CONTRIBUTIONS TO THE THEORY OF PERFECT MEASURES AND ERGODIC THEORY

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C O N T E N T S

p.t.o.

P R E F A C E

Ever since Kolmogorov's model for probability theory (see [18]) - the trinity $(X, \underline{\underline{A}}, P)$ - started gaining acceptance among workers in probability and statistics, there have been several attempts at studying nice classes of probability spaces (see [2], [12], [22], [34] and [37]). In one such attempt, the fascinating class of perfect probability spaces was introduced by Gnedenko and Kolmogorov [8].

A probability measure P on $(X, \underline{\underline{A}})$ is called perfect if for every real-valued $\underline{\underline{A}}$-measurable function f defined on X the induced measure P_f on the class of linear Borel sets is outer regular. $(X, \underline{\underline{A}}, P)$ is called a perfect probability space if P is perfect.

Although the reason given in [8] for introducing the concept of perfectness is 'to achieve complete harmony between the abstract theory of measure and the theory of measures in metric spaces' (see chapter 1, § 3 of [8]), the work that has been done involving the concept of perfectness since its introduction has brought out various other nice properties of perfect measures.

Probabilities on the Borel subsets of complete separable metric spaces are well behaved. For instance, they are approximable by compact sets from inside which is the crux of Kolmorogov's consistency theorem on product measures. But in a general probability space, to start with we do not have compactness. However we can call a collection of subsets of a set X to be a compact class if every sequence of sets from this collection having finite intersection property has nonempty intersection. Marczewski [22] introduced this notion and defined a measure P on $(X, \underset{=}{A})$ to be compact if it is approximable by a compact class. Again, a compact measure may be more than what is needed because more often the interest of a probabilist is not as such on the space he works with but mainly on the behaviour of random variables defined on that space. It suffices then to require the measure to be compact on every countably generated sub-σ-algebra. This is the notion of quasi-compactness introduced and studied by Ryll-Nardzewski [37]. Ryll-Nardzewski proved the equivalence of quasi-compactness and perfectness of a measure.

Further, as in the theory of stochastic processes, when one wants to study conditional distributions regular conditional probabilities come into picture. The existence of regular conditional probabilities, which enables such studies to be done neatly, is guaranteed in perfect spaces. Moreover, certain pathologies inherent in the theory based on Kolmogorov's model do not arise in perfect spaces (see introduction in [2]) while the concept is general enough for all applications. Thus today perfect probability spaces are, perhaps, technically the most pleasing classes of probability spaces.

The first four chapters of this thesis are devoted to studying some interesting problems on perfect measures. The fifth chapter treats some problems regarding sequences of σ-algebras and has connection with perfect probability spaces in applications. The last chapter studies the invariant measure problem in ergodic theory for a group of transformations. We give below a chapterwise summary of the thesis.

CHAPTER 1: It is shown using the notion of strongly Blackwell spaces that there is a nonperfect probability space in which the two definitions of independence are equivalent. This answers a question of Rodine [32] in the negative. Some related questions are also discussed.

<u>CHAPTER 2:</u> Let $(X, \underline{\underline{A}}, P)$ be a probability space and let $\underline{\underline{B}}$ be an atomic sub-σ-algebra of $\underline{\underline{A}}$ with respect to which there is a regular conditional probability which is proper a.s. [P]. It is shown that if P_1 is a probability measure dominated by P then there is a regular conditional probability with respect to $\underline{\underline{B}}$ in $(X, \underline{\underline{A}}, P_1)$ which is proper a.s. $[P_1]$. A decomposition of the whole space X into sets one-sheeted with respect to $\underline{\underline{B}}$ and a set not containing any one-sheeted set of positive measure is obtained. When $\underline{\underline{A}}$ is separable a necessary and sufficient condition is given for the existence of an independent complement of $\underline{\underline{B}}$.

<u>CHAPTER 3:</u> A new characterisation of Lebesgue spaces is obtained by using certain conditions on a fixed measurable partition of a separable space. It is shown that any two independent complements of a measurable partition of a Lebesgue space are isomorphic a.s. A necessary and sufficient condition is obtained for the existence of a unique independent complement.

<u>CHAPTER 4:</u> A detailed study is made of the mixture problem for perfect measures. It is shown that perfect mixtures of perfect measures need not be perfect and that a perfect

mixture of nonperfect measures can be perfect thereby settling a conjecture and a question of Rodine [33]. Perfect mixtures of discrete measures are shown to be perfect. A necessary and sufficient condition, using results of chapter 3, is given for a perfect mixture of perfect measures to be perfect.

CHAPTER 5: Some questions regarding sequences of σ-algebras which occur in a certain formulation of the theory of non-linear prediction are discussed. Some results on point separation in sequences of σ-algebras and on crossed σ-algebras are proved. Applications to product spaces are also given.

CHAPTER 6: Let G be a group of measurable, nonsingular transformations on $(X, \underline{\underline{A}}, P)$. Some results on the equivalence by countable decomposition of measurable sets are proved. Using Tarski's theory of mass functions on abstract semigroups, condition (H) which is a generalisation of Hopf's condition for cyclic group, is shown to be necessary and sufficient for the existence of a finite measure equivalent to P and invariant under G.

Some of the results of this thesis have been submitted for publication (see [27], [28], [29] and [30]).

CHAPTER 0

INTRODUCING PERFECT MEASURES

THE TERMINOLOGY AND NOTATION USED IN THIS THESIS CLOSELY FOLLOW THAT OF NEVEU [24] (ALSO REFER TO HALMOS [11]).

All measures considered in this thesis are probabilities. The word 'function', when used without qualification, will always mean 'real-valued function'.

If $\underline{\underline{C}}$ is a collection of subsets of a set X then $\text{alg}(\underline{\underline{C}})$ and $\sigma(\underline{\underline{C}})$ will denote respectively the algebra generated by $\underline{\underline{C}}$ and the σ-algebra generated by $\underline{\underline{C}}$.

Let $(X, \underline{\underline{A}})$ be a measurable space. $\underline{\underline{A}}$ is said to be countably generated if there exists a sequence of sets $\{A_n\} \subset \underline{\underline{A}}$ such that $\underline{\underline{A}} = \sigma(\{A_n\})$. $\underline{\underline{A}}$ is said to be separable if it is countably generated and contains all singletons. Let P be a measure on $(X, \underline{\underline{A}})$. A set $A \in \underline{\underline{A}}$ is said to be a P-atom if for every $B \in \underline{\underline{A}}$, $B \subset A$ either $P(B) = 0$ or $P(B) = P(A)$. The measure P is said to be discrete if there is a sequence $\{A_n\}$ of pairwise disjoint P-atoms such that $\sum_{n=1}^{\infty} P(A_n) = 1$.

We use the following definition of perfectness of a measure which is equivalent to the definition given by Gnedenko and Kolmogorov in [8]. A measure P on a measurable space $(X, \underline{\underline{A}})$ is called perfect if for every $\underline{\underline{A}}$-measurable function f on X and every subset A of the real line for which $f^{-1} A \in \underline{\underline{A}}$ there is a linear Borel set B contained in A such that $P(f^{-1} A) = P(f^{-1} B)$. We call $(X, \underline{\underline{A}}, P)$ a perfect probability space if P is a perfect measure.

We now collect some useful, well known facts regarding perfect measures (see [39]).

<u>P1.</u> A measure P on a measurable space $(X, \underline{\underline{A}})$ is perfect if and only if for every $\underline{\underline{A}}$-measurable function f on X there is a linear Borel set $B(f)$ contained in $f(X)$ such that $P(f^{-1} B(f)) = 1$.

<u>P2.</u> A measure is perfect if and only if its restriction to every countably generated sub-σ-algebra is perfect.

<u>P3.</u> The restriction to any sub-σ-algebra of a perfect measure is perfect.

<u>P4.</u> $(X, \underline{\underline{A}}, P)$ is a perfect probability space if and only if $(X, \bar{\underline{\underline{A}}}, \bar{P})$ is a perfect probability space, where $(X, \bar{\underline{\underline{A}}}, \bar{P})$ is the completion of $(X, \underline{\underline{A}}, P)$.

<u>P5.</u> A measure P on a product space $(\prod_{i \in I} X_i, \times_{i \in I} \underline{\underline{A}}_i)$ is perfect if and only if every marginal of P is perfect.

Suppose a measure μ on $(X, \underline{\underline{A}})$ is 0- 1 valued, that is, $\mu(A) = 0$ or 1 for every $A \in \underline{\underline{A}}$. Then every $\underline{\underline{A}}$-measurable function on X is essentially a constant and hence, by P1, μ is perfect. Similarly it can be shown using P1 that a discrete measure is perfect.

Let $(X, \underline{\underline{A}})$ be a measurable space with $\underline{\underline{A}}$ countably generated. Let $\{A_n\}$ be a sequence of measurable sets such that $\sigma(\{A_n\}) = \underline{\underline{A}}$. The function

$$f(x) = \sum_{n=1}^{\infty} (2/3^n)\, 1_{A_n}(x)$$

is called the Marczewski function of $\{A_n\}$. It is easy to check that $f(x_1) \neq f(x_2)$ if x_1 and x_2 belong to different atoms of $\underline{\underline{A}}$ and that $f(\underline{\underline{A}}) = \underline{\underline{B}}_{[0,1]} \cap f(X)$ where $\underline{\underline{B}}_{[0,1]}$ denotes the Borel σ-algebra of $[0, 1]$. Further the following fact can be verified using P1.

P6. A measure μ on $(X, \underline{\underline{A}})$ is perfect if and only if $(f(X), \underline{\underline{B}}_{[0,1]} \cap f(X), \mu f^{-1})$ is a perfect probability space.

Suppose X is a subset of the real line. Let $\underline{\underline{B}}_X = \{ B \cap X : B \text{ is a linear Borel set} \}$. Then we have

P7. (Lemma 3, [39]). In order that every measure on $(X, \underline{\underline{B}}_X)$ be perfect it is necessary and sufficient that the set X be universally measurable.

Let λ be the Lebesgue measure on $([0, 1], \underline{\underline{B}}_{[0, 1]})$. Let M be a subset of $[0, 1]$ such that $\lambda^*(M) = 1 \neq \lambda_*(M)$ where λ^* and λ_* denote respectively the outer measure and the inner measure induced by λ. Using P1 we can see that $(M, \underline{\underline{B}}_{[0, 1]} \cap M, \lambda^*|_M)$ is a nonperfect probability space.

We shall give exact references to other facts about perfect measures whenever we use them. Interesting results on perfect measures can be found in Ryll-Nardzewski [37], Blackwell [2], Kallianpur [16], Sazonov [39], Jiřina [15], Pfanzagl and Pierlo [26] and some others listed in the bibliography.

CHAPTER 1

PERFECT MEASURES AND INDEPENDENCE

1.0 Introduction

Let $(X, \underline{\underline{A}}, P)$ be a probability space. For an $\underline{\underline{A}}$-measurable function f defined on X let

$$\underline{\underline{B}}_f = \left\{ f^{-1}B: B \text{ is a linear Borel set} \right\}$$

and let

$$\underline{\underline{A}}_f = \left\{ f^{-1} C \in \underline{\underline{A}} : C \text{ is a linear set} \right\}.$$

Two $\underline{\underline{A}}$-measurable functions f and g are said to be

(1) independent according to Steinhaus' definition if $\underline{\underline{B}}_f$ and $\underline{\underline{B}}_g$ are independent, and

(2) independent according to Kolmogorov's definition if $\underline{\underline{A}}_f$ and $\underline{\underline{A}}_g$ are independent.

In general the two definitions are not equivalent (see [6], [14]). Doob (see appendix in [8]) has noted that the two definitions of independence are equivalent if P is a perfect measure.

The requirement of perfectness of measures is sufficient to refine the Kolmogorov model for probability theory so that it is technically pleasing (see appendix in [8] and introduction in [2]). But the necessity of perfectness for a technically pleasing model has not been investigated so far. Rodine [32] raised the following question in this direction. If the two definitions of independence are equivalent in a probability space $(X, \underline{\underline{A}}, P)$ then is P perfect? We show in this chapter, using the notion of strongly Blackwell spaces, that the answer to the above question is in the negative. We also study some related problems.

1.1 Main results

Let $(X, \underline{\underline{A}})$ be a measurable space where $\underline{\underline{A}}$ is a separable σ-algebra. $(X, \underline{\underline{A}})$ is said to be a Blackwell space if $\underline{\underline{A}}_1$ is a separable sub-σ-algebra of $\underline{\underline{A}}$ implies $\underline{\underline{A}}_1 = \underline{\underline{A}}$. $(X, \underline{\underline{A}})$ is said to be strongly Blackwell if any two countably generated sub-σ-algebras of $\underline{\underline{A}}$ with same atoms are identical. Strongly Blackwell spaces were introduced by Ashok Maitra.

For a measurable space $(X, \underline{\underline{A}})$ where the σ-algebra $\underline{\underline{A}}$ is separable, we have the following theorem.

<u>Theorem 1.1.1:</u> The following conditions are equivalent.

i) $(X, \underline{\underline{A}})$ is a strongly Blackwell space.

ii) If $\underline{\underline{B}}$ is a countably generated sub-σ-algebra of $\underline{\underline{A}}$ and $A \in \underline{\underline{A}}$ is a union of $\underline{\underline{B}}$-atoms, then $A \in \underline{\underline{B}}$.

iii) For every $\underline{\underline{A}}$-measurable function f defined on X, $\underline{\underline{B}}_f = \underline{\underline{A}}_f$.

<u>Proof:</u> (i) ⟹ (ii). If $\underline{\underline{B}}' = \sigma\left\{\underline{\underline{B}}, A\right\}$ then $\underline{\underline{B}}' = \underline{\underline{B}}$ by (i).

(ii) ⟹ (iii). The sub-σ-algebra

$$\underline{\underline{B}}_f = \left\{f^{-1} B: \ B \text{ is a linear Borel set}\right\}$$

of $\underline{\underline{A}}$ is countably generated. Clearly $\underline{\underline{B}}_f \subset \underline{\underline{A}}_f$. On the other hand every set $f^{-1} C$, where C is a linear set, is a union of atoms of $\underline{\underline{B}}_f$ and thus is in $\underline{\underline{B}}_f$ if it is in $\underline{\underline{A}}_f$, by (ii).

(iii) ⟹ (i). Let $\underline{\underline{A}}_1$, $\underline{\underline{A}}_2$ be two countably generated sub-σ-algebras of $\underline{\underline{A}}$ with same atoms. Let $\underline{\underline{A}}_1 = \sigma\left\{A_{1n}\right\}$ and let f be the Marczewski function of $\left\{A_{1n}\right\}$ (see Chapter 0). Now $\underline{\underline{A}}_2 \subset \underline{\underline{A}}_f = \underline{\underline{B}}_f$ (by (iii))

$$= \underline{\underline{A}}_1 .$$

Similarly $\underline{\underline{A}}_1 \subset \underline{\underline{A}}_2$ and (i) holds.

<u>Corollary 1.1.2:</u> Let $(X, \underline{\underline{A}})$ be a strongly Blackwell space. Let P be any measure on $(X, \underline{\underline{A}})$. Then the two definitions of independence are equivalent in $(X, \underline{\underline{A}}, P)$.

Proof: For every $\underline{\underline{A}}$-measurable function f defined on X, by 1.1.1, $\underline{\underline{B}}_f = \underline{\underline{A}}_f$. Hence whatever be the measure P on $(X, \underline{\underline{A}})$ the two definitions of independence are equivalent.

Now we present an example to show that the answer to the question raised in 1.0 is in the negative.

Example 1.1.3:

Ryll-Nardzewski [38] has shown the existence of a non-Lebesgue measurable subset X^* of the unit interval $[0, 1]$ such that if

$$\underline{\underline{A}}^* = \{ B \cap X^* : \ B \text{ is a linear Borel set} \},$$

then $(X^*, \underline{\underline{A}}^*)$ is a strongly Blackwell space. Further X^* is thick, that is, X^* has outer Lebesgue measure one. Let P^* be the trace of outer Lebesgue measure on $(X^*, \underline{\underline{A}}^*)$. Then $(X^*, \underline{\underline{A}}^*, P^*)$ is a nonperfect probability space. But in view of 1.1.2 the two definitions of independence are equivalent in $(X^*, \underline{\underline{A}}^*, P^*)$.

Remark 1.1.4:

The following question is a globalized version of the question asked in 1.0:

Suppose $(X, \underline{\underline{A}})$ is a measurable space such that whatever be the measure P on $(X, \underline{\underline{A}})$ we consider, the two definitions of independence are equivalent in $(X, \underline{\underline{A}}, P)$. Then does there exist a nonatomic, perfect measure on $(X, \underline{\underline{A}})$? Here we demand something weaker after making a stronger assumption on $(X, \underline{\underline{A}})$. Still the answer to this question is in the negative as we shall proceed to show.

A subset D of the real line is said to be a perfect set if $D = \{\text{limit points of } D\}$. Every uncountable Borel subset of the real line contains a perfect set (see [19], page 447). The following result goes in the opposite direction of P7:

Suppose X is a subset of the real line and

$$\underline{\underline{B}}_X = \{B \cap X : B \text{ is a linear Borel set}\}.$$

Every nonatomic measure on $(X, \underline{\underline{B}}_X)$ is nonperfect if and only if X does not contain any perfect set.

The proof is easily carried out using P1 and we omit the details.

The set X^* used in 1.1.3, constructed by Ryll-Nardzewski, can be taken to be such that both X^* and $[0, 1] - X^*$ do not contain any perfect set. Hence $(X^*, \underline{\underline{A}}^*)$ as in 1.1.3 is a measurable space such that the two definitions of independence are equivalent in $(X^*, \underline{\underline{A}}^*, P)$ no matter what measure P we consider, yet every nonatomic measure on $(X^*, \underline{\underline{A}}^*)$ is nonperfect. Thus the answer to our globalized question is in the negative.

We also note that $(X^*, \underline{\underline{A}}^*)$ as in 1.1.3 is a measurable space such that for every sub-σ-algebra $\underline{\underline{B}}$ of $\underline{\underline{A}}^*$ and for every measure P on $(X^*, \underline{\underline{B}})$ the two definitions of independence are equivalent in $(X^*, \underline{\underline{B}}, P)$.

Now one may raise the following question:

Suppose $(X, \underline{\underline{A}})$ is a measurable space such that

(a) $\underline{\underline{A}}$ is separable, and

(b) for every measure P on $(X, \underline{\underline{A}})$ the two definitions of independence are equivalent in $(X, \underline{\underline{A}}, P)$

then is $(X, \underline{\underline{A}})$ strongly Blackwell? The answer to this question also is in the negative as shown by the following example.

<u>Example 1.1.5:</u>

Let X_1 be a coanalytic subset of the unit interval such that if $\underline{\underline{A}}_1 = \{B \cap X_1 : B \text{ is a linear Borel set}\}$ then $(X_1, \underline{\underline{A}}_1)$ is not a Blackwell space. Such a coanalytic set X_1 has been constructed by Maitra [21]. Now by P7 every measure on $(X_1, \underline{\underline{A}}_1)$ is perfect. Hence $(X_1, \underline{\underline{A}}_1)$ is a measurable space satisfying the conditions (a) and (b) of the question since perfectness of a measure implies the equivalence of the two definitions of independence. But $(X_1, \underline{\underline{A}}_1)$ fails to be a strongly Blackwell space since it is not even a Blackwell space.

Next we modify example 1.1.3 so as to get a nonperfect probability space in which the two definitions of independence are equivalent but whose underlying measurable space is not a Blackwell space.

Example 1.1.6:

Let $(X^*, \mathcal{A}^*, P^*)$ be as in example 1.1.3. Let N be a subset of the interval $[2, 3]$ such that if $\mathcal{N} = \{B \cap N : B$ is a linear Borel set$\}$ then $(N, \mathcal{N})$ is not a Blackwell space (for instance, we can take $N = \{x + 2 : x \in X_1\}$ where X_1 is as in example 1.1.5). Let $X = X^* \cup N$,

$$\mathcal{A} = \{A^* \cup N_1 : A^* \in \mathcal{A}^* ,\ N_1 \in \mathcal{N}\},$$

and for $A \in \mathcal{A}$ let $P(A) = P^*(A \cap X^*)$. It is easy to verify that $(X, \mathcal{A})$ is not a Blackwell space and that P on $(X, \mathcal{A})$ is nonperfect. We shall show that the two definitions of independence are equivalent in $(X, \mathcal{A}, P)$.

For every $\mathcal{A}$- measurable function f defined on X let f^* = the restriction to X^* of f. Then we have

$$\mathcal{A}_f \cap X^* \subset \mathcal{A}^*_{f^*} = \{f^{*-1} C \in \mathcal{A}^* : C \text{ is a linear set}\}$$

$$= \mathcal{B}_{f^*} \text{ (by 1.1.1)}$$

$$= \mathcal{B}_f \cap X^* \subset \mathcal{A}_f \cap X^* , \qquad \text{Hence}$$

$\mathcal{A}_f \cap X^* = \mathcal{B}_f \cap X^*$. ~~Now if~~ f and g are two $\mathcal{A}$-measurable functions defined on X then $\underline{\underline{B}}_f$ and $\underline{\underline{B}}_g$ are independent

$\Rightarrow \underline{\underline{B}}_f \cap X^*$ and $\underline{\underline{B}}_g \cap X^*$ are independent

$\Rightarrow \underline{\underline{A}}_f \cap X^*$ and $\underline{\underline{A}}_g \cap X^*$ are independent

$\Rightarrow \underline{\underline{A}}_f$ and $\underline{\underline{A}}_g$ are independent.

Thus in $(X, \underline{\underline{A}}, P)$ the two definitions of independence are equivalent.

1.2 Comments

A study of properties of measures on Blackwell spaces might yield interesting results and ours seems to be the first attempt in this direction. For instance, it is not known whether in every probability space whose underlying measurable space is a Blackwell space the two definitions of independence are equivalent.

In our example 1.1.6 if we remove the P-null set N from X then the resulting space is strongly Blackwell. We do not know whether there is a nonperfect probability space in which the two definitions of independence are equivalent such that the underlying measurable space is not a Blackwell space even if we remove any null set. In this connection we may raise the following specific question.

Let E be a graph in the unit square, intersecting every closed plane set of positive Lebesgue measure. Such sets exist as is shown in chapter 4 (lemma 4.1.4) of this thesis. E is clearly non-Lebesgue measurable. Let $\underline{\underline{E}}$ be the trace on E of the Borel-σ-algebra of unit square. Let f be the projection of E onto the first coordinate. f is 1-1 and $\underline{\underline{E}}$-measurable and $\underline{\underline{B}}_f$ is a separable sub-σ-algebra of $\underline{\underline{E}}$. If $\underline{\underline{B}}_f = \underline{\underline{E}}$ it would follow that (see [19], page 489, theorem 1) E is a plane Borel set, but E is not Lebesgue-measurable. Hence $\underline{\underline{B}}_f \neq \underline{\underline{E}}$ and so $(E, \underline{\underline{E}})$ is not a Blackwell space. If λ is the trace on $(E, \underline{\underline{E}})$ of Lebesgue measure and $N \in \underline{\underline{E}}$ is such that $\lambda(N) = 0$, then by a similar argument $(E \cap N^c, \underline{\underline{E}} \cap N^c)$ is not a Blackwell space. Are the two definitions of independence equivalent in $(E, \underline{\underline{E}}, \lambda)$? *

Finally, our examples point out that a nicer model for probability theory may not be achieved by just demanding the equivalence of the two definitions of independence. Further other pathologies like the nonexistence of regular conditional probabilities also need be avoided. It will be worthwhile to investigate whether a model less restricted than perfectness exists which avoids these pathologies.

* During a discussion with K.P.S.Bhaskara Rao we discovered that the answer to this question is in the negative. For, if we take E to be a graph both ways then the projections to the first and second coordinates are independent according to (1) but not according to (2).

CHAPTER 2

EXISTENCE OF INDEPENDENT COMPLEMENTS

2.0 Introduction

Let $(X, \underline{\underline{A}}, P)$ be a probability space and let $\underline{\underline{B}}$ be a sub-σ-algebra of $\underline{\underline{A}}$. A σ-algebra $\underline{\underline{B}}^* \subset \underline{\underline{A}}$ is said to be an independent complement of $\underline{\underline{B}}$ if

i) $\underline{\underline{B}}$ and $\underline{\underline{B}}^*$ are independent and

ii) for every $A \in \underline{\underline{A}}$ there exists $A' \in \sigma\{\underline{\underline{B}}, \underline{\underline{B}}^*\}$ such that $P(A \triangle A') = 0$.

It follows by (i) that $\underline{\underline{B}} \cap \underline{\underline{B}}^* = \{X, \emptyset\}$ a.s. [P].

The problem of existence of independent complements has been studied by Rohlin [34] and later by Rosenblatt [35]. In this chapter we study this problem in detail for the case when $\underline{\underline{A}}$ is separable and $\underline{\underline{B}}$ is countably generated. First we collect some facts about regular conditional probabilities. With regular conditional probabilities as our main tool, we next employ Rohlin's techniques in his treatment of the problem for Lebesgue spaces, to give necessary and sufficient conditions for the existence of an independent complement. We apply these results to perfect probability spaces in the last section.

2.1 On regular conditional probabilities

Let $(X, \underline{\underline{A}}, P)$ be a probability space and let $\underline{\underline{B}}$ be a sub-σ-algebra of $\underline{\underline{A}}$.

A function $\mu(x, A)$ defined on $X \times \underline{\underline{A}}$ is called a regular conditional probability (hereafter denoted by r.c.p.) with respect to $\underline{\underline{B}}$ if

(CP1) $\mu(x, \cdot)$ is a probability on $\underline{\underline{A}}$ for each fixed x in X

(CP2) $\mu(\cdot, A)$ is $\underline{\underline{B}}$-measurable for each fixed A in $\underline{\underline{A}}$

and (CP3) $P(A \cap B) = \int_B \mu(x, A)\,dP$ for every $A \in \underline{\underline{A}}$, $B \in \underline{\underline{B}}$.

A r.c.p. $\mu(x, A)$ is said to be proper at $x_0 \in X$ if

(CP4) $\mu(x_0, B) = 1$ whenever $x_0 \in B \in \underline{\underline{B}}$, and $\mu(x, A)$ is said to be everywhere proper if it is proper at every $x \in X$.

The following lemma will be needed for later use.

Lemma 2.1.1: Suppose $\underline{\underline{B}}$ is countably generated and $\mu(x, A)$ is a r.c.p. with respect to $\underline{\underline{B}}$. Then $\mu(x, A)$ is proper a.s. $[P|_{\underline{\underline{B}}}]$.

Proof: Let $\underline{\underline{B}}_o$ be a countable algebra generating $\underline{\underline{B}}$. Using CP3, since $\underline{\underline{B}}_o$ is countable, we can choose $N \in \underline{\underline{B}}$ with $P(N) = 0$ such that for every $x \notin N$ we have

$$\mu(x, B_o) = 1_{B_o}(x) \quad \text{for all} \quad B_o \in \underline{\underline{B}}_o.$$

Now for every $x \notin N$, $\mu(x, \cdot)$ and $1_{(\cdot)}(x)$ are two probabilities on $\underline{\underline{B}}$ which coincide on $\underline{\underline{B}}_o$ and hence $\mu(x, B) = 1_B(x)$ for all $B \in \underline{\underline{B}}$.

The following proposition deals with the existence of r.c.p. with respect to $\underline{\underline{B}}$ in $(X, \underline{\underline{A}}, P_1)$ where P_1 is absolutely continuous with respect to P.

Proposition 2.1.2: Suppose there exists a r.c.p. $\mu(x, A)$ with respect to $\underline{\underline{B}}$ in $(X, \underline{\underline{A}}, P)$. Then there is a r.c.p. $\mu_1(x, A)$ with respect to $\underline{\underline{B}}$ in $(X, \underline{\underline{A}}, P_1)$ where P_1 is any probability absolutely continuous with respect to P. If, further, $\mu(x, A)$ is proper at $x_o \in X$, then $\mu_1(x, A)$ is also proper at x_o.

Proof: Let f be a fixed version of $\frac{dP_1}{dP}$, the Radon-Nikodym derivative of P_1 with respect to P. For every $\underline{\underline{A}}$-measurable function g on X let

$$\mu(x, g) = \int g(x')\mu(x, dx').$$

Then it is easy to check that $\mu(x, g)$ is a version of $E(g|\underline{\underline{B}})$, the conditional expectation of g with respect to $\underline{\underline{B}}$. It follows that $\mu(x, f)$ is a version of the Radon-Nikodym derivative of $P_1|_{\underline{\underline{B}}}$ with respect to $P|_{\underline{\underline{B}}}$. Now let $E = \{x: \mu(x, f) > 0\}$. Then $E \in \underline{\underline{B}}$ and $P_1(E) = 1$. Define $\mu_1(x, A)$ on $X \times \underline{\underline{A}}$ by

$$\mu_1(x, A) = \begin{cases} \dfrac{\mu(x, f1_A)}{\mu(x, f)} & \text{if } x \in E \\ \mu(x, A) & \text{if } x \notin E \end{cases}$$

clearly $\mu_1(x, A)$ satisfies CP1 and CP2. If $B \in \underline{\underline{B}}$ and $A \in \underline{\underline{A}}$ then

$$\begin{aligned} \int_B \mu_1(x, A)\,dP_1 &= \int_B \mu_1(x, A)\mu(x, f)\,dP \\ &= \int_{B \cap E} \mu(x, f1_A)\,dP \\ &= \int_{B \cap E} f1_A\,dP \\ &= P_1(A \cap B \cap E) = P_1(A \cap B). \end{aligned}$$

Thus $\mu_1(x, A)$ satisfies CP3 as well and hence $\mu_1(x, A)$ is a r.c.p. with respect to $\underline{\underline{B}}$ in $(X, \underline{\underline{A}}, P_1)$.

Finally, if $\mu(x, A)$ is proper at $x_0 \in X$, then clearly $\mu_1(x, A)$ is proper at x_0 if $x_0 \in E^c$. On the other hand if $x_0 \in E$ and $x_0 \in B \in \underline{\underline{B}}$ then

$$\mu_1(x, B) = \frac{\mu(x_o, f1_B)}{\mu(x_o, f)} = \frac{\mu(x_o, f)}{\mu(x_o, f)} = 1$$

since $\mu(x_o, B) = 1$. Hence $\mu_1(x, A)$ is proper at x_o.

Let $X_o \in \underline{\underline{A}}$ with $P(X_o) > 0$. Consider the subspace $(X_o, \underline{\underline{A}} \cap X_o, P_o)$ where $P_o(A \cap X_o) = \frac{P(A \cap X_o)}{P(X_o)}$ for $A \in \underline{\underline{A}}$. As a consequence of 2.1.2 we have the following

Corollary 2.1.3: Suppose there exists a r.c.p. $\mu(x, A)$ with respect to $\underline{\underline{B}}$ in $(X, \underline{\underline{A}}, P)$. Then there exists a r.c.p. $\mu_o(x, A \cap X_o)$ with respect to $\underline{\underline{B}} \cap X_o$ in $(X_o, \underline{\underline{A}} \cap X_o, P_o)$. If $\mu(x, A)$ is proper a.s. $[P|_{\underline{\underline{B}}}]$ then $\mu_o(x, A \cap X_o)$ is proper a.s. $[P_o|_{\underline{\underline{B}} \cap X_o}]$.

Proof: Define P_1 on $\underline{\underline{A}}$ by $P_1(A) = P_o(A \cap X_o)$, $A \in \underline{\underline{A}}$. Then P_1 is absolutely continuous with respect to P. Let $\mu_1(x, A)$ be the r.c.p. with respect to $\underline{\underline{B}}$ in $(X, \underline{\underline{A}}, P_1)$ obtained by 2.1.2. By CP3, there exists $N_1 \in \underline{\underline{B}}$ with $P_1(N_1) = 0$ such that $\mu_1(x, X_o) = 1$ for all $x \notin N_1$. Define $\mu_o(x, A \cap X_o)$ on $X_o \times \underline{\underline{A}} \cap X_o$ by

$$\mu_0(x, A \cap X_0) = \begin{cases} \mu_1(x, A \cap X_0) & \text{if } x \notin N_1 \cap X_0 \\ P_0(A \cap X_0) & \text{if } x \in N_1 \cap X_0 . \end{cases}$$

It is easy to check that $\mu_0(x, A \cap X_0)$ satisfies CP1 and CP2 and that $N_1 \cap X_0 \in \underline{\underline{B}} \cap X_0$ with $P_0(N_1 \cap X_0) = 0$. If $B \in \underline{\underline{B}}$ and $A \in \underline{\underline{A}}$ then

$$\int_{B \cap X_0} \mu_0(x, A \cap X_0) dP_0 = \int_{B \cap N_1^c \cap X_0} \mu_1(x, A \cap X_0) dP_0$$

$$= \int_{B \cap N_1^c} \mu_1(x, A \cap X_0) dP_1$$

$$= P_1(B \cap N_1^c \cap A \cap X_0)$$

$$= P_0(A \cap B \cap X_0) .$$

Hence $\mu_0(x, A \cap X_0)$ satisfies CP3 and thus is a r.c.p. with respect to $\underline{\underline{B}} \cap X_0$.

Suppose $\mu(x, A)$ is proper at every $x \notin N$, where $N \in \underline{\underline{B}}$ with $P(N) = 0$. Then $P_1(N) = 0$ and by 2.1.2, $\mu_1(x, A)$ is proper at every $x \notin N$. It can now be verified that $P_0((N \cup N_1) \cap X_0) = 0$ and that $\mu_0(x, A \cap X_0)$ is proper at every $x \notin (N \cup N_1) \cap X_0$.

One may wonder whether in corollary 2.1.3 if we start with a r.c.p. $\mu(x, A)$ which is everywhere proper then $\mu_o(x, A \cap X_o)$ can also be chosen to be everywhere proper. To show that such a choice is not always possible we shall give an example using the following two results.

Suppose there is a r.c.p. with respect to $\underline{\underline{B}}$ which is proper a.s. $[P|_{\underline{\underline{B}}}]$. We shall give a necessary and sufficient condition for the existence of an everywhere proper r.c.p. with respect to $\underline{\underline{B}}$. A function $Q(x, A)$ on $X \times \underline{\underline{A}}$ is called a transition function with respect to $\underline{\underline{B}}$ if

i) $Q(x, \cdot)$ is a probability on $\underline{\underline{A}}$ for each fixed x in X

and ii) $Q(\cdot, A)$ is $\underline{\underline{B}}$-measurable for each fixed $A \in \underline{\underline{A}}$.

The transition function $Q(x, A)$ is said to be proper at x_o if $Q(x_o, B) = 1$ whenever $x_o \in B \in \underline{\underline{B}}$ and $Q(x, A)$ is said to be everywhere proper if it is proper at every x in X.

<u>Proposition 2.1.4:</u> Suppose there exists a r.c.p. $\mu(x, A)$ with respect to $\underline{\underline{B}}$. Then there exists a r.c.p. $\mu_1(x, A)$ with respect to $\underline{\underline{B}}$ which is everywhere proper if and only if there exists a transition function $Q(x, A)$ which is everywhere proper.

Proof: Since $\mu_1(x, A)$ is itself a transition function on $X \times \underline{\underline{A}}$ the 'only if' part is obvious.

To prove the 'if' part, let $N \in \underline{\underline{B}}$ with $P(N) = 0$ be such that $\mu(x, A)$ is proper at every $x \notin N$. Define

$$\mu_1(x, A) = \begin{cases} \mu(x, A) & \text{if } x \notin N \\ Q(x, A) & \text{if } x \in N. \end{cases}$$

It is easy to check that $\mu_1(x, A)$ is a r.c.p. with respect to $\underline{\underline{B}}$ which is proper everywhere.

The next result is due to Blackwell and Ryll-Nardzewski (see [4]).

Theorem 2.1.5: Let Y be a Borel subset of a Polish space and let $\underline{\underline{A}}_Y$ be its Borel σ-algebra. Let f be a measurable function on Y and let $\underline{\underline{B}}_f = \{f^{-1}B\colon B \text{ is a linear Borel set}\}$. Then there exists a transition function $Q(y, A)$ on $Y \times \underline{\underline{A}}_Y$ with respect to $\underline{\underline{B}}_f$ which is everywhere proper if and only if there exists a $\underline{\underline{B}}_f$-measurable function g from Y into Y such that $f(g(y)) = f(y)$ for all $y \in Y$.

Example 2.1.6: Let X be the unit square and $\underline{\underline{A}}$ its Borel σ-algebra. Let $\underline{\underline{B}}$ be the σ-algebra of vertical cylinders, namely, the σ-algebra $\underline{\underline{B}}_f$ where f on X is the projection to the first coordinate. Let X_o be a Borel subset of X which does not contain any graph and whose vertical sections are all nonempty (see [3], [25]). Let P_o be any probability measure on $(X_o, \underline{\underline{A}} \cap X_o)$ and let P on $\underline{\underline{A}}$ be defined by $P(A) = P_o(A \cap X_o)$. It is known (see theorem 5, [2]) that there is a r.c.p. with respect to $\underline{\underline{B}}$ which is proper a.s. $[P|_{\underline{\underline{B}}}]$. It is also easy to check using 2.1.5 that there is a transition function on $X \times \underline{\underline{A}}$ with respect to $\underline{\underline{B}}$ which is proper everywhere. In order to show that a r.c.p. $\mu_o(x, \underline{\underline{A}} \cap X_o)$ on $X_o \times \underline{\underline{A}} \cap X_c$ with respect to $\underline{\underline{B}} \cap X_o$ can not be chosen to be proper everywhere it is now enough to show that there is no transition function $Q_o(x, A \cap X_o)$ on $X_o \times \underline{\underline{A}} \cap X_o$ with respect to $\underline{\underline{B}} \cap X_o$ which is everywhere proper. Again by 2.1.5, it is sufficient to show that there is no $\underline{\underline{B}} \cap X_o$-measurable function g from X_o into X_o such that for every $(x, y) \in X_o$, $g(x, y) \in (\{x\} \times [0, 1]) \cap X_o$. But the existence of such a g implies that $G = \{g(x, y): (x, y) \in X_o\} \subset X_o$ is a Borel graph which is a contradiction. Hence there is no r.c.p. on $X_o \times \underline{\underline{A}} \cap X_o$ with respect to $\underline{\underline{B}} \cap X_o$ which is everywhere prope

2.2 A decomposition of the whole space

Let $(X, \underline{\underline{A}}, P)$ be a probability space and let $\underline{\underline{B}}$ be an atomic sub-σ-algebra of $\underline{\underline{A}}$. We assume throughout this section the existence of a r.c.p. $\mu(x, A)$ with respect to $\underline{\underline{B}}$ which is proper a.s. $[P|_{\underline{\underline{B}}}]$, that is, there exists $N \in \underline{\underline{B}}$ with $P(N) = 0$ such that $\mu(x, A)$ is proper at every $x \notin N$.

A set $A \in \underline{\underline{A}}$ is said to be one-sheeted with respect to $\underline{\underline{B}}$ (or just one-sheeted when there can be no confusion) if for every B which is a $\underline{\underline{B}}$-atom, the set $A \cap B$ contains at most one point.

The proofs of results in this section are essentially Rohlin's proofs of analogous results on Lebesgue spaces (see § 4, No.2 in [34]) carried over with suitable modifications to the general case.

Proposition 2.2.1: Among the $\underline{\underline{A}}$-measurable one-sheeted sets there exists a set of maximal measure.

Proof: Let $\underline{\underline{A}}_o = \{A \in \underline{\underline{A}} : A \text{ is one-sheeted}\}$ and let $\beta = \sup \{P(A) : A \in \underline{\underline{A}}_o\}$. Let $\{A_n, n \geq 1\} \subset \underline{\underline{A}}_o$ be a sequence of one-sheeted sets such that $\lim_n P(A_n) = \beta$. We shall construct a sequence $\{C_n, n \geq 1\}$ of sets such that

i) $C_n \in \underline{\underline{A}}_o$ for all n

ii) $\limsup C_n \in \underline{\underline{A}}_o$

and iii) $P(C_n) \geq P(A_n)$ for all n.

From (i), (ii) and (iii) it will follow that $P(\limsup C_n) = \beta$

Let $C_1 = A_1 \in \underline{\underline{A}}_o$. Suppose $C_1, C_2, \ldots, C_{n-1}$ have been defined for $n > 1$ such that $C_i \in \underline{\underline{A}}_o$, $1 \leq i \leq n-1$. Let

$$B_{n-1} = \left\{ x : \mu(x, A_n) > \mu(x, C_{n-1}) \right\}$$

and let $C_n = (B^c_{n-1} \cap C_{n-1}) \cup (B_{n-1} \cap A_n)$. Since, by CP2, $B_{n-1} \in \underline{\underline{B}}$ it follows that $C_n \in \underline{\underline{A}}_o$. Let $\left\{C_n, n \geq 1\right\} \subset \underline{\underline{A}}_o$ be constructed in the above fashion. We note that

$$(*) \begin{cases} x \in B_{n-1} - N \Rightarrow \mu(x, C_n) = \mu(x, C_n \cap B_{n-1}) \\ \qquad = \mu(x, A_n \cap B_{n-1}) = \mu(x, A_n) > \mu(x, C_{n-1}) \\ x \in B^c_{n-1} - N \Rightarrow \mu(x, C_n) = \mu(x, C_n \cap B^c_{n-1}) \\ \qquad = \mu(x, C_{n-1} \cap B^c_{n-1}) = \mu(x, C_{n-1}) > \mu(x, A_n) \end{cases}$$

and hence for all $x \notin N$ we have $\mu(x, C_n) \geq \mu(x, C_{n-1})$. For every x, let $B(x)$ denote the $\underline{B}$-atom containing x. Let $\underline{x \notin N}$

Then $\left\{B(x) \cap C_n\right\}$ is a sequence of subsets of $B(x)$ each containing at most one point and such that $\mu(x, B(x) \cap C_n) \geq \mu(x, B(x) \cap C_{n-1})$. Further from (*) $\mu(x, B(x) \cap C_m) = \mu(x, B(x) \cap C_{m-1}) \Rightarrow x \in B^c_{m-1} - N \Rightarrow B(x) \cap C_m = B(x) \cap C_{m-1}$. Hence $\{B(x) \cap C_n\}$ contains only a finite number of distinct points. Hence there is a natural number $n(x)$ such that for $n \geq n(x)$, $B(x) \cap C_n = B(x) \cap C_{n(x)}$.

Let $M_1 = \bigcup_{x \notin N} B(x) \cap C_{n(x)}$. Then $M_1 = \limsup C_n \in \underline{\underline{A}}$. But M_1 is clearly one-sheeted and hence $M_1 \in \underline{\underline{A}}_o$. From (*) we have $\mu(x, C_n) \geq \mu(x, A_n)$ for all $x \notin N$. Hence, using CP3, for every n

$$P(C_n) = \int_{N^c} \mu(x, C_n)\,dP \geq \int_{N^c} \mu(x, A_n)\,dP = P(A_n).$$

We now prove the main result of this section.

<u>Theorem 2.2.2:</u> There is a decomposition of X' of the form

$$X = M_o \cup M_1 \cup M_2 \cup \ldots$$

where

i) M_n is an $\underline{\underline{A}}$-measurable one-sheeted set for every $n \geq 1$

ii) M_n is a set of maximal measure among all $\underline{\underline{A}}$-measurable one-sheeted subsets of $X_{n-1} = (\bigcup_{i=1}^{n-1} M_i)^c$ for every $n \geq 1$

and iii) M_0 contains no subset of positive measure which is one-sheeted with respect to $\underline{\underline{B}}$.

Proof: Let M_1 be constructed according to 2.2.1. Suppose $M_1, M_2, \ldots, M_{n-1}$ have been constructed for some $n > 1$. Let $X_{n-1} = (\bigcup_{i=1}^{n-1} M_i)^c$. If $P(X_{n-1}) = 0$ take $M_n = \emptyset$. If $P(X_{n-1}) > 0$ then, by 2.1.3, there is a r.c.p. on $X_{n-1} \times \underline{\underline{A}} \cap X_{n-1}$ with respect to $\underline{\underline{B}} \cap X_{n-1}$ which is proper a.s. $[P_{n-1}]$ where P_{n-1} on $\underline{\underline{A}} \cap X_{n-1}$ is defined by

$$P_{n-1}(A \cap X_{n-1}) = \frac{P(A \cap X_{n-1})}{P(X_{n-1})}.$$

Now we can apply 2.2.1, in the space $(X_{n-1}, \underline{\underline{A}} \cap X_{n-1}, P_{n-1})$, to the class of all measurable sets one-sheeted with respect to $\underline{\underline{B}} \cap X_{n-1}$ to get among them a set M_{n-1} of maximal measure. Plainly M_{n-1} is an $\underline{\underline{A}}$-measurable set one-sheeted with respect to $\underline{\underline{B}}$ and also has maximal measure if one considers only one-sheeted measurable subsets of X_{n-1}. Having defined $\{M_n, n \geq 1\}$ in this fashion we set $M_0 = (\bigcup_{i=1}^{\infty} M_i)^c$. Finally if $A \in \underline{\underline{A}}$ is any one-sheeted subset of M_0 then $P(A) \leq P(M_n)$ for all n and hence $P(A) = 0$.

2.3 Necessary condition

Throughout sections 2.3, 2.4 and 2.5 we let $(X, \underline{\underline{A}}, P)$ be a probability space where $\underline{\underline{A}}$ is separable and let $\underline{\underline{B}}$ be an atomic sub-σ-algebra of $\underline{\underline{A}}$. We assume further that there exists a r.c.p. $\mu(x, A)$ with respect to $\underline{\underline{B}}$ which is proper a.s. [P], that is, there exists $N \in \underline{\underline{B}}$ with $P(N) = 0$ such that $\mu(x, A)$ is proper at every $x \notin N$.

In the present section we shall give a necessary condition for the existence of an independent complement $\underline{\underline{B}}^*$ of $\underline{\underline{B}}$. We need the following lemma.

Lemma 2.3.1:

i) A set $B_1 \in \underline{\underline{A}}$ is independent of $\underline{\underline{B}}$ if and only if $\mu(x, B_1) = P(B_1)$ a.s. [P]

ii) A σ-algebra $\underline{\underline{B}}_1 \subset \underline{\underline{A}}$ is independent of $\underline{\underline{B}}$ if and only if for every $B \in \underline{\underline{B}}_1$ $\mu(x, B_1) = P(B_1)$ a.s. [P].

iii) A countably generated σ-algebra $\underline{\underline{B}}_1 \subset \underline{\underline{A}}$ is independent of $\underline{\underline{B}}$ if and only if there is a set $N_1 \in \underline{\underline{B}}$ with $P(N_1) = 0$ such that for every $x \notin N_1$

$$\mu(x, B_1) \underset{B_1 \in \underline{\underline{B}}_1}{\equiv} P(B_1) .$$

Proof: By CP 3 for every $B \in \underline{\underline{B}}$

$$P(B \cap B_1) = \int_B \mu(x, B_1) dP$$

and hence (i) follows. Consequently (ii) follows.

The sufficiency part of (iii) follows from (ii). To prove the necessity part let $\{B_n, n \geq 1\} \subset \underline{\underline{B}}_1$ be a countabl algebra generating $\underline{\underline{B}}_1$. It is clear using (i) that there exist $N_1 \in \underline{\underline{B}}_1$ with $P(N_1) = 0$ such that for every $x \notin N_1$,

$$\mu(x, B_n) \underset{n}{\equiv} P(B_n).$$

Now if $\underline{\underline{B}}_1' = \left\{C \in \underline{\underline{B}}_1 : \mu(x, C) = P(C)\right\}$ for all $x \notin N_1$

then $\underline{\underline{B}}_1'$ being a monotone class containing $\left\{B_n, n \geq 1\right\}$ we have $\underline{\underline{B}}_1' = \underline{\underline{B}}_1$.

We shall now introduce, following Rohlin, a sequence $\left\{m_n, n \geq 1\right\}$ of functions on X which, as we shall see in th section and in section 2.5, are very useful in studying the structure of the space $(X, \underline{\underline{A}}, P)$.

Let $x \notin N$. Then $(E(x) \underline{\underline{A}} \cap B(x), \mu(x, \cdot))$ is a probabi lity space where $B(x)$ denote the $\underline{\underline{B}}$-atom containing x. Let $y_1, y_2, \ldots, y_k, \ldots$ be an enumeration of points of $B(x)$ of positive $\mu(x, \cdot)$ measure such that for every $n \geq 1$, $\mu(x, \{y_k\}) \geq \mu(x, \{y_{k+1}\})$. If the sequence $\{y_k\}$ is infinite let

$$m_n(x) = \mu(x, \{y_n\}), \qquad n = 1, 2, \ldots$$

and if the sequence $\{y_k\}$ contains only r elements let

$$m_n(x) = \begin{cases} \mu(x, \{y_n\}) & \text{if } n \leq r \\ 0 & \text{if } n > r. \end{cases}$$

We have thus defined a sequence of functions $\{m_n, n \geq 1\}$ on $X - N$. Let m_n, for each $n \geq 1$, be defined to be identically zero on N.

Now $\{m_n, n \geq 1\}$ is a sequence of functions defined on X such that

a) $m_n \geq 0$ (b) $m_n \geq m_{n+1}$ and (c) $\sum_{n=1}^{\infty} m_n \leq 1$.

The following proposition gives a necessary condition for the existence of an independent complement $\underline{\underline{B}}^*$ of $\underline{\underline{B}}$.

<u>Proposition 2.3.2:</u> If $\underline{\underline{B}}$ admits an independent complement $\underline{\underline{B}}^*$ then for every $n \geq 1$, m_n = constant a.s. $[P|_{\underline{\underline{B}}}]$.

<u>Proof:</u> Since $\underline{\underline{A}}$ is countably generated we can show that there is a countably generated sub-σ-algebra $\underline{\underline{B}}^{**}$ of $\underline{\underline{B}}^{*}$ such that $\underline{\underline{B}}^{**} = \underline{\underline{B}}^{*}$ a.s. [P]. It is easy to check that $\underline{\underline{B}}^{**}$ is also an independent complement of $\underline{\underline{B}}$ and hence without loss of generality we can assume that $\underline{\underline{B}}^{*}$ is countably generated.

Let $\left\{B_1^*, B_2^*, \ldots\right\}$ be an enumeration of $\underline{\underline{B}}^*$atoms of positive measure such that $P(B_1^*) \geq P(B_2^*) \geq \ldots$ and let $B_o^* = (\bigcup_n B_n^*)^c$. By 2.3.1, there exists $N_1 \in \underline{\underline{B}}$ with $P(N_1) = 0$ such that $x \notin N_1$ and $B^* \in \underline{\underline{B}}^* \Rightarrow \mu(x, B^*) = P(B^*)$. Since $\sigma\left\{\underline{\underline{B}}, \underline{\underline{B}}^*\right\} = \underline{\underline{A}}$ a.s. [P] and since $\underline{\underline{A}}$ is countably generated we can find $X_1 \in \underline{\underline{A}}$ with $P(X_1) = 1$ such that $\sigma\left\{\underline{\underline{B}}, \underline{\underline{B}}^*\right\} \cap X_1 = \underline{\underline{A}} \cap X_1$. By CP3, $P(X_1) = 1$ implies the existence of $N_2 \in \underline{\underline{B}}$ with $P(N_2) = 0$ such that for all $x \notin N_2$, $\mu(x, X_1) = 1$.

Let $N_o = N \cup N_1 \cup N_2$ and let $x \notin N_o$. Then

$$\mu(x, B(x)) = \mu(x, B(x) \cap X_1) = \sum_{n=o}^{\infty} \mu(x, B(x) \cap B_n^*) .$$

If $y \in B(x) \cap B_o^*$ then

$$\mu(x, \{y\}) \leq \mu(x, B(x) \cap B^*(y))$$
$$= \mu(x, B^*(y)) = P(B^*(y)) = 0$$

where $B_o^*(y) \subset B_o^*$ is the $\underline{\underline{B}}^*$-atom containing y.

If $y \in B(x) - X_1$ then

$$\mu(x, \{y\}) \leq \mu(x, X_1^c) = 0.$$

Hence for every $x \notin N_o$,

$$\mu(x, \{y\}) > 0 \Rightarrow y \in B(x) \cap B_n^* \cap X_1$$

and since $\underline{\underline{A}} \cap X_1 = \sigma\left\{\underline{\underline{B}}^*, \underline{\underline{B}}\right\} \cap X_1$ we have

$B(x) \cap B_n^* \cap X_1 = \{y\}$.

Thus for every $x \notin N_o$ and for every $n \geq 1$ we have

$$m_n(x) = \mu(x, B(x) \cap B_n^* \cap X_1) = P(B_n^*)$$

or m_n = constant a.s. $[P|_{\underline{\underline{B}}}]$.

The natural question now is to ask whether the condition stated in 2.3.2 is sufficient for the existence of an independent complement. Indeed it is, and we prove this in section 2.5. In section 2.4, under the assumption that there is no one-sheeted set of positive measure, we prove the existence of an independent complement.

2.4 A sufficient condition.

The following theorem is a generalisation of a theorem of Rohlin (see § 4, No.3) on the existence of independent complements with respect to a measurable partition of a Lebesgue space which does not admit one-sheeted sets of positive measure (see chapter 5 of this thesis for relevant definitions). The proof given by Rohlin can be imitated with obvious modifications to prove our theorem and hence we refrain from repeating that proof here.

Theorem 2.4.1.: Suppose there is no one-sheeted set with respect to $\underline{\underline{B}}$ of positive measure. Then there is an independent complement $\underline{\underline{B}}^*$ of $\underline{\underline{B}}$.

Remark 2.4.2: Consider $([0, 1], \underline{\underline{B}}_{[0, 1]})$ where $\underline{\underline{B}}_{[0, 1]}$ denotes the Borel σ-algebra of $[0, 1]$. Let μ be a probability on $([0, 1], \underline{\underline{B}}_{[0, 1]})$ and let $\underline{\underline{B}}_o$ be a countably generated sub-σ-algebra of $\underline{\underline{B}}_{[0, 1]}$. Then there is a r.c.p. with respect to $\underline{\underline{B}}_o$ which is proper almost everywhere (see theorem 5 of [2]). If $\underline{\underline{B}}_o$ does not admit one-sheeted sets of positive measure then, by theorem 2.4.1, it follows that there exists an independent complement $\underline{\underline{B}}_o^*$ of $\underline{\underline{B}}_o$. This essentially is th result of Rohlin that we referred to at the beginning of this section.

Now using 2.4.2 we prove the following

Theorem 2.4.3: Let $(Z, \underline{\underline{C}}, Q)$ be a probability space where $\underline{\underline{C}}$ is separable. Let $\underline{\underline{C}}_o$ be an atomic sub-σ-algebra of $\underline{\underline{C}}$ such that there are no one-sheeted sets with respect to $\underline{\underline{C}}_o$ of positive measure. Then there is an independent complement $\underline{\underline{C}}_o^*$ of $\underline{\underline{C}}$.

Proof: By taking a generator $\{C_n, n \geq 1\}$ of $\underline{\underline{C}}$ and by using the Marczewski function of $\{C_n\}$, we can assume without loss of generality that $Z \subset [0, 1]$ and that

$$\underline{\underline{C}} = \left\{ A \cap Z : A \in \underline{\underline{B}}_{[0, 1]} \right\}.$$

Define μ on $\underline{\underline{B}}_{[0, 1]}$ by $\mu(A) = Q(A \cap Z)$, $A \in \underline{\underline{B}}_{[0, 1]}$.

Let $\underline{\underline{C}}'_o = \sigma\left\{C'_n\right\}$ be a countably generated sub-σ-algebra of $\underline{\underline{C}}_o$ such that $\underline{\underline{C}}'_o = \underline{\underline{C}}_o$ a.s. [P] (choice of $\underline{\underline{C}}'_o$ is possible since $\underline{\underline{C}}$ is countably generated). Let, for every n, $A_n \in \underline{\underline{B}}_{[0, 1]}$ be such that $A_n \cap Z = C'_n$ and let $\underline{\underline{B}}_o = \sigma\left\{A_n\right\}$.

There are no one-sheeted sets of positive Q-measure with respect to $\underline{\underline{C}}_o$ implies that there are no one-sheeted sets of positive μ-measure with respect to $\underline{\underline{B}}_o$. By 2.4.2, there is an independent complement $\underline{\underline{B}}^*_o$ of $\underline{\underline{B}}_o$. It is easy to check that $\underline{\underline{C}}^*_o = \left\{B^* \cap Z: B^* \in \underline{\underline{B}}^*_o\right\}$ is an independent complement of $\underline{\underline{C}}_o$.

Our next example shows that theorem 2.4.3 is really stronger than 2.4.1 because there is no assumption in 2.4.3 about the existence of an almost everywhere proper r.c.p.

Example 2.4.4: Let $X_1 = X_2 = [0, 1]$ and let λ on $(X_1, \underline{\underline{B}}_{[0, 1]})$ be the Lebesgue measure. Let M be a subset of $[0, 1]$ with outer Lebesgue measure one and inner Lebesgue measure zero. Define a measure

$$P_1 \text{ on } \underline{\underline{A}}_1 = \sigma\left\{\underline{\underline{B}}_{[0, 1]}, M\right\} = \left\{(B \cap M) \cup (C \cap M^c): B, C \in \underline{\underline{B}}_{[0, 1]}\right\}$$

by $P_1((B \cap M) \cup (C \cap M^c)) = \lambda(B)$. Then it is well known that there is no r.c.p. with respect to $\underline{\underline{B}}_{[0, 1]}$ on $X_1 \times \underline{\underline{A}}_1$ (see [11], p. 210).

Let $Z = X_1 \times X_2$, $\underline{\underline{C}} = \underline{\underline{A}}_1 \times \underline{\underline{B}}_{[0, 1]}$ and let $Q = P_1 \times \lambda$. It is easy to see that there is no r.c.p. with respect to $\underline{\underline{B}}_{[0, 1]} \times [0, 1]$ on $Z \times \underline{\underline{C}}$ since there is no r.c.p. with respect to $\underline{\underline{B}}_{[0, 1]}$ on $X_1 \times \underline{\underline{A}}_1$. But by Fubini's theorem there is no one-sheeted set with respect to $\underline{\underline{B}}_{[0,1]} \times [0, 1]$ of positive measure. Thus in this case there is an independent complement of $\underline{\underline{B}}_{[0,1]} \times [0, 1]$ by 2.4.3 although 2.4.1 is not applicable.

The following corollary of 2.4.1 will be needed in the next section.

<u>Corollary 2.4.5 :</u> There are no one-sheeted sets with respect to $\underline{\underline{B}}$ of positive measure if and only if almost all the measures $\{\mu(x, \cdot)\}$ are continuous.

<u>Proof:</u> If $\mu(x, \cdot)$'s are continuous for every $x \notin N_0$ where $N_0 \in \underline{\underline{B}}$ with $P(N_0) = 0$ then for any one-sheeted set $A \in \underline{\underline{A}}$, by CP3,

$$P(A) = \int \mu(x, B(x) \cap A) dP = 0.$$

If, on the other hand, there is no one-sheeted set of positive measure then, by 2.4.1, there is an independent complement $\underline{\underline{B}}^*$ of $\underline{\underline{B}}$. Since $\underline{\underline{A}}$ is separable $\underline{\underline{B}}^*$ can be assumed without loss of generality to be countably generated. By lemma 2.3.1, there exists $N_1 \in \underline{\underline{B}}$ with $P(N_1) = 0$ such that for every $B^* \in \underline{\underline{B}}^*$ $\mu(x, B^*) = P(B^*)$ for all $x \notin N_1$. Since

$\sigma\{\underline{\underline{B}}^*, \underline{\underline{B}}\} = \underline{\underline{A}}$ a.s. [P] there exists $X_1 \in \underline{\underline{A}}$ with $P(X_1) = 1$

such that $\sigma\left\{\underline{\underline{B}}^*, \underline{\underline{B}}\right\} \cap X_1 = \underline{\underline{A}} \cap X_1$. Now for every $x \notin N_I$,

$$\mu(x, \{y\}) \leq \mu(x, B^*(y)) = P(B^*(y)) = P(B^*(y) \cap X_1) = 0$$

since $B^*(y) \cap X_1$ is one-sheeted where $B^*(y)$ is the $\underline{\underline{B}}^*$-atom containing y.

2.5 A necessary and sufficient condition

In this section we prove that the necessary condition given in 2.3 for the existence of an independent complement is sufficient as well. First we prove some general results about the functions $\{m_n\}$ which were introduced in 2.3.

Let $X = M_o \cup M_1 \cup M_2 \cup \ldots$ be the decomposition of X; obtained by theorem 2.2.2 .

Proposition 2.5.1: $\mu(x, \cdot)\big|_{M_o}$ is continuous a.s. $[P|_{\underline{\underline{B}}}]$ and for every $n = 1, 2, \ldots$

$$\mu(x, M_n) \geq \mu(x, M_{n+1}) \quad \text{a.s.} \quad [P|_{\underline{\underline{B}}}].$$

Proof: If $P(M_o) = 0$ then the first assertion is trivial. If $P(M_o) > 0$ then consider the subspace $(M_o, \underline{\underline{A}} \cap M_o, P_{M_o} = \frac{P(\cdot \cap M_o)}{P(M_o)})$. Clearly in this space there is no one-shee set with respect to $\underline{\underline{B}} \cap M_o$ of positive measure. Further, by 2.1.3, there is a r.c.p. $\mu_o(x, A \cap M_o)$ on $M_o \times \underline{\underline{A}} \cap M_o$ with respect to $\underline{\underline{B}} \cap M_o$ which is proper a.s. $[P_{M_o}]$ such that

$$\mu_o(x, A \cap M_o) = \frac{\mu(x, A \cap M_o)}{\mu(x, M_o)} \quad \text{a.s. } [P_{M_o}].$$

By 2.4.5, $\mu_o(x, \cdot)$ is continuous a.s. $[P_{M_o}]$. It easily follows that $\mu(x, \cdot)\big|_{M_o}$ is continuous a.s. $[P\big|_{\underline{\underline{B}}}]$.

To prove the second assertion let

$$N_n = \left\{x : \mu(x, M_n) < \mu(x, M_{n+1})\right\}$$

and let $M_n' = (N_n^c \cap M_n) \cup (N_n \cap M_{n+1})$, then $M_n' \in \underline{\underline{A}}$ and is an one-sheeted subset of $X_{n-1} = (\bigcup_{i=1}^{n-1} M_i)^c$. If $P(N_n) > 0$ then

$$P(M_n') = \int \mu(x, M_n')dP$$

$$= \int_{N_n^c} \mu(x, M_n)dP + \int_{N_n} \mu(x, M_{n+1})dP$$

$$> \int \mu(x, M_n)dP = P(M_n)$$

which is impossible since M_n is a set of maximal measure among all measurable one-sheeted subsets of X_{n-1}. Hence $P(N_n) = 0$.

<u>Proposition 2.5.2:</u> There exists $N_o \in \underline{\underline{B}}$ with $P(N_o) = 0$ such that for all $x \notin N_o$

$$\mu(x, M_n) = m_n(x), \quad n = 1, 2, \ldots$$

In other words for all $n = 1, 2, \ldots$. m_n is $\underline{\underline{B}}$-measurable a.s. $[P|_{\underline{\underline{B}}}]$.

<u>Proof:</u> Using 2.5.1 we can get a set $N_o \in \underline{\underline{B}}$ with $P(N_o) = 0$ such that for every $x \notin N_o$, $\mu(x, \cdot)\big|_{M_o}$ is continuous and $\mu(x, M_n) \geq \mu(x, M_{n+1})$ for all $n \geq 1$. It follows that for every $x \notin N_o$, the sequence $\{B(x) \cap M_n\}$ consists of sets containing at most one point and so arranged that their measur form a non-increasing sequence. Further, this sequence contai every singleton which has positive measure. So by definition of $\{m_n\}$ we have

$$m_n(x) = \mu(x, B(x) \cap M_n) = \mu(x, M_n)$$

for all n and for every $x \notin N_o$.

<u>Theorem 2.5.3:</u> There exists an independent complement $\underline{\underline{B}}^*$ of $\underline{\underline{B}}$ if and only if for every $n \geq 1$

$$m_n = \text{constant} \qquad \text{a.s.} \ [P|_{\underline{\underline{B}}}].$$

<u>Proof:</u> 'Only if' part has been proved in 2.3.2.

To prove the 'if' part suppose the condition holds. Suppose $P(M_o) > 0$. In the subspace $(M_o, \underline{\underline{A}} \cap M_o, P_{M_o})$ there is no one-sheeted set with respect to $\underline{\underline{B}} \cap M_o$ of positive measure. By 2.1.3 and 2.4.1 (or just by 2.4.3) there exists an independent complement $\underline{\underline{B}}^*_o$ of $\underline{\underline{B}} \cap M_o$ in $(M_o, \underline{\underline{A}} \cap M_o, P_{M_o})$.

By 2.5.2 and CP3 the given condition implies that for every $n \geq 1$

$$\mu(x, M_n) = m_n(x) = P(M_n) \qquad \text{a.s.} \ [P|_{\underline{\underline{B}}}].$$

Let

$$\underline{\underline{B}}^* = \begin{cases} \sigma\left\{\underline{\underline{B}}^*_o, M_1, M_2, \ldots\right\} & \text{if } P(M_o) > 0 \\ \sigma\left\{M_1, M_2, \ldots\right\} & \text{if } P(M_o) = 0 . \end{cases}$$

It can be shown, using 2.3.1, 2.5.2 and the fact that $\underline{\underline{B}}^*_o$ is independent of $\underline{\underline{B}} \cap M_o$, that $\underline{\underline{B}}^*$ and $\underline{\underline{B}}$ are independent. Again we can check, since each M_n is one-sheeted with respect to $\underline{\underline{B}}$, that $\sigma\left\{\underline{\underline{B}}, \underline{\underline{B}}^*\right\} = \underline{\underline{A}}$ a.s. $[P]$. Thus $\underline{\underline{B}}^*$ is an independent complement of $\underline{\underline{B}}$.

2.6 Application to perfect probability spaces

Let $(X, \underline{\underline{A}}, P)$ be a perfect probability space with $\underline{\underline{A}}$ separable. Suppose $\underline{\underline{B}}$ is any sub-σ-algebra of $\underline{\underline{A}}$. Then there always exists a r.c.p. on $X \times \underline{\underline{A}}$ given $\underline{\underline{B}}$ (see [39], theorem 7). This enables us to tackle the problem of existence of independent complements in these spaces.

If $\underline{\underline{B}}$ is countably generated then because of 2.1.1 our assumptions of section 2.5 hold. Using 2.5.3 we can give a necessary and sufficient condition for the existence of an independent complement. For an arbitrary sub-σ-algebra $\underline{\underline{B}}$ of $\underline{\underline{A}}$, since $\underline{\underline{A}}$ is countably generated, we can choose a sub-σ-algebra $\underline{\underline{B}}_1$ of $\underline{\underline{B}}$ such that $\underline{\underline{B}}_1 = \underline{\underline{B}}$ a.s. [P]. Clearly the choice of $\underline{\underline{B}}_1$ is essentially unique and now we can obtain a necessary and sufficient condition for the existence of an independent complement of $\underline{\underline{B}}$ through similar conditions for $\underline{\underline{B}}_1$. We omit the details.

CHAPTER 3

ON LEBESGUE SPACES

3.0. Introduction

Lebesgue spaces are probability spaces essentially isomorphic to a subinterval of the unit interval with Lebesgue measure together with countably many points of positive measure. A systematic study of Lebesgue spaces, their isomorphisms and their factor spaces was carried out by Rohlin [34]. A Lebesgue space is always a perfect probability space (see theorem 5, [39]). In this chapter we give new characterisations of Lebesgue spaces and study the class of independent complements of a given measurable partition of a Lebesgue space.

3.1 Separable spaces, canonical system of measures and Lebesgue spaces

A probability space $(Z, \underline{\underline{C}}, P)$ is said to be a separable space if there exists a sequence of measurable sets $A_1, A_2, \ldots,$ called a basis, such that

a) $(Z, \underline{\underline{C}}, P)$ is the completion of the probability space $(Z, \sigma(\{A_n\}), P|_{\sigma(\{A_n\})})$ and

b) A_n separates points, that is, given $z, z' \in Z$, $z \neq z'$ there exists n such that $z \in A_n$ and $z' \notin A_n$ or $z \notin A_n$ and $z' \in A_n$. In other words, $\sigma(\{A_n\})$ is separable.

$\{0, 1\}^{N_0}$ denotes the unilateral countable product of the space $\{0, 1\}$ and $\underline{\underline{F}}$ denotes the product of discrete σ-algebras on the component spaces. If $\Gamma = \{G_n\}$ is a sequence of measurable sets of a measurable space $(Z, \underline{\underline{C}})$ then the Marczewski function f_Γ of the sequence Γ from Z to $\{0, 1\}^{N_0}$ is defined by

$$f_\Gamma(z) = \{1_{G_n}(z) : n = 1, 2, \ldots\}, \quad z \in Z .$$

It can be verified that f_Γ is a measurable map from $(Z, \sigma(\{G_n\}))$ onto $(f(Z), \underline{\underline{F}} \cap f(Z))$ such that forward images of measurable sets are measurable.

Rohlin has defined a separable space to be a Lebesgue space if for some basis $\Sigma = \{A_n\}$ there exists $B \in \underline{\underline{F}}$ such that $B \subset f_\Sigma(Z)$ and $P(f_\Sigma^{-1}(B)) = 1$. If $(Z, \underline{\underline{C}}, P)$ is a Lebesgue space then P is called a Lebesgue measure. It is well known (see [37], [39]) that a separable space is a

Lebesgue space if and only if it is a perfect probability space. It is not difficult to check that if P' is a measure on $(Z, \underline{\underline{C}})$ such that $(Z, \underline{\underline{C}}, P')$ is a separable space and if P' is absolutely continuous with a Lebesgue measure P, then P' is also a Lebesgue measure.

Let $(Z, \underline{\underline{C}}, P)$ be a separable space and let ξ be an arbitrary partition of Z. Let Z_ξ be the quotient space. Let H_ξ be the quotient map sending $z \in Z$ to that unique $C \in \xi$ which contains z. Let $\underline{\underline{A}}_\xi = \left\{X \subset Z_\xi : H_\xi^{-1}(X) \in \underline{\underline{C}}\right\}$. Define $P_\xi(X) = P(H_\xi^{-1}(X))$, $X \in \underline{\underline{A}}_\xi$. $(Z_\xi, \underline{\underline{A}}_\xi, P_\xi)$ is the quotient space of $(Z, \underline{\underline{C}}, P)$ with respect to the partition ξ. We shall call a partition ξ measurable if there exists a sequence $\left\{B_n\right\}$ of measurable sets, called a basis of the partition ξ, such that ξ is collection of atoms of $\sigma(\left\{B_n\right\})$. It follows that elements of a measurable partition are measurable sets. However, a partition η whose elements are measurable sets need not be a measurable partition. To see this let $Z = [0, 1]$, $\underline{\underline{C}}$ = Lebesgue measurable subsets of $[0, 1]$, P = Lebesgue measure and let η be the partition induced by the relation R defined by

$$xRy \text{ if } x-y \text{ is a rational, } x, y \in [0, 1].$$

Let ξ_1 and ξ_2 be two measurable partitions. We say that ξ_1 and ξ_2 are complementary if there exists a basis $\{B_n\}$ of ξ_1, a basis $\{B'_n\}$ of ξ_2 and a set $Z_o \in \underline{\underline{C}}$ with $P(Z_o) = 1$ such that $\{B_n \cap Z_o\}_{n\geq 1} \cup \{B'_n \cap Z_o\}_{n\geq 1}$ is a basis of the separable space $(Z_o, \underline{\underline{C}} \cap Z_o, P|_{Z_o})$. If further the σ-algebras $H^{-1}_{\xi_1}(\underline{\underline{A}}_{\xi_1})$ and $\sigma(\{B'_n\})$ are independent then we say that ξ_2 is a pseudo-independent complement of ξ_1. Rohlin has defined ξ_2 to be an independent complement of ξ_1 if ξ_1 and ξ_2 are complementary and $H^{-1}_{\xi_1}(\underline{\underline{A}}_{\xi_1})$ and $H^{-1}_{\xi_2}(\underline{\underline{A}}_{\xi_2})$ are independent. In the case of Lebesgue spaces the two definitions are equivalent. But in a general separable space the two definitions need not be equivalent. For, recall the example constructed by Doob [6] and Jessen [14] to show that the two definitions of independence (see chapter 1) are not equivalent. If ξ and ξ^* are the partitions induced by projections to first and second coordinates respectively in the space constructed by them then ξ and ξ^* are pseudo-independent complements of each other but not independent complements.

Let ξ be a measurable partition. By using the measure P_ξ it is possible to speak of almost all elements C of ξ.

<u>Definition 3.1.1:</u> Suppose for every C in ξ, there is a σ-algebra $\underline{\underline{A}}_C$ of subsets of C and a measure μ_C on $\underline{\underline{A}}_C$ such that

1) $(C, \underline{\underline{A}}_C, \mu_C)$ is a separable space for almost all C

2) if $A \in \underline{\underline{C}}$ then

   a) $A \cap C \in \underline{\underline{A}}_C$ for almost all C

   b) the function $\mu_C(A \cap C)$ on Z_ξ is $\underline{\underline{A}}_\xi$-measurable and

   c) $P(A) = \int_{Z_\xi} \mu_C(A \cap C)\, dP_\xi$

then the system $\{\mu_C\}_{C \in \xi}$ of measures is called a canonical system with respect to ξ. If we define $\mu(z, A)$ on $Z \times \underline{\underline{C}}$ by $\mu(z, A) = \mu_{C(z)}(A \cap C(z))$ where $C(z) = H_\xi(z)$ then it is easy to verify that $\mu(z, A)$ is a r.c.p. with respect to $H_\xi^{-1}(\underline{\underline{A}}_\xi)$ which is proper almost everywhere. Again it is easy to see that any two systems of measures canonical with respect to ξ are equal a.s. $[P_\xi]$. Rohlin has shown that if ξ is a measurable partition of a Lebesgue space then there exists a system $\{\mu_C\}_{C \in \xi}$ of measures canonical with respect to ξ such that μ_C is a Lebesgue measure for almost all C in ξ.

In what follows ξ is a fixed measurable partition of the separable space $(Z, \underline{\underline{C}}, P)$ such that there exists a canonical system of measures with respect to ξ. A set $A \subset Z$ is said to be one-sheeted with respect to ξ if $A \cap C$ contains at most one point for every C in ξ. The following two theorems are direct translations of 2.2.2 and 2.4.1 to the present set-up.

<u>Theorem 3.1.2:</u> There exists a partition of Z of the form $Z = \bigcup_{n=0}^{\infty} M_n$ where M_n, $n \geq 1$ are measurable sets one-sheeted with respect to ξ such that M_n is a set of maximal measure among all measurable one-sheeted subsets of the set $\left(\bigcup_{k=1}^{n-1} M_k\right)^c$ and M_o does not contain any subset of positive measure which is one-sheeted with respect to ξ.

<u>Theorem 3.1.3:</u> If there does not exist any set of positive measure one-sheeted with respect to ξ then ξ admits a pesu independent complement.

Let the measurable partition ξ be without one-sheeted sets of positive measure and let ξ^* be a pseudo-independent complement of ξ which is guaranteed by theorem 3.1.3.

Consequently there exist bases $\{B_n\}$ of ξ and $\{B_n^*\}$ of ξ^* and $Z_o \in \underline{\underline{C}}$ with $P(Z_o) = 1$ such that $\{B_n \cap Z_o\} \cup \{B_n^* \cap Z_o\}$ is a basis of $(Z_o, \underline{\underline{C}} \cap Z_o, P|_{Z_o})$ and the σ-algebras $H_\xi^{-1}(\underline{\underline{A}}_\xi)$ and $\sigma(\{B_n^*\})$ are independent. Let $\underline{\underline{B}}_{\xi^*} = \overline{\sigma\{H_{\xi^*}(B_n^*)\}}$, the completion of $\sigma\{H_{\xi^*}(B_n^*)\}$ with respect to P_{ξ^*}. Then $(Z_{\xi^*}, \underline{\underline{B}}_{\xi^*}, P_{\xi^*})$ is a separable space. It is not difficult to check that $H_\xi^{-1}(\underline{\underline{A}}_\xi)$ and $H_{\xi^*}^{-1}(\underline{\underline{B}}_{\xi^*})$ are independent. Now let $Z' = Z_\xi \times Z_{\xi^*}$, $\underline{\underline{C}}' = \overline{\underline{\underline{A}}_\xi \times \underline{\underline{B}}_{\xi^*}}$ and $P' = \overline{P_\xi \times P_{\xi^*}}$. Define the map $h_\xi : Z \to Z'$ by $h_\xi(z) = (C(z), C^*(z))$ where $C(z)$ and $C^*(z)$ are respectively the elements C in ξ and C^* in ξ^* which contain z.

<u>Theorem 3.1.4:</u> <u>Let ξ be without one-sheeted sets of positive measure.</u> Then h_ξ is measurable and $Ph_\xi^{-1} = P'$. Further if $Z_o' = h_\xi(Z_o)$ then the probability spaces $(Z_o, \underline{\underline{C}} \cap Z_o, P|_{Z_o})$ and $(Z_o', \underline{\underline{C}}' \cap Z_o', P'^*|_{Z_o'})$ are isomorphic.

<u>Proof:</u> Let $X \in \underline{\underline{A}}_\xi$ and $X^* \in \underline{\underline{B}}_{\xi^*}$. Consider the measurable rectangle $X \times X^*$. Then

$$h_\xi^{-1}(X \times X^*) = H_\xi^{-1}(X) \cap H_{\xi^*}^{-1}(X^*)$$

and since $H_\xi^{-1}(\underline{\underline{A}}_\xi)$ and $H_{\xi^*}^{-1}(\underline{\underline{B}}_{\xi^*})$ are independent we have

$$\begin{aligned} P'(X \times X^*) &= P_\xi(X) \cdot P_{\xi^*}(X^*) \\ &= P(H_\xi^{-1}(X)) \cdot P(H_{\xi^*}^{-1}(X^*)) \\ &= P(H_\xi^{-1}(X) \cap H_{\xi^*}^{-1}(X^*)) \\ &= Ph_\xi^{-1}(X \times X^*). \end{aligned}$$

Thus it follows that h_ξ is measurable and $P' = Ph_\xi^{-1}$.

Clearly h_ξ is 1-1 on Z_o. In order to prove that $(Z_o, \underline{\underline{C}} \cap Z_o, P|_{Z_o})$ and $(Z'_o, \underline{\underline{C}}' \cap Z'_o, P'^*|_{Z'_o})$ are isomorphic it remains to show that $h_\xi(\underline{\underline{C}} \cap Z_o) \subset \underline{\underline{C}}' \cap Z'_o$. But

$$h_\xi(B_n \cap Z_o) = (H_\xi(B_n) \times Z_{\xi^*}) \cap Z'_o$$

and $h_\xi(B_n^* \cap Z_o) = (Z_\xi \times H_{\xi^*}(B_n^*)) \cap Z'_o$ and so $h_\xi(\{B_n \cap Z_o\} \cup \{B_n^* \cap Z_o\}) \subset \underline{\underline{C}}' \cap Z'_o$. It follows that $h_\xi(\underline{\underline{C}} \cap Z_o) \subset \underline{\underline{C}}' \cap Z'_o$.

<u>Definition 3.1.5:</u> Let ξ be without one-sheeted sets of positive measure. Then ξ is said to resolve $(Z, \underline{\underline{C}}, P)$ as a product space if $h_\xi(Z) \in \underline{\underline{C}}'$.

<u>Theorem 3.1.6:</u> Suppose ξ is without one-sheeted sets of positive measure. Then $(Z, \underline{\underline{C}}, P)$ is a Lebesgue space if and only if

i) $(Z_\xi, \underline{\underline{A}}_\xi, P_\xi)$ is a Lebesgue space.

ii) μ_C is a Lebesgue measure for almost all C in ξ

and iii) ξ resolves $(Z, \underline{\underline{C}}, P)$ as a product space.

<u>Proof:</u> The necessity of conditions (i), (ii) and (iii) follows from theorems of Rohlin on Lebesgue spaces (see [34] § 3 and § 4).

To prove the sufficiency let ξ^* be a pseudo-independent complement of ξ. Let $\{B_n\}, \{B_n^*\}$ and Z_o be defined as before. Let $Y \in \underline{\underline{B}}_{\xi^*}$. For any $X \in \underline{\underline{A}}_\xi$ we have by 2(c) of definition 3.1.1 and independence

$$\int_X \mu_C(C \cap H_{\xi^*}^{-1}(Y))\,dP_\xi$$

$$= P(H_\xi^{-1}(X) \cap H_{\xi^*}^{-1}(Y))$$

$$= P(H_\xi^{-1}(X)) \cdot P(H_{\xi^*}^{-1}(Y))$$

$$= \int_X P_{\xi^*}(Y)\,dP_\xi .$$

Hence $P_{\xi^*}(Y) = \mu_C(C \cap H_{\xi^*}^{-1}(Y))$ for almost all C. Let $\Sigma^* = \text{alg}\left\{H_{\xi^*}(B_n^*)\right\}$. Since Σ^* is countable we can find a set $N_1 \in \underline{\underline{A}}_\xi$ with $P_\xi(N_1) = 0$ such that $C \notin N_1$ implies $C \cap H^{-1}_{\xi*}(X^*) \in \underline{\underline{A}}_G$ and $\mu_C(C \cap H^{-1}_{\xi*}(X^*)) = P_{\xi*}(X^*)$ for all X^* in Σ^*. Thus for $C \notin N_1$, the map $H_{\xi*}$ from C to the separable space $(Z_{\xi*}, \underline{\underline{B}}_{\xi*}, P_{\xi*})$ is a measurable, measure-preserving map. Since

$$\int_{Z_\xi} \mu_C(C \cap Z_o) = P(Z_o) = 1$$

there exists $N_2 \in \underline{\underline{A}}_\xi$ with $P_\xi(N_2) = 0$ such that for $C \notin N_2$, $C \cap Z_o \in \underline{\underline{A}}_C$ and $\mu_C(C \cap Z_o) = 1$. By condition (ii) there exists $N_3 \in \underline{\underline{A}}_\xi$ with $P_\xi(N_3) = 0$ such that for $C \notin N_3$, μ_C is a Lebesgue measure. Let $N = N_1 \cup N_2 \cup N_3$. Then $N \in \underline{\underline{A}}_\xi$ and $P_\xi(N) = 0$ and for $C \notin N$, $H_{\xi*}$ is 1-1 on $C \cap Z_o$. Under a 1-1 measurable, measure-preserving map the image of a Lebesgue space in a separable space, is a Lebesgue space ([34], § 2, No.5). So it is enough to look at some $C \notin N$ to conclude that $(Z_{\xi*}, \underline{\underline{B}}_{\xi*}, P_{\xi*})$ is a Lebesgue space, since for such a C, $(C, \underline{\underline{A}}_C, \mu_C)$ is a Lebesgue space.

Now $(Z', \underline{\underline{C}}', P')$ is a Lebesgue space since it is a product of two Lebesgue spaces. So by condition (iii) $(Z, \underline{\underline{C}}, P)$ can be imbedded in a Lebesgue space as a measurable subspace of measure one which means $(Z, \underline{\underline{C}}, P)$ is a Lebesgue space.

We shall give an example in chapter 4 (see remark 4.3.7) to show that condition (i) and (ii) of theorem 3.1.6 are not sufficient to ensure that $(Z, \underline{\underline{C}}, P)$ is a Lebesgue space.

The following lemma can be proved by arguments analogous to those used in the proof of theorem 2.1.2 and so we omit the proof.

<u>Lemma 3.1.7:</u> Suppose $Z_1 \in \underline{\underline{C}}$ with $P(Z_1) > 0$. Let $X = \left\{C : \mu_C(C \cap Z_1) > 0\right\}$. Then for each $C_1 = C \cap Z_1$, $C \in X$ if we define $\underline{\underline{A}}_{C_1} = \underline{\underline{A}}_C \cap Z_1$ and $\mu_{C_1}(A \cap C_1) = \mu_C(A \cap Z_1 \cap C) / \mu_C(C_1)$ and for $C \notin X$ if we define $\underline{\underline{A}}_{C_1}$ and μ_{C_1} arbitrarily the system $\left\{\mu_{C_1}\right\}_{C_1 \in \xi_1}$, where $\xi_1 = \xi\big|_{Z_1}$ is the restriction of ξ to Z_1, is a canonical system of measures with respect to ξ_1.

<u>Definition 3.1.8:</u> We say that ξ is a product type partition if, whenever $P(M_o) > 0$, ξ_o resolves the subspace

$$(M_o, \underline{\underline{C}} \cap M_o, P_{M_o} = \frac{P(\,\cdot\,)}{P(M_o)})$$

as a product space where the measurable partition ξ_o is the restriction of ξ to M_o.

<u>Theorem 3.1.9:</u> $(Z, \underline{\underline{C}}, P)$ is a Lebesgue space if and only if

i) $(Z_\xi, \underline{\underline{A}}_\xi, P_\xi)$ is a Lebesgue space

ii) μ_C is a Lebesgue measure for almost all C in ξ and

iii) ξ is a product type partition.

<u>Proof:</u> Let $Z = \bigcup_{n=o}^{\infty} M_n$ be the partition of Z given by theorem 3.1.2. For $n \geq 1$ let $X_n = \left\{C : \mu_C(C \cap M_n) > 0\right\}$ and let $Y_n = H_\xi^{-1}(X_n)$. Then

$$P(Y_n \cap M_n) = \int_{X_n} \mu_C(C \cap M_n)dP_\xi = P(M_n).$$

Since M_n is one-sheeted H_ξ is a 1-1, measurable map from $(Y_n \cap M_n, \underline{\underline{C}} \cap Y_n \cap M_n)$ to $(X_n, \underline{\underline{A}}_\xi \cap X_n)$. Further if $Y \in \underline{\underline{C}} \cap Y_n \cap M_n$ observe that $H_\xi(Y) = \left\{C : \mu_C(C \cap Y) > 0\right\} \in \underline{\underline{A}}_\xi \cap X_n$. Hence $H_\xi : Y_n \cap M_n \longrightarrow X_n$ is a 1-1, bimeasurable map. Further the measures $P_{Y_n \cap M_n}$ and $P_\xi\big|_{X_n}$ are mutually absolutely continuous. Now from (i) it follows that $(X_n, \underline{\underline{A}}_\xi \cap X_n, P_\xi\big|_{X_n})$ is a Lebesgue space. So it follows that

$(Y_n \cap M_n, \underline{\underline{C}} \cap Y_n \cap M_n, P_{Y_n \cap M_n})$ and hence $(M_n, \underline{\underline{C}} \cap M_n, P_{M_n})$ are Lebesgue spaces.

If $P(M_o) > 0$, let $X = \{C : \mu_C(C \cap M_o) > 0\}$, and let $Y = H_\xi^{-1}(X)$. Let $Y_o = Y \cap M_o$. Then $P(Y_o) = P(M_o)$. Let $\underline{\underline{C}}_o = \underline{\underline{C}} \cap Y_o$ and $\eta = \xi \Big|_{Y_o}$. Consider $(X, \underline{\underline{A}}_\xi \cap X)$. Let $\phi : X \longrightarrow Y_{o\eta}$ be defined by

$$\phi(C) = H_\eta(H_\xi^{-1}(C) \cap M_o).$$

Then ϕ is 1-1 and measurable. Further the measure $P_\eta \phi$ is absolutely continuous with respect to the measure $P_\xi \Big|_X$. It follows from (i) that $P_\xi \Big|_X$ is a Lebesgue measure and hence $(Y_{o\eta}, \underline{\underline{C}}_{o\eta}, P_\eta)$ is a Lebesgue space. By lemma 3.1.7 and (ii), in the separable space $(Y_o, \underline{\underline{C}} \cap Y_o, P_{Y_o})$ there exists a system $\{\mu_{C_o}\}_{C_o \in \eta}$ of measures canonical with respect to η such that almost all measures μ_{C_o} are Lebesgue measures. Since M_o does not contain one-sheeted subsets of positive measure with respect to ξ, η is without one-sheeted sets of positive measure. Hence by (iii) and theorem 3.1.6, $(Y_o, \underline{\underline{C}}_o, P_{Y_o})$ and so $(M_o, \underline{\underline{C}} \cap M_o, P_{M_o})$ are Lebesgue spaces.

Finally let Γ be a basis of $(Z, \underline{\underline{C}}, P)$ and let f_Γ be the Marczewski function from Z to $\{0, 1\}^{N_o}$ corresponding to Γ. Let $\Gamma_n = \Gamma|_{M_n}$, $n = 0, 1, 2, \ldots$. Then Γ_n is a basis of $(M_n, \underline{\underline{C}} \cap M_n, P_{M_n})$, $n = 0, 1, 2, \ldots$. Since $f_\Gamma(Z) = \bigcup_{n=o}^{\infty} f_{\Gamma_n}(M_n)$ and since $(M_n, \underline{\underline{C}} \cap M_n, P_{M_n})$ is a Lebesgue space for each $n = 0, 1, 2, \ldots$ we have for every n, $B_n \in \underline{\underline{F}}$, $B_n \subset f_{\Gamma_n}(M_n)$ with $P(f_{\Gamma_n}^{-1}(B_n)) = 1$ or $P(f_\Gamma^{-1}(B_n)) = P(M$ B_n's are disjoint since M_n's are . Therefore $B = \bigcup_{n=o}^{\infty} B_n \in \underline{\underline{F}}$ and $P(f_\Gamma^{-1}(B)) = \sum_{n=o}^{\infty} P(M_n) = 1$. Hence $(Z, \underline{\underline{C}}, P)$ is a Lebesgue space.

<u>Lemma 3.1.10:</u> ξ does not admit one-sheeted sets of positive measure if and only if μ_C is continuous for almost all C in ξ.

<u>Proof:</u> By 2(c) of definition 3.1.1 for any $A \in \underline{\underline{C}}$, $P(A) = \int_{Z_\xi} \mu_C(A \cap C)\,dP_\xi$. If almost all μ_C are continuous and if $A \in \underline{\underline{C}}$ is one-sheeted then, since $A \cap C$ is at most a singleton for every C in ξ, $P(A) = 0$.

Suppose ξ does not admit one-sheeted sets of positive measure. Let ξ^* be a pseudo-independent complement of ξ. Proceeding as in the proof of 3.1.6 we can find a P_ξ-null set $N \in \underline{\underline{A}}_\xi$ and a set $Z_o \in \underline{\underline{C}}$ with $P(Z_o) = 1$ such that for $C \notin N$ we have

(1) $H_{\xi^*} : C \longrightarrow Z_{\xi^*}$ is measurable and measure-preserving

(2) $C \cap Z_o \in \underline{\underline{A}}_C$, $\mu_C(C \cap Z_o) = 1$ and

(3) H_{ξ^*} is 1-1 on $C \cap Z_o$.

In the separable space $(Z_{\xi^*}, \underline{\underline{A}}_{\xi^*}, P_{\xi^*})$ P_{ξ^*} is a continuous measure, for, if C^* is a point in Z_{ξ^*} then

$$P_{\xi^*}(\{C^*\}) = P(C^*) = P(C^* \cap Z_o) = 0$$

as $C^* \cap Z_o$ is a measurable set one-sheeted with respect to ξ.

Now if $z \in C \notin N$ then

$$\mu_C(\{z\}) = 0 \text{ if } z \notin C \cap Z_o, \text{ and}$$

$$\mu_C(\{z\}) = P_{\xi^*}(\{H_{\xi^*}(z)\})$$

$$= 0 \text{ if } z \in C \cap Z_o.$$

Thus μ_C is continuous for almost all C in ξ.

Theorem 3.1.11: If $(Z_\xi, \underline{\underline{A}}_\xi, P_\xi)$ is a Lebesgue space and if μ_C is discrete for almost all C in ξ, then $(Z, \underline{\underline{C}}, P)$ is a Lebesgue space.

Proof: We shall show that if μ_C is discrete for almost all C in ξ then $P(M_o) = 0$ where M_o, given by 3.1.2, does not contain any subset of positive measure one-sheeted with respect to ξ. By theorem 3.1.9, $P(M_o) = 0$ implies that $(Z, \underline{\underline{C}}, P)$ is a Lebesgue space.

Suppose $P(M_o) > 0$. Then the set $X = \{C : \mu_C(C \cap M_o) > 0\}$ is of positive P_ξ-measure. By Lemma 3.1.7, for each $C_o = C \cap M_o$, $C \in X$, if we define $\underline{\underline{A}}_{C_o} = \underline{\underline{A}}_C \cap M_o$ and

$$\mu_{C_o}(A \cap C_o) = \mu_C(A \cap M_o \cap C) \Big/ \mu_C(C \cap M_o), \quad A \in \underline{\underline{A}}_C \text{ and for}$$

and for each $C_o = C \cap M_o$, $C \notin X_o$ if we define $\underline{\underline{A}}_{C_o}$ and μ_{C_o} arbitrarily, then the system $\{\mu_{C_o}\}_{C_o \in \xi_o}$ is a canonical system of measures with respect to ξ_o, the restriction of ξ to M_o. Now ξ_o does not admit one-sheeted subsets of positive measure. Therefore, by lemma 3.1.10, μ_{C_o} must be continuous for almost all C_o in ξ_o, say for all $C_o \in X_o$, $P_{\xi_o}(X_o) = 1$. Then

$$P(H_{\xi_o}^{-1}(X_o)) = P(M_o) = P(H_{\xi}^{-1}(X) \cap M_o) .$$

Let $Y = H_{\xi}^{-1}(X) \cap M_o \cap H_{\xi_o}^{-1}(X_o)$ and let $X' = \{C : \mu_C(C \cap Y) > 0\}$. Then $P(Y) = P(M_o) > 0$ and so $P_{\xi}(X') > 0$. Further $X' \subseteq X$. Thus we have for all $C \in X'$, $P_{\xi}(X') > 0$, μ_C is continuous on the set $C \cap M_o$ with $\mu_C(C \cap M_o) > 0$. So μ_C is not discrete for $C \in X'$, $P_{\xi}(X') > 0$, which is a contradiction. Hence $P(M_o) = 0$.

3.2. On the collection of independent complements of a measurable partition

In this section we let ξ be a fixed measurable partition of a Lebesgue space $(Z, \underline{\underline{C}}, P)$. We shall show that any two independent complements of ξ are isomorphic a.s. in a sense to be stated precisely later and we conclude the section with a necessary and sufficient condition for the existence of a unique independent complement.

Suppose ξ^* is an independent complement of ξ. Proceeding as in the proof of 3.1.6 we can show that there exists $N \in \underline{\underline{A}}_{\xi}$ with $P_{\xi}(N) = 0$ and $Z_o \in \underline{\underline{C}}$ with $P(Z_o) = 1$ such that for all $C \notin N$,

i) $H_{\xi*}$ is a measurable, measure-preserving map from $(C, \underline{\underline{A}}_C, \mu_C)$ to $(Z_{\xi*}, \underline{\underline{A}}_{\xi*}, P_{\xi*})$ (as noted earlier in the case of a Lebesgue space $\underline{\underline{B}}_{\xi*}$ defined earlier is the same as $\underline{\underline{A}}_{\xi*}$)

ii) $\mu_C(C \cap Z_o) = 1$ and

iii) $H_{\xi*}$ is 1-1 on $C \cap Z_o$.

Suppose $\xi*$ and $\xi**$ are two independent complements of ξ. Then it is easy to see that conditions (i), (ii) and (iii) are simultaneously satisfied by both $H_{\xi*}$ and $H_{\xi**}$. Again since we are dealing with Lebesgue spaces (see [34], §2, No.5) it follows that $H_{\xi*}$ and $H_{\xi**}$ are isomorphisms a.s. Hence the following proposition holds.

<u>Proposition 3.2.1</u> There exists a map f from $(Z_{\xi*}, \underline{\underline{A}}_{\xi*}, P_{\xi*})$ to $(Z_{\xi**}, \underline{\underline{A}}_{\xi**}, P_{\xi**})$ such that f is an isomorphism a.s. $[P_{\xi*}]$, that is, f is measurable, measure preserving and 1-1 except on a $P_{\xi*}$- null set.

The next proposition is a direct consequence of 3.2.1.

<u>Proposition 3.2.2:</u> The function g from $(Z_\xi \times Z_{\xi*}, \overline{\underline{\underline{A}}_\xi \times \underline{\underline{A}}_{\xi*}}, \overline{P_\xi \times P_{\xi*}})$ to $(Z_\xi \times Z_{\xi**}, \overline{\underline{\underline{A}}_\xi \times \underline{\underline{A}}_{\xi**}}, \overline{P_\xi \times P_{\xi**}})$ defined by $g(C, C*) = (C, f(C*))$ is an

isomorphism a.s. $[P_\xi \times P_{\xi*}]$.

Now we have the following theorem on the isomorphism of the two independent complements ξ^* and ξ^{**} of ξ.

Theorem 3.2.3: There exists a map $\emptyset$ from $(Z, \underline{\underline{C}}, P)$ to $(Z, \underline{\underline{C}}, P)$ which is measurable, measure-preserving and such that

a) $\emptyset$ is 1-1 on a set Z_o with $P(Z_o) = 1$

b) $\emptyset(A \cap Z_o) = A \cap \emptyset(Z_o)$ for all $A \in H_\xi^{-1}(\underline{\underline{A}}_\xi)$ and

c) $\emptyset(\underline{\underline{A}}_{\xi*} \cap Z_o) = \underline{\underline{A}}_{\xi**} \cap \emptyset(Z_o)$.

Proof: Let $h_1 : Z \to Z_\xi \times Z_{\xi*}$, $h_2 : Z \to Z_\xi \times Z_{\xi**}$ be defined by

$$h_1(z) = (C(z), C^*(z))$$

$$h_2(z) = (C(z), C^{**}(z))$$

where $C(z)$, $C^*(z)$, $C^{**}(z)$ are respectively the elements of ξ, ξ^*, ξ^{**} containing z. An argument similar to the proof of 3.1.4 will show that there exists $Z_i \in \underline{\underline{C}}$ with $P(Z_i) = 1$ such that $(Z_i, \underline{\underline{C}} \cap Z_i, P_{Z_i})$ is isomorphic to the subspace induced by $h_i(Z_i)$, $i = 1,2$. Since all the spaces we are dealing with happen to be Lebesgue spaces forward images under h_i are measurable. Let g be the mapping defined in 3.2.2 and let $Z' \in \underline{\underline{A}}_{\xi} \times \underline{\underline{A}}_{\xi^*}$ with $P_{\xi} \times P_{\xi^*}(Z') = 1$ be such that g is 1-1 on Z'.

Let $Z_o = Z_1 \cap h_1^{-1}(g^{-1}h_2(Z_2) \cap Z')$ and let $\emptyset$ be defined by

$$\emptyset = \begin{cases} h^{-1} g\, h_1 & \text{on } Z_o \\ z_o & \text{on } Z_o^c \text{ where } z_o \in Z_o^c . \end{cases}$$

$\emptyset$ can be verified to satisfy the conditions (a), (b) and (c) of the theorem.

To obtain the necessary and sufficient condition for the existence of a unique independent complement we use some results of Rohlin regarding the structure of the measurable partition ξ of the Lebesgue space $(Z, \underline{\underline{C}}, P)$. As we did in the earlier chapter (section 2.3) we define for almost all C in ξ, $m_n(C) = \mu_C(\{y_n(C)\})$ where $y_1(C), y_2(C), \ldots$ is an enumeration of points of C of positive μ_C-measure arranged in such a way that their measures form a non-increasing sequence (as before we define $m_n(C) = 0$ for $n > p$ if this sequence contains only p many points). Now for each $n \geq 1$, m_n is a function defined on Z_{ξ}. Theorem 2.5.3 can now be restated as

Theorem 3.2.4: ξ admits an independent complement if and only if for every $n \geq 1$, m_n = constant a.s. $[P_\xi]$.

Let ξ admit an independent complement. By results of Rohlin (see [34], § 4, No.1) and 3.2.4, we can, without loss of generality, take $(Z, \underline{\underline{C}}, P)$ and ξ to be as follows:

Let $C_1, C_2, \ldots$ be a sequence of points of Z_ξ containing all points of Z_ξ of positive P_ξ-measure and such that $P_\xi(C_1) \geq P_\xi(C_2) \geq \ldots$. Let $\ell_k = P_\xi(C_k)$, $k \geq 1$, and let $\ell_o = 1 - \sum_{k=1}^{\infty} \ell_k$. Since ξ admits an independent complement m_n = constant a.s. $[P_\xi]$ for all n and let m_n itself denote that constant value. Then $m_n \geq m_{n+1} \geq 0$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} m_n \leq 1$. Let $m_o = 1 - \sum_{n=1}^{\infty} m_n$.

Let

$$Z_1 = [0, \ell_o] \cup \left\{x_k = 1 + \frac{1}{k}\right\}_{k \geq 1}$$

$$\underline{\underline{A}}_1 = \sigma \left\{ \text{Lebesgue measurable subsets of } [0, \ell_o], \ \{x_k, k \geq 1\} \right\}$$

$$P_1(A_1) = \lambda(A_1 \cap [0, \ell_o]) + \sum_{k=1}^{\infty} \ell_k 1_{A_1}(x_k)$$

for every $A_1 \in \underline{\underline{A}}_1$ where λ denotes the usual Lebesgue measure on [0, 1]

$$Z_2 = [0, m_o] \cup \left\{x_k = 1 + \frac{1}{k}\right\}_{k \geq 1}$$

$$\underline{\underline{A}}_2 = \sigma \left\{ \text{Lebesgue measurable subsets of } [0, m_o] \text{ and } \{x_k\}_{k \geq 1} \right\}$$

$$P_2(A_2) = \lambda(A_2 \cap [0, m_o]) + \sum_{k=1}^{\infty} m_k 1_{A_2}(x_k)$$

for every $A_2 \in \underline{\underline{A}}_2$.

Now take

$$Z = Z_1 \times Z_2, \ \underline{\underline{C}} = \overline{\underline{\underline{A}}_1 \times \underline{\underline{A}}_2}, \ P = \overline{P_1 \times P_2} \text{ and}$$

$$\xi = \left\{ \{x\} \times Z_2 : x \in Z_1 \right\}.$$

If ξ happens to be the partition of $(Z, \underline{\underline{C}}, P)$ consisting of $\{Z, \emptyset\}$ then clearly the only independent complement of ξ is the partition of Z consisting of singletons. Hence let us assume that the partition ξ is nontrivial a.s. $[P]$. Let $R = \{k \geq 1 : m_k > 0\}$.

<u>Theorem 3.2.5:</u> ξ admits a unique independent complement if and only if $m_o = 0$ and $m_k > m_{k+1}$ for all $k \in R$.

<u>Proof:</u> The necessity of the condition is obvious. To prove the sufficiency first observe that $\xi^* = \left\{Z_1 \times \{x\}, \ x \in Z_2\right\}$ is an independent complement of ξ. Now by the given condition $Z_2 = \{x_k, k \geq 1\}$. Suppose ξ^{**} is another independent

complement of ξ. Let $Z' \in \underline{\underline{C}}$ with $P(Z') = 1$ be such that $\{C^{**} \cap Z'\}$ is one-sheeted with respect to ξ for every element C^{**} of ξ^{**}. Let C_1^{**}, $C_2^{**}, \ldots$ be the elements of ξ^{**} of positive measure.

Let

$$C_k = Z_1 \times \{x_k\}, \quad \lambda_k = P(C_k \cap C_1^{**} \cap Z')$$

$$A_k = \{x \in Z_1 : (x, x_k) \in C_k \cap C_1^{**} \cap Z'\}$$

and $$r_k = P(A_k) \quad \text{for} \quad k \in R.$$

Then $$C_k \cap C_1^{**} \cap Z' = C_k \cap (A_k \times Z_2)$$
$$= C_1^{**} \cap Z' \cap (A_k \times Z_2)$$

and hence by independence

$$\lambda_k = m_k r_k = \Big(\sum_{k \in R} \lambda_k\Big) r_k .$$

Therefore $\Big(\sum_{k \in R} m_k^2 r_k\Big) = \Big(\sum_{k \in R} m_k r_k\Big)^2$.

Now r_k's are such that $r_k \geq 0$ and $\sum_{k \in R} r_k = 1$ and $m_k > m_{k+1}$ for all $k \in R$. So by Cauchy-Schwarz's inequality we have $r_j = 1$ for some $j \in R$. Hence $\lambda_j = m_j$ or $C_j = C_1^{**}$ a.s. for some $j \in R$. Similarly for every k, $C_k^{**} = C_{k_j}$ a.s. [P] for some $k_j \in R$. Finally ξ^{**} being a complement of ξ it follows that there is a k_j such that $C_k = C_{k_j}^{**}$, for every k.

In other words $\xi^* = \xi^{**}$ a.s. [P]

CHAPTER 4

MIXTURES OF PERFECT MEASURES

4.0 Introduction

Let $(X, \underline{\underline{A}})$ and $(Y, \underline{\underline{B}})$ be two measurable spaces and let $\mu(x, B)$ be a transition probability on $X \times \underline{\underline{B}}$, that is, $\mu(x, B)$ is a function defined on $X \times \underline{\underline{B}}$ taking values in $[0, 1]$ such that for every x in X, $\mu(x, \cdot)$ is a probability on $\underline{\underline{B}}$ and for every $B \in \underline{\underline{B}}$, $\mu(\cdot, B)$ is $\underline{\underline{A}}$-measurable. Let λ be a measure on $\underline{\underline{A}}$. The set function μ defined on $\underline{\underline{B}}$ by

$$\mu(B) = \int_X \mu(x, B)\, d\lambda, \quad B \in \underline{\underline{B}}$$

is a measure on $\underline{\underline{B}}$. μ is called the λ-mixture of $\mu(x, \cdot)$'s. λ is called mixing measure, $\mu(x, \cdot)$'s are called mixand measures and μ is called mixture measure. In general the properties of a mixture measure depend on those of the mixing and mixand measures. We shall study in detail, in this chapter, the role of perfectness of measures in the mixture problem.

We call a mixture measure perfect mixture or nonperfect mixture according as the mixing measure is perfect or non-perfect. Rodine [32], [33] initiated the study of mixtures of perfect measures and showed that mixtures of perfect measures, in general, need not be perfect. Rodine conjectured that perfect mixtures of perfect measures are perfect and also raised the following question: If μ is perfect, does it follow that the $\mu(x, \cdot)$'s are perfect except possibly for x's in a λ-null set? In section 4.1 we give examples to show that the conjecture is false and that the answer to the above question is in the negative. Later we study at length perfect mixtures of perfect measures. Finally we treat the mixture problem for compact measures.

4.1 The Examples

Rodine's example (example 4.1.1) consisted of a non-perfect mixture of perfect measures which is nonperfect. Then he made the conjecture and raised the question which are stated in 4.0 and which are answered by our examples 4.1.3 and 4.1.5. The following table lists all the cases that can arise in the mixture problem and our examples illustrating each case.

No.	Mixing measure	Mixand measures	Mixture measure	Example
1	perfect	perfect	perfect	4.1.2
2	perfect	nonperfect	perfect	4.1.5
3	perfect	perfect	nonperfect	4.1.3
4	perfect	nonperfect	nonperfect	4.1.7
5	nonperfect	perfect	perfect	4.1.6
6	nonperfect	nonperfect	perfect	4.1.9
7	nonperfect	perfect	nonperfect	4.1.1
8	nonperfect	nonperfect	nonperfect	4.1.10

Example 4.1.1 (Rodine): Let $(X, \underline{\underline{A}}, \lambda)$ be a nonperfect probability space. Let $Y = X$, $\underline{\underline{B}} = \underline{\underline{A}}$ and let $\mu(x, B)$ on $X \times \underline{\underline{B}}$ be defined by $\mu(x, B) = 1_B(x)$. Then each $\mu(x, \cdot)$, being a $0-1$ valued measure, is perfect. The mixture $\mu = \lambda$ and is nonperfect.

Example 4.1.2: Let $(X, \underline{\underline{A}}, \lambda)$ be a perfect probability space. Let $Y = X$, $\underline{\underline{B}} = \underline{\underline{A}}$ and let $\mu(x, B)$ on $X \times \underline{\underline{B}}$ be defined by $\mu(x, B) = 1_B(x)$. Then the mixture $\mu = \lambda$ and is perfect.

Example 4.1.3: Let $X = [0, 1]$, the unit interval, $\underline{\underline{A}}$ = the Borel σ-algebra of X and λ = the Lebesgue measure on $(X, \underline{\underline{A}})$

Consider the product space $(X \times X, \mathcal{A} \times \mathcal{A}, \lambda \times \lambda)$. For our purposes it is enough to consider a set E contained in $X \times X$, which meets every closed subset of positive $\lambda \times \lambda$ measure and no three of whose points are collinear, constructed by Sierpiński [40] . But we use the following lemma.

Lemma 4.1.4: There exists a subset E of $X \times X$ such that

(a) E intersects every closed subset of $X \times X$ of positive $\lambda \times \lambda$ measure

and (b) E is a graph, that is, for every x in X the set $E_x = \{y \in X : (x, y) \in E\}$ is exactly a singleton.

Proof: Let w_c be the first ordinal corresponding to c, the cardinality of the continuum. Let $\{A_\alpha : \alpha < w_c\}$ be a well ordering of all closed subsets of $X \times X$ of positive $\lambda \times \lambda$ measure. We shall define a transfinite sequence $\{p_\alpha = (x_\alpha, y_\alpha) : \alpha < w_c\}$ as follows. Take $p_1 = (x_1, y_1) \in A_1$. Suppose $\{p_\alpha = (x_\alpha, y_\alpha) : \alpha < \beta\}$ have been defined for $\beta < w_c$. The set $\{x : \lambda((A_\beta)_x) > 0\}$ is an uncountable Borel set and hence has cardinality c. So we can find x_β in $\{x : \lambda((A_\beta)_x) > 0\} - \{x_\alpha : \alpha < \beta\}$.

Take a $y_\beta \in (A_\beta)_{x_\beta}$. Let $p_\beta = (x_\beta, y_\beta)$. Let
$X_o = X - \{x_\alpha : \alpha < w_c\}$ and let $E = \{p_\alpha : \alpha < w_c\} \cup X_o \times \{0\}$.
Then E has the required properties.

The set E given by 4.1.4 is such that both E and $X \times X - E$ intersect every closed subset of positive $\lambda \times \lambda$ measure. Hence, letting $Y = X \times X - E$, we have $(\lambda \times \lambda)^*(E) = (\lambda \times \lambda)^*(Y) = 1$ where $(\lambda \times \lambda)^*$ is the outer measure induced by $\lambda \times \lambda$. Let $\underline{\underline{B}} = \underline{\underline{A}} \times \underline{\underline{A}} \cap Y = \{C \cap Y : C \in \underline{\underline{A}} \times \underline{\underline{A}}\}$. Since $(\lambda \times \lambda)^*(Y) = 1$, if we define $\mu(B) = (\lambda \times \lambda)^*(B)$, $B \in \underline{\underline{B}}$, then μ is a measure on $\underline{\underline{B}}$. Define $\mu(x, B)$ on $X \times \underline{\underline{B}}$ by $\mu(x, B) = \lambda(B_x)$. If $B = C \cap Y$, $C \in \underline{\underline{A}} \times \underline{\underline{A}}$ then $\mu(x, B) = \lambda(B_x) = \lambda(C_x \cap Y_x) = \lambda(C_x)$ since, for each $x \in X$, Y_x differs from X by exactly one point. It is now easy to see that $\mu(x, B)$ is well-defined and is $\underline{\underline{A}}$-measurable for every $B \in \underline{\underline{B}}$. Since our definition of $\mu(x, B)$ is such that $\mu(x, \cdot)$ is a measure on $\underline{\underline{B}}$ for every x in X, $\mu(x, B)$ turns out to be a transition probability on $X \times \underline{\underline{B}}$. For any $B \in \underline{\underline{B}}$ [redacted] we have

$$\int_X \mu(x, B)\, d\lambda = \int_X \lambda(C_x)\, d\lambda = \lambda \times \lambda(C) = \mu(B)$$

where $C \in \underline{\underline{A}} \times \underline{\underline{A}}$ such that $B = C \cap Y$. In other words μ is a λ-mixture of $\mu(x, \cdot)$'s.

λ and $\mu(x, \cdot)$'s are all perfect. Let g be the restriction to Y of a 1-1, $\underline{\underline{A}} \times \underline{\underline{A}}$-measurable function on $X \times X$. For every linear Borel set contained in $g(Y)$, $g^{-1} B$ is an $\underline{\underline{A}} \times \underline{\underline{A}}$-measurable set contained in Y and since $(\lambda \times \lambda)^*(E) = 1$, $\mu(g^{-1}B) = \lambda \times \lambda(g^{-1}B) = 0$. Thus, by P1, μ is nonperfect. Hence a perfect mixture of perfect measures need not be perfect.

<u>Example 4.1.5:</u> Let X be any uncountable set and let $\underline{\underline{A}}$ be the class of countable subsets of X and their complements. For each x in X let $(Y_x, \underline{\underline{B}}_x, P_x)$ be a nonperfect probability space. Let $Y = \prod_{x \in X} Y_x$, the product space and $\underline{\underline{B}} = \bigotimes_{x \in X} \underline{\underline{B}}_x$, the product σ-algebra. Fix f_o in Y. Let $f_o^x = f_o|X - \{x\}$, the restriction of f_o to $X - \{x\}$. For every $B \in \underline{\underline{B}}$ define $B_x = f_o^x$-th section of B

$$= \left\{ g(x) \in Y_x : g \in B \text{ and } g = f_o \text{ on } X - \{x\} \right\}.$$

It can be easily verified that $\underline{\underline{B}}_x = \left\{ B_x : B \in \underline{\underline{B}} \right\}$. Define $\mu(x, B)$ on $X \times \underline{\underline{B}}$ as $\mu(x, B) = P_x(B_x)$. For each x in X, $\mu(x, \cdot)$ is a measure on $\underline{\underline{B}}$. P_x on $\underline{\underline{B}}_x$ is nonperfect implies that $\mu(x, \cdot)$ is nonperfect on $\underline{\underline{B}}'_x = \left\{ B_x \times \prod_{z \in X - \{x\}} Y_z : B_x \in \underline{\underline{B}}_x \right\}$. Therefore by P3 $\mu(x, \cdot)$ is nonperfect on $\underline{\underline{B}}$ for every x in X. Since every B in $\underline{\underline{B}}$ is a countable dimensional cylinder it follows that either $B_x = \emptyset$ for all but a countable number of x's or $B_x = Y_x$ for all but a countable number of x's. So for each fixed B in $\underline{\underline{B}}$ either $\mu(x, B) = 0$ for all but a countable number of x's or $\mu(x, B) = 1$ for all but a countable number of x's. Hence $\mu(\cdot\ , B)$ is $\underline{\underline{A}}$-measurable for every B in $\underline{\underline{B}}$. Let λ be the 0-1 valued measure on $\underline{\underline{A}}$ defined by $\lambda(A) = 0$ or 1 according as A is countable or co-countable. λ is perfect. The mixture μ is such that $\mu(B) = 0$ or 1 according as $f_0 \notin B$ or $f_0 \in B$. Therefore μ is perfect as well. By choice $\mu(x, \cdot)$ is nonperfect for every x in X.

The above example is only a modification of the one constructed by K. P. S. Bhaskara Rao and M. Bhaskara Rao [1] to show that mixtures of nonatomic measures need not be non-atomic. The set-up in both examples is the same but instead of taking $\mu(x, \cdot)$'s to be nonatomic, here we take them to be nonperfect.

The σ-algebra $\underline{\underline{B}}$ considered in this example is not countably generated. When $\underline{\underline{B}}$ happens to be countably generated, Rodine's question has an affirmative answer as we shall show in theorem 4.2.2.

Example 4.1.6: In order to show that a nonperfect mixture of perfect measures can be perfect it is sufficient to consider a perfect measure μ on $(Y, \underline{\underline{B}})$, a nonperfect space $(X, \underline{\underline{A}}, \lambda)$ and define $\mu(x, B)$ on $X \times \underline{\underline{B}}$ by taking $\mu(x, B) = \mu(B)$ for all x in X. But in this case the transition function $\mu(x, B)$ is actually measurable with respect to the trivial sub-σ-algebra $\{X, \emptyset\}$ of $\underline{\underline{A}}$. Clearly the restriction of λ to this σ-algebra is perfect and thus in this case μ turns out, in fact, to be a perfect mixture of perfect measures. We give below a nontrivial example of a perfect measure which is a nonperfect mixture of perfect measures.

Let Y be the unit interval and $\underline{\underline{B}}$ the Borel subsets of Y. Let X be a subset of the unit interval with outer Lebesgue measure one and inner Lebesgue measure zero. Let $\underline{\underline{A}} = \underline{\underline{B}} \cap X$, the trace of $\underline{\underline{B}}$ on X and let λ on $\underline{\underline{A}}$ be the trace of outer Lebesgue measure. $(X, \underline{\underline{A}}, \lambda)$ is a nonperfect probability space. Define a transition probability $\mu(x, B)$ on $X \times \underline{\underline{B}}$ by $\mu(x, B) = 1_B(x)$. Plainly $\underline{\underline{A}}$ is the smallest σ-algebra on X with respect to which the transition function is measurable. Each $\mu(x, \cdot)$ is perfect and the mixture, being a measure on the Borel subsets of the unit interval, is perfect by P7 though λ is nonperfect.

Example 4.1.7: Let $X = \{0\}$, $\underline{\underline{A}} = \{X, \emptyset\}$ and let λ on $(X, \underline{\underline{A}})$ be such that $\lambda(X) = 1$, $\lambda(\emptyset) = 0$. Let $(Y, \underline{\underline{B}}, \mu)$ be a nonperfect probability space. Define $\mu(x, B) = \mu(B)$ for all x in X. Then μ is a nonperfect measure which is a perfect mixture of nonperfect measures.

For the construction of our next two examples we need the following result:

Let for each $i = 1, 2$ $(X_i, \underline{\underline{A}}_i, \lambda_i)$ be two probability spaces, $(Y_i, \underline{\underline{B}}_i)$ two measurable spaces and $\mu_i(x_i, \underline{\underline{B}}_i)$ on $X_i \times \underline{\underline{B}}_i$ two transition probabilities. Let μ_i be the λ_i-mixture of $\mu_i(x_i, \cdot)$'s. If we define $X = X_1 \times X_2$, $\underline{\underline{A}} = \underline{\underline{A}}_1 \times \underline{\underline{A}}_2$, $\lambda = \lambda_1 \times \lambda_2$, $Y = Y_1 \times Y_2$, $\underline{\underline{B}} = \underline{\underline{B}}_1 \times \underline{\underline{B}}_2$ and $\mu((x_1, x_2), \cdot) = \mu_1(x_1, \cdot) \times \mu_2(x_2, \cdot)$ then the following lemma can be easily established.

<u>Lemma 4.1.8</u>: $\mu((x_1, x_2), \cdot)$ is a transition probability on $X \times \underline{B}$ such that $\mu = \mu_1 \times \mu_2$ where μ is the λ-mixture of the $\mu((x_1, x_2), \cdot)$'s.

We shall call μ as the product mixture of μ_1 and μ_2.

<u>Example 4.1.9</u>: Let μ_1 and μ_2 be obtained as in examples 4.1.5 and 4.1.6 respectively. Then, by 4.1.8 and P5, it follows that the product mixture is a perfect measure which is a nonperfect mixture of nonperfect measures.

Example 4.1.10: Let μ_1 and μ_2 be obtained as in examples 4.1.1 and 4.1.5 respectively. Then, by 4.1.8 and P5, the product mixture is a nonperfect measure which is a nonperfect mixture of nonperfect measures.

4.2 Perfect mixtures of discrete measures

Let $(X, \underline{\underline{A}}, \lambda)$ be a probability space. Let $(Y, \underline{\underline{B}})$ be a measurable space and let $\mu(x, B)$ be a transition probability on $X \times \underline{\underline{B}}$. Let μ be the λ-mixture of $\mu(x, \cdot)$'s and let $\lambda\mu$ on $(X \times Y, \underline{\underline{A}} \times \underline{\underline{B}})$ be the product probability measure defined by

$$\lambda\mu(C) = \int_X \mu(x, C_x)\,d\lambda, \quad C \in \underline{\underline{A}} \times \underline{\underline{B}} .$$

We have the following result on perfect mixtures.

Theorem 4.2.1: Suppose λ is perfect. Then μ is perfect if and only if $\lambda\mu$ is perfect.

Proof: If μ is perfect then both the marginals of $\lambda\mu$ are perfect and so $\lambda\mu$ is perfect by P5. If $\lambda\mu$ is perfect then by P3 the marginal of $\lambda\mu$ on $X \times \underline{\underline{B}}$ is perfect and hence μ is perfect.

Before considering perfect mixtures of discrete measures we note that our example 4.1.6 shows that a theorem of Bodine (theorem 2.12 in [32]) is incorrect. We now directly prove the following modified version of Rodine's theorem.

<u>Theorem 4.2.2:</u> If $\underline{\underline{B}}$ is countably generated and if μ is perfect then $\mu(x, \cdot)$ is perfect for almost all x in X.

<u>Proof:</u> Since $\underline{\underline{B}}$ is countably generated using the Marczewski function (see Chapter 0) we can assume that Y is a subset of [0, 1] and that $\underline{\underline{B}}$ is the trace on Y of the Borel σ-algebra of [0, 1]. μ is perfect implies that there is a Borel subset B_0 of [0, 1] such that $B_0 \subset Y$ and $\mu(B_0) = 1$. Hence $\mu(x, B_0) = 1$ for all $x \notin N_0$ where $N_0 \in \underline{\underline{A}}$ with $\lambda(N_0) = 0$. For each $x \notin N_0$, $\mu(x, \cdot)$ being a measure on the Borel subsets of B_0, by P7 is perfect.

Using 4.2.1 and 4.2.2 we have the following corollary.

<u>Corollary 4.2.3:</u> Let $\underline{\underline{B}}$ be countably generated. Then $\lambda\mu$ is perfect if and only if λ is perfect, μ is perfect and $\mu(x, \cdot)$ is perfect for almost all x in X.

We shall now proceed to show that perfect mixtures of discrete measures are perfect. The present proof is a direct one and in section 4.3 we shall give yet another proof of this result using results of chapter 3.

Let $\underline{\underline{B}}_{[0, 1]}$ denote the Borel σ-algebra of [0, 1].

Lemma 4.2.4: Let $X \in \underline{\underline{B}}_{[0, 1]}$ and let $\underline{\underline{A}} = \underline{\underline{B}}_{[0, 1]} \cap X$. If $\mu(x, B)$ is a transition probability on $X \times \underline{\underline{B}}_{[0, 1]}$ then the set

$$D = \left\{ (x, y) : \mu(x, \{y\}) > 0 \right\}$$

is a Borel subset of the unit square.

Proof: First note (starting from indicators of cubes etc.) that for any measurable map $\emptyset$ from $X \times [0, 1] \times [0, 1]$ to $[0, 1]$ the function $\emptyset^*$ from $X \times [0, 1]$ to $[0, 1]$ defined by

$$\emptyset^* (x, y) = \int \emptyset(x, y, z) \, \mu(x, dz)$$

is $\underline{\underline{A}} \times \underline{\underline{B}}_{[0, 1]}$ - measurable. Taking

$$\emptyset(x, y, z) = \begin{cases} 1 & \text{if} \quad y = z \\ 0 & \text{if} \quad y \neq z \end{cases}$$

we get $\emptyset^*(x, y) = \mu(x, \{y\})$ is $\underline{\underline{A}} \times \underline{\underline{B}}_{[0, 1]}$-measurable and hence D is a Borel subset of the unit square.

Lemma 4.2.5: Suppose $(X, \underline{\underline{A}})$ and $(Y, \underline{\underline{B}})$ are two measurable spaces with $X \in \underline{\underline{B}}_{[0, 1]}$, Y a subset of $[0, 1]$, $\underline{\underline{A}} = \underline{\underline{B}}_{[0, 1]} \cap X$ and $\underline{\underline{B}} = \underline{\underline{B}}_{[0, 1]} \cap Y$. If $\mu(x, B)$ is a transition probability on $X \times \underline{\underline{B}}$ then the set

$$D = \left\{(x, y): \mu(x, \{y\}) > 0\right\}$$

is a Borel subset of the unit square.

Proof: Define $\mu'(x, B)$ on $X \times \underline{\underline{B}}_{[0, 1]}$ by $\mu'(x, B) = \mu(x, B \cap Y)$. It is easy to check that $\mu'(x, B)$ is a transition probability on $X \times \underline{\underline{B}}_{[0, 1]}$. Hence by 4.2.4 the set

$$D' = \left\{(x, y): \mu'(x, \{y\}) > 0\right\}$$

is a Borel subset of the unit square. But $D' = D$.

Theorem 4.2.6: Let $(X, \underline{\underline{A}}, \lambda)$ be a perfect probability space with $\underline{\underline{A}}$ countably generated. Let $(Y, \underline{\underline{B}})$ be a measurable space with $\underline{\underline{B}}$ countably generated and let $\mu(x, B)$ be a transition probability on $X \times \underline{\underline{B}}$ such that $\mu(x, \cdot)$ is discrete for almost every x in X. Then μ, the λ-mixture of $\mu(x, \cdot)$'s, is perfect.

Proof: By using the Marczewski function and P6 and by suitably redefining the transition probability we can, without loss of generality, assume X to be a Borel subset of $[0, 1]$ with $\underline{\underline{A}} = \underline{\underline{B}}_{[0, 1]} \cap X$ and Y to be a subset of $[0, 1]$ with $\underline{\underline{B}} = \underline{\underline{B}}_{[0, 1]} \cap Y$. By 4.2.5, the set

$$D = \left\{(x, y) : \mu(x, \{y\}) > 0\right\}$$

is a Borel subset of the unit square. Since $\mu(x, \cdot)$ is discrete for almost every x in X

$$\lambda\mu(D) = \int \mu(x, D_x)\, d\lambda = 1$$

Thus $(D, \underline{\underline{A}} \times \underline{\underline{B}} \cap D, \lambda\mu|_D)$ is a perfect probability space (see [39], section 2.5) and hence $\lambda\mu$ is perfect. By 4.2.1, μ is perfect.

<u>Theorem 4.2.7:</u> Perfect mixtures of discrete measures are perfect.

<u>Proof:</u> Let $(X, \underline{\underline{A}}, \lambda)$ be a perfect probability space, $(Y, \underline{\underline{B}})$ a measurable space and $\mu(x, B)$ a transition probability on $X \times \underline{\underline{B}}$ such that $\mu(x, \cdot)$ is discrete for almost every x in X. Let μ be the λ-mixture of the $\mu(x, \cdot)$'s. Let $\underline{\underline{B}}'$ be a countably generated sub-σ-algebra of $\underline{\underline{B}}$ and let $\mu'(x, B')$ be the restriction of $\mu(x, B)$ on $X \times \underline{\underline{B}}'$. Then $\mu'(x, B')$ is a transition probability on $X \times \underline{\underline{B}}'$ and $\underline{\underline{A}}'$, the smallest sub-$\sigma$-algebra of $\underline{\underline{A}}$ with respect to which $\{\mu'(\cdot, B') : B' \in \underline{\underline{B}}'\}$ are all measurable, is countably generated. Thus if $\mu' = \mu|_{\underline{\underline{B}}'}$ and $\lambda' = \lambda|_{\underline{\underline{A}}'}$ then μ' is the λ'-mixture of $\mu'(x, \cdot)$'s where $\mu'(x, \cdot)$ is discrete for almost every x in X. By P2, λ' is perfect. Hence, by 4.2.6, μ' is perfect. Again by P2, μ is perfect.

4.3 Perfect mixtures of perfect measures

In this section, given a mixture problem where the σ-algebras considered are countably generated we construct an associated separable space and a measurable partition of this space which possesses a canonical system of measures. Sazonov has proved (see [39], theorem 5) that a separable space is a perfect probability space if and only if it is a Lebesgue space. Using this fact and the results of chapter 3, we shall study the perfectness of perfect mixtures of perfect measures through the associated separable space.

Let μ be the λ-mixture of $\mu(x, \cdot)$'s where $(X, \underline{\underline{A}}, \lambda)$ is a probability space with $\underline{\underline{A}}$ countably generated, $(Y, \underline{\underline{B}})$ is a measurable space with $\underline{\underline{B}}$ countably generated and $\mu(x, B)$ is a transition probability on $X \times \underline{\underline{B}}$. We denote by $(\overset{*}{X}, \overset{*}{\underline{\underline{A}}}, \overset{*}{\lambda})$ the completion of the quotient space of $(X, \underline{\underline{A}}, \lambda)$ with respect to the partition of X by atoms of $\underline{\underline{A}}$ and by $(\overset{*}{Y}, \overset{*}{\underline{\underline{B}}}, \overset{*}{\mu})$ the quotient space of $(Y, \underline{\underline{B}}, \mu)$ with respect to the partition of Y by atoms of $\underline{\underline{B}}$. For $x \in X$ let $\bar{x}$ denote the $\underline{\underline{A}}$-atom containing x and for $B \in \underline{\underline{B}}$ let $\bar{B}$ denote the corresponding set in $\overset{*}{\underline{\underline{B}}}$. Define $\overset{*}{\mu}(\bar{x}, \bar{B})$ on $\overset{*}{X} \times \overset{*}{\underline{\underline{B}}}$ by $\overset{*}{\mu}(\bar{x}, \bar{B}) = \mu(x, B)$. Then $\overset{*}{\mu}(\bar{x}, \bar{B})$ is well-defined and is a transition probability on $\overset{*}{X} \times \overset{*}{\underline{\underline{B}}}$. Let $Z_\mu = \overset{*}{X} \times \overset{*}{Y}$, $\underline{\underline{C}} = \overline{\overset{*}{\underline{\underline{A}}} \times \overset{*}{\underline{\underline{B}}}}$ and $P_\mu = \bar{P}$ where P on

$\underline{\underline{\overset{*}{A}}} \times \underline{\underline{\overset{*}{B}}}$ is defined by

$$P(C) = \int \overset{*}{\mu}(\bar{x}, C_{\bar{x}}) d\overset{*}{\lambda}, \quad C \in \underline{\underline{\overset{*}{A}}} \times \underline{\underline{\overset{*}{B}}} .$$

The following facts can be directly verified.

<u>4.3.1:</u> $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ is a separable space.

<u>4.3.2:</u> $(\overset{*}{Y}, \underline{\underline{\overset{*}{B}}}, \overset{*}{\mu})$ and $(Z_\mu, \overset{*}{X} \times \underline{\underline{\overset{*}{B}}}, P_\mu|_{\overset{*}{X} \times \underline{\underline{\overset{*}{B}}}})$ are isomorphic probability spaces.

<u>4.3.3:</u> Let ξ_μ be the partition of Z_μ of the form $\left\{\{\bar{x}\} \times \overset{*}{Y}\right\}_{\bar{x} \in \overset{*}{X}}$. ξ_μ is a measurable partition such that the quotient space of $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ with respect to ξ_μ is isomorphic to $(\overset{*}{X}, \underline{\underline{\overset{*}{A}}}, \overset{*}{\lambda})$.

<u>4.3.4:</u> Define $\overset{*}{\mu}_{\bar{x}}(\{\bar{x}\} \times \bar{B}) = \overset{*}{\mu}(\bar{x}, \bar{B})$ for $\bar{x} \in \overset{*}{X}, \bar{B} \in \underline{\underline{\overset{*}{B}}}$. For each $\bar{x} \in \overset{*}{X}$, $(\{\bar{x}\} \times \overset{*}{Y}, \{\bar{x}\} \times \underline{\underline{\overset{*}{B}}}, \overset{*}{\mu}_{\bar{x}})$ is isomorphic to $(\overset{*}{Y}, \underline{\underline{\overset{*}{B}}}, \overset{*}{\mu}(\bar{x}, \cdot))$ and the completion of $(\{\bar{x}\} \times \overset{*}{Y}, \{\bar{x}\} \times \underline{\underline{\overset{*}{B}}}, \overset{*}{\mu}_{\bar{x}})$ is a separable space.

<u>4.3.5:</u> Let us denote for each $\bar{x}$ in $\overset{*}{X}$ the completion of $\overset{*}{\mu}_{\bar{x}}$ on $\{\bar{x}\} \times \underline{\underline{\overset{*}{B}}}$ by $\overset{*}{\mu}_{\bar{x}}$ itself. Then the collection $\left\{\overset{*}{\mu}_{\bar{x}}\right\}_{\bar{x} \in \overset{*}{X}}$ is a canonical system of measures with respect to ξ_μ.

We call $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ the separable space associated with μ and we have the following theorem which is another form of theorem 4.2.1 for the countably generated case.

<u>Theorem 4.3.6:</u> Suppose λ is perfect. Then μ is perfect if and only if the associated separable space $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ is a Lebesgue space.

<u>Proof:</u> If μ is perfect, then $(\overset{*}{Y}, \overset{*}{\underline{\underline{B}}}, \overset{*}{\mu})$ and hence $(Z_\mu, \overset{*}{X} \times \overset{*}{\underline{\underline{B}}}, P_\mu|_{\overset{*}{X} \times \overset{*}{\underline{\underline{B}}}})$ are perfect probability spaces. Since λ is perfect, the separable space $(\overset{*}{X}, \overset{*}{\underline{\underline{A}}}, \overset{*}{\lambda})$ is a Lebesgue space by P1 and P4. It follows that in $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ which is actually a product space, both marginals are Lebesgue measures. Hence (see [34], § 3, No.4) in fact $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ is a Lebesgue space.

If $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ is a Lebesgue space then, by P3, $(Z_\mu, \overset{*}{X} \times \overset{*}{\underline{\underline{B}}}, P_\mu|_{\overset{*}{X} \times \overset{*}{\underline{\underline{B}}}})$ is a perfect probability space. Hence $(\overset{*}{Y}, \overset{*}{\underline{\underline{B}}}, \overset{*}{\mu})$ is perfect or, in other words, μ is perfect.

<u>Remark 4.3.7:</u> Now we are in a position to show that in theorem 3.1.6 of chapter 3 conditions (i) and (ii) are not sufficient to ensure that $(Z, \underline{\underline{C}}, P)$ is a Lebesgue space. It

is easy to observe that $\mu(x, \cdot)$ is perfect if and only if $\overset{*}{\mu}(\bar{x}, \cdot)$ is perfect and so if and only if $\overset{*}{\mu}_{\bar{x}}$ is a Lebesgue measure. Consider the set-up of example 4.1.3 of section 4.1. Construct $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ the separable space associated with μ. λ is perfect implies the quotient space of $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ with respect to ξ_μ is a Lebesgue space. Each $\mu(x, \cdot)$ is perfect implies that each $\overset{*}{\mu}_{\bar{x}}$ is a Lebesgue measure. Thus conditions (i) and (ii) of theorem 3.1.6 hold. But μ is nonperfect and so by theorem 4.3.6 $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ is not a Lebesgue space.

Let us now turn our attention to the general mixture problem. We are given a probability space $(X, \underline{\underline{A}}', \lambda')$, a measurable space $(Y, \underline{\underline{B}}', \lambda')$, a transition probability $\mu(x, B')$ on $X \times \underline{\underline{B}}'$. Let μ' be the λ'-mixture of $\mu(x, \cdot)$'s. Suppose $\underline{\underline{B}}$ is a countably generated sub-σ-algebra of $\underline{\underline{B}}'$ and $\mu = \mu'|_{\underline{\underline{B}}}$. We shall denote by $\underline{\underline{A}}$ the smallest sub-σ-algebra of $\underline{\underline{A}}'$ with respect to which $\{\mu(\cdot, B), \ B \in \underline{\underline{B}}\}$ are all measurable. It is easy to check that $\underline{\underline{A}}$ is countably generated. We denote $\lambda'|_{\underline{\underline{A}}}$ by λ. So μ is in fact a λ-mixture of $\mu(x, \cdot)$'s and the σ-algebras considered now are countably generated. We can now associate a separable space $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ and a measurable partition ξ_μ of $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ with μ. Thus in the general mixture problem whenever μ denotes the restriction of the mixture measure to a countably generated sub-σ-algebra, we can associate a separable space $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ and a measurable partition ξ_μ of this space with μ. If in addition the mixing measure and almost all mixand measures are perfect, then the quotient space of $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ with respect to ξ_μ is a Lebesgue space and in the system of measures canonical with respect to ξ_μ almost every measure is Lebesgue. Hence we have the following theorem on perfect mixtures of perfect measures.

<u>Theorem 4.3.8:</u> Let the measure μ' on $(Y, \underline{\underline{B}}')$ be a perfect mixture of perfect measures. In order that μ' is perfect it is necessary and sufficient that for every countably generated sub-σ-algebra $\underline{\underline{B}}$ of $\underline{\underline{B}}'$ the measurable partition ξ_μ of the separable space $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ is of product type, where $\mu = \mu'\big|_{\underline{\underline{B}}}$.

<u>Proof:</u> If μ' is perfect then, by P3, μ is perfect where $\mu = \mu'\big|_{\underline{\underline{B}}}$ and $\underline{\underline{B}}$ is countably generated. By 4.3.6, $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ is a Lebesgue space. By 3.1.9, ξ_μ is of product type.

Suppose the condition holds. By 3.1.9 and 4.3.6, it follows that the restriction of μ' to every countably generated sub-σ-algebra of $\underline{\underline{B}}'$ is perfect. By P2, μ' is perfect.

Remark 4.3.9: We are in a position to give an alternative proof to theorem 4.2.7 which says that perfect mixtures of discrete measures are perfect.

Proof: Let the set-up be as in theorem 4.3.8 with the additional assumption that almost all mixand measures are discrete. Let $\underline{\underline{B}}$ be a countably generated sub-σ-algebra of $\underline{\underline{B}}'$ and let $\mu = \mu'\Big|_{\underline{\underline{B}}}$. Consider the measurable partition ξ_μ of the associated separable space $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$. Since the mixing measure is perfect the quotient space of $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ with respect to ξ_μ is a Lebesgue space. It is easy to check, since almost all mixand measures are discrete, that in the system of measures canonical with respect to ξ_μ almost all measures are discrete. Hence, by 3.1.11, $(Z_\mu, \underline{\underline{C}}_\mu, P_\mu)$ is a Lebesgue space. By 4.3.6, μ is perfect. Now it follows by P2 that μ' is perfect.

4.4 Mixture problem for compact measures

We shall make an attempt, in this section, at studying the mixture problem for compact measures. Marczewski [22] introduced the notion of compact measures, which are closely related to perfect measures. We are not able to obtain general results in this section and we will be raising more questions than we answer.

A collection $\underline{\underline{K}}$ of subsets of a set X is called a compact class if any sequence $\{K_n\}_n$ of sets from the class $\underline{\underline{K}}$ having the property $\bigcap_{j=1}^{n} K_j \neq \emptyset$, $n = 1, 2, \ldots,$ also has the property $\bigcap_{j=1}^{\infty} K_j \neq \emptyset$. A measure P on a measurable space $(X, \underline{\underline{A}})$ is said to be compact if there is a compact class $\underline{\underline{K}}$ of subsets of X approximating $\underline{\underline{A}}$ with respect to P, that is, for every $A \in \underline{\underline{A}}$ and for every $\epsilon > 0$, there exist $K_\epsilon \in \underline{\underline{K}}$, $A_\epsilon \in \underline{\underline{A}}$ such that $A_\epsilon \subset K_\epsilon \subset A$ and $P(A - A_\epsilon) < \epsilon$.

Let $(X, \underline{\underline{A}})$ be a measurable space. The following results are well known (see [22], [37] and [39]):

C1. Every compact measure is perfect.

C2. Suppose $\underline{\underline{A}}$ is countably generated. Then a measure P on $(X, \underline{\underline{A}})$ is compact if and only if it is perfect.

C3. If $(X_i, \underline{\underline{A}}_i, P_i)$, $i = 1, 2$ are two probability spaces where each P_i is compact then $P_1 \times P_2$ on $(X_1 \times X_2, \underline{\underline{A}}_1 \times \underline{\underline{A}}_2)$ is compact.

Every 0-1 valued measure P on $(X, \underline{\underline{A}})$ can be checked to be compact, $\underline{\underline{A}}$ being approximated with respect to the measure P by the compact class $\{\{A \in \underline{\underline{A}} : P(A) = 1\}, \emptyset\}$.

By C1 every nonperfect measure is noncompact. Using C1, C2 and C3 it is easy to verify that the examples we have given in section 4.1 also illustrate the cases that can arise in the mixture problem for compact measures. The following table lists all the cases and our examples illustrating each case. The only modification needed is in 4.1.2 where we take λ on $(X, \underline{\underline{A}})$ to be a compact measure.

No.	Mixing measure	Mixand measures	Mixture measure	Example
1	compact	compact	compact	4.1.2
2	compact	noncompact	compact	4.1.5
3	compact	compact	noncompact	4.1.3
4	compact	noncompact	noncompact	4.1.7
5	noncompact	compact	compact	4.1.6
6	noncompact	noncompact	compact	4.1.9
7	noncompact	compact	noncompact	4.1.1
8	noncompact	noncompact	noncompact	4.1.10

When the σ-algebras are countably generated, since compactness and perfectness are equivalent by C 2, we can recast results of section 4.2 and 4.3 replacing the word 'perfect' by the word 'compact'. We omit the details. We shall prove only the following two results.

<u>Proposition 4.4.1:</u> Let μ be the λ-mixture of $\mu(x, .)$'s. Suppose λ is compact. Then μ is compact only if $\lambda\mu$ is compact.

<u>Proof:</u> If μ is compact then both the marginals of $\lambda\mu$ are compact and hence $\lambda\mu$ is compact (see [22], section 6).

Theorem 4.2.6 together with C1 yields the following

<u>Theorem 4.4.2:</u> Compact mixtures of discrete measures are perfect.

We now mention three questions for which we do not know the answers:

Q1. Is the restriction of a compact measure to any sub-σ-algebra compact?

Q2. Does there exist a noncompact perfect measure?

Q3. Is every compact mixture of discrete measures compact?

Since we do not know the answer to Q1 we are unable to assert in 4.4.1 that if $\lambda\mu$ is compact then λ and μ are compact.

These three questions are related. A negative answer to Q1 or Q3 will answer Q2 in the affirmative because of P3 or 4.4.2. A negative answer to Q2 will show that perfectness and compactness are equivalent and hence Q1 and Q3 will be answered in the affirmative. An affirmative answer to any of these questions will leave the other two unanswered.

CHAPTER 5

ON SEQUENCES OF σ-ALGEBRAS

5.0 Introduction

The objective of this chapter is to prove some results on sequences of σ-algebras which occur in certain formulation of the theory of non-linear prediction due to Wiener and Kallianpur [17] and Rosenblatt [35], [36]. A succinct account of this theory together with some unsolved problems is given by Masani (page 89 of [23]). Contents of this chapter have their origin in some of these problems.

5.1 Point separation in σ-algebras

Let $\underline{\underline{A}}$ be a σ-algebra of subsets of a set X. Two subsets A and B of X are said to be separated in $\underline{\underline{A}}$ if there exists a subset $C \in \underline{\underline{A}}$ such that $A \subset C$ and $B \subset X-C$. We say that $\underline{\underline{A}}$ separates points if any two distinct points of X are separated in $\underline{\underline{A}}$. For any finite collections $\underline{\underline{A}}_1, \underline{\underline{A}}_2, \ldots, \underline{\underline{A}}_n$ of σ-algebras of subsets of X we write $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2 \vee \ldots \vee \underline{\underline{A}}_n$ and $\underline{\underline{A}}_1 \wedge \underline{\underline{A}}_2 \wedge \ldots \wedge \underline{\underline{A}}_n$ to denote the σ-algebras generated by $\underline{\underline{A}}_1 \cup \ldots \cup \underline{\underline{A}}_n$ and $\underline{\underline{A}}_1 \cap \ldots \cap \underline{\underline{A}}_n$ respectively. $\bigvee_{n=1}^{\infty} \underline{\underline{A}}_n$ and $\bigwedge_{n=1}^{\infty} \underline{\underline{A}}_n$ have similar meanings.

We shall need the following theorem due to Blackwell [2] and Mackey [20] .

Blackwell-Mackey Theorem: Let X be an analytic set in a complete separable metric space and let $\underline{\underline{B}}$ denote its Borel σ-algebra. If $\underline{\underline{A}}$ is a countably generated sub-σ-algebra of $\underline{\underline{B}}$ which separates points then $\underline{\underline{A}} = \underline{\underline{B}}$.

Proposition 5.1.1: Let $\{\underline{\underline{C}}_n, n \geq 0\}$, $\{\underline{\underline{D}}_n, n \geq 0\}$ be two sequences of σ-algebras of subsets of a set X and assume that the sequence $\{\underline{\underline{D}}_n, n \geq 0\}$ is decreasing.

Let $\underline{\underline{C}} = \bigvee_{n=0}^{\infty} \underline{\underline{C}}_n$, $\underline{\underline{D}} = \bigwedge_{n=0}^{\infty} \underline{\underline{D}}_n$. Let x and y be two distinct points of X. Then

(i) x and y are separated in $\underline{\underline{C}}_n \vee \underline{\underline{D}}_n$ for every n implies that x and y are separated in $\underline{\underline{C}} \vee \underline{\underline{D}}$.

(ii) x and y are separated in $\underline{\underline{C}} \vee \underline{\underline{D}}$ implies that for some n, x and y are separated in $\underline{\underline{C}}_n \vee \underline{\underline{D}}_n$.

Proof: (i) Since x and y are separated in $\underline{\underline{C}}_n \vee \underline{\underline{D}}_n$ for every n, for any given n it is true that x and y are separated either in $\underline{\underline{C}}_n$ or in $\underline{\underline{D}}_n$. If x and y are separated in $\underline{\underline{C}}_n$ for some n, then clearly x and y are separated in $\underline{\underline{C}}$ and hence in $\underline{\underline{C}} \vee \underline{\underline{D}}$.

Suppose it happens that x and y are not separated in any $\underline{\underline{C}}_n$, then x and y are separated in $\underline{\underline{D}}_n$ for every n. Hence for every $n \geq 0$, there is an $A_n \in \underline{\underline{D}}_n$ such that $x \in A_n$ and $y \notin A_n$. Let $A = \limsup A_n$. Then $x \in A$ and $y \notin A$. Since $\{\underline{\underline{D}}_n, \ n \geq 0\}$ is a decreasing sequence, $A \in \underline{\underline{D}}_n$ for every $n \geq 0$ and hence $A \in \underline{\underline{D}}$. Thus x and y are separated in $\underline{\underline{D}}$ and hence in $\underline{\underline{C}} \vee \underline{\underline{D}}$.

(ii) If x and y are separated in $\underline{\underline{C}} \vee \underline{\underline{D}}$, then they are separated either in $\underline{\underline{C}}$ or in $\underline{\underline{D}}$. If they are separated in $\underline{\underline{C}}$, then they are separated in some $\underline{\underline{C}}_n$ (for if not, then the collection $\{A : x, y \in A \text{ or } x, y \in A^c\}$ is a σ-algebra which contains every $\underline{\underline{C}}_n$ and hence $\underline{\underline{C}}$ and thus x and y are not separated in $\underline{\underline{C}}$!). Hence x and y are separated in some $\underline{\underline{C}}_n \vee \underline{\underline{D}}_n$. If x and y are separated in $\underline{\underline{D}}$, then, since $\underline{\underline{D}} \subset \underline{\underline{D}}_n$ for all n, x and y are separated in $\underline{\underline{D}}_n$ and so in $\underline{\underline{C}}_n \vee \underline{\underline{D}}_n$ for every n.

Some immediate consequences of this proposition are the following:

<u>Corollary 5.1.2:</u> Assume, in addition to the hypothesis of 5.1.1, that $\{\underline{\underline{C}}_n, \ n \geq 0\}$ is an increasing sequence. Then

(i) x and y are separated in $\underline{\underline{C}}_n \vee \underline{\underline{D}}_n$ for all but finitely many n if and only if x and y are separated in $\underline{\underline{C}} \vee \underline{\underline{D}}$.

ii) $\mathcal{C}_n \vee \mathcal{D}_n$ separates points for all but finitely many n implies that $\mathcal{C} \vee \mathcal{D}$ separates points.

<u>Corollary 5.1.3:</u> Assume, in addition to the hypothesis of proposition 5.1.1, that X is an analytic subset of a complete separable metric space and that for each n, $\mathcal{C}_n$ and $\mathcal{D}_n$ are countably generated sub-σ-algebras of the Borel σ-algebra $\mathcal{B}$ of X. Then

i) $\mathcal{C}_n \vee \mathcal{D}_n = \mathcal{B}$ for all n implies $\mathcal{C} \vee \mathcal{D}$ separates points

ii) $\mathcal{C}_n \vee \mathcal{D}_n = \mathcal{B}$ for all n and $\mathcal{D}$ is countably generated implies $\mathcal{C} \vee \mathcal{D} = \mathcal{B}$.

Part (ii) of 5.1.3 follows from Blackwell-Mackey Theorem.

It may be conjectured that if $\{\mathcal{C}_n, n \geq 0\}$, $\{\mathcal{D}_n, n \geq 0\}$ are respectively increasing and decreasing sequences of countably generated sub-σ-algebras of the Borel-σ-algebra $\mathcal{B}$ of an analytic subset X of a complete separable metric space such that $\mathcal{C}_n \vee \mathcal{D}_n = \mathcal{B}$ for all n, then $\mathcal{C} \vee \mathcal{D} = \mathcal{B}$. This however is not true as illustrated by the following example, method of which is connected with the problem of complementation of σ-algebras considered by B. V. Rao [31].

Example 5.1.4: Let $X = \{0, 1\}^{N_o}$ where $\{0, 1\}$ is the additive group of two elements with discrete topology and N_o is the set of non-negative integers. Regard X as a compact group with coordinatewise addition and product topology. Let Q be the subgroup of X consisting of elements with finitely many ones. Let $\underline{\underline{D}}_n$ be the smallest σ-algebra with respect to which all the coordinate variables $\{f_k, k \geq n\}$ are measurable. Then $\underline{\underline{D}}_{n+1} \subset \underline{\underline{D}}_n$ and $\underline{\underline{D}} = \bigwedge_{n=0}^{\infty} \underline{\underline{D}}_n$ is the sub-σ-algebra of Borel sets of X invariant under translation by elements of Q. $\underline{\underline{D}}$ is not countably generated, $\underline{\underline{D}}$ is atomic and the atoms of $\underline{\underline{D}}$ are precisely the cosets of Q. Further by Kolmogorov's zero-one law every element of $\underline{\underline{D}}$ has Haar measure zero or one.

Let $\underline{\underline{F}}$ be any other sub-σ-algebra of $\underline{\underline{B}}$. Then $\underline{\underline{F}} \vee \underline{\underline{D}} \subset \{B \in \underline{\underline{B}} : h(B \triangle F) = 0$ for some $F \in \underline{\underline{F}}\}$ where h stands for the Haar measure on X. Now let $\alpha \in X - Q$, say $\alpha = (1,1,1,\ldots)$. Let $A = \{x \in X : x_o = 0\}$ where x_o denotes the 0^{th} coordinate of x. Then $A + \alpha = \{x : x_o = 1\} = A^c$. Now let $\underline{\underline{C}} = \{B \cup B + \alpha : B$ a Borel subset of $A\}$. Then $\underline{\underline{C}}$ is a countably generated sub-σ-algebra of $\underline{\underline{B}}$ and the atoms of $\underline{\underline{C}}$ are of the type $\{x, x + \alpha\}$, $x \in A$. Since $\alpha \notin Q$, x and $x + \alpha$ are separated in $\underline{\underline{D}}$ for every x. Thus $\underline{\underline{C}} \vee \underline{\underline{D}}$ separates points. But every element of $\underline{\underline{C}} \vee \underline{\underline{D}}$ differs from a set in $\underline{\underline{C}}$ by a set of Haar measure zero. Thus the set A can never belong to $\underline{\underline{C}} \vee \underline{\underline{D}}$. Now let for all n, $\underline{\underline{C}}_n = \underline{\underline{C}}$. Then $\underline{\underline{C}}_n \vee \underline{\underline{D}}_n$ is countably generated (since $\underline{\underline{C}}_n$ and $\underline{\underline{D}}_n$ are countably generated) and separates points (since $\underline{\underline{D}} \subset \underline{\underline{D}}_n$) for every n. Hence for each n, $\underline{\underline{C}}_n \vee \underline{\underline{D}}_n = \underline{\underline{B}} \neq \underline{\underline{C}} \vee \underline{\underline{D}}$.

<u>Remark 5.1.5:</u> Instead of sequences of σ-algebras, if we consider collections $\{\underline{\underline{C}}_t, \ t \geq 0\}$, $\{\underline{\underline{D}}_t, \ t \geq 0\}$ of σ-algebras then it is easy to check that propositions analogous to 5.1.1, 5.1.2 and 5.1.3 hold in this case.

<u>5.2 Crossed σ-algebras</u>

Let $(X, \underline{\underline{A}}, P)$ be a probability space and let $\underline{\underline{A}}_1$ be a sub-σ-algebra of $\underline{\underline{A}}$. In the theory of non-linear prediction one seeks, as a first step, conditions on $\underline{\underline{A}}_1$ under which there is an independent complement $\underline{\underline{A}}_2$ of $\underline{\underline{A}}_1$. We have treated extensively in chapter 2 the problem of existence of independent complements. The example given under remark 4.3.7, together with theorem 3.1.6, shows the following:

Given a probability space $(X, \underline{\underline{A}}, P)$ and independent σ-algebras $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ such that $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2 = \underline{\underline{A}}$ a.s. [P], it is not necessarily true that $(X, \underline{\underline{A}}, P)$ can be identified with a product space $(X_1 \times X_2, \underline{\underline{B}}_1 \times \underline{\underline{B}}_2, P_1 \times P_2)$ a.s. $[P_1 \times P_2]$ through a bimeasurable, measure-preserving point map T such that $T^{-1}(\underline{\underline{B}}_1 \times X_2) = \underline{\underline{A}}_1$ and $T^{-1}(X_1 \times \underline{\underline{B}}_2) = \underline{\underline{A}}_2$.

In this section we consider atomic σ-algebras $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ in the special case where $(X, \underline{\underline{A}}_1 \vee \underline{\underline{A}}_2)$ can be identified as a product of two spaces and $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ as σ-algebras of coordinate variables.

By an automorphism of a σ-algebra $\underline{\underline{A}}$ of subsets of a set X we mean a 1-1 mapping of $\underline{\underline{A}}$ onto $\underline{\underline{A}}$ which preserves countable unions and complements. Let $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ be sub-σ-algebras of $\underline{\underline{A}}$. We say that $\underline{\underline{A}}_1$ is isomorphic to $\underline{\underline{A}}_1'$ modulo $\underline{\underline{A}}_2$ in $\underline{\underline{A}}$ if there is an automorphism T of $\underline{\underline{A}}$ such that $T\underline{\underline{A}}_1 = \underline{\underline{A}}_1'$ and $T\underline{\underline{A}}_2 = \underline{\underline{A}}_2$.

If $\underline{\underline{A}}_1$ is isomorphic to $\underline{\underline{A}}_1'$ modulo $\underline{\underline{A}}_2$ in $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$ under T then $\underline{\underline{A}}_1' \vee \underline{\underline{A}}_2 = T\underline{\underline{A}}_1 \vee T\underline{\underline{A}}_2 = T(\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2) = \underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$. Suppose $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ are atomic σ-algebras. Following Rohlin we say that $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ are crossed if an atom of $\underline{\underline{A}}_1$ and an atom of $\underline{\underline{A}}_2$ always have a nonempty intersection. It is easy to see, if $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ are crossed, that

i) $\underline{\underline{A}}_1 \wedge \underline{\underline{A}}_2 = \{X, \emptyset\}$,

ii) $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$ is atomic and its atoms are obtained by taking intersections of atoms of $\underline{\underline{A}}_1$ with atoms of $\underline{\underline{A}}_2$, and

iii) Consider the map $f(x) = (A_1(x), A_2(x))$ where $A_i(x)$ is the $\underline{\underline{A}}_i$-atom containing x $(i = 1,2)$. Let for $i = 1,2$ $(X_i, \underline{\underline{B}}_i)$ denote the quotient space of $(X, \underline{\underline{A}})$ with respect to the partition of X by $\underline{\underline{A}}_i$-atoms. Then f is an isomorphism from $(X, \underline{\underline{A}})$ onto $(X_1 \times X_2, \underline{\underline{B}}_1 \times \underline{\underline{B}}_2)$ such that $f^{-1}(\underline{\underline{B}}_1 \times X_2) = \underline{\underline{A}}_1$ and $f^{-1}(X_1 \times \underline{\underline{B}}_2) = \underline{\underline{A}}_2$.

<u>Proposition 5.2.1:</u> Let $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ be crossed σ-algebras of subsets of a set X. Let $\underline{\underline{A}}_1$ be isomorphic to $\underline{\underline{A}}_1'$ modulo $\underline{\underline{A}}_2$ in $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$. Then

i) $\underline{\underline{A}}_1'$ is atomic,

ii) $\underline{\underline{A}}_1'$ and $\underline{\underline{A}}_2$ are crossed, and

iii) there is an automorphism T of $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$ such that $T\underline{\underline{A}}_1 = \underline{\underline{A}}_1'$ and T is identity on $\underline{\underline{A}}_2$.

<u>Proof:</u> i) Since $\underline{\underline{A}}_1$ is isomorphic to $\underline{\underline{A}}_1'$ and $\underline{\underline{A}}_1$ is atomic, $\underline{\underline{A}}_1'$ is atomic.

ii) Let T_o be an automorphism of $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$ such that $T_o\underline{\underline{A}}_1 = \underline{\underline{A}}_1'$ and $T_o\underline{\underline{A}}_2 = \underline{\underline{A}}_2$. Let ξ, η be atoms of $\underline{\underline{A}}_1'$ and $\underline{\underline{A}}_2$ respectively. Then $T_o^{-1}\xi$ and $T_o^{-1}\eta$ are atoms of $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ respectively and since $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ are crossed $T_o^{-1}\xi \cap T_o^{-1}\eta \neq \emptyset$. Hence $\xi \cap \eta = T_o(T_o^{-1}\xi \cap T_o^{-1}\eta) \neq \emptyset$.

iii) Fix an atom ξ of $\underline{\underline{A}}_2$ and let $E = \{\xi \cap \alpha : \alpha$ an atom of $\underline{\underline{A}}_1\}$. Now any atom of $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$ is the intersection of an atom α_1 of $\underline{\underline{A}}_1$ and an atom α_2 of $\underline{\underline{A}}_2$. Let α_1' be the atom of $\underline{\underline{A}}_1'$ such that $\alpha_1 \cap \xi = \alpha_1' \cap \xi$, and define $T(\alpha_1 \cap \alpha_2) = \alpha_1' \cap \alpha_2$. It is easy to check that $T\alpha_2 = \alpha_2$ and $T\alpha_1 = \alpha_1'$. Thus T is identity on $\underline{\underline{A}}_2$. Since $\underline{\underline{A}}_1$, $\underline{\underline{A}}_1'$ and $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$ induce same σ-algebra on ξ we see that $T\underline{\underline{A}}_1 = \underline{\underline{A}}_1'$. Finally if $A_1 \in \underline{\underline{A}}_1$ and $A_2 \in \underline{\underline{A}}_2$ then $T(A_1 \cap A_2) = TA_1 \cap TA_2$ from which it follows that T is an automorphism of $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$.

<u>Proposition 5.2.2:</u> Let $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ be crossed σ-algebras and let p, q be atoms of $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$ which belong to distinct atoms of $\underline{\underline{A}}_2$. Then there exists an $\underline{\underline{A}}_1'$ isomorphic to $\underline{\underline{A}}_1$ modulo $\underline{\underline{A}}_2$ in $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$ such that p and q belong to the same atom of $\underline{\underline{A}}_1'$.

<u>Proof</u>: Let π and ρ be atoms of $\underline{\underline{A}}_1$ containing p and q respectively. Define $\underline{\underline{A}}_1'$ as follows: its atoms are same as those of $\underline{\underline{A}}_1$ except that the atom π is replaced by $(\pi \cup q) - (\eta \cap \pi)$ and ρ is replaced by $(\rho - q) \cup (\eta \cap \pi)$ where η is the atom of $\underline{\underline{A}}_2$ which intersects ρ in q.

The elements of $\underline{\underline{A}}'_1$ are those sets in $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$ which are unions of atoms described above. Since p, q belong to distinct atoms of $\underline{\underline{A}}_2$ we see that

$$p, q \in (\pi \cup q) - (\eta \cap \pi)$$

$$p, q \notin (\varrho - q) \cup (\eta \cap \pi).$$

Thus p and q belong to the same atom of $\underline{\underline{A}}'_1$. Finally the automorphism

$$Tx = \begin{cases} x & \text{if} \quad x \neq q \text{ or } \eta \cap \pi \\ \eta \cap \pi & \text{if} \quad x = q \\ q & \text{if} \quad x = \eta \cap \pi \end{cases}$$

sets up an isomorphism of $\underline{\underline{A}}_1$ onto $\underline{\underline{A}}'_1$ modulo $\underline{\underline{A}}_2$ in $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$.

5.3 Application to Product spaces

Now let $\underline{\underline{A}}_0, \underline{\underline{A}}_1, \underline{\underline{A}}_2 \dots$ be point separating σ-algebras of subsets of $X_0, X_1, X_2, \dots$ respectively. Let

$X = X_0 \times X_1 \times X_2 \times \dots$, $\underline{\underline{A}} = \underline{\underline{A}}_0 \times \underline{\underline{A}}_1 \times \underline{\underline{A}}_2 \times \dots$. We denote by $\underline{\underline{A}}_k$ also the σ-algebra

$$X_0 \times X_1 \times \dots \times X_{k-1} \times \underline{\underline{A}}_k \times X_{k+1} \times \dots .$$

Let $\underline{\underline{D}}_n = \underline{\underline{A}}_{n+1} \vee \underline{\underline{A}}_{n+2} \vee \dots$

$\underline{\underline{C}}_n = \underline{\underline{A}}_0 \vee \underline{\underline{A}}_1 \vee \dots \vee \underline{\underline{A}}_{n-1}$

Then $\underline{\underline{C}}_n \vee \underline{\underline{D}}_n = \underline{\underline{A}}$. Let $\underline{\underline{A}}'_k$ be isomorphic to $\underline{\underline{A}}_k$ modulo $\underline{\underline{D}}_{k+1}$ in $\underline{\underline{D}}_k$ and write $\underline{\underline{C}}'_n = \underline{\underline{A}}'_o \vee \ldots \vee \underline{\underline{A}}'_{n-1}$. Then $\underline{\underline{A}} = \underline{\underline{C}}_n \vee \underline{\underline{D}}_n = \underline{\underline{C}}'_n \vee \underline{\underline{D}}_n$. The proposition that $\bigvee_{n=o}^{\infty} \underline{\underline{C}}_n = \bigvee_{n=o}^{\infty} \underline{\underline{C}}'_n$ is not true in general as shown by the following example.

<u>Example 5.3.1.</u>: Let for every $n \geq 0$, $X_n = \{0, 1\}$ and $\underline{\underline{A}}_n$ = the discrete σ-algebra on $\{0, 1\}$. Now take $p, q \in X$ which belong to distinct atoms of $\underline{\underline{D}}_\infty = \bigcap_{n=o}^{\infty} \underline{\underline{D}}_n$ (see example 5.1.4). Since $\underline{\underline{D}}_n \supset \underline{\underline{D}}_\infty$, atoms of $\underline{\underline{D}}_n$ are contained in atoms of $\underline{\underline{D}}_\infty$ and so p, q belong to distinct atoms of $\underline{\underline{D}}_n$ for each n. Let for every n, π_n and ρ_n be the atoms of $\underline{\underline{D}}_n$ containing p and q respectively. Then π_n and ρ_n are subsets of distinct atoms of $\underline{\underline{D}}_{n+1}$. Hence, by proposition 5.2.2. for every n there exists a σ-algebra $\underline{\underline{A}}'_n$ isomorphic to $\underline{\underline{A}}_n$ modulo $\underline{\underline{D}}_{n+1}$ in $\underline{\underline{D}}_n$ such that π_n and ρ_n belong to the same atom of $\underline{\underline{A}}'_n$. Having thus defined $\underline{\underline{A}}'_n$ for every n, we see that p and q are not separated in $\bigvee_{n=o}^{\infty} \underline{\underline{A}}'_n$ by 5.1.1(ii) whereas they are separated in $\bigvee_{n=o}^{\infty} \underline{\underline{A}}_n$. Hence

$$\bigvee_{n=o}^{\infty} \underline{\underline{C}}'_n = \bigvee_{n=o}^{\infty} \underline{\underline{A}}'_n \neq \bigvee_{n=o}^{\infty} \underline{\underline{A}}_n = \bigvee_{n=o}^{\infty} \underline{\underline{C}}_n .$$

In case $(X_i, \underline{\underline{A}}_i)$ are complete separable metric spaces with their Borel σ-algebras, we have the following

Proposition 5.3.2:

$$\bigvee_{n=0}^{\infty} \underline{\underline{C}}_n = \bigvee_{n=0}^{\infty} \underline{\underline{C}}'_n \iff \underline{\underline{D}}_\infty \subset \bigvee_{n=0}^{\infty} \underline{\underline{C}}'_n .$$

Proof: '$\Rightarrow$ part' is obvious since

$$\underline{\underline{D}}_\infty \subset \bigvee_{n=0}^{\infty} \underline{\underline{C}}_n .$$

To prove '$\Leftarrow$ part' we note that for every n, since $\underline{\underline{C}}'_n \vee \underline{\underline{D}}_n = \underline{\underline{A}}$, $\underline{\underline{C}}'_n \vee \underline{\underline{D}}_n$ separates points. Hence by 5.1.3 (i), $\bigvee_{n=0}^{\infty} \underline{\underline{C}}'_n \vee \underline{\underline{D}}_\infty$ separates points. If in addition $\underline{\underline{D}}_\infty \subset \bigvee_{n=0}^{\infty} \underline{\underline{C}}'_n$ then $\bigvee_{n=0}^{\infty} \underline{\underline{C}}'_n$ separates points. Since $\bigvee_{n=0}^{\infty} \underline{\underline{C}}'_n$ is countably generated, by Blackwell-Mackey theorem, $\bigvee_{n=0}^{\infty} \underline{\underline{C}}'_n = \underline{\underline{A}} = \bigvee_{n=0}^{\infty} \underline{\underline{C}}_n$.

Though $\bigvee_{n=0}^{\infty} \underline{\underline{C}}_n = \bigvee_{n=0}^{\infty} \underline{\underline{C}}'_n$ is not true in general one may suspect that $\bigvee_{n=0}^{\infty} \underline{\underline{C}}_n \vee \underline{\underline{D}}_\infty = \bigvee_{n=0}^{\infty} \underline{\underline{C}}'_n \vee \underline{\underline{D}}_\infty$. Here we give an example to show that this also need not be true. We shall make use of example 5.1.4.

Example 5.3.3:

Let $X_k = \{0, 1\}$, $\underline{\underline{A}}_k$ = the discrete σ-algebra on X_k ($k \geq 0$). Denote also by $\underline{\underline{A}}_k$ the σ-algebra

$X_o \times X_1 \times \ldots \times X_{k-1} \times \underline{\underline{A}}_k \times X_{k+1} \times \ldots$.

Let $\underline{\underline{D}}_k = X_o \times \ldots \times X_{k-1} \times \underline{\underline{A}}_k \times \underline{\underline{A}}_{k+1} \times \ldots$.

Let $X = X_0 \times X_1 \times \ldots$,

$A_{k1} = X_0 \times X_1 \times \ldots \times X_{k-1} \times \{0\} \times \{1\} \times X_{k+2} \times \ldots$,

$A_{k2} = X_0 \times X_1 \times \ldots \times X_{k-1} \times \{1\} \times \{0\} \times X_{k+2} \times \ldots$

Define $\underline{\underline{A}}'_k = \{\emptyset, X, A_k, X - A_k\}$ where $A_k = A_{k1} \cup A_{k2}$ and f_k by

$f_k(x_0, x_1, \ldots, x_{k-1}, 0, 1, x_{k+2}, \ldots)$

$= (x_0, x_1, \ldots, x_{k-1}, 1, 1, x_{k+2}, \ldots)$

$f_k(x_0, x_1, \ldots, x_{k-1}, 1, 0, x_{k+2}, \ldots)$

$= (x_0, x_1, \ldots, x_{k-1}, 1, 0, x_{k+2}, \ldots)$

$f_k(x_0, x_1, \ldots, x_{k-1}, 1, 1, x_{k+2}, \ldots)$

$= (x_0, x_1, \ldots, x_{k-1}, 0, 1, x_{k+2}, \ldots)$

$f_k(x_0, x_1, \ldots, x_{k-1}, 0, 0, x_{k+2}, \ldots)$

$= (x_0, x_1, \ldots, x_{k-1}, 0, 0, x_{k+2}, \ldots)$.

It is easily checked that $f_k : X \to X$ sets up an isomorphism between $\underline{\underline{A}}'_k$ and $\underline{\underline{A}}_k$ modulo $\underline{\underline{D}}_{k+1}$ in $\underline{\underline{D}}_k$. Since $\underline{\underline{A}}_k + \alpha = \underline{\underline{A}}_k$, the σ-algebra $\bigvee_{n=0}^{\infty} \underline{\underline{A}}'_n$, which is the same as $\bigvee_{n=0}^{\infty} \underline{\underline{C}}'_n$, is contained in $\underline{\underline{C}}$ where α and $\underline{\underline{C}}$ are as in example 5.1.4. Thus $\bigvee_{n=0}^{\infty} \underline{\underline{C}}'_n \vee \underline{\underline{D}}_\infty \subset \underline{\underline{C}} \vee \underline{\underline{D}}_\infty \neq$ Borel σ-algebra on $X = \bigvee_{n=0}^{\infty} \underline{\underline{C}}_n \vee \underline{\underline{D}}_\infty$. It is worth noting that the collection $\{\underline{\underline{A}}'_n, n \geq 0\}$ is independent and $\underline{\underline{A}}'_n$ and $\underline{\underline{A}}_n$ are both independent complements of $\underline{\underline{D}}_{n+1}$ in $\underline{\underline{D}}_n$ with respect to Haar measure.

CHAPTER 6

HOPF'S THEOREM ON INVARIANT MEASURES FOR A GROUP OF TRANSFORMATIONS

6.0 Introduction

Let $(X, \underline{\underline{A}}, P)$ be a probability space and let $G = \{g\}$ be a group of measurable and nonsingular transformations defined on $(X, \underline{\underline{A}}, P)$. The problem of existence of finite invariant measures for G has been studied for a long time and is of interest in ergodic theory. Hopf [13] gave a necessary and sufficient condition for the existence of a finite invariant measure μ on $\underline{\underline{A}}$ equivalent to P for the case when G is a cyclic group.

Tarski [41] studied the existence and properties of mass functions on abstract semigroups and used his results to give a necessary and sufficient condition for the existence of a finitely additive measure λ on the class of all subsets of a metric space X such that

i) $\lambda(E) = 1$ for a specified subset E of X, and

ii) $\lambda(A) = \lambda(B)$ if A and B are isometrically isomorphic subsets of X.

During a seminar on Tarski's paper Professor Ashok Maitra raised the question whether Hopf's theorem on invariant measures follows from the general results of Tarski. We show in this chapter that the answer is in the affirmative. In fact we prove using Tarski's results that condition (H) (see section 6.2), the analogue of Hopf's condition when G is a general group of transformations, is necessary and sufficient for the existence of a finite, equivalent measure invariant under every $g \in G$.

The only alternative proof known (see Hajian and Ito [10]) of Hopf's theorem for a group of transformations is through the equivalence of condition (H) and the condition of nonexistence of weakly wandering sets of positive measure introduced by Hajian and Kakutani. While Hajian and Ito use functional analytic methods in their proof we use Tarski's results. In section 6.1 we give some definitions and notation. In section 6.2 we consider condition (H) and derive some consequences. In section 6.3 we present the results of Tarski and in section 6.4 we prove our main theorem.

6.1 Definitions and notation

Let $(X, \underline{\underline{A}}, P)$ be a probability space. A finitely additive measure μ on $\underline{\underline{A}}$ is said to be equivalent to P if

μ and P have the same sets of measure zero. A measurable map g of X into itself is called nonsingular if for every $A \in \underline{\underline{A}}$, $P(A) > 0$ implies $P(g^{-1}A) > 0$.

By a group of measurable and nonsingular transformations $G = \{g\}$ defined on $(X, \underline{\underline{A}}, P)$ we mean that each $g \in G$ is a measurable and nonsingular transformation defined on $(X, \underline{\underline{A}}, P)$ and such that the multiplication in G is defined as follows: if $g, g' \in G$ then $g \cdot g'(x) = g(g'(x))$ for every x in X. We note that in this case every $g \in G$ is a 1-1 map of X onto itself such that both g and g^{-1} are measurable and nonsingular transformations defined on $(X, \underline{\underline{A}}, P)$.

Let $G = \{g\}$ be a group of measurable and nonsingular transformations defined on $(X, \underline{\underline{A}}, P)$. A measure μ on $\underline{\underline{A}}$ is said to be invariant under the group G if for every $g \in G$ and for every $E \in \underline{\underline{A}}$ we have $\mu(gE) = \mu(E)$. A set $E \in \underline{\underline{A}}$ is called an almost invariant set if $P(gE - E) = 0$ for every $g \in G$. It is easy to check that the collection

$$\underline{\underline{A}}^* = \left\{ E \in \underline{\underline{A}} : E \text{ is almost invariant} \right\}$$

is a sub-σ-algebra of $\underline{\underline{A}}$ and that given any $A \in \underline{\underline{A}}$ there is a minimal almost invariant set A_1 containing A in the sense that if $A_2 \in \underline{\underline{A}}^*$ and $A_2 \supset A$ then $P(A_1) \leq P(A_2)$. In what follows if $A \in \underline{\underline{A}}$ then A^* will denote a minimal almost invariant set containing A. If $A, B \in \underline{\underline{A}}$ and $A \subset B \subset A^*$ then it is easy to see that $A^* = B^*$.

If $\underline{\underline{C}}$ is a σ-algebra of subsets of a set Z and if $A \in \underline{\underline{C}}$ then we use the notation $A = C + D$, $A = \sum_{j=1}^{n} A_j$ and $A = \sum_{j=1}^{\infty} A_j$ to mean respectively that A is the union of disjoint $\underline{\underline{C}}$ - measurable sets C and D, A is the union of pairwise disjoint sets $\{A_1, A_2, \ldots, A_n\} \subset \underline{\underline{C}}$ and A is the union of pairwise disjoint sets $\{A_j, j = 1, 2, \ldots\} \subset \underline{\underline{C}}$.

6.2 Equivalence of measurable sets and condition (H)

Two $\underline{\underline{A}}$- measurable sets E and F are said to be equivalent if $E = \sum_{j=1}^{\infty} E_j$, $F = \sum_{j=1}^{\infty} F_j$ and there is a sequence $\{g_j\} \subset G$ such that $F_j = g_j E_j$ for every j; and in this case we say that E is equivalent to F under $\{g_j\}$. A set E is called bounded if it is not equivalent to a measure theoretically proper subset of itself, that is, E is equivalent to F and $F \subset E$ implies $P(E - F) = 0$. The group G is bounded if X is bounded. We now consider the following.

<u>CONDITION (H):</u> G IS BOUNDED.

This condition, introduced by Cotlar and Ricabarra [5], reduces to Hopf's condition when G is a cyclic group.

Lemma 6.2.1: If (H) holds then every $A \in \underline{\underline{A}}$ is bounded.

Lemma 6.2.2: If $E, F \in \underline{\underline{A}}$ are equivalent then there is a 1-1, bimeasurable map h from E onto F such that for every $E_1 \in \underline{\underline{A}}$, $E_1 \subset E$, E_1 and $h(E_1)$ are equivalent.

Proof: $E = \sum_{j=1}^{\infty} E_j$, $F = \sum_{j=1}^{\infty} F_j$ and there is a sequence $\{g_j\} \subset G$ such that $g_j E_j = F_j$. The function h on E defined by $h = g_j$ on E_j, $j = 1, 2, \ldots,$ satisfies the required properties.

Lemma 6.2.3: Let $E, F \in \underline{\underline{A}}$ be equivalent under $\{g_j\}$. Then for every $C \in \underline{\underline{A}}$ and for every countable subgroup G' of G such that $\{g_j\} \subset G'$, $E \cap (\bigcup_{G'} gC)$ is equivalent to $F \cap (\bigcup_{G'} gC)$ and $E - (\bigcup_{G'} gC)$ is equivalent to $F - (\bigcup_{G'} gC)$.

Proof: It is easy to see that

$$h(E \cap (\bigcup_{G'} gC)) = F \cap (\bigcup_{G'} gC) \quad \text{and}$$

$$h(E - (\bigcup_{G'} gC)) = F - (\bigcup_{G'} gC) \quad \text{where}$$

h is given by lemma 6.2.2.

The following lemma is a generalisation of lemma 6 of [9].

<u>Lemma 6.2.4:</u> Let $A \in \underline{\underline{A}}$ and let A^* be fixed. Then there exist $C, D \in \underline{\underline{A}}$, $C^*, D^* \in \underline{\underline{A}}^*$ such that

i) $A = C + D$, $A^* = C^* + D^*$

ii) C is equivalent to a subset of $C^* - C$

and iii) $D^* - D$ is equivalent to a subset of D.

<u>Proof:</u> Let $B = A^* - A$ and let
$\alpha_1 = \sup \left\{ P(g^{-1}(gA \cap B)) : g \in G \right\}$. Choose $A_1 \in \underline{\underline{A}}$ and $g_1 \in G$ such that $A_1 \subset A$, $B_1 = g_1 A_1 \subset B$ and $P(A_1) \geq \alpha_1/2$. Let n be a positive integer. Suppose pairwise disjoint sets $\left\{ A_1, A_2, \ldots, A_n \right\} \subset \underline{\underline{A}}$, pairwise disjoint sets $\left\{ B_1, B_2, \ldots, B_n \right\} \subset \underline{\underline{A}}$ and $\left\{ g_1, g_2, \ldots, g_n \right\} \subset G$ have been chosen such that for $1 \leq k \leq n$,

$$A_k \subset A - \sum_{j=1}^{k-1} A_j, \quad B_k = g_k A_k \subset B - \sum_{j=1}^{k-1} B_j$$

and $P(A_k) \geq \alpha_k/2$ where

$$\alpha_k = \sup \left\{ P(g^{-1}[g(A - \sum_{j=1}^{k-1} A_j) \cap (B - \sum_{j=1}^{k-1} \cap B_j)]) : g \in G \right\}.$$

Let $\alpha_{n+1} = \sup \left\{ P(g^{-1}[g(A - \sum_{j=1}^{n} A_j) \cap (B - \sum_{j=1}^{n} B_j)]) : g \in G \right\}$

and choose $A_{n+1} \in \underline{\underline{A}}$, $g_{n+1} \in G$ such that

$$A_{n+1} \subset A - \sum_{j=1}^{n} A_j,\ B_{n+1} = g_{n+1}A_{n+1} \subset B - \sum_{j=1}^{n} B_j$$

and $P(A_{n+1}) \geq \alpha_{n+1}/2$.

Let $A_o = \sum_{n=1}^{\infty} A_n$ and $B_o = \sum_{n=1}^{\infty} B_n$. Then A_o is equivalent to B_o under $\{g_n\}$. If $g \in G$ then $P(g^{-1}[g(A - A_o) \cap (B - B_o)]) \leq \alpha_n/2 \leq \alpha_1/2^n$ for every n and so $P(g^{-1}[g(A - A_o) \cap (B - B_o)]) = 0$ for every $g \in G$. Since every $g \in G$ is nonsingular $P(g(A - A_o) \cap (B - B_o)) = 0$ for every $g \in G$. Let G' be the countable subgroup generated by $\{g_n, n \geq 1\}$ and let $(A - A_o)^*$ be fixed. It follows that $(A \cup B_o) \cap (\bigcup_{G'} g(A - A_o)^*) \in \underline{\underline{A}}^*$ and is a minimal almost invariant set containing $(A - A_o)$. Let $D = A \cap (\bigcup_{G'} g(A - A_o)^*)$ $= (A - A_o) \cup (A_o \cap (\bigcup_{G'} g(A - A_o)^*))$. Since $(A - A_o) \subset D$, take $D^* = D \cup (B_o \cap (\bigcup_{G'} g(A - A_o)^*)) = (A \cup B_o) \cap (\bigcup_{G'} g(A - A_o)^*) \in \underline{\underline{A}}^*$. By lemma 6.2.3, $B_o \cap (\bigcup_{G'} g(A - A_o)^*)$ and $A_o \cap (\bigcup_{G'} g(A - A_o)^*)$ are equivalent and so $D^* - D$ is equivalent to a subset of D. Now let $C = A_o - (\bigcup_{G'} g(A - A_o)^*)$. Again by lemma 6.2.3, C is equivalent to $B_o - (\bigcup_{G'} g(A - A_o)^*)$. Let $C^* = A^* - D^*$ $= C \cup (B_o - (\bigcup_{G'} g(A - A_o)^*)) \cup (B - B_o)$ to complete the proof.

<u>Proposition 6.2.5:</u> Let $A \in \underline{\underline{A}}$ and let A^* be fixed. Then A and A^* admit decompositions of the form

$$A = D_{-1} + D_{\infty} + \sum_{k \geq 0} D_k$$

$$A^* = D^*_{-1} + D^*_{\infty} + \sum_{k \geq 0} D^*_k$$

where

i) $P(D^*_{-1}) = 0$

ii) For each $k = 0, 1, 2, \ldots$

D^*_k contains 2^k pairwise disjoint measurable sets $D^1_k, \ldots, D^{2^k}_k$ each equivalent to D_k and $E_k = D^*_k - \sum_{j=1}^{2^k} D^j_k$

is equivalent to a subset of $\sum_{j=1}^{2^k} D^j_k$

and iii) D^*_{∞} contains infinitely many pairwise disjoint sets each equivalent to D_{∞} .

<u>Proof:</u> First we shall show by inductive application of lemma 6.2.4 that there is a sequence $D_0, D_1, \ldots, D_k, \ldots$ of subsets of A and a sequence of subsets $D^*_0, D^*_1, \ldots, D^*_k, \ldots$ of A^* satisfying (ii) of the proposition and

iv) for every $k = 0, 1, 2, \ldots$

$$A^* - \sum_{j=0}^{k} D^*_j = F^1_{k+1} + \ldots + F^{2^{k+1}}_{k+1} + H_{k+1}$$

where F_{k+1}^{i} is equivalent to $F_{k+1}^{1} = A - \sum_{j=o}^{k} D_j$, $1 \leq i \leq 2^{k+1}$. We shall also denote F_{k+1}^{1} by F_{k+1}.

Let $F_o^1 = A$. Apply lemma 6.2.4 to A and A^* and let $D_o = D_o^1 = D$, $F_1 = F_1^1 = C$, F_1^2 = a subset of $C^* - C$ equivalent to C and $H_1 = C^* - (F_1^1 + F_1^2)$. Thus (ii) and (iv) hold when $k = 0$. Suppose for $k = n$ (a nonnegative integer), $D_o, D_1, \ldots, D_n$, $D_o^*, D_1^*, \ldots, D_n^*$ have been constructed such that $D_k \subset A$ for all i, $D_k \cap D_j = \emptyset = D_k^* \cap D_j^*$ if $k \neq j$ and (ii) and (iv) hold for every $0 \leq k \leq n$. Let $A_{n+1} = \sum_{j=1}^{2^{n+1}} F_{n+1}^{j}$, $A_{n+1}^* = A^* - \sum_{j=o}^{n} D_j^*$. Clearly $A_{n+1}^* = F_{n+1}^*$. Proceeding as in the proof of lemma 6.2.4 we have $\underline{\underline{A}}$- measurable sets $A'_{n+1} \subset A_{n+1}$, $B_{n+1} \subset A_{n+1}^* - A_{n+1}$ and $\{g_m^{n+1}\ m \geq 1\} \subset G$ such that A'_{n+1} is equivalent to B_{n+1} under $\{g_m^{n+1}\}$. Further $D'_{n+1} = (A_{n+1} - A'_{n+1})$ is such that for every $g \in G$, $P(gD'_{n+1} - (A_{n+1} \cup B_{n+1})) = 0$. Let F_{n+1}^{j} be equivalent to F_{n+1} under the countable set $G_{n+1}^{j} \subset G$ for $1 \leq j \leq 2^{n+1}$. Let G_{n+1} be a countable subgroup of G containing $(\bigcup_{j=1}^{2^{n+1}} G_{n+1}^{j}) \cup \{g_m^{n+1}\}$ and let D'^*_{n+1} be fixed. Let $D''^*_{n+1} = (A_{n+1} \cup B_{n+1}) \cap (\bigcup_{G_{n+1}} gD'^*_{n+1})$. By 6.2.3, $D''^*_{n+1} \cap A'_{n+1}$ is equivalent to $D''^{*}_{n+1} \cap B_{n+1}$ and $A'_{n+1} - D''^{*}_{n+1}$ is equivalent to $B_{n+1} - D''^{*}_{n+1}$. Let, for $1 \leq j \leq 2^{n+1}$,

$D^{j}_{n+1} = D''^{*}_{n+1} \cap F^{j}_{n+1} = F^{j}_{n+1} \cap (\bigcup_{G_{n+1}} gD'^{*}_{n+1})$, $D_{n+1} = D^{1}_{n+1}$, and let $D^{*}_{n+1} = D''^{*}_{n+1}$. Since $\{F^{j}_{n+1}, 1 \leq j \leq 2^{n+1}\}$ are all equivalent to F_{n+1} and since $G_{n+1} \supset \bigcup_{j=1}^{2^{n+1}} G^{j}_{n+1}$ it follows by lemma 6.2.3 that $\{D^{j}_{n+1}, 1 \leq j \leq 2^{n+1}\}$ are all equivalent to D_{n+1}. Let $E_{n+1} = D^{*}_{n+1} - \sum_{j=1}^{2^{n+1}} D^{j}_{n+1}$. By construction, $E_{n+1} = D''^{*}_{n+1} - A_{n+1}$ $= D''^{*} \cap B_{n+1}$ which is equivalent to $D''^{*}_{n+1} \cap A'_{n+1} \subset D''^{*}_{n+1} \cap A_{n+1} =$ $= \sum_{j=1}^{2^{n+1}} D^{j}_{n+1}$. Thus condition (ii) of the proposition holds for $k = n+1$.

Let $F^{j}_{n+2} = (A'_{n+1} - D''^{*}_{n+1}) \cap F^{j}_{n+1} = F^{j}_{n+1} - D''^{*}_{n+1}$, $1 \leq j \leq 2^{n+1}$ and let $F_{n+2} = F^{1}_{n+2}$. Clearly $F_{n+2} = F_{n+1} - D_{n+1} = A - \sum_{j=o}^{n+1} D_{j}$. Again by lemma 6.2.3, since $G_{n+1} \supset \bigcup_{j=1}^{2^{n+1}} G^{j}_{n+1}$ it follows that $\{F^{j}_{n+2}, 1 \leq j \leq 2^{n+1}\}$ are all equivalent to F_{n+2}. Since $\sum_{j=1}^{2^{n+1}} F^{j}_{n+2} = A'_{n+1} - D''^{*}_{n+1}$ is equivalent to $B_{n+1} - D''^{*}_{n+1}$ we can write, using 6.2.2, $B_{n+1} - D''^{*}_{n+1} = \sum_{j=2^{n+1}+1}^{2^{n+2}} F^{j}_{n+2}$, where $F^{2^{n+1}+j}_{n+2}$

is equivalent to F^j_{n+1} for $1 \leq j \leq 2^{n+1}$. Let

$$H_{n+2} = A^* - \sum_{j=o}^{n+1} D^*_j - \sum_{j=1}^{2^{n+2}} F^j_{n+2}.$$ Thus (iv) holds for $k = n+1$.

Hence, by induction, there is a sequence $\{D_k\}$ of subsets of A and a sequence $\{D^*_k\}$ of subsets of A^* satisfying (ii) and (iv).

Let G_o be a countable subgroup of G containing $\bigcup_{k=o}^{\infty} \bigcup_{j=1}^{2^{k+1}} G^j_{k+1}$, where for every $k \geq 0$, F^j_{k+1} is equivalent to under G^j_{k+1} $(1 \leq j \leq 2^{k+1})$. Let $C^*_\infty = \bigcup_{G_o} g(\sum_{k \geq o} D^*_k)$ and let

$$D_\infty = A - C^*_\infty, \quad D_{-1} = A - (D_\infty + \sum_{k \geq o} D_k)$$

$$D^*_\infty = A^* - C^*_\infty, \quad D^*_{-1} = A^* - (D^*_\infty + \sum_{k \geq o} D^*_k).$$

Since $\sum_{k \geq o} D^*_k \in \underline{\underline{A}}^*$, $C^*_\infty - \sum_{k \geq o} D^*_k = D^*_{-1}$ is such that $P(D^*_{-1}) = 0$.

Now let

$$D^k_\infty = F^k_k \cap D^*_\infty = F^k_k - C^*_\infty, \quad k = 1, 2, \ldots$$

For each $k \geq 1$, $D_\infty \subset F^1_k$, $D^k_\infty \subset F^k_k$ and F^1_k is equivalent to F^k_k under elements from G_o. Hence by 6.2.3, $D_\infty = F^1_k - C^*_\infty$ is equivalent to $F^k_k - C^*_\infty = D^k_\infty$ for every k. Since by construction $\{F^k_k, k \geq 1\}$ is a pairwise disjoint family we have $D_\infty^m \cap D_\infty^n = \emptyset$ if $m \neq n$ and the proposition is proved.

Let $Y = N \times X$ where $N = \{1, 2, \ldots\}$ and let $\underline{\underline{B}} = \left\{ \bigcup_{n=1}^{\infty} \{n\} \times A_n : A_n \in \underline{\underline{A}} \text{ for each } n \in N \right\}$. Two sets $A, B \in \underline{\underline{B}}$ are called equivalent if $A = \sum_{j=1}^{\infty} \{n_j\} \times A_j$, $B = \sum_{j=1}^{\infty} \{m_j\} \times B_j$ and A_j is equivalent to B_j for all j (the n_j's need not be distinct for different j's and the m_j's need not be distinct for different j's).

<u>Lemma 6.2.6:</u> If $A, B \in \underline{\underline{B}}$ are equivalent then there is a 1-1, bimeasurable map h from A onto B such that for every $A_1 \in \underline{\underline{B}}$, $A_1 \subset A$, A_1 and $h(A_1)$ are equivalent.

<u>Proof:</u> $A = \sum_{j=1}^{\infty} \{n_j\} \times A_j$, $B = \sum_{j=1}^{\infty} \{m_j\} \times B_j$ and A_j is equivalent to B_j for every j. Let, for each j, h_j be the map from A_j onto B_j given by 6.2.2. Let h on A be defined by $h(n_j, x) = (m_j, h_j(x))$ on $\{n_j\} \times A_j$ for every $j = 1, 2, \ldots$. Then h has the required properties.

<u>Lemma 6.2.7:</u> Let $A, B, C \in \underline{\underline{B}}$ be such that $A \cap B = \emptyset = A \cap C$. Suppose $A+B$ is equivalent a subset D of $A+C$. Then B admits a decomposition

$$B = B_1 + B_2 \quad \text{where}$$

i) B_1 is equivalent to a subset A_1 of A such that A contains infinitely many pairwise disjoint sets each equivalent to A_1, and

ii) B_2 is equivalent to a subset of C.

Proof: Let h be the map from $A+B$ onto D given by lemma 6.2.7. Let

$$D_1 = h(B) \cap C \qquad F_1 = h^{-1}(D_1)$$

$$D_2 = h^2(B - F_1) \cap C \qquad F_2 = h^{-2}(D_2)$$

$$\cdots\cdots\cdots \qquad \cdots\cdots$$

$$\cdots\cdots\cdots \qquad \cdots\cdots$$

$$D_k = h^k(B - \sum_{j=1}^{k-1} F_j) \cap C \qquad F_k = h^{-k}(F_k)$$

$$\cdots\cdots\cdots \qquad \cdots\cdots$$

Since h is 1-1 from $A+B$ onto D it is easy to verify that $D_1, D_2, \ldots$ is a sequence of pairwise disjoint subsets of C and that $F_1, F_2, \ldots$ is a sequence of pairwise disjoint subsets of B. Further, by lemma 6.2.6, F_k is equivalent to $h^k(F_k) = D_k$ for every $k = 1, 2, \ldots$. Let $B_1 = B - \sum_{j=1}^{\infty} F_j$ and $B_2 = \sum_{j=1}^{\infty} F_j$. Clearly B_2 is equivalent to $\sum_{j=1}^{\infty} D_j \subset C$.

By construction, $h^k(B_1) \subset A$ for all $k \geq 1$. B_1 and $h(B_1)$ are disjoint. Since h is 1-1, $\{B_1, h(B_1), h^2(B_1)\}$ are pairwise disjoint. Similar argument shows that $\{h^k(B_1), k \geq 1\}$ is a pairwise disjoint sequence of subsets of A. By lemma 6.2.6, $h^k(B_1)$ is equivalent to $h(B_1) = A_1$ for every $k \geq 1$.

<u>Lemma 6.2.8:</u> (H) holds if and only if for every sequence $\{A_n\}$ of $\underline{\underline{A}}$-measurable, pairwise disjoint, mutually equivalent sets $P(A_n) = 0$ for all n.

<u>Proof:</u> 'if part'. Suppose (H) does not hold. Let $A \in \underline{\underline{A}}$ with $P(A) < 1$ be such that X is equivalent to A. Let h from X onto A be the map given by 6.2.2. Let for every $n = 1, 2, \ldots$ $A_n = h^n(X - A)$. Since every $g \in G$ is nonsingular, from the properties of h it follows that $\{A_n\} \subset \underline{\underline{A}}$ is a sequence of pairwise disjoint mutually equivalent sets such that $P(A_n) > 0$ for all n.

'only if' part. Suppose there is a sequence $\{A_n\}$ of $\underline{\underline{A}}$-measurable, pairwise disjoint, mutually equivalent sets such that $P(A_n) > 0$ for some n. Since every $g \in G$ is non-singular, it follows that $P(A_n) > 0$ for every n. Hence $\sum_{n=1}^{\infty} A_n$ is equivalent to the measure theoretically proper subset $\sum_{n=2}^{\infty} A_n$ and so $\sum_{n=1}^{\infty} A_n$ is not bounded. By 6.2.1, (H) does not hold.

<u>Proposition 6.2.9:</u> Let (H) hold. Then for every $E \in \underline{\underline{A}}$ with $P(E) > 0$ and for every positive integer k, $\sum_{j=1}^{k+1} \{j\} \times E$ is not equivalent to any subset of $\sum_{j=1}^{k} \{j\} \times E$.

<u>Proof:</u> By 6.2.1, the proposition is true for $k = 1$. Suppose the proposition is true for $k = n \geq 1$. Let, if possible, $F \in \underline{\underline{A}}$ with $P(F) > 0$ be such that $\sum_{j=1}^{n+2} \{j\} \times F$ is equivalent to a subset of $\sum_{j=1}^{n+1} \{j\} \times F$. Take $A = \{1\} \times F$, $B = \sum_{j=2}^{n+2} \{j\} \times F$, $C = \sum_{j=2}^{n+1} \{j\} \times F$ and apply lemma 6.2.7. We get $B = B_1 + B_2$ where B_2 is equivalent to a subset C_2 of C and there exists a sequence $A_1, A_2, \ldots$ of pairwise disjoint measurable subsets of F such that B_1 is equivalent to $\{1\} \times A_m$ for every m. By lemma 6.2.8, $P(A_m) = 0$ for all m. Since B_2 is equivalent to C_2, we have $B_2 = \sum_{i=1}^{\infty} \{n_i\} \times B_{2i}$, $C_2 = \sum_{i=1}^{\infty} \{m_i\} \times C_{2i}$ and B_{2i} is equivalent to C_{2i} under $\{g_m^i, m \geq 1\} \subset G$. Since B_1 is equivalent to $\{1\} \times A_1$ we have $B_1 = \sum_{i=1}^{\infty} \{n_i'\} \times B_{1i}$, $\{1\} \times A_1 = \sum_{i=1}^{\infty} \{1\} \times A_{1i}$ and B_{1i} is equivalent to A_{1i} under $\{g_m'^i, m \geq 1\} \subset G$. Let $G_o \supset \bigcup_{i=1}^{\infty} \left\{ \{g_m^i\} \cup \{g_m'^i\} \right\}$

be a countable subgroup of G and let $N = \bigcup_{G_o} g A_1$. Since every $g \in G$ is nonsingular and since $P(A_1) = 0$ we have $P(N) = 0$. Let $E = F - N \in \underline{\underline{A}}$. Then $P(E) > 0$. Now $\sum_{j=2}^{n+2} \{j\} \times E = B - \sum_{j=2}^{n+2} \{j\} \times N$. Further, since B_{1i} is equivalent to A_{1i} under $\{g'^{i}_{m}\}$ and since $G_o \supset \bigcup_{i=1}^{\infty} \{g'^{i}_{m}\}$ we have $B_1 \subset \sum_{j=2}^{n+2} \{j\} \times N$. Therefore

$$\sum_{j=2}^{n+2} \{j\} \times E = B_2 - \sum_{j=2}^{n+2} \{j\} \times N = \sum_{i=1}^{\infty} \{n_i\} \times (B_{2i} - N).$$

But $(B_{2i} - N)$ is equivalent to $(C_{2i} - N)$ for every i, by 6.2.3, since $G_o \supset \bigcup_{i=1}^{\infty} \{g^{i}_{m}\}$. So $\sum_{i=1}^{\infty} \{n_i\} \times (B_{2i} - N)$ is equivalent to $\sum_{i=1}^{\infty} \{m_i\} \times (C_{2i} - N) = C_2 - \sum_{j=2}^{n+1} \{j\} \times N$

$\subset C - \sum_{j=2}^{n+1} \{j\} \times N = \sum_{j=2}^{n+1} \{j\} \times E$. This means that $\sum_{j=1}^{n+1} \{j\} \times E$ is equivalent to a subset of $\sum_{j=1}^{n} \{j\} \times E$ where $E \in \underline{\underline{A}}$ with $P(E) > 0$ contradicting our induction assumption. Hence, by induction, the proposition is true.

6.3 Mass functions on semigroups

Let $(S, +)$ be a commutative semigroup. The proofs of propositions stated in this section are elementary and so after each proposition we only hint at the preceding definition and propositions from which it follows.

$\alpha, \beta, \partial, \xi, \eta, \ldots$ denote elements of S. k, m, n denote positive integers. Let e be a fixed element of S.

<u>Definition 6.3.1:</u> $\alpha \leq \beta \Longleftrightarrow$ either $\alpha = \beta$ or there is a ξ in S such that $\alpha + \xi = \beta$.

<u>Definition 6.3.2:</u>

$$\alpha_1 + \alpha_2 + \ldots + \alpha_k = \alpha_1 \quad \text{if} \quad k = 1$$

$$\alpha_1 + \alpha_2 + \ldots + \alpha_k = (\alpha_1 + \ldots + \alpha_{k-1}) + \alpha_k \quad \text{if} \quad k > 1.$$

<u>Definition 6.3.3:</u> $k\alpha = \alpha$ if $k = 1$.

$$k\alpha = (k-1)\alpha + \alpha \quad \text{if} \quad k > 1.$$

General commutativity and associativity of addition $\alpha_1 + \ldots + \alpha_k$ and associativity and distributivity of multiplication $k\alpha$ can be easily verified.

<u>Definition 6.3.4:</u> $E(e) = \{\alpha \in S : \text{ for some } k \geq 1 \ \alpha \leq ke\}$.

<u>Proposition 6.3.5:</u> $e \in E(e)$ (6.3.3, 6.3.4).

<u>Proposition 6.3.6:</u> If $\alpha + \partial = \beta + \partial$ and $\partial \in E(e)$ then there is a k such that $\alpha + ke = \beta + ke$. If $\alpha + \partial \leq \beta + \partial$ and $\partial \in E(e)$ then there is a k such that $\alpha + ke \leq \beta + ke$.

(6.3.1, 6.3.3, 6.3.4)

Definition 6.3.7: α is called normal if there does not exist any k such that $(k + 1)\alpha \leq k\alpha$.

Proposition 6.3.8: if α is normal then

$$k\alpha \leq m\alpha \Rightarrow k \leq m \quad \text{and}$$

$$k\alpha = m\alpha \Rightarrow k = m \qquad (6.3.1, 6.3.3, 6.3.7)$$

Definition 6.3.9: A function f from a subset J of S to $[0, \infty)$ is called a mass function on J in case

i) $e \in J \subset E(e)$

ii) $f(e) = 1$ and

iii) if $\xi_1, \xi_2, \ldots, \xi_k, \eta_1, \ldots, \eta_m$ are elements of J such that

$$\xi_1 + \xi_2 + \ldots + \xi_k \leq \eta_1 + \ldots + \eta_m \text{ then}$$

$$f(\xi_1) + f(\xi_2) + \ldots + f(\xi_k)$$

$$\leq f(\eta_1) + f(\eta_2) + \ldots + f(\eta_m).$$

In what follows in this section f is a mass function on J.

Proposition 6.3.10: If $\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_1 + \ldots + \alpha_k$ are all in J then

$$f(\alpha_1 + \ldots + \alpha_k) = f(\alpha_1) + \ldots + f(\alpha_k) \qquad (6.3.9)$$

Proposition 6.3.11: If $\alpha, k\alpha \in J$ then $f(k\alpha) = kf(\alpha)$

(6.3.9)

<u>Definition 6.3.12</u>: Let $\alpha \in E(e)$. Then

$$f_i^J(\alpha) = \sup\{z: z = (1/n)[f(\xi_1) + \ldots + f(\xi_k) - f(\eta_1) - \ldots - f(\eta_m)]\}$$

where $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_m \in J$ and $\xi_1 + \ldots + \xi_k \leq \eta_1 + \ldots + \eta_m + n\alpha$

$$f_o^J(\alpha) = \inf\{z: z = (1/n)[f(\xi_1) + \ldots + f(\xi_k) - f(\eta_1) - \ldots - f(\eta_m)]\}$$

where $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_m \in J$ and

$$\eta_1 + \ldots + \eta_m + n\alpha \leq \xi_1 + \ldots + \xi_k.$$

<u>Proposition 6.3.13</u>: Let $\alpha \in E(e)$. Then

$$0 \leq f_i^J(\alpha) \leq f_o^J(\alpha)$$ (6.3.4, 6.3.9, 6.3.12)

<u>Proposition 6.3.14</u>: Let $\alpha \in E(e)$. Let $f(\alpha)$ be chosen such that $f_i^J(\alpha) \leq f(\alpha) \leq f_o^J(\alpha)$. Then f is a mass function on $J \cup \{\alpha\}$. (6.3.6, 6.3.9, 6.3.12, 6.3.13)

<u>Proposition 6.3.15</u>: Let $J = \{e\}$. Then h is a mass function on J if and only if e is normal and $h(e) = 1$. (6.3.7, 6.3.8, 6.3.9)

<u>Proposition 6.3.16</u>: Let $J_0, J_1, \ldots, J_\pi, \ldots$ be a (finite or) transfinite sequence of subsets of S such that $e \in J_\pi \subset J_\rho$

for $\pi \leq \varrho < \lambda$ where λ is a given ordinal number > 0. Then h is a mass function om $\bigcup_{\pi < \lambda} J_\pi$ if and only if h is a mass function on J_π for every $\pi < \lambda$. (6.3.9)

Theorem 6.3.17: Let $J \subset V \subset E(e)$. Then there is a mass function h on V such that $h = f$ on J.

The above important extension theorem follows from 6.3.14, 6.3.15, the well ordering theorem and 6.3.16.

Theorem 6.3.18 (Fundamental theorem of Tarski):

There is a mass function on $E(e)$ if and only if e is normal. (6.3.15, 6.3.17)

6.4 Hopf's theorem on invariant measures for a group of transformations

Let $S = \{[A] : A \in \underline{\underline{B}}\}$ where $[A] = \{B \in \underline{\underline{B}} : A$ is equivalent to $B\}$. For $[A], [B] \in S$ define $[A] + [B] = [A \cup B]$ where $A \in [A], B \in [B]$ and $A \cap B = \emptyset$. Then the operation + is well defined and $(S, +)$ is a commutative semigroup. (It is for this reason we constructed Y as + may not be well defined if we consider $S' = \{[A] : A \in \underline{\underline{A}}\}$).

Let $S_o = \left\{\alpha_A \in S: \alpha_A = [\{1\} \times A],\ A \in \underline{\underline{A}}\right\}$. We assume throughout this section that (H) holds.

In view of definition 6.3.7 we can restate proposition 6.2.9 as

<u>Proposition 6.4.1:</u> Let $\alpha_A \in S_o$ be such that $P(A) > 0$. Then α_A is normal.

Let $S^* = \left\{\alpha_A \in S_o : A \in \underline{\underline{A}}^*\right\}$. If $\alpha_A, \alpha_B \in S^*$ and $\alpha_A = \alpha_B$ it follows that $P(A) = P(B)$. If $\alpha_A \in S^*$, $1 \leq j \leq n$ and $A_i \cap A_j = \emptyset$ for $i \neq j$ then $\alpha_{A_1} + \ldots + \alpha_{A_n} = \alpha_{A_1 + \ldots + A_n}$. Now define f^* on S^* by $f^*(\alpha_A) = P(A)$, $\alpha_A \in S^*$. f^* is well defined and if $\alpha_{A_1}, \ldots, \alpha_{A_n} \in S^*$, $A_i \cap A_j = \emptyset$ if $i \neq j$, then $f^*(\alpha_{A_1 + \ldots + A_n}) = \sum_{j=1}^{n} f^*(\alpha_{A_j})$. Let $\alpha_X = e$.

<u>Proposition 6.4.2:</u> f^* is a mass function on S^*.

<u>Proof:</u> i) $e = \alpha_X \in S^* \subset E(e)$.

ii) $f^*(e) = 1$, $f^* \geq 0$.

iii) Suppose $\alpha_{A_i}, \alpha_{B_j} \in S^*$, $1 \leq i \leq k$, $1 \leq j \leq m$ and $\alpha_{A_1} + \ldots + \alpha_{A_k} \leq \alpha_{B_1} + \ldots + \alpha_{B_m}$, that is, $\sum_{i=1}^{k} \{i\} \times A_i$ is equivalent to a subset of $\sum_{j=1}^{m} \{j\} \times B_j$.

Let G_o be a countable subgroup of G containing all the elements of G involved in this equivalence. Replacing A_i by $\bigcap_{G_o} gA_i$ and B_j by $\bigcup_{G_o} gB_j$ if necessary, we can assume that A_i and B_j are invariant under G_o for every i and j.

Let $D_1, D_2, \ldots, D_n$ be the partition of $(\bigcup_{i=1}^{k} A_i) \cup (\bigcup_{j=1}^{m} B_j)$ induced by $\{A_i, B_j, 1 \leq i \leq k, 1 \leq j \leq m\}$. Then $D_i \in \underline{\underline{A}}^*$ and D_i is invariant under G_o for every i. Letting $\partial_i = \alpha_{D_i}$ we have positive integers r_i, s_i such that

$$\begin{aligned} r_1\partial_1 + \ldots + r_n\partial_n &= \alpha_{A_1} + \ldots + \alpha_{A_k} \\ &\leq \alpha_{B_1} + \ldots + \alpha_{B_m} \\ &= s_1\partial_1 + \ldots + s_n\partial_n . \end{aligned}$$

Since $\sum_{i=1}^{k} \{i\} \times A_i$ is equivalent to a subset of $\sum_{j=1}^{m} \{j\} \times B_j$ under G_o and since every D_i is invariant under G_o it is easy to see that $r_i\partial_i \leq s_i\partial_i$ for every i. Now if $P(D_i) = 0$ then $f^*(\partial_i) = 0$ and so $0 = r_i f^*(\partial_i) \leq s_i f^*(\partial_i)$. If $P(D_i) > 0$ then, by 6.4.1, ∂_i is normal and so by 6.3.8,

$$r_i\partial_i \leq s_i\partial_i \Rightarrow r_i \leq s_i \Rightarrow r_i f^*(\partial_i) \leq s_i f^*(\partial_i).$$

Thus

$$\sum_{i=1}^{k} f^*(\alpha_{A_i}) = \sum_{i=1}^{n} r_i f^*(\partial_i) \leq \sum_{i=1}^{n} s_i f^*(\partial_i) = \sum_{j=1}^{m} f^*(\alpha_{B_j}).$$

Thus f^* is a mass function on S^*.

By Tarski's extension theorem 6.3.17, f^* admits an extension f to the whole of $E(e)$ as a mass function. In particular f is an extension of f^* as a mass function to S_o.

Proposition 6.4.3: Let $A \in \underline{\underline{A}}$. Then

$$f(\alpha_A) > 0 \iff P(A) > 0.$$

Proof: $P(A) = 0 \implies P(A^*) = 0$

$$\implies f^*(\alpha_{A^*}) = 0$$

$$\implies f(\alpha_A) + f(\alpha_{A^*-A}) = 0 \quad \text{by} \quad (6.3.10)$$

$$\implies f(\alpha_A) = 0.$$

Now suppose $P(A) > 0$. Let A^* be fixed. Let

$$A = D_{-1} + D_\infty + \sum_{k \geq o} D_k \quad \text{and}$$

$$A^* = D^*_{-1} + D^*_\infty + \sum_{k \geq o} D^*_k$$

be the decompositions of A and A^* according to proposition 6.2.5. $P(D^*_{-1}) = 0$ by 6.2.5 and $P(D_\infty) = 0$ by lemma 6.2.8.

Therefore $P(D_k) > 0$ for some $k \geq 0$. Further $D_k^* = D_k^1 + \ldots + D_k^{2^k} + E_k$, D_k^i is equivalent to D_k for all i and E_k is equivalent to a subset of $D_k^* - E_k$ (see 6.2.5). So $\alpha_{D_k^i} = \alpha_{D_k}$ for all i and $\alpha_{E_k} \leq 2^k \alpha_{D_k}$. We have

$f(\alpha_{D_k^*}) = f^*(\alpha_{D_k^*}) = P(D_k^*) \geq P(D_k) > 0$. But

$$f(\alpha_{D_k^*}) = 2^k f(\alpha_{D_k}) + f(\alpha_{E_k})$$

where $f(\alpha_{E_k}) \leq 2^k f(\alpha_{D_k})$ and so $f(\alpha_{D_k}) > 0$. Hence $f(\alpha_A) \geq f(\alpha_{D_k}) > 0$.

Theorem 6.4.4 (Hopf's theorem for a group of transformations):

There is a finite measure invariant under G and equivalent to P if and only if (H) holds.

Proof: The necessity part is easy to verify. To prove the sufficiency part first we note that the existence of a finite measure equivalent to P and invariant under G implies and is implied by the existence of a finite, finitely additive measure μ invariant under G and equivalent to P (see [7], theorem 3.13). Let (H) hold and let f be the mass function on S_o obtained at the end of proposition 6.4.2 by extending the mass finition f^* on S^*. Define for every $A \in \underline{\underline{A}}$,

$\mu(A) = s(\alpha_A)$. Then μ is finite, finitely additive and invariant under G. By 6.4.3, μ is equivalent to P and the proof of the sufficiency part is complete.

Theorem 6.4.5: There is a finite measure invariant under G and equivalent to P if and only if for every countable subgroup G_o of G there is a finite measure invariant under G_o and equivalent to P.

Proof: The necessity part is trivial. On the other hand it is easy to see that the given condition implies that G is bounded and hence, by theorem 6.4.4, the sufficiency part follows.

Remark 6.4.6: If we can obtain a direct proof of theorem 6.4.5 then the proof of theorem 6.4.4 given here can be simplified considerably.

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