})$ is a matrix with row sums $b_1, b_2, \ldots, b_m$, and column sums $a_1, a_2, \ldots, a_n$. So, A is a member of $\underline{\underline{A}}$. Call this matrix the pseudo-product matrix.

<u>Theorem 6.3.2</u>. The pseudo-product matrix A is an extreme point of $\underline{\underline{A}}$ if and only if either $a_j = a$ for some j or $b_i = a$ for some i.

<u>Proof</u>. If either $a_j = a$ for some j or $b_i = a$ for some i, then $\underline{\underline{A}}$ will contain only the pseudo-product matrix A, and consequently, A is an extreme point of $\underline{\underline{A}}$. If $a_j \neq a$ for every j and $b_i \neq a$ for every i, we can find a_q, a_s, b_p and b_r all positive, where $q \neq s$ and $p \neq r$. Then, $[a_{pq}, a_{ps}, a_{rs}, a_{rq}]$ is a positive loop in A.

<u>Remark.</u> Compare this theorem with Theorem 3 of [15, Kemp, pp. 1356].

4. <u>Supports of extreme points.</u>

<u>Proposition 6.4.1.</u> Let A be B be two distinct extreme

points of $\underline{\underline{A}}$. Then the support of A neither contains nor is contained in the support of B properly.

Proof. Suppose support of A is contained properly in the support of B. Let (i_1, j_1) be such that $b_{i_1 j_1} > 0 = a_{i_1 j_1}$. There exists $j_2 \neq j_1$ such that $b_{i_1 j_2} < a_{i_1 j_2}$. Clearly, $b_{i_1 j_2}$ is positive. For, otherwise, $a_{i_1 j_2}$ will be equal to zero. There exists $i_2 \neq i_1$ such that $b_{i_2 j_2} > a_{i_2 j_2}$ which implies $b_{i_2 j_2}$ is positive. Thus we can construct an infinite sequence $b_{i_1 j_1}, b_{i_1 j_2}; b_{i_2 j_2}, b_{i_2 j_3}; \ldots$ of positive elements in B with the property that any two consecutive suffixes are distinct. By Lemma 6.2.2, there exists a positive loop in B giving rise to a contradiction. A similar argument shows that the support of B is not contained in the support of A properly.

Proposition 6.4.2. Let A and B be two distinct extreme points of $\underline{\underline{A}}$. Then the supports of A and B are distinct.

Proof. Suppose supports of A and B are equal. Since A and B are distinct, there exists a pair (i_1, j_1) such that $a_{i_1j_1} \neq b_{i_1j_1}$. Assume, without loss of generality, $a_{i_1j_1} > b_{i_1j_1}$ which obviously implies $a_{i_1j_1}$ is positive. There exists $j_2 \neq j_1$ such that $a_{i_1j_2} < b_{i_1j_2}$. Since supports are equal, $a_{i_1j_2}$ is positive. There exists $i_2 \neq i_1$ such that $a_{i_2j_2} > b_{i_2j_2}$, and so on. Thus we can construct an infinite sequence $a_{i_1j_1}, a_{i_1j_2}, a_{i_2j_2}, a_{i_2j_3}, \ldots$ of positive elements in A with the property that any two consecutive suffixes are distinct. By Lemma 6.2.2, there exists a positive loop in A, giving rise to a contradiction.

Corollary 6.4.3. The number of extreme points of $\underline{\underline{A}}$ is finite.

Proof. The set of all subsets of $\{(i, j) : i = 1$ to m and $j = 1$ to $n\}$ is finite.

5. <u>The general case.</u> In the general set-up of probability spaces $(X, \underline{\underline{B}}, \mu)$ and $(Y, \underline{\underline{C}}, \lambda)$, let $F_o = \{I_{B \times Y} : B \text{ in } \underline{\underline{B}}\} \cup \{I_{X \times C} : C \text{ in } \underline{\underline{C}}\}$. Let F be the linear manifold spanned by F_o, i.e., finite linear combinations of functions from F_o.

It is easy to verify that if ξ and η are in $\underline{\underline{M}}(\mu, \lambda)$, then $\int f d\xi = \int f d\eta$ for every f in F.

The proof of the following theorem is based on the proof of Theorem 1 of [9, Douglas, pp. 243].

<u>Theorem 6.5.1.</u> A probability measure ξ on $\underline{\underline{B}} \times \underline{\underline{C}}$ is an extreme point of $\underline{\underline{M}}(\mu, \lambda)$ if and only if F is dense in $L_1(X \times Y, \underline{\underline{B}} \times \underline{\underline{C}}, \xi)$.

<u>Proof.</u> Suppose ξ is not an extreme point of $\underline{\underline{M}}(\mu, \lambda)$. Let $\xi = \frac{\eta + \varphi}{2}$ for some η and φ in $\underline{\underline{M}}(\mu, \lambda)$. This implies $2\xi \geq \eta \geq 0$ and by the Radon-Nikodym theorem, there exists a function h in $L_\infty(X \times Y, \underline{\underline{B}} \times \underline{\underline{C}}, \xi)$ such that

$$\frac{d\eta}{d\xi} = h \qquad \text{a.e. } [\xi], \text{ and}$$

$$1 - h \neq 0 \qquad \text{a.e. } [\xi].$$

The function $1-h$ is orthogonal to F, i.e.,

$$\int f(1-h)d\xi = \int f d\xi - \int f h d\xi$$
$$= \int f d\xi - \int f d\eta = \int f d\xi - \int f d\xi$$
$$= 0 \qquad \text{for every } f \text{ in } F.$$

This clearly demonstrates that F is not dense in $L_1(X \times Y, \underline{\underline{B}} \times \underline{\underline{C}}, \xi)$. For, if F were to be dense, since the dual $L_1^*(X \times Y, \underline{\underline{B}} \times \underline{\underline{C}}, \xi)$ of $L_1(X \times Y, \underline{\underline{B}} \times \underline{\underline{C}}, \xi)$ is $L_\infty(X \times Y, \underline{\underline{B}} \times \underline{\underline{C}}, \xi)$, the linear functional induced by $1-h$ on L_1 vanishes identically on F would imply $1-h = 0$ a.e. $[\xi]$, a contradiction.

Conversely, suppose F is not dense in $L_1(X \times Y, \underline{\underline{B}} \times \underline{\underline{C}}, \xi)$. There exists an essentially non-zero function h in $L_\infty(X \times Y, \underline{\underline{B}} \times \underline{\underline{C}}, \xi)$ orthogonal to F.

Set

$$\eta(E) = \int_E h \, d\xi, \quad \text{for } E \text{ in } \underline{\underline{B}} \times \underline{\underline{C}}.$$

Since $\int hf\, d\xi = 0$ for every f in F, we have, in particular, $\int h\, d\xi = 0$. This implies $\eta\,(X \times Y) = 0$. Define

$$\varrho_1(E) = \int_E \left[1 + \frac{h}{\|h\|_\infty}\right] d\xi\,, \text{ and}$$

$$\varrho_2(E) = \int_E \left[1 - \frac{h}{\|h\|_\infty}\right] d\xi\,, \text{ for } E \in \underline{\underline{B}} \times \underline{\underline{C}},$$

where $\|h\|_\infty$ is L_∞- norm of h. Since $1 \pm [h/\|h\|_\infty] \geq 0$ a. e. $[\xi]$, the set function ϱ_1 and ϱ_2 are nonnegative. In fact, as can be easily checked, $\varrho_1, \varrho_2 \in \underline{\underline{M}}\,(\mu, \lambda)$. We can write $\xi = \frac{\varrho_1 + \varrho_2}{2}$. This completes the proof.

We conclude this section by constructing some extreme points of $\underline{\underline{M}}\,(\mu, \lambda)$ using measure preserving transformations.

Let T be a measurable transformation from X to Y preserving the measures μ and λ, i.e.,

$$T^{-1} C \in \underline{\underline{B}}\,, \text{ and}$$

$$\mu\,(T^{-1} C) = \lambda\,(C) \quad \text{for every } C \text{ in } \underline{\underline{C}}.$$

Let D be the graph of T, i.e.,

$$D = \{(x, Tx) : x \text{ in } X\}.$$

Let P_1 be the projection map from $X \times Y$ to X, i.e.,

$$P_1(x, y) = x \quad \text{for} \quad (x, y) \text{ in } X \times Y.$$

For every E in $\underline{\underline{B}} \times \underline{\underline{C}}$, $P_1(E \cap D)$ is available in $\underline{\underline{B}}$. For, let $\underline{\underline{E}} = \{E \text{ in } \underline{\underline{B}} \times \underline{\underline{C}} : P_1(E \cap D) \in \underline{\underline{B}}\}$. Rectangle sets are available in $\underline{\underline{E}}$, for $P_1[(B \times C) \cap D] = B \cap T^{-1} C$. $\underline{\underline{E}}$ is closed under countable unions, and complementation, for $P_1(E^c \cap D) = X - P_1(E \cap D)$. Hence $\underline{\underline{E}} = \underline{\underline{B}} \times \underline{\underline{C}}$. Define a set function ξ on $\underline{\underline{B}} \times \underline{\underline{C}}$ as follows.

$$\xi(E) = \mu[P_1(E \cap D)], \quad \text{for } E \text{ in } \underline{\underline{B}} \times \underline{\underline{C}}.$$

ξ is a probability measure on $\underline{\underline{B}} \times \underline{\underline{C}}$ with marginals μ and λ:

$$\begin{aligned}
\xi(B \times Y) &= \mu(P_1[(B \times Y) \cap D]) \\
&= \mu(B \cap T^{-1} Y) = \mu(B). \\
\xi(X \times C) &= \mu(P_1[(X \times C) \cap D]) \\
&= \mu(X \cap T^{-1} C) = \mu(T^{-1} C) \\
&= \lambda(C), \quad \text{since } T \text{ is}
\end{aligned}$$

measure preserving.

D is a thick subset of $X \times Y$ under ξ, i.e., the outer measure, $\xi^{*\prime}(D) = 1$. For, if E is any set in $\underline{\underline{B}} \times \underline{\underline{C}}$ containing D, then $P_1(E \cap D) = P_1(D) = X$. Loosely speaking, ξ is concentrated on the graph of T.

ξ is an extreme point of $\underline{\underline{M}}(\mu, \lambda)$. Suppose $\xi = \frac{\eta + \varphi}{2}$ for some η and φ in $\underline{\underline{M}}(\mu, \lambda)$. It is obvious that $\eta^*(D) = 1 = \varphi^*(D)$. Further, for any E in $\underline{\underline{B}} \times \underline{\underline{C}}$, $\eta^*(E \cap D) = \eta(E)$ and $\varphi^*(E \cap D) = \varphi(E)$. If $B \times C \in \underline{\underline{B}} \times \underline{\underline{C}}$, $(B \times C) \cap D \subset (B \cap T^{-1} C) \times Y$. Consequently

$$\begin{aligned}\eta(B \times C) \leq \eta[(B \cap T^{-1} C) \times Y] &= \mu(B \cap T^{-1} C) \\ &= \xi(B \times C),\end{aligned}$$

and

$$\begin{aligned}\varphi(B \times C) \leq \varphi[(B \cap T^{-1} C) \times Y] &= \mu(B \cap T^{-1} C) \\ &= \xi(B \times C).\end{aligned}$$

Hence $\xi(B \times C) = \varphi(B \times C) = \eta(B \times C)$ for every B in $\underline{\underline{B}}$ and C in $\underline{\underline{C}}$. Therefore, $\xi = \varphi = \eta$.

<u>Remark.</u> Theorem 6.5.1 is a generalisation of Theorem 1 of [19, Lindenstrauss, pp. 379].

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