SOME COMBINATORIAL ARRANGEMENTS AND INCOMPLETE BLOCK DESIGNS THROUGH THEM

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C O N T E N T S

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1. GENERAL INTRODUCTION AND SUMMARY

The use and importance, in Statistical Experiments, of Incomplete Block Designs, particularly, Balanced Incomplete Block (BIB) Designs, Doubly Balanced Incomplete Block (DBIB) Designs and Partially Balanced Incomplete Block (PBIB) Designs are well known. Several combinatorial arrangements, including the incidence matrices of these Incomplete Block Designs and association matrices associated with PBIB Designs are known to be of use in Design of Experiments. In this thesis, we consider the construction problems pertaining to some of these combinatorial arrangements and take up the problem of construction of BIB, DBIB and PBIB Designs through them. The combinatorial arrangements studied in the thesis have, of course, other important uses besides their relevance in obtaining some BIB, DBIB and PBIB Designs or proving the non-existence of some of them. But, it is the construction of Incomplete Block Designs that has mostly prompted the author to study the construction of the combinatorial arrangements included in the thesis. A brief summary of the work undertaken in the thesis is provided below.

Hadamard matrices always give rise to certain BIB and DBIB Designs. In Chapter 2, we give methods of construction of some infinite series of Hadamard matrices, based on orthogonal matrices with zero diagonal and ± 1 elsewhere. The results proved are essentially generalisations and extensions of the results given by Williamson (1944, '47), Wallis (1969) and Turyn (1972a).

In Chapter 3, some results are proved regarding general Orthogonal Arrays (OA's) and Orthogonal Arrays of strength two and three. Systematic methods are developed for constructing OA's of strength 2 and 3 for different indices, with comparatively large number of constraints in most of the cases. Methods are also given for obtaining OA : II 's of strength 2 of Rao (1961a), when the number of levels, s is not a prime power. OA's and OA : II's of strength 2 are utilised for the purpose of constructing series of BIB Designs, Group Divisible (GD) Designs and PBIB Designs with three associate classes. Some of the results of chapter 3 about the construction of BIB Designs through OA's and OA : II's of strength 2 have been published (Mukhopadhyay, 1972 a).

A Balanced Orthogonal Design (BOD) is a combinatorial arrangement first considered by Rao (1966). In Chapter 4, we give a method for constructing an infinite series of BOD's, given any odd prime power. The first two BOD's of this series have been known to exist from Paley (1933), Rao and Das (1969) and Rao (1970). Exploiting these BOD's, series of BIB and GD Designs can be constructed by applying the methods of Rao (1970).

A Partially Balanced Array (PBA) is a combinatorial arrangement introduced by Chakravarty (1956) for fractional factorial experiments. In Chapter 5, is considered the construction of some general and two level PBA's. PBA's of strength 2 and satisfying certain properties are utilised for the purpose of constructing PBIB Designs with three associate rectangular association scheme. Some of the results of this chapter giving the construction of PBA's have been published (Mukhopadhyay, 1971).

(r, λ)- and λ- systems are certain categories of Incomplete Block Designs. In Chapter 6, we prove an inequality concerning the number of blocks in such systems and disprove a conjecture of Mullin and Stanton (1966). (r, λ)- systems are utilised to prove the non-existence of some PBIB Designs. The content of this chapter has been published (Mukhopadhyay, 1972b).

Shrikhande and Singh (1962) considered the construction of some series of symmetrical BIB Designs from association matrices of PBIB association schemes with two classes. Blackwelder (1969) considered a more general treatment of the problem and we, in chapter 7, generalise Blackwelder's (1969) method of constructing BIB Designs from the association matrices of PBIB association schemes. The potentiality of the general result proved in the chapter is illustrated by obtaining with its help three BIB Designs, two of which are indicated as unsolved in Sprott's (1962) list and one is missing in Rao's (1961b) list. The BIB Designs constructed from association matrices have got certain properties which are exploited to obtain further series of BIB, DBIB and GD Designs.

In Chapter 8, we define a new property of Incomplete Block Designs, termed 'near resolvability' which is more general than 'resolvability'. Methods are given for obtaining new Incomplete Block Designs with the help of Designs which are 'nearly resolvable'.

In Chapter 9 are given some miscellaneous methods of constructing PBIB Designs. The existence of two series of PBIB Designs with GD and rectangular association schemes is proved. Methods are given for constructing PBIB Designs with the help of certain types of matrices with elements 0 and $\pm$ 1. The paracyclic association scheme of Adhikary (1969b) is generalised to higher associate classes and a method is given for constructing PBIB Designs with higher associate paracyclic association schemes.

2. HADAMARD MATRICES

2.1 Introduction :

Hadamard Matrices, besides being a very important category of Combinatorial Arrangements, are of immense use in Statistical Design of Experiments. The existence of a Hadamard matrix of order 4t, viz., H_{4t} is known to be equivalent to the existence of a Balanced Incomplete Block Design with parameters $v = b = 4t - 1$, $r = k = 2t - 1$, $\lambda = t - 1$ (Hall, 1967). An H_{4t} is also equivalent to an Orthogonal Array with size 4t, constraints 4t-1, strength 2 and levels 2, denoted by OA [4t, 4t-1, 2, 2] and as such is useful as a fractional multifactor experiment (Plackett and Burman, 1946). It is known that if a Hadamard matrix H_n exists, it provides an optimum weighing Design for the problem of weighing n objects in n weighings on a chemical balance. The optimality lies in giving the minimum variance of each of the estimated weights obtained on n weighings. Also, Hadamard matrices or a more general version of it, viz., Balanced Orthogonal Designs have been made use of by Rao (1970) in obtaining solutions to new Incomplete Block Designs from known solutions of some Incomplete Block Designs.

It has been conjectured that H_n exists for all $n \equiv 0 \pmod 4$. The conjecture has been neither proved nor disproved. But, conditions have been provided by various authors under which an H matrix exists and H matrices of various orders have been constructed. Many series of Hadamard matrices are in this manner known. An exhaustive list of available H matrices of orders ≤ 4000 is to be found in Wallis (1972). The main contents of the present chapter consist in providing improvements on some results by Williamson (1944, 1947), Wallis (1969) and Turyn (1972a), which prove the existence of infinitely more series of Hadamard matrices than those known so far.

2.2 Definitions, Notations and Preliminaries :

For any (not necessarily square) matrix A, A' will denote its transpose. E_{mn} will denote an $m \times n$ matrix with all elements + 1, O_{mn} a null matrix of type $m \times n$ and I_n an identity matrix of order n.

If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is a $p \times q$ matrix, the direct product or kronecker product $A \otimes B$ is the $mp \times nq$ matrix given by

$$A \otimes B = \begin{bmatrix} a_{11} B & a_{12} B & \cdots\cdots & a_{1n} B \\ a_{21} B & a_{22} B & \cdots\cdots & a_{2n} B \\ \vdots & \vdots & & \vdots \\ a_{m1} B & a_{m2} B & \cdots\cdots & a_{mn} B \end{bmatrix} \qquad \ldots\ldots (2.2.1)$$

Let A and C be square matrices of order m, and B and D be square matrices of order n. Then, it is easy to see that

$$(A \otimes B)' = A' \otimes B' \qquad \ldots \quad \ldots \quad (2.2.2)$$

$$(A \otimes B)(C \otimes D) = AC \otimes BD \qquad \ldots \quad \ldots \quad (2.2.3)$$

A Hadamard matrix H_n of order n is an $n \times n$ matrix of + 1's and - 1's such that $H_n H_n' = n I_n$. We shall throughout use the notation H_n to denote a Hadamard matrix of order n.

It is known that if H_n exists, then $n = 2$ or necessarily a multiple of 4. If H_n exists, n will be called an H number.

Following Williamson (1944) we have that:

(a) An nth order symmetric matrix having elements ± 1 off the diagonal and zeros in the diagonal is called an S_n matrix if

$$\left.\begin{array}{ll} \text{(i)} & S_n E_{n1} = O_{n1} \\ \text{and (ii)} & S_n S_n' = n . I_n - E_{nn} \end{array}\right\} \qquad \ldots \quad \ldots \quad (2.2.4)$$

It is known that S_n exists for all n, a prime power $\equiv 1 \pmod 4$. It can be easily shown that if S_n exists, then

$$T_{n+1} = \begin{bmatrix} 0 & E_{1n} \\ E_{n1} & S_n \end{bmatrix} \quad \text{is a symmetric matrix}$$

of order n+1 with the property

$$T_{n+1} \; T'_{n+1} = n \, I_{n+1} \qquad \ldots \quad \ldots \quad \ldots \quad (2.2.5)$$

(b) An nth order skew symmetric matrix having elements $\pm$ 1's off the diagonal and zeros in the diagonal is called Σ_n matrix if

$$\left.\begin{array}{l} \text{(i)} \;\; \Sigma_n E_{n1} = \Sigma'_n E_{n1} = 0_{n1} \\ \text{and (ii)} \;\; \Sigma_n \Sigma'_n = \Sigma'_n \Sigma_n = n\, I_n - E_{nn} \end{array}\right\} \qquad \ldots \quad \ldots \quad (2.2.6)$$

If Σ_n exists, $T^*_{n+1} = \begin{bmatrix} 0 & E_{1n} \\ -E_{n1} & \Sigma_n \end{bmatrix}$ is a skew symmetric matrix of order $n+1$ with the property

$$T^*_{n+1} \; T^{*\prime}_{n+1} = T^{*\prime}_{n+1} \; T^*_{n+1} = n \, I_{n+1} \qquad \ldots \quad \ldots \quad \ldots \quad (2.2.7)$$

An H_n is said to be of skew type if

$$H_n = I_n + T^*_n \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.2.8)$$

If H_n is of skew type, it can always be written (upto isomorphism) as

$$H_n = \begin{bmatrix} 1 & E_{1,n-1} \\ -E_{n-1,1} & Q_{n-1} \end{bmatrix}, \qquad \ldots \quad \ldots\ldots \quad (2.2.9)$$

where $Q_{n-1} = I_{n-1} + \Sigma_{n-1}$

and $Q_{n-1} Q'_{n-1} = Q'_{n-1} Q_{n-1}$

$$= n I_{n-1} - E_{n-1, n-1}$$

Obviously, the existence of Σ_{n-1} implies and is implied by the existence of a skew type H_n.

It is known that a skew type H matrix of order n exists for all n of the form $2^t k_1 \ldots\ldots k_s$, where each k_i = some prime power + 1 $\equiv 0 \pmod 4$ (Hall, 1967). If n is of the form $n = 2^t k_1 \ldots\ldots k_s$, where, either $k_i = p_i^{r_i} + 1 \equiv 0 \pmod 4$ or $k_i = 2 (p_i^{u_i} + 1)$, $p_i^{u_i} \equiv 1 \pmod 4$, p_i standing for a prime number, $i = 1, 2, \ldots\ldots s$, there exists a symmetric H matrix of order n (Hall, 1967).

A Balanced Incomplete Block (BIB) Design with v treatments, b blocks, r replications, block size k and λ = the number of blocks in which any pair of treatments occur together will be denoted by BIB (v, b, r, k, λ) or in short BIB (v, k, λ). Moreover, in the BIB Design if any triplet of treatments occur together in a constant number, say μ, of blocks, it is called a Doubly Balanced Incomplete Block (DBIB) Design and the notation used for this will be DBIB $(v, b, r, k, \lambda, \mu)$. The use of DBIB Designs in Statistical experiments was first pointed out by Calvin (1954).

2.3 Improvements on Some Series of H matrices by Williamson (1944, '47) :

The relevant results of Williamson (1944, '47) which are generalised in this section can be stated neatly in the form of theorems as :

Theorem 2.3.1 : The existence of a skew type H_n implies the existence of $H_{(n-1)n}$.

Theorem 2.3.2 : The existence of a skew type H_n and a symmetrical H_{n+4} implies the existence of $H_{n(n+3)}$.

Theorem 2.3.3 : The existence of S_n implies the existence of $H_{n_1 n_2 n(n+1)}$, where n_1 and n_2 are H numbers.

Theorem 2.3.4 : The existence of S_n and S_{n+4} implies the existence of $H_{n_1 n_2 (n+1)(n+4)}$, where n_1 and n_2 are H numbers.

Goethals and Seidel (1967) slightly improved the theorems 2.3.3 and 2.3.4 as

Theorem 2.3.3' : The existence of S_n implies the existence of $H_{n_1 n(n+1)}$, where n_1 is an H-number.

Theorem 2.3.4' : The existence of S_n and S_{n+4} implies the existence of $H_{n_1 (n+1)(n+4)}$, where n_1 is an H-number $>$ 2.

Before proceeding to enunciate and prove the generalised versions of the above theorems, we prove two important lemmas from which the required results can be deduced easily.

Lemma 2.3.1 : Let L_1 and L_2 be two square matrices of order p with elements ± 1 satisfying the conditions

(i) $L_1 L_2' = L_2 L_1'$

and (ii) T_n^* exists and $L_3 = L_2 \otimes I_n + L_1 \otimes T_n^*$ is an H - matrix of order pn.

Then, writing

$$L_4 = L_2 \otimes I_{n-1} + L_1 \otimes \Sigma_{n-1}$$

and $L_5 = L_1 \otimes E_{n-1,\, n-1}$,

$L_6 = L_5 \otimes I_n + L_4 \otimes T_n^*$ is an H matrix of order $p(n-1)n$, and

$$L_4 L_5' = L_5 L_4'$$

Proof : From the given conditions,

$$L_3 L_3' = L_2 L_2' \otimes I_n + L_1 L_1' \otimes (n-1) I_n$$

$$[\because L_1 L_2' = L_2 L_1' \text{ and } T_n^{*\prime} = - T_n^*]$$

$$= p\, n\, I_{pn}$$

$$= p\, n\, I_p \otimes I_n .$$

Hence, $L_2 L_2' + (n-1) L_1 L_1' = p\, n\, I_p$ ... ... (2.3.11)

Since, $L_1 L_2' = L_2 L_1'$, so obviously

$$L_4 L_5' = L_5 L_4' \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.3.12)$$

Now, $L_4 L_4' = L_2 L_2' \otimes I_{n-1} + L_1 L_1' \otimes \Sigma_{n-1} \Sigma_{n-1}'$

$$\left[\because \Sigma_{n-1}' = - \Sigma_{n-1} \right]$$

$$= L_2 L_2' \otimes I_{n-1} + L_1 L_1' \otimes (\overline{n-1}\, I_{n-1} - E_{n-1,n-1})$$

and $L_5 L_5' = L_1 L_1' \otimes (n-1) E_{n-1,\, n-1}$

$$\therefore L_6 L_6 = L_5 L_5' \otimes I_n + L_4 L_4' \otimes (n-1) I_n$$

$$= (n-1) L_1 L_1' \otimes E_{n-1,n-1} \otimes I_n$$

$$+ (n-1) \left\{ L_2 L_2' \otimes I_{n-1} + (n-1) L_1 L_1' \otimes I_{n-1} - L_1 L_1' \otimes E_{n-1,n-1} \right\} \otimes I_n$$

$$= (n-1) \left\{ L_2 L_2' + (n-1) L_1 L_1' \right\} \otimes I_{n-1} \otimes I_n$$

$$= (n-1)\, p\, n\, I_p \otimes I_{n-1} \otimes I_n$$

$$= p\, (n-1) n\, I_p \otimes I_{n-1} \otimes I_n$$

Hence, L_6 is a Hadamard matrix and the lemma is proved.

Lemma 2.3.2 : Let L_1 and L_2 be two square matrices of order p with elements ± 1, satisfying the following conditions :

(i) $L_1 L_2' = - L_2 L_1'$,

and (ii) T_n exists and

$$L_3 = L_2 \otimes I_n + L_1 \otimes T_n$$ is an H matrix of order pn.

Then, writing

$$L_4 = L_2 \otimes I_{n-1} + L_1 \otimes S_{n-1}$$

and $L_5 = L_1 \otimes E_{n-1,n-1}$,

$$L_6 = L_5 \otimes I_n + L_4 \otimes T_n$$ is an H-matrix of order p(n-1)n,

and

$$L_4 L_5' = - L_5 L_4' .$$

Proof : Similar to that of lemma 2.3.1.

The following theorems 2.3.1a, 2.3.2a, 2.3.3'a and 2.3.4'a provide the required improvements on the theorems 2.3.1, 2.3.2, 2.3.3' and 2.3.4'. The two lemmas deduced already are made use of in proving these theorems.

Theorem 2.3.1a : The existence of a skew type H_n implies the existence of a series of H matrices or orders $(n-1)^r n$, r any positive integer.

Proof :

Let $H_n = I_n + T_n^* = \begin{bmatrix} 1 & E_{1,n-1} \\ -E_{n-1,1} & Q_{n-1} \end{bmatrix}$,

where $Q_{n-1} = I_{n-1} + \Sigma_{n-1}$

Then, $E_{n-1,n-1} Q_{n-1}' = Q_{n-1} E_{n-1,n-1} = E_{n-1,n-1}$ ... ... (2.3.13)

It is easy to see that

$L_2 \otimes I_n + L_1 \otimes T_n^*$, where

$L_1 = Q_{n-1}$ and $L_2 = E_{n-1,n-1}$, is an H matrix of order

$(n-1)n$.

And $L_1 L_2' = L_2 L_1'$ from (2.3.13).

Hence, by making use of Lemma 2.3.1 repeatedly, H matrices exist for all orders $(n-1)^r n$, r any positive integer.

Theorem 2.3.2a : The existence of H_n of skew type and H_{n+4} symmetrical implies the existence of a series of H matrices of orders $(n-1)^r n (n+3)$, where $r = 0$ or any positive integer.

Proof :

H_{n+4} after suitably multiplying some rows and columns by -1, if necessary, can always be written as

$$H_{n+4} = \begin{bmatrix} 1 & E_{1,n+3} \\ E_{n+3,1} & D_{n+3} \end{bmatrix},$$

where D_{n+3} is symmetrical,

$$D_{n+3}^2 = \overline{n+4}\; I_{n+3} - E_{n+3,n+3} \quad \ldots \quad \ldots \quad (2.3.14)$$

and $E_{n+3,n+3}\; D_{n+3} = D_{n+3}\; E_{n+3,n+3} = -E_{n+3,n+3}$ ... (2.3.15)

Also, $H_n = I_n + T_n^* = \begin{bmatrix} 1 & E_{1,n-1} \\ E_{n-1,1} & Q_{n-1} \end{bmatrix}$,

where $Q_{n-1} = I_{n-1} + \Sigma_{n-1}$

Let $F_{n+3} = 2\, I_{n+3} - E_{n+3,n+3}$. Then,

clearly

$$F_{n+3}\; D_{n+3} = D_{n+3}\; F_{n+3} \quad \ldots \quad \ldots \quad (2.3.16)$$

[By (2.3.15)]

It is easy to show that

$L_2 \otimes I_n + L_1 \otimes T_n^*$, where

$L_2 = F_{n+3}$ and $L_1 = D_{n+3}$, is an H matrix of order $n(n+3)$.

From (2.3.16),

$$L_1\, L_2' = L_2\, L_1'$$

Hence, by making use of Lemma 2.3.1 repeatedly, the theorem is proved.

Theorem 2.3.3'a : The existence of S_n implies the existence of a series of H matrices of orders $n_1 \; n^r(n+1)$, where n_1 is an H number and $r = 0$ or any positive integer.

Proof :

Let H_{n_1} be the H matrix of order n_1 and $P_{n_1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times I_{n_1/2}$, the generalised permutation matrix defined by Williamson (1944).

Then P_{n_1} is skew symmetric i.e.,

$$P'_{n_1} = -P_{n_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.3.17)$$

$$\text{and} \quad P_{n_1} P'_{n_1} = I_{n_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.3.18)$$

The existence of S_n obviously implies the existence of T_{n+1}.

Let us write

$$L_3 = L_2 \otimes I_{n+1} + L_1 \otimes T_{n+1}$$

$$\text{where} \quad L_2 = H_{n_1} \otimes E_{nn}$$

$$\text{and} \quad L_1 = H_{n_1} \otimes S_n + P_{n_1} H_{n_1} \otimes I_n .$$

By (2.3.17) and (2.3.18),

$$\text{(i)} \quad L_1 L_2' = - L_2 L_1' ,$$

$$\text{(ii)} \quad L_2 L_2' = n_1 I_{n_1} \otimes n E_{nn}$$

$$= n_1 n I_{n_1} \otimes E_{nn} ,$$

$$\text{(iii)} \quad L_1 L_1' = n_1 I_{n_1} \otimes (n I_n - E_{nn})$$

$$+ n_1 I_{n_1} \otimes I_n .$$

So $L_3 L_3' = L_2 L_2' \otimes I_{n+1} + L_1 L_1' \otimes T_{n+1} T_{n+1}'$

$$= n_1 n I_{n_1} \otimes E_{nn} \otimes I_{n+1}$$

$$+ \left\{ n_1 I_{n_1} \otimes (n I_n - E_{nn}) + n_1 I_{n_1} \otimes I_n \right\} \otimes n I_{n+1}$$

$$= n_1 n (n+1) I_{n_1} \otimes I_n \otimes I_{n+1}$$

Thus, L_3 is a Hadamard matrix.

Hence, by making use of Lemma 2.3.2 repeatedly, the theorem is proved.

Theorem 2.3.4'a : The existence of S_n and S_{n+4} implies the existence of a series of H matrices of orders $n_1 n^r (n+1)(n+4)$, where n_1 is an H number > 2 and $r = 0$ or any positive integer.

Proof :

Let H_{n_1} be an H matrix of order $n_1 > 2$. Let K_{n_1} and L_{n_1} be the generalised permutation matrices of Goethals and Seidel (1967). In fact, $K_{n_1} = \bar{K}_4 \times I_{n_1/4}$ and $L_{n_1} = \bar{L}_4 \times I_{n_1/4}$, where $\bar{K}_4$ and $\bar{L}_4$ are the quarternion matrices of order 4 defined by Williamson (1944).

Then, K_{n_1} and L_{n_1} are skew and satisfy

$$K_{n_1} K_{n_1}' = L_{n_1} L_{n_1}' = I_{n_1} \quad \ldots \quad \ldots \quad \ldots (2.3.19)$$

$$\text{and } K_{n_1} L_{n_1}' = - L_{n_1} K_{n_1}' \quad \ldots \quad \ldots \quad \ldots \quad \ldots (2.3.20)$$

Obviously, S_n implies T_{n+1}.

Let $L_3 = L_2 \otimes I_{n+1} + L_1 \otimes T_{n+1}$,

where $L_2 = L_{n_1} H_{n_1} \otimes (2\, I_{n+4} - E_{n+4,\, n+4})$

and $L_1 = H_{n_1} \otimes S_{n+4} + K_{n_1} H_{n_1} \otimes I_{n+4}$.

Then, by (2.3.19) and (2.3.20).

$$\text{(i)} \quad L_1 L_1' = n_1 I_{n_1} \otimes (\overline{n+4}\, I_{n+4} - E_{n+4,n+4})$$

$$+ \, n_1 I_{n_1} \otimes I_{n+4}$$

$$\text{(ii)} \quad L_2 L_2' = n_1 I_{n_1} \otimes (4\, I_{n+4} + n\, E_{n+4,\, n+4})$$

(iii) $L_1 L_2' = - L_2 L_1'$

So, $L_3 L_3' = n_1 I_{n_1} \otimes (4 I_{n+4} + n E_{n+4,\ n+4}) \otimes I_{n+1}$

$+ n_1 I_{n_1} \otimes (\overline{n+5}\ I_{n+4} - E_{n+4,n+4}) \otimes n I_{n+1}$

$= n_1 (n+1)(n+4) I_{n_1} \otimes I_{n+4} \otimes I_{n+1}$.

Thus, L_3 is a Hadamard matrix.

Hence making use of Lemma 2.3.2 repeatedly, the theorem is proved.

One more theorem is proved in this section on similar lines. A result given in Hall (1967) is required in the proof of the theorem and hence is stated below in the form of a lemma.

Lemma 2.3.3 : If there exists an H matrix A of order n_1, there exist two more H-matrices B and C of order n_1 such that $AB' = - BA'$, $AC' = CA'$ and $BC' = CB'$.

Theorem 2.3.5a : The existence of Σ_n and Σ_{n+4} implies the existence of a series of H matrices of orders $n_1 n^r (n+1)(n+4)$, where n_1 is an H number and $r = 0$ or any positive integer.

Proof :

As n_1 is an H number, there exist three H matrices A, B and C of order n_1 with the property $AB' = -BA'$, $AC' = CA'$ and $BC' = CB'$ (by Lemma 2.3.3).

Let $L_1 = C \otimes I_{n+4} + B \otimes \Sigma_{n+4}$

and $L_2 = A \otimes (2\, I_{n+4} - E_{n+4,\, n+4})$.

Then, $L_1 L_1' = n_1 I_{n_1} \otimes (I_{n+4} + \overline{n+4}\; I_{n+4} - E_{n+4,n+4})$.

$$= n_1 I_{n_1} \otimes (\overline{n+5}\; I_{n+4} - E_{n+4,\, n+4}),$$

$$L_2 L_2' = n_1 I_{n_1} \otimes (4\, I_{n+4} + n\, E_{n+4,n+4}) ,$$

and $L_1 L_2' = L_2 L_1'$.

Writing $L_3 = L_2 \otimes I_{n+1} + L_1 \otimes T^*_{n+1}$, we have

$$L_3 L_3' = L_2 L_2' \otimes I_{n+1} + L_1 L_1' \otimes n\, I_{n+1}$$

$$= n_1 I_{n_1} \otimes (4\, I_{n+4} + n\, E_{n+4,n+4}) \otimes I_{n+1}$$

$$+ n_1 I_{n_1} \otimes (\overline{n+5}\; I_{n+4} - E_{n+4,n+4}) \otimes n\, I_{n+1}$$

$$= n_1 (n+1)(n+4)\; I_{n_1} \otimes I_{n+4} \otimes I_{n+1} .$$

Thus L_3 is an H matrix of order $n_1(n+1)(n+4)$ and $L_1 L_2' = L_2 L_1'$.

Hence, by making use of Lemma 2.3.2 repeatedly, the theorem is proved.

2.4 Improvements on a Series by Wallis (1969) :

Wallis (1969) proved that if S_{n+1} and a skew type H_n exist, there always exists an H matrix of order $2n(n+1)$. The purpose of the present section is to improve upon the result of Wallis (1969) and provide a proof for the existence of two distinct types of infinite series of H matrices (The results are stated in the form of two theorems and proved).

Theorem 2.4.1 : The existence of S_{n+1} and a skew type H_n implies the existence of a series of H matrices of orders $2(n-1)^r\, n\, (n+1)$, where $r = 0$ or any positive integer.

Proof :

$$\text{Let } H_n = I_n + T_n^* = \begin{bmatrix} 1 & E_{1,n-1} \\ -E_{n-1,1} & Q_{n-1} \end{bmatrix},$$

$$\text{where } Q_{n-1} = I_{n-1} + \Sigma_{n-1}.$$

Let us first show that the matrix P defined as :

$$P = \begin{bmatrix} A & : & B \\ C & : & D \end{bmatrix}, \qquad \ldots \quad \ldots \quad (2.4.1)$$

where

$$A = E_{n+1,n+1} \otimes I_n + (I_{n+1} + S_{n+1}) \otimes T_n^*$$

$$B = (E_{n+1,n+1} - 2\, I_{n+1}) \otimes I_n + (I_{n+1} - S_{n+1}) \otimes T_n^*$$

$$C = -(E_{n+1,n+1} - 2\, I_{n+1}) \otimes I_n + (I_{n+1} - S_{n+1}) \otimes T_n^*$$

and $D = E_{n+1,n+1} \otimes I_n - (I_{n+1} + S_{n+1}) \otimes T_n^*$, is an H matrix of order $2n\,(n+1)$.

We see that

$$AA' = DD' = (n+1)\, E_{n+1,n+1} \otimes I_n + (I_{n+1} + S_{n+1})^2 \otimes (n-1)\, I_n \quad \ldots\ \ldots \quad (2.4.2)$$

$$BB' = CC' = \left\{ 4\, I_{n+1} + (n-3)\, E_{n+1,n+1} \right\} \otimes I_n + (I_{n+1} - S_{n+1})^2 \otimes (n-1)\, I_n \quad \ldots \quad (2.4.3)$$

$$BD' = -AC' = (n-1)\, E_{n+1,n+1} \otimes I_n + 2\,(E_{n+1,n+1} - I_{n+1} - S_{n+1}) \otimes T_n^* - (I_{n+1} - S_{n+1}^2) \otimes T_n^*\, T_n^{*\prime} \quad \ldots\ \ldots \quad (2.4.4)$$

So, $AA' + BB' = CC' + DD'$

$$= 4\, I_{n+1} \otimes I_n + 2\,(n-1)\, E_{n+1,n+1} \otimes I_n + 2\,(I_{n+1} + S_{n+1}^2) \otimes (n-1)\, I_n$$

[by (2.4.2) & (2.4.3)]

$$= 4\, I_{n+1} \otimes I_n + 2\,(n-1)\, E_{n+1,n+1} \otimes I_n + 2(n-1)(\overline{n+2}\; I_{n+1} - E_{n+1,n+1}) \otimes I_n$$

$$= 2\, n\,(n+1)\; I_{n+1} \otimes I_n .$$

Also $AC' + BD' = 0$, a null matrix by (2.4.4).

Hence, P is an H matrix of order $2n(n+1)$.

Let us next prove that if

$$X = \begin{bmatrix} L_{12} \otimes I_n + L_{11} \otimes T_n^* & : & L_{22} \otimes I_n + L_{21} \otimes T_n^* \\ -L_{32} \otimes I_n + L_{31} \otimes T_n^* & : & L_{42} \otimes I_n - L_{41} \otimes T_n^* \end{bmatrix} \quad \ldots (2.4.5)$$

is an H matrix of order 2pn where L_{ij}'s are all $p \times p$ matrices with elements ± 1 with the properties

(i) $L_{i1} L_{i2}' = L_{i2} L_{i1}'$, $i = 1, 2, 3, 4.$

(ii) $L_{1j} L_{3j}' = L_{2j} L_{4j}'$, $j = 1, 2.$

and $L_{11} L_{32}' + L_{12} L_{31}' = L_{21} L_{42}' + L_{22} L_{41}'$,

then,

$$Y = \begin{bmatrix} L_{14} \otimes I_n + L_{13} \otimes T_n^* & : & L_{24} \otimes I_n + L_{23} \otimes T_n^* \\ -L_{34} \otimes I_n + L_{33} \otimes T_n^* & : & L_{44} \otimes I_n - L_{43} \otimes T_n^* \end{bmatrix} , \ldots (2.4.6)$$

where $L_{i3} = L_{i2} \otimes I_{n-1} + L_{i1} \otimes \Sigma_{n-1}$, $L_{i4} = L_{i1} \otimes E_{n-1,n-1}$,

for $i = 1, 2.$

and $L_{i3} = L_{i2} \otimes I_{n-1} - L_{i1} \otimes \Sigma_{n-1}$, $L_{i4} = L_{i1} \otimes E_{n-1,n-1}$,

for $i = 3, 4.$

is an H matrix of order $2p(n-1)n$ and

$$\text{(i)} \quad L_{i3} L_{i4}' = L_{i4} L_{i3}', \quad i = 1, 2, 3, 4.$$

and $$\text{(ii)} \quad L_{1j} L_{3j}' = L_{2j} L_{4j}', \quad j = 3, 4.$$

and $$L_{13} L_{34}' + L_{14} L_{33}' = L_{23} L_{44}' + L_{24} L_{43}'.$$

This is so because

(a) $L_{i1} L_{i2}' = L_{i2} L_{i1}'$ implies

$$(L_{i2} \otimes I_n \pm L_{i1} \otimes T_n^*)(L_{i2} \otimes I_n \pm L_{i1} \otimes T_n^*)'$$

$$= (L_{i2} L_{i2}' + (n-1) L_{i1} L_{i1}') \otimes I_n$$

and hence we obtain

$$(L_{i4} \otimes I_n \pm L_{i3} \otimes T_n^*)(L_{i4} \otimes I_n \pm L_{i3} \otimes T_n^*)'$$

$$= (L_{i2} L_{i2}' + (n-1) L_{i1} L_{i1}') \otimes I_{n-1} \otimes I_n.$$

and $L_{i3} L_{i4}' = L_{i4} L_{i3}'$, for $i = 1, 2, 3, 4$.

[all these results are obtained easily from a close scrutiny of the proof of Lemma 2.3.1]

(b) X is an H matrix and so

$$(L_{12} L_{12}' + (n-1) L_{11} L_{11}') + (L_{22} L_{22}' + (n-1) L_{21} L_{21}')$$

$$= (L_{32} L_{32}' + (n-1) L_{31} L_{31}') + (L_{42} L_{42}' + (n-1) L_{41} L_{41}')$$

$$= 2pn I_p.$$

(c) $L_{13} L_{34}' + L_{14} L_{33}'$

$$= (L_{12} \otimes I_{n-1} + L_{11} \otimes \Sigma_{n-1}) (L_{31} \otimes E_{n-1,n-1})'$$

$$+ (L_{11} \otimes E_{n-1,n-1}) (L_{32} \otimes I_{n-1} - L_{31} \otimes \Sigma_{n-1})'$$

$$= L_{12} L_{31}' \otimes E_{n-1,n-1} + L_{11} L_{32}' \otimes E_{n-1,n-1}$$

$$= (L_{12} L_{31}' + L_{11} L_{32}') \otimes E_{n-1,n-1}$$

Similarly,

$$L_{23} L_{44}' + L_{24} L_{43}'$$

$$= (L_{22} \otimes I_{n-1} + L_{21} \otimes \Sigma_{n-1})(L_{41} \otimes E_{n-1,n-1})'$$

$$+ (L_{21} \otimes E_{n-1,n-1})(L_{42} \otimes I_{n-1} - L_{41} \otimes \Sigma_{n-1})'$$

$$= (L_{22} L_{41}' + L_{21} L_{42}') \otimes E_{n-1,n-1}$$

and thus, $L_{12} L_{31}' + L_{11} L_{32}' = L_{22} L_{41}' + L_{21} L_{42}'$ implies

$L_{13} L_{34}' + L_{14} L_{33}' = L_{23} L_{44}' + L_{24} L_{43}'$.

(d) $L_{14} L_{34}' = L_{12} L_{32}' \otimes (n-1) E_{n-1,n-1}$

$L_{24} L_{44}' = L_{22} L_{42}' \otimes (n-1) E_{n-1,n-1}$

and so $L_{12} L_{32}' = L_{22} L_{42}'$ implies.

$$L_{14} L_{34}' = L_{24} L_{44}'$$

Now
$$L_{13} L_{33}' = (L_{12} \otimes I_{n-1} + L_{11} \otimes \Sigma_{n-1})(L_{32} \otimes I_{n-1} - L_{31} \otimes \Sigma_{n-1})'$$

$$= L_{12} L_{32}' \otimes I_{n-1} - L_{11} L_{31}' \otimes \Sigma_{n-1} \Sigma_{n-1}'$$

$$+ (L_{11} L_{32}' + L_{12} L_{31}') \otimes \Sigma_{n-1}$$

Similarly,

$$L_{23} L_{43}' = L_{22} L_{42}' \otimes I_{n-1} - L_{21} L_{41}' \otimes \Sigma_{n-1} \Sigma_{n-1}'$$

$$+ (L_{21} L_{42}' + L_{22} L_{41}') \otimes \Sigma_{n-1}$$

and so, $L_{11} L_{32}' = L_{22} L_{42}'$, $L_{11} L_{32}' = L_{21} L_{41}'$

and $L_{11} L_{32}' + L_{12} L_{31}' = L_{21} L_{41}' + L_{22} L_{41}'$ together imply

$$L_{13} L_{33}' = L_{23} L_{43}'$$

Statements (a) through (d) cited above prove that, if X is an H matrix, Y too is an H matrix with respective properties stated.

Now, the matrix (2.4.1) is already of the form (2.4.5) and satisfies all the properties stated for (2.4.5). Hence, by mathematical induction, the theorem follows.

Theorem 2.4.2 : The existence of symmetrical H_{n+2} and T_n (i.e. S_{n-1}) implies the existence of a series of H matrices of orders $2\, n_1\, (n-1)^r\, n\, (n+1)$, where n_1 is an H-number and $r = 0$ or any positive integer.

Proof : By Lemma 2.3.3, if n_1 is an H-number there exist two H matrices A and B of order n_1 such that $AB' = -BA'$.

H_{n+2} after multiplying some rows and columns by -1, if necessary, can always be written as

$$H_{n+2} = \begin{bmatrix} 1 & E_{1,n+1} \\ E_{n+1,1} & D_{n+1} \end{bmatrix}, \quad \text{where } D_{n+1} \text{ is}$$

symmetrical and $D^2_{n+1} = \overline{n+2}\; I_{n+1} - E_{n+1,\, n+1}$.

As in theorem 2.4.1, we can show easily that

$$P = \begin{bmatrix} A \otimes E_{n+1,n+1} \otimes I_n + B \otimes D_{n+1} \otimes T_n & : & A \otimes (E_{n+1,n+1} - 2\, I_{n+1}) \otimes I_n + B \otimes D_{n+1} \otimes T_n \\ -A \otimes (E_{n+1,n+1} - 2\, I_{n+1}) \otimes I_n + B \otimes D_{n+1} \otimes T_n & : & A \otimes E_{n+1,n+1} \otimes I_n - B \otimes D_{n+1} \otimes T_n \end{bmatrix} \qquad \ldots \ldots (2.4.7)$$

is an H matrix of order $2\, n_1\, n\, (n+1)$. Also, it can be shown as in theorem 2.4.1 that

if
$$X = \begin{bmatrix} L_{12} \otimes I_n + L_{11} \otimes T_n & : & L_{22} \otimes I_n + L_{21} \otimes T_n \\ -L_{32} \otimes I_n + L_{31} \otimes T_n & : & L_{42} \otimes I_n - L_{41} \otimes T_n \end{bmatrix} \quad \ldots (2.4.8)$$

where L_{ij}'s are all $p \times p$ matrices with elements ± 1, is an H matrix of order $2pn$ with the properties.

$$\text{(i)} \quad L_{i1} L_{i2}' = - L_{i2} L_{i1}' \quad , \quad i = 1, 2, 3, 4$$

$$\text{and (ii)} \quad L_{1j} L_{3j}' = L_{2j} L_{4j}' \quad , \quad j = 1, 2$$

$$\text{and} \quad L_{12} L_{31}' - L_{11} L_{32}' = L_{22} L_{41}' - L_{21} L_{42}'$$

then,

$$Y = \begin{bmatrix} L_{14} \otimes I_n + L_{13} \otimes T_n & : & L_{24} \otimes I_n + L_{23} \otimes T_n \\ -L_{34} \otimes I_n + L_{33} \otimes T_n & : & L_{44} \otimes I_n - L_{43} \otimes T_n \end{bmatrix} \quad \ldots (2.4.9)$$

where $L_{i3} = L_{i2} \otimes I_{n-1} + L_{i1} \otimes S_{n-1}$, $L_{i4} = L_{i1} \otimes E_{n-1,n-1}$

for $i = 1, 2$

$L_{i3} = L_{i2} \otimes I_{n-1} - L_{i1} \otimes S_{n-1}$, $L_{i4} = L_{i1} \otimes E_{n-1,n-1}$

for $i = 3, 4$,

is an H matrix of order $2p(n-1)n$ with the properties

(i) $L_{i3} L_{i4}^{'} = L_{i4} L_{i3}^{'}$, $i = 1, 2, 3, 4.$

and (ii) $L_{1j} L_{3j}^{'} = L_{2j} L_{4j}^{'}$, $j = 3, 4$

and $L_{14} L_{33}^{'} - L_{13} L_{34}^{'} = L_{24} L_{43}^{'} - L_{23} L_{44}^{'}$

The matrix (2.4.7) is obviously of the form (2.4.8).

Hence, by mathematical induction, the theorem follows.

2.5 A series of H matrices based on Turyn's result (1972b) :

Williamson (1947) showed that if A, B, C and D are symmetric matrices, each of same order, say t, with elements ± 1 and if they are pairwise commutative, the matrix H defined as

$$H = \begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix} \quad \ldots \quad \ldots \quad (2.5.1)$$

is a Hadamard matrix of order $4t$, provided

$$A^2 + B^2 + C^2 + D^2 = 4t\ I_t \quad \ldots \quad \ldots \quad \ldots \quad (2.5.2)$$

The matrix of the type (2.5.1) has been used to generate some Hadamard matrices. Any H_{4t} which can be written in the form (2.5.1) with A, B, C and D pairwise commutative, symmetric and satisfying the condition (2.5.2), is said to be of Williamson type. Baumert and Hall (1965) showed that a H_{4t} of Williamson type implies the existence of H_{12t}. Welch (1970) showed that it also implies the existence of H_{20t}. Later, Turyn (1972b) proved that an H_{4t} of Williamson type implies the existence of H matrices of orders 4bt and 20bt, where $b \in \{ i : i$ is an odd integer, $i \leq 23$ or $i = 29$ or $i = 1 + 2^x 10^y 26^z$, where x, y and z are non-negative integers $\}$.

Turyn (1972a) gave the first infinite series of H matrices of these types. He showed that if q is a prime power $\equiv 1 \pmod 4$, there always exists an H matrix of Williamson type of order 2(q+1) and by Baumert and Hall (1965), also an H matrix of order 6(q+1). In the present section, we give methods for constructing a large number of such infinite series of H matrices, based on the result of Turyn (1972b) and what has been proved in section 2.3. The series given in this section provide essentially an extension of the series of H matrices given by Turyn (1972a).

Now, it is easily seen that if any H_{4t} has a $t \times 4t$ submatrix which can be partitioned as $[A : B : C : D]$, where A, B, C and D are symmetric and pairwise commutative, there exists an H_{4t} of Williamson type.

Paley (1933) first gave the method of construction of S_n and Σ_n (i.e. T_{n+1} and T^*_{n+1}) matrices when n is an odd prime power. Goethals and Seidel (1967) proved the following result regarding T_{n+1} obtained by following Paley's construction procedure. The result is stated below in the form of a theorem.

Theorem 2.5.1 : If $n+1 \equiv 2 \pmod 4$, n a prime power, then there exists a T_{n+1} of the form $\begin{bmatrix} P & Q \\ Q & -P \end{bmatrix}$ with square symmetric circulant submatrices P and Q.

In the light of Turyn's (1972b) result and theorem 2.5.1, let us scrutinise the construction procedure suggested in theorem 2.3.3'a.

In theorem 2.3.3'a, the final H matrix of order $n_1 n^r (n+1)$ can always be written in the form

$$H_{n_1} \otimes X_1 \otimes I_{n+1} + P_{n_1} H_{n_1} \otimes X_2 \otimes I_{n+1} + H_{n_1} \otimes X_3 \otimes T_{n+1} + P_{n_1} H_{n_1} \otimes X_4 \otimes T_{n+1} ,$$

where X_i's are symmetric matrices with elements ± 1 and 0 , and each X_i can be written as a linear combination of r matrices each of order $n \times n$. We can write in fact

$$X_i = \sum_{j=1}^{n_i} \epsilon_{ij} \; Y_{ij1} \otimes Y_{ij2} \otimes \cdots\cdots \otimes Y_{ijr} , \quad i = 1,2,3,4,$$

for some number n_i, $\epsilon_{ij} = +1$ or -1 for $j = 1,2,\ldots\ldots n_i$

and $i = 1,2,3,4.$

Any particular Y_{ijk} is one of the four matrices S_n, E_{nn}, I_n and O_{nn} for $i = 1,2,3,4$; $j = 1,2, \ldots\ldots n_i$; and $k = 1,2,\ldots\ldots r.$

Now, S_n, E_{nn}, I_n and O_{nn} are all symmetric and pairwise commutative. Hence, X_1, X_2, X_3 and X_4 too are symmetric and pairwise commutative.

Let $n_1 = 2$ and by theorem 2.5.1, let us choose $T_{n+1} = \begin{bmatrix} P & : & Q \\ Q & : & -P \end{bmatrix}$,

where P and Q are symmetric (and obviously commutative because of the orthogonality property of T_{n+1}). Then, the first $n^r(n+1)/2$ rows of the $H_{2n^r(n+1)}$ can be written as $[A : B : C : D]$, where A, B, C and D can respectively be written as

$$A = \epsilon_1 X_1 \otimes I_{(n+1)/2} + \epsilon_2 X_2 \otimes I_{(n+1)/2}$$

$$+ \epsilon_1 X_3 \otimes P + \epsilon_2 X_4 \otimes P$$

$$B = \epsilon_1 X_1 \otimes O_{\frac{n+1}{2}, \frac{n+1}{2}} + \epsilon_2 X_2 \otimes O_{\frac{n+1}{2}, \frac{n+1}{2}}$$

$$+ \epsilon_1 X_3 \otimes Q + \epsilon_2 X_4 \otimes Q .$$

$$C = \epsilon_3 \; X_1 \otimes I_{(n+1)/2} + \epsilon_4 \; X_2 \otimes I_{(n+1)/2}$$

$$+ \epsilon_3 \; X_3 \otimes P + \epsilon_4 \; X_4 \otimes P$$

$$D = \epsilon_3 \; X_1 \otimes 0_{\frac{n+1}{2}, \frac{n+1}{2}} + \epsilon_4 \; X_2 \otimes 0_{\frac{n+1}{2}, \frac{n+1}{2}}$$

$$+ \epsilon_3 \; X_3 \otimes Q + \epsilon_4 \; X_4 \otimes Q, \quad \text{where } \epsilon_i = +1 \text{ or } -1,$$

$$i = 1,2,3,4.$$

Evidently, $I_{(n+1)/2}$, $0_{\frac{n+1}{2}, \frac{n+1}{2}}$, P and Q are pairwise commutative and symmetric. Thus A, B, C and D are pairwise commutative and symmetric. Hence, there exists an H matrix of Williamson type of order $2n^r(n+1)$ for all $r = 0$ or any positive integer, when n is a prime power $\equiv 1 \pmod 4$. Then, by Turyn (1972b), we can prove the following theorem.

Theorem 2.5.2 : If n is a prime power $\equiv 1 \pmod 4$, there always exists a series of H matrices of orders $2n_1 \, n^r \, (n+1)$ and $10 n_1 n^r (n+1)$, for $r = 0$ or any positive integer and $n_1 \in \{ i : i$ is an odd integer, $i \leq 23$ or $i = 29$ or $i = 1 + 2^x \, 10^y \, 26^z$, where x, y and z are non-negative integers $\}$.

We note the existence of the following Hadamard matrices, indicated as unsolved in the list provided by Wallis (1972).

(i) H_{404}, H_{808}, H_{1616}, H_{3232} and H_{2322} by Turyn (1972b), the last one because there exists an H_{44} of Williamson type.

and (ii) H_{1212}, H_{2756}, H_{2828} and H_{3636} by theorem 2.5.2

2.6 Incomplete Block Designs Through Hadamard Matrices :

It has already been pointed out in the introduction to this chapter that the existence of H_{4t} is equivalent to the existence of a BIB Design with parameters $v = b = 4t-1$, $r = k = 2t-1$, $\lambda = t-1$. By block intersection (Bose, 1939) from the BIB $(4t-1, 2t-1, t-1)$ we obtain BIB Design with parameters

(i) $v = 2k = 2t$, $b = 2r = 2(2t-1)$, $\lambda = t-1$

and (ii) $b = 2v = 2(2t-1)$, $r = 2k = 2(t-1)$, $\lambda = t-2$.

We give below two lemmas which are useful in obtaining series of DBIB Designs from known H matrices.

Lemma 2.6.1 : If N is the incidence matrix of a BIB Design with parameters $v = b = 4t-1$, $r = k = 2t-1$, $\lambda = t-1$ and $\bar{N} = E_{vv} - N$ is the incidence matrix of its complement, then

$$M_1 = \begin{bmatrix} N & : & \bar{N} \\ E_{1v} & : & 0_{1v} \end{bmatrix}$$ is the incidence matrix of a DBIB Design

with parameters $v = 4t$, $b = 8t-2$, $r = 4t-1$, $k = 2t$, $\lambda = 2t-1$, $\mu = t-1$.

Proof : That M_1 is a BIB Design has been proved by Bhat and Shrikhande (1970). That it is also a DBIB can be seen easily.

Lemma 2.6.2 : If N is the incidence matrix of a BIB Design with parameters $v = 2k = 2t$, $b = 2r = 2(2t-1)$, $\lambda = t-1$, then $M_2 = [N : \bar{N}]$ is the incidence matrix of a DBIB Design with parameters $v = 2t$, $b = 4(2t-1)$, $r = 2(2t-1)$, $k = t$, $\lambda = 2(t-1)$, $\mu = t-2$.

The proof is easy and so is omitted.

Hence, all the results on the existence of series of H matrices proved in section 2.3 to 2.5, can be recast in the following form : The existence of the series of

(i) BIB (4t-1, 4t-1, 2t-1, 2t-1, t-1)

(ii) BIB (2t, 2(2t-1), 2t-1, t, t-1)

(iii) BIB (2t-1, 2(2t-1), 2(t-1), t-1, t-2)

(iv) DBIB (4t, 8t-2, 4t-1, 2t, 2t-1, t-1)

and (v) DBIB (2t, 4(2t-1), 2(2t-1), t, 2(t-1), t-2) is always implies by the existence of

(i) H_n of skew type, for $t = n_1 (n-1)^r n$, where $r = 0$ or any positive integer and $n_1 = 1$ or any H-number.

(ii) H_n of skew type and H_{n+4} symmetrical, for $t = n_1(n-1)^r n(n+3)$, where $r = 0$ or any positive integer and $n_1 = 1$ or any H number.

(iii) S_n, for $t = n_2 n_1 n^r (n+1)$, where $r = 0$ or any positive integer, n_1 = any H number and $n_2 = 1$ or any H number.

(iv) S_n and S_{n+4}, for $t = n_2 n_1 n^r (n+1)(n+4)$, where $r = 0$ or any positive integer, n_1 = any H number > 2 and $n_2 = 1$ or any H number.

(v) S_{n+1} and H_n of skew type, for $t = 2n_1(n-1)^r n(n+1)$, where $r = 0$ or any positive integer, $n_1 = 1$ or any H number.

(vi) S_{n-1} and symmetrical H_{n+2}, for $t = 2n_2 n_1 (n-1)^r n(n+1)$, where $r = 0$ or any positive integer, n_1 any H number and $n_2 = 1$ or any H number.

(vii) Σ_n and Σ_{n+4}, for $t = n_2 n_1 n^r (n+1)(n+4)$, where $r = 0$ or any positive integer, n_1 any H number and $n_2 = 1$ or any H number.

(viii) n, a prime power $\equiv 1 \pmod 4$ for $t = 2n_2 n_1 n^r (n+1)$ and $10n_2 n_1 n^r (n+1)$, where $r = 0$ or any positive integer, $n_1 \in \{ i : i$ is an odd integer, $1 \leq i \leq 23$ or $i = 29$ or $i = 1+2^x 10^y 26^z$, where x,y and z are non-negative integers$\}$ and $n_2 = 1$ or any H number.

3. ORTHOGONAL ARRAYS AND COMBINATORIAL ARRANGEMENTS ANALOGOUS TO ORTHOGONAL ARRAYS

3.1 Introduction

Orthogonal Arrays (in short OA's) were first introduced by Rao (1946). The use of Orthogonal Arrays as fractional factorial experiments is too well known to be repeated. Efforts have been made by several authors, including Rao (1946), Bose and Bush (1952), Bush (1952a,b), Bose (1947), Seiden (1954), Seiden and Zemack(1966), Addelman and Kempthorne (1961), Shrikhande (1964), Bose and Srivastava (1964), Gulati and Kounias (1970), and Gulati (1971) - to name **only** some - to construct OA's of various strengths and indices and to provide suitable upper and lower bounds to the constraints for an array with a given index, level and strength. Bose (1960) first pointed out the interrelationship between Orthogonal Arrays and error correcting codes. Importance of the study of properties and construction of OA's has been considerably enhanced because of their application in error correcting and error detecting codes.

Rao (1961a)introduced the combinatorial arrangements analogous to OA, particularly OA : II.

In the present chapter, in section 3.3, we prove some results regarding general OA's. The results are generalisations of Bush's results (1952) which were proved when the number of levels is a prime power. In the same section, we also give methods of constructing OA's of strength 2 and 3. The methods are simple and by these methods OA's with even very large indices can be constructed easily and systematically. Although the number of constraints obtained in these methods do not in some cases coincide with any of the existing upper bounds, yet the number is quite considerable and in many cases appreciably more than those given by the OA's known so far. The method of construction given here has certain additional properties which will be exploited in chapter 4 in the construction of Balanced Orthogonal Designs. In section 3.4, we consider the construction problems of OA : II's. In sections 3.5 and 3.6 is considered the construction of Incomplete Block Designs through OA's and OA : II's.

Some definitions and notations which will be required in proving the results of this chapter are described in section 3.2.

3.2 Definitions and Notations :

The definition of an OA as given by Bose and Bush (1952) is :

A $r \times N$ matrix A with entries from a set Σ of $s \geq 2$ elements, is called an Orthogonal Array of strength t, r constraints size N and s levels if each possible $t \times N$ submatrix of A contains each possible $t \times 1$ column vector, called t-tuple, with the same frequency λ. The array may be denoted by OA $[N, r, s, t]$. The number λ is called the index of the array. Clearly $N = \lambda s^t$. We shall also use the notation OA (r, t) to denote an OA of strength t with r constraints, where from the context N and s are known. The maximum number of constraints of an OA of size λs^t, levels s and strength t will be denoted by $f(\lambda s^t, s, t)$.

Extending the definition of completely resolvable array of strength 2 in Bose and Bush (1952), we define an OA $[\lambda s^t, r, s, t]$ to be completely resolvable, if it is the juxtaposition of s different arrays $[\lambda s^{t-1}, r, s, t-1]$, $t \geq 2$.

We shall also call an OA $[\lambda s^t, r, s, t]$, $t \geq 2$ completely decomposable, if it is the juxtaposition of λs^{t-1} different arrays $[s, r, s, 1]$.

OA : II defined by Rao (1961a) is as follows :

An $r \times N$ array with entries from a set Σ of s elements is defined to be an Orthogonal Array of type II, strength t, constraints r and index λ and represented by (N, r, s, t) : II, if in every set of t rows, the N columns contain each of the $\binom{s}{t}$ combinations of s elements taken t at a time, with order ignored, λ times, $s > t \geq 2$.

Let $\mathcal{M}$ be a finite module of s elements, viz., the null element, e_0 and other elements $e_1, e_2, \ldots\ldots e_{s-1}$. For $t \geq 2$, let us consider the s^t distinct t-tuples formed by the elements of $\mathcal{M}$. They can be divided into s^{t-1} sets, viz., $M_1, M_2, \ldots\ldots,$ $M_{s^{t-1}}$ each consisting of s distinct t-tuples such that given any t-tuple in a set, say M_i, all the s t-tuples in the set can be obtained by adding successively the elements $e_0, e_1, \ldots\ldots, e_{s-1}$ of $\mathcal{M}$ to each element of the given t-tuple. Suppose that it is possible to find a scheme B of r rows and $n = \mu s^{t-1}$ columns with elements belonging to $\mathcal{M}$ such that in every t-rowed submatrix of B, the number of t-tuples belonging to each M_i is the same and equals μ. Such an array B will be denoted by $S_t [\mu s^{t-1}, r, s]$, $t \geq 2$. Moreover, if the array B is also orthogonal of strength t-1, we will denote it as $S_t^* [\mu s^{t-1}, r, s]$. The shorter notations $S_t(r)$ and $S_t^*(r)$ will also be used where there is no scope for confusion. The arrays of the type $S_t(r)$ and $S_t^*(r)$ have been considered by Seiden (1954) and Seiden and Zemach (1966).

Two matrix operations which will be frequently used in the construction of OA's are described below :

$$\text{Let } A = \begin{bmatrix} a_{11} & \ldots\ldots & a_{1n_1} \\ \vdots & & \vdots \\ a_{r_1 1} & \ldots\ldots & a_{r_1 n_1} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & \ldots\ldots & b_{1n_2} \\ \vdots & & \vdots \\ b_{r_2 1} & \ldots\ldots & b_{r_2 n_2} \end{bmatrix}$$

be two matrices with elements in the finite module, $\mathcal{M}$.

(i) Let $r_1 = r_2 = r$. Then we define $A \oplus B$ as a $r \times n_1 n_2$ matrix where for any column of A, say $\alpha_i = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ri} \end{pmatrix}$ and any column of B, say $\beta_j = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{pmatrix}$, we define a column

$$\alpha_i + \beta_j = \begin{pmatrix} a_{1i} + b_{1j} \\ a_{2i} + b_{2j} \\ \vdots \\ a_{ri} + b_{rj} \end{pmatrix}, \qquad \begin{array}{l} i = 1, 2, \ldots\ldots n_1 , \\ j = 1, 2, \ldots\ldots n_2 , \end{array}$$

in $A \oplus B$.

(ii) The kronecker sum, $A \circledcirc B$ denotes a $r_1 r_2 \times n_1 n_2$ matrix of the form

$$\begin{bmatrix} A(11) & A(12) & \ldots\ldots\ldots\ldots & A(1n_2) \\ \vdots & & & \vdots \\ A(r_2 1) & A(r_2 2) & \ldots\ldots\ldots & A(r_2 n_2) \end{bmatrix},$$

where $A(ij)$ is obtained by adding the element b_{ij} of B to every element of A, $i = 1, 2, \ldots. r_2$; $j = 1, 2, \ldots.. n_2$.

As in chapter 2, BIB (v, b, r, k, λ) or in short BIB(v,k, λ) will denote a BIB Design with v treatments, b blocks, r replications, block size k and λ = No. of blocks in which any pair of treatments occur together.

Bose and Nair (1939) introduced Partially Balanced Incomplete Block (PBIB) Designs and the related association schemes. The necessary algebra of association matrices has been considered extensively by Bose and Mesner (1959).

Association Scheme : Given v treatments 1,2, ... v, a relation satisfying the following conditions is said to be an association scheme with m classes.

(i) Any two treatments are either 1st, 2nd, ... or mth associates, the relation of association being symmetrical; i.e., if the treatment θ is the ith associate of the treatment φ, then φ is the ith associate of θ.

(ii) Each treatment θ has n_i ith associates, the number n_i being independent of θ.

(iii) If any two treatments θ and φ are ith associates of each other, then the number of treatments which are jth associates of θ and kth associates of φ is p^i_{jk} and is independent of the pair of ith associates θ and φ.

The numbers v, n_i ($i = 1, 2, \ldots\ldots m$) and p^i_{jk} ($i,j,k = 1,2,\ldots\ldots\ldots m$) are called the parameters of the association scheme.

<u>PBIB Designs</u> : Given an association scheme with m classes for v treatments, we define a PBIB Design with m associate classes if the v treatments are arranged into b blocks of size k ($< v$) such that

(i) Every treatment occurs in exactly r blocks,

(ii) If two treatments θ and φ are ith associates, then they occur together in λ_i blocks, the number λ_i being independent of the particular pair of ith associates, θ and φ.

v, b, r, k, λ_i ($i = 1, 2, \ldots\ldots m$) are called the parameters of the design.

Association schemes with two associate classes were studied extensively by Bose and Shimamoto (1952). Mesner (1964) and Adhikary (1969b)showed that the association schemes introduced by Bose and Shimamoto do not exhaust all association schemes with two associate classes. However, the available PBIB Designs provided by Bose, Clatworthy and Shrikhande (1954) and Clatworthy (1956) may be made use of in constructing the generlised PBIB Designs with three associate classes considered in section 3.6.

The only type of PBIB Design with two associate classes, whose construction through OA's is taken up in the present chapter pertains to a particular association scheme, called Group Divisible (GD) scheme. GD Scheme can be described as :

Suppose, for integers $m \geq 2$ and $n \geq 2$, there is a set of $v = mn$ treatments. Let the treatments be divided into m groups of n treatments each. Call two treatments which appear together in a group as first associates; if two treatments are in different groups they are second associates. Then, the parameters of the association scheme are obviously

$$v = m\,n \;, \qquad n_1 = n - 1 \;, \qquad n_2 = n\,(m - 1)$$

$$(p^1_{jk}) = \begin{pmatrix} n-2 & 0 \\ 0 & n(m-1) \end{pmatrix} \;, \qquad (p^2_{jk}) = \begin{pmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{pmatrix}$$

A PBIB Design with GD association scheme will be referred to as a GD Design. A GD Design with parameters $v, b, r, k, \lambda_1, \lambda_2$, m groups of n treatments each will be denoted as GD $(v, b, r, k, \lambda_1, \lambda_2 ; m, n)$ or GD $(m, n, k, \lambda_1, \lambda_2)$.

Given an association scheme with two associate classes, Adhikary(1966) has considered two types of Generalisations to three associate classes. Construction of PBIB Designs with three associate classes of Adhikary(1966) through OA's is taken up in section 3.6. We describe below the generalised three associate association schemes of Adhikary (1966) :

Let G_2 represent an Abelian Group of n elements for which a two class association scheme is defined with parameters

$$v = n , \quad n_1 = n_1' , \quad n_2 = n_2' , \quad (p^1_{jk}) , \quad (p^2_{jk}) ,$$

j, k = 1, 2.

Moreover, for any element $\theta \in G_2$, let the n_1' first associates of θ be denoted by A_θ and the n_2' second associates of θ be denoted by B_θ . Let G_1 represent another Abelian Group of m elements. Then, the direct product $G = G_1 (x) G_2$ defines a group of m n elements. As G is obtained as the direct product of G_1 and G_2, for any element $\theta \in G$, let the corresponding elements in G_1 and G_2 be respectively θ_1 and θ_2. Then, the following two types of three class association schemes were defined by Adhikary (1966) :

(I) First Type of Generalisation :

For any $\theta \in G$,

the 1st associates of θ are $(G_1 - \{1\}) (x) \theta_2$

2nd associates of θ are $G_1 (x) A_{\theta_2}$

3rd associates of θ are $G_1 (x) B_{\theta_2}$

The parameters of the association scheme are :

$$v = mn , \quad n_1 = m - 1, \quad n_2 = m\, n_1' , \quad n_3 = m\, n_2' ,$$

$$(p_{jk}^{1*}) = \begin{pmatrix} m-2 & 0 & 0 \\ 0 & m\,n_1' & 0 \\ 0 & 0 & m\,n_2' \end{pmatrix} , \quad (p_{jk}^{2*}) = \begin{pmatrix} m-1 & 0 & 0 \\ 0 & mp_{11}^1 & mp_{12}^1 \\ 0 & mp_{12}^1 & mp_{22}^1 \end{pmatrix}$$

$$(p_{jk}^{3*}) = \begin{pmatrix} m-1 & 0 & 0 \\ 0 & mp_{11}^2 & mp_{12}^2 \\ 0 & mp_{12}^2 & mp_{22}^2 \end{pmatrix} .$$

(II) Second Type of Generalisation :

For any $\theta \in G$,

the 1st associates of θ are θ_1 (x) A_{θ_2}

2nd associates of θ are θ_1 (x) B_{θ_2}

3rd associates of θ are $(G_1 - \{1\})$ (x) G_2

The parameters of the association scheme are :

$$v = mn, \quad n_1 = n_1' , \quad n_2 = n_2' , \quad n_3 = (m-1)\, n ,$$

$$(p_{jk}^{1**}) = \begin{pmatrix} p_{11}^1 & p_{12}^1 & 0 \\ p_{12}^1 & p_{22}^1 & 0 \\ 0 & 0 & (m-1)n \end{pmatrix} , \quad (p_{jk}^{2**}) = \begin{pmatrix} p_{11}^2 & p_{12}^2 & 0 \\ p_{12}^2 & p_{22}^2 & 0 \\ 0 & 0 & (m-1)n \end{pmatrix}$$

$$(p_{jk}^{3**}) = \begin{pmatrix} 0 & 0 & n_1' \\ 0 & 0 & n_2' \\ n_1' & n_2' & (m-2)n \end{pmatrix}$$

A BIB or PBIB Design is said to be resolvable, if the blocks in the design can be divided into r sets of $\frac{b}{r}$ (an integer) blocks each, such that in each set each treatment occurs once and only once.

3.3 Construction of Orthogonal Arrays :

Let us start with 3 lemmas which will be needed to prove the main results of this section.

Lemma 3.3.1 : Given an array B with elements of $\mathcal{M}$, which is $S_t [\mu s^{t-1} , r, s]$ and a vector $\alpha' = (\alpha_1, \alpha_2 , \ldots \alpha_m)$ with $m = qs$ such that among the elements of α' each element of $\mathcal{M}$ occurs q times,

$$A = \alpha' \textcircled{U} B \text{ gives an } OA [q \mu s^t , r, s, t].$$

Proof : The result is obvious from the definition of $S_t[\mu s^{t-1},r,s]$ and it first appeared in Seiden (1954).

Lemma 3.3.2 : If A is an $OA [\lambda s^t, t, s, t]$ and $\alpha' = (\alpha_1, \alpha_2, \ldots \ldots , \alpha_t)$ is any t-tuple with all elements $\in \mathcal{M}$, $\alpha \oplus A$ is also an $OA [\lambda s^t, t, s, t]$.

Proof : If all the elements of α are null, $\alpha \oplus A$ gives A itself. Let us suppose that only one element of α, say, $\alpha_i \neq e_o$, all other elements of α being null. Then in $\alpha \oplus A$, all the rows excepting ith row are same as the corresponding rows of A. Excluding ith row, in the other (t-1) rows of A, each (t-1)-tuple occurs λ s times. Let us consider λs t-tuples of A with any fixed (t-1)-tuple obtained by omitting the ith element in the t-tuples. For a given (t-1)-tuple given by the other rows, in the ith row each one of the elements $e_o, e_1, \ldots\ldots, e_{s-1}$ occurs λ times. When a fixed non-null element is added to the ith row, keeping the other rows unaltered, these λs t-tuples will again give each one of the elements $e_o, e_1, \ldots\ldots, e_{s-1}$, λ times in the ith position. Thus, in this case when only one of the elements of α is non-null, $\alpha \oplus A$ is OA (t, t). When more than one elements, say p elements in α are non-null, α is the sum of p t-tuples where each t-tuple has exactly one non-null element. Let these components be denoted by $\beta_1, \beta_2, \ldots\ldots, \beta_p$ so that $\alpha = \beta_1 + \beta_2 + \ldots\ldots + \beta_p$, where exactly one element in each β_i is non-null. Then, writing

$$\beta_1 \oplus A = A_1$$

and $$\beta_i \oplus A_{i-1} = A_i, \quad i = 2, 3, \ldots\ldots, p,$$

by the argument already explained, each A_i is OA (t, t), $i = 1,2,\ldots$ $\ldots$ p. Obviously, $\alpha \oplus A = A_p$ and so is OA (t,t).

Lemma 3.3.3 : Let A be a $t \times n$ array ($n = \lambda_1 s^{t-1}$) where first (t-1)-rows constitute an OA (t-1, t-1) and the tth row is identical with the (t-1)th row. Let B be a $t \times m$ array ($m = \lambda_2 s$) the last two rows of which constitute an $S_2(2)$. All the elements of the arrays $\in \mathcal{M}$. Then,

$$B \oplus A \quad \text{is} \quad OA\ (t,\ t)$$

Proof : Let us consider any column β of B and let its last two elements be b_{t-1} and b_t. In $\beta \oplus A$, the first t-1 rows constitute an OA (t-1, t-1) by Lemma 3.3.2 and in all the t-tuples of $\beta \oplus A$ the t-th element differs from (t-1)th element by $b_t - b_{t-1}$. As the last two rows of B constitute an S_2 (2), in the difference series of the last two rows of B (tth row minus (t-1)th row), each element of $\mathcal{M}$ occurs λ_2 times. Hence, $B \oplus A$ is an OA (t, t) in which each t-tuple occurs $\lambda_1 \lambda_2$ times.

Now, we prove some theorems concerning general OA's and OA's of strength 2 and 3 with the help of the above lemmas.

Theorem 3.3.1 : $f\ (s^t, s, t) \geq t+1$, for $t \geq 2$ and any s.

Proof : For any s, let $\mathcal{M}$ be the class of residues Mod (s). We know that $f\ (s^2, s, 2) \geq 3$. This array with 3 constraints can in fact be written as :

$$A = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & (s-1) & (s-1) & \ldots & (s-1) \\ 0 & 1 & \ldots & (s-1) & 0 & 1 & \ldots & (s-1) & \ldots & 0 & 1 & \ldots & (s-1) \\ 0 & 1 & \ldots & (s-1) & 1 & 2 & \ldots & 0 & \ldots & (s-1) & 0 & \ldots & (s-2) \end{bmatrix}$$

Let us assume $f(s^t, s, t) \geq t+1$ and let B, an array of order $(t+1) \times s^t$ with elements of $\mathcal{M}$ represent OA $[s^t, t+1, s, t]$. Let us write $C = [B_0 : B_1 : \ldots\ldots : B_{s-1}]$, where B_θ is a $(t+1) \times s^t$ matrix obtained from B by retaining the first t rows of B unaltered and adding θ Mod (s) to each element of the last row, $\theta = 0, 1, \ldots\ldots, (s-1)$.

C is obviously an OA $(t+1, t+1)$. Let us consider B_θ for any given θ. A t-rowed submatrix of B_θ containing the last row and $(t-1)$ rows from the first t rows is an OA (t, t) by Lemma 3.3.2. The first t rows of B_θ certainly constitute an OA (t, t) by assumption. So, B_θ is an OA $(t+1, t)$, $\forall\ \theta$. Hence, C is completely resolvable. So, we can add one more row to C by writing θ for each column of B_θ, $\forall\ \theta$, and the augmented array is OA$(t+2, t+1)$. Thus, by mathematical induction, the theorem is proved.

It was proved by Bush (1952a) that $f(s^t, s, t) \leq t+1$ for $s \leq t$. Combinining this result with theorem 3.3.1, we have

Corollary 3.3.1 : $f(s^t, s, t) = t + 1$ for $s \leq t$.

From theorem 3.3.1, it obviously follows that

$$f(s^{t+p}, s, t) \geq t + p + 1 \quad \text{for } p \geq 0 \quad \ldots \quad \ldots(3.3.1)$$

An OA $[\lambda s^{t+p}, r+p, s, t]$ can be obtained from an OA $[\lambda s^t, r, s, t]$ easily by a successive application of the following procedure :

An OA $[Ns, r+1, s, t]$ can be obtained from an OA $[N, r, s, t]$ by juxtaposing s times the $r \times N$ array with elements in the class of residues Mod (s) for the latter OA and adding a row with 0 for all the N columns of the first array, 1 for all the columns of the second array, (s-1) for all the columns of the last array.

Theorem 3.3.2 : The existence of an OA $[\lambda s^2, r_1, s, 2]$ and an $S_2 [\mu s, r_2, s]$ implies the existence of an OA $[\lambda \mu s^3, r_1 r_2+1, s, 2]$.

Proof : Let the arrays be written with elements from $\mathcal{M}$. Let the array A with r_1 rows and $m = \lambda s^2$ columns represent the OA $[\lambda s^2, r_1, s, 2]$ and let the array B with r_2 rows and $n = \mu s$ columns represent the $S_2 [\mu s, r_2, s]$.

Let C = A Ⓤ B . Then,

$$C = \begin{bmatrix} C(1) \\ C(2) \\ \vdots \\ C(r_2) \end{bmatrix}, \quad \text{where } C(i) = [A(i1) : A(i2) \ldots : A(in)],$$

$$i = 1, 2, \ldots, r_2$$

$A(ij)$ is obtained from A by adding b_{ij} (i.e., the element in the cell (i, j) of B) to each element of A, $i = 1,2,\ldots, r_2$, $j = 1,2,\ldots, n$.

Let a 2-rowed submatrix of C be selected. If both the rows are from the same C(i) for some i, they obviously constitute an OA (2, 2). If the two rows are from two different C(i)'s, say, $C(i_1)$ and $C(i_2)$, $i_1 \neq i_2$, let the two rows be j_1th row of $C(i_1)$ and j_2th row of $C(i_2)$, $j_1, j_2 = 1, 2, \ldots, r_1$. Now two cases may arise, viz., (i) $j_1 \neq j_2$ and (ii) $j_1 = j_2$. In case (i) by lemma 3.3.2 and in case (ii) by lemma 3.3.3, the two-rowed submatrix chosen is OA (2, 2).

Hence, C is an OA $(r_1 r_2, 2)$. But C is the juxtaposition of μs arrays each of which is an OA $(r_1 r_2, 1)$. So, one more row can be added to C by writing e_0 for all the columns of the first μ arrays, e_1 for all the columns of the 2nd μ arrays and so on, e_{s-1} for all the columns of the last μ arrays. The resulting array is OA $(r_1 r_2 + 1, 2)$.

Theorem 3.3.3 : The existence of an OA $[\lambda s^3, r_1, s, 3]$ and an $S_3 [\mu s^2, r_2, s]$ implies the existence of an OA $[\lambda\mu s^5, r_1 r_2, s, 3]$.

Proof : All the arrays are written with elements from $\mathcal{M}$. Let the array A with r_1 rows and $m = \lambda s^3$ columns represent the OA $[\lambda s^3, r_1, s, 3]$ and B with r_2 rows and $n = \mu s^2$ columns represent the $S_3 [\mu s^2, r_2, s]$.

Let C = A Ⓤ B . Then,

$$C = \begin{bmatrix} C(1) \\ C(2) \\ \vdots \\ C(r_2) \end{bmatrix}, \quad \text{where } C(i) = [A(i1) : A(i2) : \ldots\ldots : A(in)], \quad i = 1, 2, \ldots\ldots, r_2,$$

and $A(ij)$ is obtained by adding b_{ij} (the element in the cell (i,j) of B) to each element of A, $i = 1,2, \ldots\ldots r_2$; $j = 1,2, \ldots\ldots n$.

Let a 3-rowed submatrix of C be chosen. If all the 3 rows are from the same $C(i)$ for some i, they obviously constitute an OA (3, 3). If the rows are from two different $C(i)$'s, say 2 rows from $C(i_1)$ and one row from $C(i_2)$, $i_1 \neq i_2$, let the two rows from $C(i_1)$ be j_1th and j_1' th rows (obviously $j_1 \neq j_1'$) and the row selected from $C(i_2)$ be j_2th row. Now, two cases may arise.

Case (i) $j_2 \neq j_1 \neq j_1'$. In this case the submatrix is OA(3, 3) by lemma 3.3.2. Case (ii) j_2 is equal to one of j_1 and j_1', say j_1. In this case writing j_1th row of $C(i_1)$ and the j_2th row of $C(i_2)$ as respectively the 2nd and 3rd rows of the submatrix, by lemma 3.3.3 the submatrix is OA (3, 3).

If the 3 rows of the submatrix are from 3 different C(i)'s, say, $C(i_1)$, $C(i_2)$ and $C(i_3)$, $i_1 \neq i_2 \neq i_3$, let the rows selected be j_1th row of $C(i_1)$, j_2th row of $C(i_2)$ and j_3th row of $C(i_3)$. In this situation, three cases may arise. Case (i) $j_1 = j_2 = j_3$. Then by lemma 3.3.1 the submatrix is OA (3, 3). Case (ii), $j_1 \neq j_2 \neq j_3$. By lemma 3.3.2, the submatrix is OA (3, 3). Case (iii), two of the j_k's are equal, say $j_1 \neq j_2$, $j_2 = j_3$. In this case writing the j_2th row of $C(i_2)$ and j_3th row of $C(i_3)$ as respectively the 2nd and 3rd rows of the 3-rowed submatrix and applying lemma 3.3.3, the submatrix is OA (3, 3). [Since any $S_3(r)$ is also $S_2(r)$].

Hence, the resulting array C is OA $(r_1 r_2, 3)$.

Now, given an $S_t^*[\mu s^{t-1}, r_2, s]$, $t \geq 1$, an $S_t[\mu s^{t-1}, r_2+1, s]$ can be obtained by adding one more row to the former with all elements e_o. Hence, we have

Corollary 3.3.2 : The existence of an OA $[\lambda s^2, r_1, s, 2]$ and an $S_2^*[\mu s, r_2, s]$ implies the existence of an OA $[\lambda \mu s^3, r_1(r_2+1)+1, s, 2]$.

Corollary 3.3.3 : The existence of an OA $[\lambda\, s^3, r_1, s, 3]$ and an $S_3^* [\mu\, s^2, r_2, s]$ implies the existence of an OA $[\lambda\mu s^3, r_1(r_2+1), s, 3]$.

Construction of S_t^*'s :

If s is a prime power > 2, let $e_o, e_1, \ldots\ldots, e_{s-1}$ be the elements of GF (s). In this case we can always construct an $S_2^* [s, s-1, s]$ as follows :

$$\begin{bmatrix} e_1 \cdot e_o & e_1 \cdot e_1 & \cdots\cdots\cdots & e_1 \cdot e_{s-1} \\ e_2 \cdot e_o & e_2 \cdot e_1 & \cdots\cdots\cdots & e_2 \cdot e_{s-1} \\ \vdots & \vdots & & \vdots \\ e_{s-1} \cdot e_o & e_{s-1} \cdot e_1 & \cdots\cdots\cdots & e_{s-1} \cdot e_{s-1} \end{bmatrix} \quad \ldots \quad (3.3.2)$$

In general, when s is a prime power, by modifying slightly the method of construction of OA $[s^t, s+1, s, t]$, $s \geq t$ given in Bush (1952a), we can obtain an $S_t^* [s^{t-1}, s-1, s]$. The modification to be done is as follows :

Let us consider s^{t-1} polynomials $y_j(x) = a_{t-1}\, x^{t-1} + a_{t-2}\, x^{t-2} + \ldots\ldots + a_1\, x$, where the coefficients range over the field GF(s). Column subscript j ranges from 0 to $(s^{t-1} - 1)$ and the row subscript i ranges from 1 to (s-1).

An (s-1) by s^{t-1} array is formed by writing in the ith row and jth column the integer u, where

$$y_j\,(e_i) = e_u\ .$$

The resulting array is obviously S_t (s-1). Because $x \neq e_o$, by following the line of proof in Bush (1952a), the (s-1) by s^{t-1} array so constructed can be shown to be an OA of strength t-1. Thus, the array is S_t^* (s-1). So, we have the following result :

For s a prime power $\geq$ t , we can always construct

$$S_t^* [\, s^{t-1},\ s-1,\ s\,], \quad t \geq 2 \qquad \ldots \quad \ldots \quad \ldots \ (3.3.3)$$

The following theorem will be useful in obtaining S_t^*'s when s is not a prime power.

Theorem 3.3.4 : The existence of $S_t^* [\mu_i\, s_i^{t-1},\ r_i,\ s_i\,]$, $i = 1, 2, \ldots, m$ implies the existence of $S_t^* [\mu\, s^{t-1},\ r,\ s\,]$, where $\mu = \mu_1 \mu_2 \ldots \mu_m$, $s = s_1 s_2 \ldots s_m$, $r = \text{Min}\,(r_1, r_2, \ldots, r_m)$.

Proof :

$$\text{Let } A_i = \begin{bmatrix} a_{11}^{(i)} & a_{12}^{(i)} & \ldots & a_{1\,n_i}^{(i)} \\ \vdots & & & \\ a_{r1}^{(i)} & a_{r2}^{(i)} & \ldots & a_{r\,n_i}^{(i)} \end{bmatrix}, \quad n_i = \mu_i\, s_i^{t-1},$$

be the array for $S_t^* [\mu_i\ s_i^{t-1}, r, s_i]$ with elements from the finite module $\mathcal{M}_i$ of s_i elements. Also, let the elements of $\mathcal{M}_i$ be $e_o^{(i)}, e_1^{(i)}, \ldots\ldots\ e_{s_{i-1}}^{(i)}$, $i = 1, 2, \ldots\ldots, m$.

Let $m = 2$. From A_1 and A_2, we can construct the array B of r rows and n_1n_2 columns with elements given by the ordered pairs of numbers of the type $(a_{ij}^{(1)}, a_{kl}^{(2)})$:

$$B = [B_1 : B_2 : \ldots\ldots : B_{n_2}] , \text{ where}$$

$$B_i = \begin{bmatrix} (a_{11}^{(1)}, & a_{1i}^{(2)}) & \ldots\ldots & (a_{1n_1}^{(1)}, & a_{1i}^{(2)}) \\ & \vdots & & & \\ (a_{r1}^{(1)}, & a_{ri}^{(2)}) & \ldots\ldots & (a_{rn_1}^{(1)}, & a_{ri}^{(2)}) \end{bmatrix} ,$$

$$i = 1, 2, \ldots\ldots, n_2 .$$

When so constructed, B is of strength t-1 as proved in Bush (1952b). Sum of any two elements $(e_{i_1}^{(1)}, e_{i_2}^{(2)})$ and $(e_{j_1}^{(1)}, e_{j_2}^{(2)})$ is defined as usual by $(e_{i_1}^{(1)} + e_{j_1}^{(1)}, e_{i_2}^{(1)} + e_{j_2}^{(2)})$. In ξ_k, η_k and ζ defined below, ordering of elements is ignored, although the pairs representing elements have, of course, to be ordered.

Let $\xi_k = [(e_o^{(1)}, e_k^{(2)}), (e_1^{(1)}, e_k^{(2)}), \ldots\ldots, (e_{s_1-1}^{(1)}, e_k^{(2)})]$

$$k = 0, 1, 2, \ldots\ldots, s_2-1.$$

and $\eta_k = [(e_k^{(1)}, e_o^{(2)}), (e_k^{(1)}, e_1^{(2)}), \ldots\ldots, (e_k^{(1)}, e_{s_2-1}^{(2)})]$,

$k = 0, 1, 2, \ldots\ldots, s_1-1.$

Also let

$$\begin{aligned} \xi &= [\xi_o : \xi_1 : \ldots\ldots\ldots : \xi_{s_2-1}] \\ &= [\eta_o : \eta_1 : \ldots\ldots\ldots : \eta_{s_1-1}] \\ &= \xi_o \text{ Ⓤ } \eta_o = \eta_o \text{ Ⓤ } \xi_o , \end{aligned}$$

ordering of elements being totally ignored.

Now, let the s_i^t t-tuples formed from the elements of $\mathcal{M}_i$ be divided into s_i^{t-1} sets, viz., $M_1^{(i)}, M_2^{(i)}, \ldots\ldots, M_{s_i^{t-1}}^{(i)}$ each set consisting of s_i distinct t-tuples such that given any t-tuple in a set, say $M_j^{(i)}$, all the t-tuples in the set can be obtained by adding successively the elements $e_o^{(i)}, e_1^{(i)}, \ldots\ldots, e_{s_i-1}^{(i)}$ of $\mathcal{M}_i$ to each element of the given t-tuple; $i = 1, 2.$

Now, let us consider a t rowed submatrix of B, say $B(t) = [B_1(t) : \ldots\ldots : B_{n_2}(t)]$ and let $A_1(t)$ and $A_2(t)$ be the corresponding t-rowed submatrices of A_1 and A_2 giving rise to B(t).

Obviously, considering the first co-ordinates and second co-ordinates of elements separately, ξ_o Ⓤ B(t) is an OA(r, t) with regard to the first co-ordinates of elements and an $S_t^*(r)$ with respect to the second co-ordinates of elements. Hence, $\xi \circledU B(t) = \eta_o \circledU [\xi_o \circledU B(t)]$ is OA (r, t) with regard to both the first and second co-ordinates of elements, considered separately.

Again, because A_1 is an $S_t^*(r)$, if τ_2 is any t-tuple formed by the elements of $\mathcal{M}_2$ occurring p times in $A_2(t)$, each of the s_1^t t-tuples of the elements of $\mathcal{M}_1$ will occur in the first co-ordinates of elements μ_1 p times with τ_2 as the t-tuple of the second co-ordinates in $\xi_o \circledU B(t)$. Obviously, the same result holds if $B(t)$ is replaced by $\eta_o \circledU B(t)$ and there are p t-tuples $\in M_j^{(2)}$ which occur in $A_2(t)$, when $\tau_2 \in M_j^{(2)}$ for some j, $j = 1, 2, \ldots\ldots s_2^{t-1}$. Now, A_2 being S_t^*, number of t-tuples belonging to $M_j^{(2)}$, which occur in $A_2(t)$ is μ_2, $\forall\ j = 1, 2, \ldots\ldots, s_2^{t-1}$. So, in $\xi \circledU B(t)$

$= \xi_o \circledU [\eta_o \circledU B(t)]$, each of the s_1^t t-tuples of the first co-ordinates of elements occurs $\mu_1 \mu_2$ times with any of the s_2^t t-tuples of the second co-ordinates of elements.

Thus $\xi \circledU B(t)$ is OA (t, t) i.e. $B(t)$ is $S_t^*(t)$. So, B is $S_t^*(r)$.

Once the result is proved for $m = 2$, it follows easily for all $m \geq 2$.

From the result (3.3.3) and theorem 3.3.4, we have :

If $s = p_1^{n_1} \; p_2^{n_2} \; \ldots\ldots \; p_k^{n_k}$ be the prime power decomposition of s and $s_o = \text{Min}\,(p_1^{n_1}, \; p_2^{n_2}, \; \ldots\ldots, \; p_k^{n_k})$, we can always construct an $S_t^* [s^{t-1}, s_o - 1, s]$, $t \geq 2$, when $s_o > t$. ... (3.3.4)

With the help of (3.3.4), the following results can be derived from theorems 3.3.2 and 3.3.3, Corollaries 3.3.2 and 3.3.3.

Corollary 3.3.4 : The existence of an OA $[\lambda s^2, r, s, 2]$ implies the existence of OA $[\lambda s^3, r s_o + 1, s, 2]$ and by repeated applications of the procedure, implies the existence of OA$[\lambda s^{p+2}, r s_o^p + s_o^{p-1} + \ldots\ldots + s_o + 1, s, 2]$, for all $p \geq 1$, where $s \geq 2$ and s_o is as defined in (3.3.4).

Clearly $s_o = s$ if s is a prime power. Note that the case $s = 2$ can be included here in the general result by writing $S_2^* (2, 1, 2) = (0, 1)$.

Corollary 3.3.5 : The existence of an OA $[\lambda s^3, r, s, 3]$ implies the existence of an OA $[\lambda s^5, r s_o, s, 3]$ and by successive applications of the same procedure, implies the existence of an OA $[\lambda s^{3+2p}, r s_o^p, s, 3]$, $s \geq 3$ and s_o is as defined in (3.3.4), for all $p \geq 1$.

Obviously, an OA $[\lambda s^t, r, s, t]$ is also $S_t^* [\lambda s^t, r, s]$. Suppose there exist OA $[\lambda_i s^2, r_i, s, 2]$, $i = 1, 2$. Treating the first OA as OA and the second as S_2^* and making use of Corollary 3.3.2, we have by writing A for the first array and B for the second array,

$$C = A \circledcirc B \text{ is OA } [\lambda_1 \lambda_2 s^4, r_1 (r_2+1), s, 2].$$

Some more rows can be added to C in the following manner.

Let us write

$$D = \begin{bmatrix} B \otimes E_{1n} \\ C \end{bmatrix}, \quad \text{where } n = \lambda_1 s^2$$

[E_{mn} is an $m \times n$ matrix with all elements 1 and the direct product of two matrices has already been defined in Chapter 2].

D can be easily shown to be OA $[\lambda_1 \lambda_2 s^4, (r_1+1)(r_2+1)-1, s, 2]$. Hence we have,

Corollary 3.3.6 : The existence of OA $[\lambda_i s^2, r_i, s, 2]$, $i = 1,2$, implies the existence of OA $[\lambda_1 \lambda_2 s^4, r, s, 2]$, where $r = (r_1+1)(r_2+1) - 1$.

It may be noted that Corollary 3.3.6 is an improvement upon Theorem 4 of Shrikhande (1964).

In the same manner we obtain, by making use of Corollary 3.3.3, the following corollary dealing with OA's of strength 3.

Corollary 3.3.7 : The existence of OA $[\lambda_i s^3, r_i, s, 3]$, $i = 1, 2$, implies the existence of OA $[\lambda_1 \lambda_2 s^6, r, s, 3]$, where $r = \text{Max}\ (r_1 (r_2+1),\ r_2 (r_1+1))$.

If s is a prime power, it is proved in Bose and Bush (1952) that we can always construct an

$$OA\ [s^4,\ s^2+1, s, 3] \quad \ldots \quad \ldots \quad (3.3.5)$$

and from Bush (1952a), it is known that

$$f\ (s^3, s, 3) = s+1, \text{ if } s \text{ is odd}; \quad = s+2, \text{ if } s \text{ is even} \quad \ldots (3.3.6)$$

Hence, using results (3.3.5) and (3.3.6) and applying corollaries 3.3.5 and 3.3.7, we can construct the following OA's for s, a prime power ≥ 3 and $p \geq 1$.

(i) $OA\ [s^{4p}, (s^2+1)(s^2+2)^{p-1}, s, 3]$

(ii) $OA\ [s^{4p+3}, (s^2+1)(s^2+2)^{p-1}(s+2), s, 3]$, if s is odd

and $OA\ [s^{4p+2}, (s^2+1)(s^2+2)^{p-1}(s+3), s, 3]$, if s is even

(iii) $OA\ [s^{4p+2}, (s^2+1)(s^2+2)^{p-1} s, s, 3]$

... (3.3.7)

Theorem 3.3.5 : The existence of an OA $[\lambda s^3, r, s, 3]$ implies the existence of OA $[\lambda s^4, 2r, s, 3]$.

Proof : Let $\mathcal{M}$ be the class of residues Mod (s). Let A be the array for OA $[\lambda s^3, r, s, 3]$ with elements from $\mathcal{M}$.

$$\text{Let } B = \begin{bmatrix} 0 & 0 & \ldots\ldots & 0 \\ 0 & 1 & \ldots\ldots & s-1 \end{bmatrix}$$

Define $C = A \circledcirc B$.

Then, applying lemma 3.3.2 and 3.3.3 as in theorem 3.3.3, it is easy to see that C is OA (2r, 3).

From theorem 3.3.5, we get the following results :

(i) From an OA $[\lambda 2^3, r, 2, 3]$, it is always possible to construct an OA $[\lambda . 2^4, 2r, 2, 3]$ and more generally an OA $[\lambda . 2^{p+3}, 2^p . r, 2, 3]$, for $p \geq 1$.

From Seiden and Zemach (1966), it is known that

$$f(\lambda . 2^3, 2, 3) \leq 4\lambda$$

Incidentally, Seiden (1954) and Seiden and Zemach (1966) give a simple and elegant method for construction of OA $[\lambda . 2^3, r, 2, 3]$ from a given OA $[\lambda . 2^2, r-1, 2, 2]$.

(ii) From an OA $[\lambda . 3^3, r, 3, 3]$, we can always construct an OA $[\lambda . 3^4, 2r, 3, 3]$. It being known from Bose and Bush (1952) that

$$f(3^4, 3, 3) \geq 10,$$

by a repeated application of Theorem 3.3.5, we can always construct an

$$OA [3^{4+p}, 10.2^p, 3, 3], \quad p \geq 0 \quad \ldots \quad \ldots \quad \ldots (3.3.8)$$

(iii) If s is a prime power, then applying the result (3.3.5) along with corollary 3.3.7 and theorem 3.3.5, we find that we can always construct an

$$OA [s^{4p+1}, 2(s^2+1)(s^2+2)^{p-1}, s, 3] \quad \ldots \quad \ldots \quad \ldots (3.3.9)$$

(3.3.9) is the only series missing in (3.3.7).

3.4 Existence of OA : II's of Strength Two :

The two theorems in this section give methods for constructing OA : II's in the case when the number of levels, s is not a prime power. In later sections of this chapter, OA's and OA : II's of strength 2 have been utilised for constructing Incomplete Block Designs.

Theorem 3.4.1 : The existence of $(\lambda_i s_i (s_i-1)/2, t_i, s_i, 2)$: II and OA $[\lambda_i' s_i^2, t_i', s_i, 2]$, $i = 1, 2, \ldots\ldots p$, implies the existence of $(\lambda s(s-1)/2,\ t,\ s,\ 2) : II$, where $\lambda = \prod_{i=1}^{p} (\lambda_i \lambda_i^{'})$,

$s = \prod_{i=1}^{p} s_i$, $t = \min (t_1, \ldots\ldots, t_p,\ t_1^{'}, \ldots\ldots, t_p^{'})$.

Proof :

Let the array $A_i = \begin{bmatrix} a_{11}^{(i)} & a_{12}^{(i)} & \ldots\ldots\ldots\ldots & a_{1m_i}^{(i)} \\ \vdots & \vdots & & \vdots \\ a_{t1}^{(i)} & a_{t2}^{(i)} & \ldots\ldots\ldots\ldots & a_{tm_i}^{(i)} \end{bmatrix}$,

$$m_i = \lambda_i^{'} s_i^2$$

represent $OA\,[\,\lambda_i^{'} s_i^2,\ t,\ s_i,\ 2\,]$

and the array $B_i = \begin{bmatrix} b_{11}^{(i)} & b_{12}^{(i)} & \ldots\ldots\ldots\ldots & b_{1n_i}^{(i)} \\ \vdots & \vdots & & \vdots \\ b_{t1}^{(i)} & b_{t2}^{(i)} & \ldots\ldots\ldots\ldots & b_{tn_i}^{(i)} \end{bmatrix}$

$$n_i = \lambda_i s_i (s_i-1)/2$$

represent $(\lambda_i s_i (s_i-1)/2,\ t,\ s_i,\ 2) : II$. For both the arrays the elements denoting the s_i levels are taken to be $0, 1, 2, \ldots\ldots s_i-1$; $i = 1, 2, \ldots\ldots p$. First let $p = 2$ and $t = \min (t_1, t_2, t_1^{'}, t_2^{'})$.

Define $C = [A_1(1) : A_1(2) : \ldots\ldots\ldots : A_1(n_2)]$,

$$\text{where } A_1(j) = \begin{bmatrix} (a_{11}^{(1)}, b_{1j}^{(2)}) & \ldots\ldots & (a_{1m_1}^{(1)}, b_{1j}^{(2)}) \\ \vdots & & \vdots \\ (a_{t1}^{(1)}, b_{tj}^{(2)}) & \ldots\ldots & (a_{tm_1}^{(1)}, b_{tj}^{(2)}) \end{bmatrix},$$

$j = 1, 2, \ldots\ldots n_2$.

and $D = [B_1(0) : B_1(1) : \ldots\ldots\ldots B_1(s_2-1)]$,

$$\text{where } B_1(j) = \begin{bmatrix} (b_{11}^{(1)}, j) & \ldots\ldots\ldots & (b_{1n_1}^{(1)}, j) \\ \vdots & & \vdots \\ (b_{t1}^{(1)}, j) & \ldots\ldots\ldots & (b_{tn_1}^{(1)}, j) \end{bmatrix},$$

$j = 0, 1, 2, \ldots\ldots, s_2-1$.

Let $C^* = [C : C : \ldots\ldots : C]$ be the juxtaposition of $\lambda_1 \lambda_2'$, C's

$D^* = [D : D : \ldots\ldots : D]$ be the juxtaposition of $\lambda_1' \lambda_2 \lambda_2'$ D's

and $E = [C^* : D^*]$.

Then, E is an array with t rows and $\lambda s(s-1)/2$ columns, $\lambda = \lambda_1 \lambda_1' \lambda_2 \lambda_2'$, $s = s_1 s_2$. The elements of E are the $s = s_1 s_2$ ordered pairs of numbers (i, j), $i = 0, 1, \ldots s_1-1$; $j = 0, 1, \ldots, s_2-1$. Let us denote the set of $s_1 s_2$ elements formed in this manner by Σ.

Let a two-rowed submatrix of E be chosen. In the part of the submatrix obtained from C^*, each unordered pair of elements of Σ with second co-ordinates unequal occurs λ times. In the part of the submatrix obtained from D^*, each unordered pair of elements of Σ with second co-ordinates same occurs λ times.

Hence, E is an OA : II of strength 2.

Obviously, by Bush's result (1952b), OA $[\lambda_i' s_i^2, t, s_i, 2]$, $i = 1, 2$, implies OA $[\lambda_1' \lambda_2' s^2, t, s, 2]$ which implies OA $[\lambda s^2, t, s, 2]$. So, the theorem is proved for $p = 2$. Hence, by repeated application of the result, the theorem is true for all $p \geq 2$.

Now, the existence of GF (s) implies OA $[s^2, s+1, s, 2]$ and by theorem 1 of Rao (1961a).

The existence of GF(s) implies the existence of $(s(s-1)/2, s, s, 2)$: II, when s is odd. Hence, we obtain the following corollary to theorem 3.4.1.

Corollary 3.4.1 : If s is an odd number > 1 and $s = p_1^{n_1} \times p_2^{n_2} \times \ldots \ldots \times p_m^{n_m}$ is the prime power decomposition of s, then we can always construct $(s(s-1)/2, s_o, s, 2)$: II with $s_o = \min(p_1^{n_1}, p_2^{n_2}, \ldots\ldots, p_m^{n_m})$.

OA : II's can be obtained from pairwise balanced designs also[Definition of a pairwise balanced design is given in section 5.1 of chapter 5].

Theorem 3.4.2 : The existence of a pairwise balanced design with parameters $(v; k_1, \ldots\ldots k_l; b_1, \ldots\ldots, b_l; \lambda)$ with k_i odd, $\forall i$, implies the existence of $(\lambda v(v-1)/2, t, v, 2)$: II, where the prime power decomposition of k_i is $k_i = p_{i1}^{n_{i1}} \times p_{i2}^{n_{i2}} \times \ldots \ldots\ldots \times p_{im_i}^{n_{im_i}}$, $i = 1, 2, \ldots\ldots, l$ and $t = \min\limits_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m_i}} (p_{ij}^{n_{ij}})$.

Proof : The existence is proved by constructing an OA : II from each block of the pairwise balanced design.

5 Construction of BIB and GD Designs Through OA and OA : II of Strength Two

Theorem 3.5.1 : The existence of $(\lambda s(s-1)/2, r, s, 2)$: II implies the existence of the series of BIB $(s, k, \lambda k(k-1)/2)$, $k \leq \min(r, s-1)$.

Proof : Given $(\lambda s(s-1)/2, r, s, 2)$: II, a k-rowed submatrix of the array can be chosen, $k \leq r$ and $< s$. An Incomplete Block Design can be constructed by treating each column of the submatrix as a block. The Incomplete Block Design so constructed is obviously a BIB $(s, k, \lambda k (k-1)/2)$.

The series of BIB $(s, k, \lambda k (k-1)/2)$ will be hereafter, referred to as (s, λ) - series (This is for the purpose of reference only).

From Corollary 3.4.1, we immediately obtain as a particular case of theorem 3.5.1, the result of Ramanuja Charyulu (1966) which may be restated in the following lines -

If s is any odd number > 1 and $s = p_1^{n_1} \times p_2^{n_2} \times \ldots \times p_m^{n_m}$ is the prime power decomposition of s, then we can always construct a $(s, 1)$-series with $k \leq s_o$ and $< s$, where $s_o = \min(p_1^{n_1}, p_2^{n_2}, \ldots, p_m^{n_m})$.

It is to be noted that the existence of $(s, 1)$-series was first proved by Gassner (1965) for $k \leq p$ and $< s$, where p is the minimum prime which divides s. Gassner's result was generalised by Ramanuja Charyulu (1966) in the form given above. But, our theorem 3.5.1 is more general than the result of Ramanuja Charyulu (1966), because an OA : II may have more constraints than are implied by the Corollary 3.4.1 of the preceding section e.g., BIB (21, 5, 1) implies (210, 5, 21, 2) : II and by theorem 3.4.2 implies

(21, 1) - series with $k \leq 5$, whereas by Gassner's (1965) and Ramanuja Charyulu's (1966) results, the existence is known for $k \leq 3$ only.

The following Theorem is similar to Theorem 2A of Shrikhande and Bose (1960) with the conditions of the theorem slightly relaxed.

Theorem 3.5.2 : The existence of BIB (v_1, k, λ_1), BIB (v_2, k, λ_3) and OA $[\lambda_2 v_2^2, k, v_2, 2]$, implies the existence of BIB $(v_1 v_2, k, \lambda)$, where λ is divisible by $\lambda_1 \lambda_2$ and λ_3.

Proof : Let the numbers of blocks in (i) BIB (v_1, k, λ_1) and (ii) BIB (v_2, k, λ_3) be b_1 and b_2 respectively.

Let the b_1 blocks of (i) be written as

$$(a_{i1}, a_{i2}, \ldots\ldots, a_{ik}),$$
$$i = 1, 2, \ldots\ldots, b_1,$$

and the b_2 blocks of (ii) as

$$(b_{i1}, b_{i2}, \ldots\ldots, b_{ik}),$$
$$i = 1, 2, \ldots\ldots\ldots, b_2,$$

where in (i) the treatments are denoted by the v_1 symbols $0, 1, 2, \ldots \ldots\ldots v_1-1$ and in (ii) the treatments are denoted by $0, 1, 2, \ldots\ldots, v_2-1$.

Let OA $[\lambda_2 v_2^2, k, v_2, 2]$ be represented by the array

$$\begin{bmatrix} c_{11} & c_{12} & \cdots\cdots\cdots & c_{1n} \\ \vdots & \vdots & & \vdots \\ c_{k1} & c_{k2} & \cdots\cdots\cdots & c_{kn} \end{bmatrix}, \quad n = \lambda_2 v_2^2 ,$$

where v_2 levels are the numbers 0, 1, 2, ... , v_2-1.

Let the following sets of blocks be constructed with $v_1 v_2$ treatments, a treatment being represented by an ordered pair of numbers, say (i, j), $i = 0, 1, \ldots\ldots v_1-1$; $j = 0, 1, \ldots\ldots, v_2-1$.

(a) $p_1 \lambda_2 v_2^2 b_1$ blocks, where $p_1 = \lambda / \lambda_1 \lambda_2$.

The blocks are

$$((a_{i1}, c_{1j}) , \quad (a_{i2} , c_{2j}) \ldots\ldots\ldots\ldots (a_{ik} , c_{kj}))$$

$$i = 1, 2, \ldots\ldots\ldots , b_1$$

$$j = 1, 2, \ldots\ldots\ldots\ldots, n \; (= \lambda_2 v_2^2),$$

each block repeated $p_1 = \lambda / \lambda_1 \lambda_2$ times.

(b) $p_2 v_1 b_2$ blocks, where $p_2 = \lambda / \lambda_3$.

The blocks are

$$((j, b_{i1}) , \quad (j, b_{i2}), \ldots\ldots\ldots\ldots , (j, b_{ik})) ,$$

$$i = 1, 2, \ldots\ldots\ldots , b_2$$

$$j = 0, 1, \ldots\ldots\ldots, v_1-1 ,$$

each block repeated $p_2 = \lambda / \lambda_3$ times.

The two sets of blocks (a) and (b) together obviously constitute a BIB $(v_1 v_2, k, \lambda)$.

In the theorem 3.5.2, if $v_2 = k$, BIB (v_2, k, λ_3) can be replaced by a single block design (0, 1, 2, ... , k-1). Theorem 3.5.2 is then reduced to the following corollary.

Corollary 3.5.2a : The existence of BIB (v_1, k, λ_1) and OA $[\lambda_2 k^2, k, k, 2]$ implies the existence of BIB $(v_1 k, k, \lambda)$, where λ is divisible by both λ_1 and λ_2.

But, the existence of a completely resolvable OA $[k^2, k, k, 2]$ is always ensured when k is a prime power. Thus, we have,

Corollary 3.5.2b : The existence of BIB (v, k, λ), where k is a prime power, implies the existence of BIB (vk, k, λ) and by repeated application of the same procedure, implies the existence of a series of BIB (vk^n, k, λ), n any positive integer. And, if BIB(v, k, λ) is resolvable, BIB (vk^n, k, λ) constructed with the help of completely resolvable OA $[k^2, k, k, 2]$, is resolvable, for n any positive integer.

By Ramanuja Charyulu's result (1966), the existence of (s, 1)-series is not known for any even s. By following Theorem 3.5.2 and Corollaries 3.5.2a and 3.5.2b, the existence can be established for some selective k's when s is even, e.g., (i) (5,1)-series exists for

$k \leq 4$ and so (20, 1)-series exists for $k = 4$ by Corollary 3.5.2b.

(ii) (5, 1)-series exists for $k \leq 4$ and BIB (8, 4, 3) is known to exist. Hence, by Corollary 3.5.2a, the (40, 1)-series exists for $k = 4$.

Let us have a close look at the conditions of theorem 3.5.2 and the method of construction of the design described in the proof. In the proof of the theorem, suppose the series of blocks (a) is considered without repeating each block p_1 times when $p_1 > 1$ as is done there. Thus, each block considered once, the series of blocks (a) will obviously constitute a GD $(v_1, v_2, k, 0, \lambda_1 \lambda_2)$. Moreover, if the BIB (v_1, k, λ_1) is resolvable and the OA $[\lambda_2 v_2^2, k, v_2, 2]$ is completely decomposable, the resulting GD constructed as explained will be resolvable. We can state the result in the form of a theorem as :

Theorem 3.5.3 : The existence of BIB (v_1, k, λ_1) and OA $[\lambda_2 v_2^2, k, v_2, 2]$, implies the existence of GD $(v_1, v_2, k, 0, \lambda_1 \lambda_2)$. Moreover, if the given BIB is resolvable and the given OA completely decomposable, the resulting GD will be resolvable.

Now the following result is known :

For s a prime power, $f(s^2, s, 2) = s + 1$ (3.5.1)

and there exists a completely decomposable OA $[s^2, s, s, 2]$.

So, we have,

Corollary 3.5.3a : The existence of BIB (v_1, k, λ_1) implies the existence of GD $(v_1, s, k, 0, \lambda_1)$, if s is a prime power $\geq$ k-1. And, the existence of a resolvable BIB (v_1, k, λ_1) and s, a prime power $\geq$ k , implies the existence of a resolvable GD $(v_1, s, k, 0, \lambda_1)$.

In case $v_1 = k$ and $\lambda_1 = 1$, i.e. BIB (v_1, k, λ_1) is replaced by a single complete block design (0, 1, 2, ... , k-1), the theorem 3.5.3 reduces to

Corollary 3.5.3b : The existence of OA $[\lambda_2 v_2^2 , k, v_2, 2]$, implies the existence of GD $(k, v_2, k, 0, \lambda_2)$. Moreover, if the OA is completely decomposable, the resulting GD is resolvable.

Let us closely scrutinise the method of constructing OA's of strength 2 explained in section 3.3 of chapter 3.

The method of construction employed in the proof of theorem 3.3.2 shows that if A giving the OA $[\lambda s^2, r_1, s, 2]$ is completely decomposable and B is the array for $S_2 [\mu s, r_2, s]$, then $C = A \textcircled{U} B$ giving the OA $[\lambda \mu s^3, r_1 r_2 , s, 2]$ will also be completely decomposable.

If s is a prime power, there always exists a completely decomposable OA $[s^2, s, s, 2]$ and an $S_2 [s, s, s]$. Hence, a repeated application of the same construction procedure as employed in the proof of theorem 3.3.2 ensures the existence of a completely decomposable OA $[s^{p+2}, s^{p+1}, s, 2]$ for $p = 0$ or any positive integer, when s is a prime power. Hence, with the help of Corollary 3.5.3b, we have,

Corollary 3.5.3c : There always exists a resolvable GD (ts, s^{p+2}, s^{p+1}, t ; $\lambda_1 = 0$, $\lambda_2 = s^p$, $m = t$, $n = s$) for all $t \leq s^{p+1}$ and $p = 0$ or any positive integer, when s is a prime power.

Again, theorem 3.5.2 will still hold, if BIB(v_1, k, λ_1) is replaced by a single complete block (0, 1, ... , k-1) with k treatments, i.e. BIB (v_1, k, λ_1) with $v_1 = k$ and $\lambda_1 = 1$. Hence, we obtain

Theorem 3.5.4 : The existence of BIB (v_2, k, λ_3) and OA $[\lambda_2 v_2^2 . k , v_2, 2]$, implies the existence of BIB $(k v_2, k, \lambda)$, where λ is divisible by both λ_2 and λ_3 .

Now, because of (3.5.1), the following Corollary is immediate from theorem 3.5.4.

Corollary 3.5.4 : The existence of BIB(v, k, λ), implies the existence of BIB(kv, k, λ) for v a prime power. Moreover, if the given BIB(v, k, λ) is resolvable, the resulting BIB(kv, k, λ) is also resolvable.

3.6 Construction of PBIB Designs with Three Associate Classes of Adhikary's Association Schemes (1966) :

In this section we show how PBIB Designs, with three associate classes, of the first and second types of generalisations of Adhikary (1966) can be constructed through suitable OA's of strength 2.

Theorem 3.6.1 : The existence of a PBIB Design with two associate classes with parameters $v_1, b_1, r_1, k, \lambda_1', \lambda_2', n_1', n_2', p_{j\ell}^1$ and $p_{j\ell}^2$, $j, \ell = 1, 2$, an OA $[\lambda_3' v_2^2, k, v_2, 2]$ and a BIB Design with parameters $v_2, b_2, r_2, k, \lambda_2''$, implies the existence of a PBIB Design, with three associate classes, of the first type of generalisation of Adhikary with parameters $v = v_1 v_2$,

$b = \lambda_3' v_2^2 b_1 + v_1 b_2$, $r = r_1 \lambda_3 v_2 + r_2$, k; $\lambda_1 = \lambda_2''$,

$\lambda_2 = \lambda_1' \lambda_3'$, $\lambda_3 = \lambda_2' \lambda_3'$, $n_1 = v_2 - 1$, $n_2 = n_1' v_2$, $n_3 = n_2' v_2$.

Proof : Let the given PBIB Design be written with symbols 0, 1, 2, ..., v_1-1, the given OA and BIB Design with symbols 0, 1, 2, ..., v_2-1.

Let the $v_1 v_2$ treatments of the derived design be written as the ordered pairs (i, j), $i = 0, 1, 2, \ldots\ldots, v_1-1$,

$j = 0, 1, 2, \ldots\ldots, v_2-1$.

Suppose the blocks of the given PBIB Design are $(a_{i1}, a_{i2}, \ldots \ldots, a_{ik})$, $i = 1, 2, \ldots\ldots, b_1$, the blocks of the given BIB are $(b_{i1}, b_{i2}, \ldots\ldots, b_{ik})$, $i = 1, 2, \ldots\ldots, b_2$ and the ith column of the given OA is $(c_{i1}, c_{i2}, \ldots, c_{ik})'$, $i = 1,2,\ldots, \lambda_3' v_2^2$.

Now, let us construct the following sets of blocks with $v_1 v_2$ treatments :

(i) $((a_{i1}, c_{j1}), (a_{i2}, c_{j2}), \ldots\ldots, (a_{ik}, c_{jk}))$,

$$i = 1, 2, \ldots\ldots, b_1$$
$$j = 1, 2, \ldots\ldots, \lambda_3' v_2^2$$

and (ii) $((i, b_{j1}), (i, b_{j2}), \ldots\ldots, (i, b_{jk}))$,

$$i = 0, 1, \ldots\ldots, v_1 - 1$$
$$j = 1, 2, \ldots\ldots, b_2$$

It is easy to see that the two sets of blocks (i) and (ii) together constitute the required PBIB Design, with three associate classes, of the first type of generalisation of Adhikary.

The non-existence of any BIB Design with the parameters $v_2, b_2, r_2, k, \lambda_2''$ in the theorem 3.6.1 makes $\lambda_1 = 0$. Also, we can see that if k is a prime power, OA $[k^2, k, k, 2]$ always exists and the BIB with parameters $v_2, b_2, r_2, k, \lambda''_2$ in the theorem 3.6.1 can be replaced by a single complete block (0, 1, 2, ... , k-1). In this case, $\lambda'_3 = \lambda''_2 = 1$, $b_2 = r_2 = 1$ and $v_2 = k$. Thus, we have the following corollary.

Corollary 3.6.1 : The existence of a PBIB Design with two associate classes with parameters $v_1, b_1, r_1, k, \lambda'_1, \lambda'_2, n'_1, n'_2, p^1_{j\ell}$ and $p^2_{j\ell}$, $j, \ell = 1, 2$, where k is a prime power, implies the existence of a PBIB Design, with three associate classes, of the first type of generalisation of Adhikary with parameters $v = v_1 k$, $b = b_1 k^2 + v_1$, $r = r_1 k + 1$, k; $\lambda_1 = 1$, $\lambda_2 = \lambda'_1$, $\lambda_3 = \lambda'_2$, $n_1 = k-1$, $n_2 = n'_1 k$, $n_3 = n'_2 k$.

Theorem 3.6.2 : The existence of a BIB Design with parameters $v_1, b_1, r_1, k, \lambda''_1$, an OA $[\lambda'_3 v_2^2, k, v_2, 2]$ and a PBIB Design with parameters v_2, b_2, r_2, k; $\lambda'_1, \lambda'_2, n'_1, n'_2, p^1_{j\ell}$ and $p^2_{j\ell}$, $j, \ell = 1,2$, implies the existence of a PBIB Design, with three associate classes, of the second type of generalisation of Adhikary with parameters $v = v_1 v_2$, $b = \lambda'_3 v_2^2 b_1 + v_1 b_2$, $r = \lambda'_3 r_1 + r_2$, k; $\lambda_1 = \lambda'_1$, $\lambda_2 = \lambda'_2$, $\lambda_3 = \lambda''_1 \lambda'_3$, $n_1 = n'_1$, $n_2 = n'_2$, $n_3 = (v_1-1) v_2$.

Proof : As in theorem 3.6.1, let the symbols used for the given BIB Design be $0, 1, 2, \ldots, v_1-1$. Let the symbols used for the given OA and PBIB Design be $0, 1, 2, \ldots, v_2-1$. Then, suppose the blocks of the given BIB Design are written as $(a_{i1}, a_{i2}, \ldots, a_{ik})$, $i = 1, 2, \ldots, b_1$, the blocks of the given PBIB Design are $(b_{i1}, b_{i2}, \ldots, b_{ik})$, $i = 1, 2, \ldots b_2$ and the ith column of the given OA is

$$(c_{i1} , c_{i2} , \ldots , c_{ik})' , \quad i = 1, 2, \ldots , \lambda_3' v_2^2 .$$

The two sets of blocks

$$\text{(i)} \quad ((a_{i1}, c_{j1}) , (a_{i2}, c_{j2}) , \ldots , (a_{ik}, c_{jk})) ,$$

$$i = 1, 2, \ldots , b_1$$

$$j = 1, 2, \ldots , \lambda_3' v_2^2$$

and

$$\text{(ii)} \quad ((i, b_{j1}) , (i, b_{j2}), \ldots , (i, b_{jk})) ,$$

$$i = 0, 1, \ldots , v_1-1$$

$$j = 1, 2, \ldots , b_2$$

constitute the required PBIB Design, with three associate classes, of the second type of generalisation of Adhikary.

Again, in case $v_1 = k$, the given BIB in theorem 3.6.2 can be replaced by a single complete block $(0, 1, \ldots, k-1)$ and hence $\lambda_1'' = 1$. Let v_2 be a prime power so that OA $[v_2^2, k, v_2, 2]$ exists for all $k \leq v_2+1$. So, in that case, theorem 3.6.2 can be modified as

Corollary 3.6.2 : The existence of a PBIB Design with parameters $v_2, b_2, r_2, k, \lambda_1', \lambda_2', n_1', n_2', p_{j\ell}^1$ and $p_{j\ell}^2$, $j, \ell = 1, 2$, where v_2 is a prime power, implies the existence of a PBIB Design, with three associate classes, of the second type of generalisation of Adhikary with parameters $v = v_2 k$, $b = v_2^2 + b_2 k$, $r = r_2+1$, k; $\lambda_1 = \lambda_1'$, $\lambda_2 = \lambda_2'$, $\lambda_3 = 1$, $n_1 = n_1'$, $n_2 = n_2'$, $n_3 = (k-1) v_2$.

Theorems 3.6.1 and 3.6.2, and Corollaries 3.6.1 and 3.6.2 can be exploited to construct innumerable PBIB Designs with three associate classes from known PBIB Designs with two associate classes. Adhikary (1969a) has given a number of methods for constructing PBIB Designs with three associate classes. In this section, we have given a new approach, that of obtaining solutions through known OA's of strength 2 and known PBIB Designs with two associate classes.

4. BALANCED ORTHOGONAL DESIGNS

4.1 Introduction :

Balanced Orthogonal Design (BOD) is an extension of Hadamard matrix. BOD's were introduced by Rao (1966) and were exploited to give methods of construction of some families of BIB and GD Designs by Rao (1966, 1970). Also, BOD's can be looked upon as Incomplete Weighing Designs admitting orthogonal estimation of all the weights. Rao and Das (1969) and Rao (1970) have given methods for constructing a BOD with parameters $(s^2 + s + 1, s^2, s^2 - s)$, when s is an odd prime power. In the present chapter, we prove the existence of an infinite series of BOD's with parameters $((s^{p+3}-1)/(s-1), s^{p+2}, s^{p+2}-s^{p+1})$ for all $p \geq -1$, when s is an odd prime power.

4.2 Definitions and Notations :

The definition of a BOD is as follows :

A matrix of order v x b with entries +1, -1, 0, is said to be a BOD if it satisfies the following conditions -

(i) inner product of any two rows is zero.

(ii) when all -1's in the matrix are changed to 1's, it becomes the incidence matrix of a BIB Design.

Let X be a BOD of order $v \times b$. Let N be the incidence matrix of the BIB Design obtained from X, by replacing -1's by 1's in X. Let (v, b, r, k, λ) be the parameters of the BIB Design. Then,

$X X' = r I_v$. When $b = v = r$, the BOD becomes a Hadamard matrix of order v. A BOD with the parameters of the corresponding BIB Design (obtained by changing -1's to 1's in the BOD) as v, b, r, k, λ will be denoted by BOD (v, k, λ).

T_n and T_n^* considered in chapter 2 are obviously BOD's. Σ and S matrices have been introduced in chapter 2. We continue to use the notations and definitions introduced in chapters 2 and 3.

4.3 Construction of BOD's :

Corollary 3.5.3 c in the preceding chapter ensures the existence of a resolvable GD $(s^{p+2}, s^{p+2}, s^{p+1}, s^{p+1}; \lambda_1 = 0, \lambda_2 = s^p, m = s^{p+1}, n = s)$, for all $p \geq 0$, when s is a prime power. Let $N^{(p+2)}$ be the incidence matrix of this resolvable GD Design. We can always write $N^{(p+2)}$ as

$$N^{(p+2)} = \begin{bmatrix} N_{1.} \\ N_{2.} \\ \vdots \\ N_{s^{p+1}.} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1,s^{p+1}} \\ N_{21} & N_{22} & \cdots & N_{2,s^{p+1}} \\ \vdots & & & \vdots \\ N_{s^{p+1},1} & N_{s^{p+1},2} & \cdots & N_{s^{p+1},s^{p+1}} \end{bmatrix},$$

where every N_{ij} is an $s \times s$ matrix with a single unity in each row and column and the remaining elements 0 ; $i,j = 1,2, \ldots , s^{p+1}$. The inner product of any two rows of $N_{i.}$ is zero, $i = 1,2,\ldots, s^{p+1}$ and the inner product of any row of $N_{i.}$ and any row $N_{i'.}$ is s^p, $i \neq i'$, $i, i' = 1, 2, \ldots\ldots, s^{p+1}$.

It has been observed in chapter 2 that when s is an odd prime power there always exists an $s \times s$ matrix, $S^{(1)}$ with diagonal elements 0 and off diagonal elements +1 or -1 such that $(S^{(1)})(S^{(1)})' = s I_s - E_{ss}$. $S^{(1)}$ is obviously either Σ_s or S_s according as $s \equiv 3$ (Mod 4) or 1 (Mod 4).

Let S_{ij} be obtained from N_{ij}, $i,j = 1,2, \ldots\ldots, s^{p+1}$ of $N^{(p+2)}$ in the following manner :

If the (f, g)th cell of N_{ij} contains unity, f^{th} row of N_{ij} is to replaced by g^{th} row of $S^{(1)}$.

Let us write

$$S^{(p+2)} = \begin{bmatrix} S_{11} & S_{12} & \cdots\cdots & S_{1,\, s^{p+1}} \\ S_{21} & S_{22} & \cdots\cdots & S_{2,\, s^{p+1}} \\ \vdots & & & \\ S_{s^{p+1},1} & S_{s^{p+1},2} & \cdots\cdots & S_{s^{p+1},\, s^{p+1}} \end{bmatrix}$$

Then, it is easy to see that

$$(S^{(p+2)}) (S^{(p+2)})' = s^{p+1} I_{s^{p+1}} \otimes (s I_s - E_{ss}) \quad \ldots \quad (4.3.1)$$

Thus, an $S^{(p+2)}$ matrix with the property (4.3.1) always exists corresponding to an $N^{(p+2)}$. $S^{(p+2)}$ matrices so obtained are $s^{p+2} \times s^{p+2}$ matrices with number of 1's (+ or -) in each row and column equal to s^{p+1} (s-1) and exist for all $p \geq -1$, when s is an odd prime power.

Let us consider the matrix

$$X^{(p+2)} = \begin{bmatrix} X^{(p+2)}_{p+2} \\ X^{(p+2)}_{p+1} \\ \vdots \\ X^{(p+2)}_{1} \\ X^{(p+2)}_{o} \end{bmatrix} , \text{ where}$$

$$X^{(p+2)}_j = [S^{(j)} \otimes E_{1\, s^{p+2-j}} : S^{(j-1)} \otimes E_{s\, s^{p+2-j}} : \ldots\ldots\ldots$$

$$\ldots : S^{(0)} \otimes E_{s^j\, s^{p+2-j}} : O_{s^j\, s^{p+1-j}} : O_{s^j\, s^{p-j}} : \ldots\ldots$$

$$\ldots : O_{s^j\, 1}] , \quad j = 0, 1, 2, \ldots\ldots, p+2$$ and $S^{(0)}$ is the scalar 1.

$X^{(1)}$ is obviously a BOD (s+1, s, s-1). Rao (1970) has proved that $X^{(2)}$ is a BOD $(s^2+s+1,\ s^2,\ s^2-s)$. The purpose of the present chapter is to prove that $X^{(p+2)}$ is a BOD for all $p \geq -1$.

$X^{(p+2)}$ is a $(s^{p+2} + s^{p+1} \ldots\ldots + s+1) \times (s^{p+2} + s^{p+1} \ldots + s+1)$ matrix and the number of 1's (+ or -) in each row and each column of $X^{(p+2)}$ is obviously s^{p+2}.

Let us assume

$$[S^{(m)} : S^{(m-1)} \otimes E_{s1} : \ldots\ldots : S^{(1)} \otimes E_{s^{m-1},1} : S^{(0)} \otimes E_{s^m,1}]$$

$$[S^{(m)} : S^{(m-1)} \otimes E_{s1} : \ldots\ldots : S^{(1)} \otimes E_{s^{m-1},1} : S^{(0)} \otimes E_{s^m,1}]'$$

$$= s^m I_{s^m} \quad \text{for all} \quad m \leq p+1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.3.2)$$

Then,

$$(X^{(p+2)}_{p+2})\ (X^{(p+2)}_{p+2})'$$

$$= (S^{(p+2)})\ (S^{(p+2)})' + [S^{(p+1)} : S^{(p)} \otimes E_{s1} : \ldots\ldots\ldots :$$

$$S^{(1)} \otimes E_{s^p 1} : S^{(0)} \otimes E_{s^{p+1},1}]\ [S^{(p+1)} : S^{(p)} \otimes E_{s1} :$$

$$\ldots\ldots : S^{(1)} \otimes E_{s^p 1} : S^0 \otimes E_{s^{p+1},1}]' \otimes E_{s1} E'_{s1}$$

$$= s^{p+1} I_{s^{p+1}} \otimes (s I_s - E_{ss}) + s^{p+1} I_{s^{p+1}} \otimes E_{ss}$$

[from (4.3.1) and (4.3.2)]

$$= s^{p+2} I_{s^{p+2}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.3.3)$$

and

$$(X_j^{(p+2)}) (X_j^{(p+2)})' \quad \text{for} \quad j \leq p+1$$

$$= [S^{(j)} \otimes E_{1 s^{p+2-j}} : \ldots\ldots\ldots : S^{(1)} \otimes E_{s^{j-1}, s^{p+2-j}} :$$

$$S^{(0)} \otimes E_{s^j s^{p+2-j}}] [S^{(j)} \otimes E_{1 s^{p+2-j}} : \ldots\ldots\ldots\ldots :$$

$$S^{(1)} \otimes E_{s^{j-1} s^{p+2-j}} : S^{(0)} \otimes E_{s^j s^{p+2-j}}]'$$

$$= [S^{(j)} : S^{(j-1)} \otimes E_{1s} \ldots\ldots : S^{(1)} \otimes E_{s^{j-1} 1} :$$

$$S^{(0)} \otimes E_{s^j 1}] [S^{(j)} : S^{(j-1)} \otimes E_{1s} : \ldots\ldots\ldots\ldots :$$

$$S^{(1)} \otimes E_{s^{j-1} 1} : S^{(0)} \otimes E_{s^j 1}]' \otimes E_{1 s^{p+2-j}} E'_{1 s^{p+2-j}}$$

$$= s^j I_{s^j} \otimes s^{p+2-j} E_{11}$$

[from (4.3.2)]

$$= s^{p+2} I_{s^j} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.3.4)$$

$$(S^{(j)}) (S^{(j-k)} \otimes E_{m, s^k})' \quad \text{for any} \quad k < j \quad \text{and any } m$$

$$= 0_{s^j, m s^{(j-k)}}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.3.5)$$

from the property

$$S^{(1)} E_{s1} = 0 .$$

Hence, by (4.3.5)

$$(X_j^{(p+2)}) (X_k^{(p+2)})'$$

$$= 0_{s^j \, s^k} \qquad \text{for} \quad j \neq k \;, \quad j,k \leq p+2 \qquad \ldots \quad \ldots \quad \ldots \qquad (4.3.6)$$

So, assuming (4.3.2) we get

$$(X^{(p+2)}) (X^{(p+2)})'$$

$$= s^{p+2} \; I_n \;, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.3.7)$$

where $n = s^{p+2} + s^{p+1} \;\ldots\ldots + s + 1 = \dfrac{s^{p+3} - 1}{s - 1}$

Thus, assuming (4.3.2) to be true for all $m \leq p+1$, it is found to be true for all $m \leq p+2$. But (4.3.2) is known to be true for $m \leq 2$ from Rao's result (1970). Hence, by mathematical induction (4.3.2) is true for all m. And (4.3.6) is true for all $p \geq 0$. Hence, (4.3.7) is true for all $p \geq -1$.

It is easy to see that if -1's are all replaced by 1's in $X^{(p+2)}$, the resulting matrix will be the incidence matrix of a BIB Design with parameters $v = b = (s^{p+3} - 1)/(s-1)$, $r = k = s^{p+2}$ and $\lambda = s^{p+2} - s^{p-1}$. Hence, we have the following theorem.

Theorem 4.3.1 : If s is an odd prime power, there always exists a series of BOD's ($(s^{p+3} - 1)/(s-1)$, s^{p+2} , $s^{p+2} - s^{p+1}$) , for all $p \geq -1$.

Moreover, as a consequence of theorem 4.3.1, Theorem 4.1 of Rao (1970) can be easily extended to

Theorem 4.3.2 : The existence of BOD ($(s^{p+3} - 1)/(s-1)$, s^{p+2} , $s^{p+2} - s^{p+1}$) and BIB (s+1, 2s, s, (s+1)/2 , (s-1)/2), implies the existence of BIB ($(s+1)(s^{p+3} - 1)/(s-1)$, $2s(s^{p+3} - 1)/(s-1)$, s^{p+3} , $s^{p+2}(s+1)/2$, $s^{p+2}(s-1)/2$).

Thus, the series of BIB Designs in Theorem 4.3.2 is ensured for all $p \geq -1$, when s is an odd prime power and BIB(s+1, 2s, s, (s+1)/2, (s-1)/2) exists.

5. PARTIALLY BALANCED ARRAY

5.1 Introduction :

Partially Balanced Array (PBA) was first introduced by Chakravarty (1956) as a substitute for OA, both serving the purpose of fractional replicates of factorial experiments. It has been shown by Chakravarty (1956) that by considering a PBA in place of an OA, it may be possible to reduce the size without sacrifycing the essential orthogonality property of the estimates of factorial effects satisfied by the corresponding OA, although estimation may be a bit more complicated. The definition of a PBA as occurring in Chakravarty (1956) is as follows :

A PBA $[N, k, s, d]$ of strength d, size N with k constraints or factors and s levels for each factor is a subset of N treatment combinations from an s^k factorial experiment with the property that for any group of d factors ($d \leq k$), a combination of the levels of d factors $(i_1, i_2, \ldots, i_d)$ $[i_j = 0, 1, \ldots s-1, \forall\, i_j]$ occurs $\lambda_{i_1 i_2 \ldots i_d}$ (>0) times, where $\lambda_{i_1 i_2 \ldots i_d}$ remains the same for all permutations of a given set $(i_1, i_2, \ldots, i_d)$ and for any group of d factors. If in the definition of PBA, $\lambda_{i_1 i_2 \ldots i_d}$ is a constant, not depending on the levels $i_1, i_2, \ldots, i_d$, the resulting array is an OA.

Some more arrangements considered in this ~~section~~ chapter are described here for the sake of completeness.

Pairwise balanced design ~~has already been introduced in chapter 3~~ is an Incomplete Block Design in which each pair of treatments occurs together in a constant number of blocks, say λ. It is called equireplicate, if each treatment occurs in a constant number of blocks, say r. The parametric notation we shall use for an equireplicate pairwise balanced design is $(v; k_1, k_2, \ldots\ldots k_l ; b_1, b_2, \ldots\ldots , b_l ; r, \lambda)$, where the design has b_i blocks of size k_i each, $i = 1,2, \ldots\ldots l$. The total number of blocks $b = \sum_{i=1}^{l} b_i$. An equireplicate triple-wise balanced design is an equireplicate pairwise balanced design where any three treatments occur together in a constant number, say μ of blocks and may be represented as the design $(v; k_1, \ldots\ldots , k_l ; b_1, \ldots\ldots , b_l ; r, \lambda , \mu)$.

5.2 General PBA's of Strength 3 :

Lemma 5.2.1 : The existence of an equireplicate pairwise balanced design with parameters $(v; k_1, \ldots , k_l ; b_1, \ldots, b_l ; r, \lambda)$, implies the existence of another pairwise balanced design with parameters $(v ; v-k_1, \ldots\ldots v - k_l ; b_1, \ldots\ldots , b_l ; b - r , b - 2r + \lambda)$ such that the two sets of blocks given by the two designs together constitute a triple wise balanced design of v treatments in 2b blocks, with each pair of treatments occurring in $(b - 2r + 2 \lambda)$ blocks and each triplet occurring in $(b - 3r + 3 \lambda)$ blocks.

Proof : The second design is obtained from the first as follows. To each block of the first design, there corresponds a block of the second design consisting of all the treatments absent in the former. The parameters of the second design are obvious and hence the number of times a pair of treatments occurs together in the combined design is $(b - 2r + 2\lambda)$. Now, let μ be the number of times a given triplet of treatment occurs together in the first design. Then, the number of times, the triplet occurs together in the second design is $(b - 3r + 3\lambda - \mu)$. So, in the combined design the triplet occurs $(b - 3r + 3\lambda)$ times.

Theorem 5.2.1 : The existence of an equireplicate triplewise balanced design with parameters $(v; k_1, \ldots.. k_l ; b_1, \ldots. b_l ; r , \lambda, \mu)$ and OA's $[p\ k_i^3 , t_i, k_i, 3]$, $i = 1,2, \ldots.. l$, implies the existence of PBA $[p \sum_{i=1}^{l} k_i^3\ b_i , t, v, 3]$, where $t = \min (t_1, t_2, \ldots.. , t_l)$ and $\lambda_{iii} = pr, \forall\ i$; $\lambda_{iij} = \lambda_{iji} = \lambda_{jii} = p\lambda, \forall\ i \neq j$; $\lambda_{ijk} = \mu, \forall\ i \neq j \neq k$.

Proof : Similar to that of theorem 4.1 of Chakravarty (1961).

Theorem 5.2.2 : The existence of an equireplicate pairwise balanced design with parameters $(v ; k_1, \ldots. k_l ; b_1, \ldots. , b_l ; r, \lambda)$, OA's $[p\ k_i^3 , t_i, k_i, 3]$ and OA's $[p\ (v - k_i)^3, t_i' ,(v-k_i), 3]$,

$i = 1, 2, \ldots\ldots l$, implies the existence of a PBA $\lceil p \sum_{i=1}^{l} b_i \{ k_i^3 + (v - k_i)^3 \}, t, v, 3 \rfloor$, where $t = \min(t_1, \ldots, t_l, t_1', \ldots t_l')$ and $\lambda_{iii} = p\,b, \forall i$; $\lambda_{iij} = \lambda_{iji} = \lambda_{jii} = p(b - 2r + 2\lambda)$, $\forall i \neq j$; $\lambda_{ijk} = p(b - 3r + 3\lambda)$, $\forall i \neq j \neq k$.

Proof : The result follows from lemma 5.2.1 and theorem 5.2.1.

The following corollary follows obviously from theorem 5.2.2.

Corollary 5.2.2 : The existence of BIB (v, b, r, k, λ), OA $\lceil p\,k^3, t, k, 3 \rfloor$ and OA $\lceil p\,(v - k)^3, t', (v-k), 3 \rfloor$, implies the existence of a PBA $\lceil pb \{ k^3 + (v-k)^3 \}, t, v, 3 \rfloor$, where $t = \min(t, t')$ and λ-parameters same as in theorem 5.2.2.

5.3. Two Level PBA's :

Let x and y be two $t \times 1$ column vectors with $x' = (x_1, \ldots\ldots, x_t)$, $y' = (y_1, \ldots\ldots, y_t)$, where $x_i \neq y_i$ and x_i's and y_i's are elements in the class of residues Mod 2, $\forall i$. Then, y may be called complement of x.

From the definition of PBA, in any $t \times N$ submatrix of the matrix of a PBA $\lceil N, k, 2, t \rfloor$, the number of times a $t \times 1$ column vector x occurs depends only on the number of 1's in the vector and not on their arrangement.

Lemma 5.3.1 : Let in a $t \times N$ matrix with elements in the class of residues mod 2, which is a PBA of strength t-1, a $t \times 1$ column vector x occur μ times. Then its complement y occurs $c + (-1)^t \mu$ times, where c depends only on the number of 1's in x.

Proof : The number of times y occurs is evidently $c + (-1)^t \mu$, where

$$c = N - \sum_{i_1=1}^{t} N_{i_1} + \sum_{\substack{i_1, i_2 = 1 \\ i_1 < i_2}}^{t} N_{i_1 i_2} \ldots\ldots\ldots\ldots\ldots \\ + (-1)^{t-1} \sum_{\substack{i_1, i_2, \ldots i_{t-1} = 1 \\ i_1 < i_2 < \ldots < i_{t-1}}}^{t} N_{i_1 i_2 \ldots i_{t-1}} \quad \ldots (5.3.1)$$

where $N_{i_1 i_2 \ldots i_k}$ $(i_1 < i_2 < \ldots\ldots < i_k, \quad k = 1,2,\ldots\ldots, t-1)$ denotes the number of columns in the matrix with i_j th element same as that of x, viz., x_{i_j}, $\forall\, i_j$, $j = 1,2, \ldots\ldots k$.

The matrix being a PBA of strength (t-1), $N_{i_1 i_2 \ldots i_k}$ in (5.3.1) depends only on the number of x_{i_j}'s equal to 1, $\forall\, i_j$. Thus, c depends only on the number of 1's in x.

Theorem 5.3.1 : The existence of a PBA [N, k, 2, 2p], implies the existence of a PBA [2N, k, 2, 2p+1], for any positive integer p.

Proof : Let A_1, a $k \times N$ matrix, give the PBA $[N, k, 2, 2p]$ with 2 levels 0 and 1. Let A_2 be another $k \times N$ matrix, obtained from A_1 by writing 1 for each 0 and 0 for each 1 in A_1. Then, $A = [A_1 : A_2]$ gives the required PBA of strength $2p+1 = t$. The number of times a $t \times 1$ column vector x occurs in any $t \times 2N$ submatrix of A is from lemma 5.3.1, $c + (-1)^t \mu + \mu = c$, which depends only on the number of 1's in x.

Corollary 5.3.1 : The existence of a PBA $[N, k, 2, 2]$ with

$$\lambda_{x_1x_2} = \lambda_1, \quad x_1 = x_2 = 0$$

$$= \lambda_2, \quad x_1 = x_2 = 1$$

$$= \lambda_3, \quad \text{one of } x_1 \text{ and } x_2 = 1 \text{ and the other } 0,$$

implies the existence of a PBA $[2N, k, 2, 3]$ with

$$\lambda_{x_1x_2x_3} = N - 3\lambda_3, \quad x_1 = x_2 = x_3 = 0$$

$$\text{or } x_1 = x_2 = x_3 = 1$$

$$= N - \lambda_1 - \lambda_2 - \lambda_3, \quad \text{one of } x\text{'s is } 1, \text{ others } 0$$

$$\text{or one of } x\text{'s is } 0, \text{ others } 1.$$

Theorem 5.3.2 : An equireplicate pairwise balanced design with v treatments, b blocks and r replications, in which each pair of treatments occurs together in λ blocks, is equivalent to a

PBA [b, v, 2, 2] with

$$\lambda_{x_1 x_2} = \lambda \quad , \quad x_1 = x_2 = 1$$

$$= b - 2r + \lambda \quad , \quad x_1 = x_2 = 0$$

$$= r - \lambda \text{ , one of } x\text{'s is 1, other 0.}$$

Proof : Given the equireplicate pairwise balanced design, suppose M is its incidence matrix. It is easy to see that M is PBA with λ - parameters given in the theorem. Conversely, let M, written with elements 0 and 1 give the PBA with the λ-parameters given in the theorem. It can be easily shown that M is the incidence matrix of an equireplicate pairwise balanced design with v treatments in b blocks.

By dint of theorem 5.3.2 and the result $b \geq v$, already known regarding pairwise balanced designs, we can state that for a PBA [N, k, 2, 2] with none of the λ parameters zero, $N \geq k$. A PBA [N, N, 2, 2] is the incidence matrix of a symmetrical BIB Design.

From a PBA of strength 2 with two levels, obtained as the incidence matrix of a pairwise balanced design, a PBA of strength 3 with two levels can be obtained by virtue of theorem 5.3.1. In general, theorem 2.1 of Chakravarty (1961) can be improved to include the following result :

The existence of a Tactical configuration in which every subset of μ treatments occurs together in a constant number of blocks, implies the existence of a PBA of strength $\mu + 1$ with 2 levels under the conditions of theorem 2.1 of Chakravarty (1961), provided μ is an even number.

5.4 PBIB Design Through PBA :

In this section, the use of a certain type of PBA's of strength 2 is indicated in constructing PBIB Designs with three associate classes belonging to a particular type of association scheme, known as rectangular association scheme. For the sake of completeness, we describe below this association scheme which was first introduced by Vartak (1959).

Suppose we have a set of $v = mn$ elements for some integers $m, n \geq 2$. Then, we can arrange the v elements in a rectangular array with m rows and n columns. If two elements appear in the same row, call them first associates, if they appear in the same column, call them second associates; otherwise, call them third associates. Then, we have a three class association scheme, with parameters

$$v = mn, \quad n_1 = n-1, \quad n_2 = m-1, \quad n_3 = (m-1)(n-1)$$

$$(p^1_{jk}) = \begin{pmatrix} n-2 & 0 & 0 \\ 0 & 0 & m-1 \\ 0 & m-1 & (m-1)(n-2) \end{pmatrix}, \quad (p^2_{jk}) = \begin{pmatrix} 0 & 0 & n-1 \\ 0 & m-2 & 0 \\ n-1 & 0 & (m-2)(n-1) \end{pmatrix}$$

$$(p^3_{jk}) = \begin{pmatrix} 0 & 1 & n-2 \\ 1 & 0 & m-2 \\ n-2 & m-2 & (m-2)(n-2) \end{pmatrix}$$

Theorem 5.4.1 : The existence of two BIB Designs with parameters $v_1, b_1, r_1, k, \lambda'_1$ and $v_2, b_2, r_2, k, \lambda''_2$ respectively and a PBA $[N, k, v_2, 2]$, every two-rowed submatrix of which contains an ordered pair $\binom{i}{j}$, λ'_2 times, $\forall\, i = j$ and λ'_3, $\forall\, i \neq j$, implies the existence of a PBIB Design with three associate rectangular association scheme with parameters $v = v_1 v_2$, $b = Nb_1 + v_1 b_2$, $r = r_1(\lambda'_2 + \lambda'_3\ \overline{v_2 - 1}) + r_2$, k ; $\lambda_1 = \lambda'_1 \lambda'_2$, $\lambda_2 = \lambda''_2$, $\lambda_3 = \lambda'_1 \lambda'_3$, $n_1 = v_1 - 1$, $n_2 = v_2 - 1$, $n_3 = (v_1-1)(v_2-1)$.

Proof : Similar to those of theorems 3.6.1 and 3.6.2.

Non-existence of a BIB Design with parameters v_2, b_2, r_2, k, λ_2'' in Theorem 6.4.1 makes $\lambda_2 = 0$ when $v_2 \neq k$. When $v_2 = k$, the mentioned BIB can be replaced by a single block design (0, 1, ... , k-1) to make $\lambda_2 = 1$ in the resulting design. Other manipulations as in theorems 3.6.1 and 3.6.2 are also possible in the case of the theorem 5 .4.1.

6. λ AND (r, λ) - SYSTEMS

6.1 Introduction :

Definitions of λ and (r, λ)- systems occur in Mullin and Stanton (1966). A λ-System is an Incomplete Block Design in which each block size is > 1 and each pair of treatments occurs together in λ blocks. Thus, λ-system is a pairwise balanced design considered in chapters 3 and 5 with the condition added that each block size is > 1. An Incomplete Block Design is called an (r, λ) - System if each treatment occurs in r blocks and each pair of treatments occurs together in λ blocks. An (r, λ)-system is the same as the equireplicate pairwise balanced design of chapter 5.

λ-systems do not include (r, λ)-systems as blocks consisting of a single treatment are not permitted in the former, while they are permitted in the latter. Let us consider a larger class of systems covering both λ- and (r, λ)-systems of Mullin and Stanton (1966). Let us define an $(n * \lambda)$ system as follows :

A collection of b subsets (called blocks) of a set V of v treatments is said to form a $(n * \lambda)$- system, when the following axioms are satisfied :

I : every pair of treatments occurs in precisely λ blocks and there is at least one block with block size $< v$.

II : sum of the block sizes giving the total number of design points is n.

Associated with every $(n * \lambda)$- system there is a sequence of non-negative integers $B = (b_1, b_2, b_3, \ldots\ldots)$, where b_i is the number of blocks containing exactly i treatments $i \geq 1$, b_i's being all zero after a certain stage. Also associated is a sequence of non-negative integers $(r_1, r_2, \ldots\ldots, r_v)$, where r_i is the number of blocks which contain ith treatment (also called the replication of the ith treatment), $r_i \geq \lambda$; $i = 1,2,\ldots\ldots v$, the inequality being strict for at least one i.

Obviously for a $(n * \lambda)$- system with v treatments

$$\sum_{i=1}^{\infty} i\, b_i = \sum_{i=1}^{v} r_i = n \quad \ldots \quad \ldots \quad (6.1.1)$$

and

$$\sum_{i=1}^{\infty} \binom{i}{2} b_i = \lambda \binom{v}{2} \quad \ldots \quad \ldots \quad (6.1.2)$$

An $(n * \lambda)$- system becomes an (r, λ)- system of Stanton and Mullin (1966), when $r_1 = r_2 \ldots\ldots = r_v = r$. For an $(n * \lambda)$- system let us define average replication per treatment as

$$\bar{r} = \frac{\sum_{i=1}^{v} r_i}{v} = \frac{n}{v} .$$

6.2 Results on λ - and (r, λ)- systems :

Theorem 6.2.1 : In an $(n * \lambda)$- system with v treatments, the total no. of blocks, b satisfies the inequality, $b \geq \frac{n}{k_o}$, where $k_0 = \frac{\lambda(v-1)}{\bar{r}} + 1$. Equality implies the system is a BIB Design with parameters $v, b, r = \bar{r}, k = k_o, \lambda$.

Proof : For an $(n * \lambda)$- system with v treatments, we have

$$\sum_{i=1}^{\infty} b_i = b \quad \ldots \quad \ldots \quad \ldots \quad (6.2.1)$$

and from (6.1.1) and (6.1.2) ,

$$\sum_{i=1}^{\infty} i\, b_i = n = v\,\bar{r} \quad \ldots \quad \ldots \quad \ldots \quad (6.2.2)$$

$$\begin{aligned} \sum_{i=1}^{\infty} i^2\, b_i &= \lambda\, v(v-1) + v\,\bar{r} \\ &= v\,\bar{r}\,(k_o - 1) + v\,\bar{r} \\ &= v\,\bar{r}\, k_o \quad \ldots \quad \ldots \quad \ldots \quad (6.2.3) \end{aligned}$$

By Cauchy - Schwartz inequality ,

$$\left(\sum_{i=1}^{\infty} b_i\right)\left(\sum_{i=1}^{\infty} i^2\, b_i\right) \geq \left(\sum_{i=1}^{\infty} i\, b_i\right)^2 ,$$

which on simplification gives

$$b \geq \frac{v\,\bar{r}}{k_o} = \frac{n}{k_o} \quad \ldots \quad \ldots \quad \ldots \quad (6.2.4)$$

Equality in (6.2.4) implies $\frac{i\sqrt{b_i}}{\sqrt{b_i}} = i$ is constant for all $i \geq 1$, which is impossible unless i takes only one value, say k. In that case $b_i = 0$ for all $i \neq k$ and $b_k = b$ i.e. $k\, b_k = v\,\bar{r}$ and $k^2\, b_k = v\,\bar{r}\,k_o$. Hence, $k = k_o$. This implies that k_o must be a positive integer. The resulting $(n * \lambda)$- system with v treatments is such that there are b blocks, each of same size k_o and each pair of treatments occurs together in precisely λ blocks. Then, by theorem 2 of Mullin and Stanton (1966), the system is a BIB Design with replication for each treatment $\bar{r}$. This implies again that $\bar{r}$ is a positive integer.

It is to be noted that we did not assume $\bar{r}$ and k_o to be positive integers, but the equality in (6.2.4) implies that they are so.

Corollary 6.2.1 : Non-existence of a BIB Design with parameters v, b, r and λ, implies the non- existence of a $(vr * \lambda)$- system, with v treatments in b blocks and in particular, the non-existence of an (r, λ)- system with v treatments and b blocks.

Following Mullin and Stanton (1966), we can define an $(n * \lambda)$- system to be elliptic, parabolic or hyperbolic according as the expression $\lambda\,(v-1) - \bar{r}\,(\,\bar{r} - 1\,)$ is negative, zero or positive.

Theorem 6.2.2 : A non- hyperbolic $(n * \lambda)$- system with v treatments and b blocks is a symmetrical BIB Design if $b = v$.

Proof : As the $(n * \lambda)$- system is non-hyperbolic,

$$\bar{r}(\bar{r} - 1) \geq \lambda(v - 1) \quad \ldots\ldots \quad \ldots\ldots \quad (6.2.5)$$

Defining $k_o = \dfrac{\lambda(v - 1)}{\bar{r}} + 1$,

$$\bar{r}(k_o - 1) = \lambda(v - 1) \leq \bar{r}(\bar{r} - 1)$$

$$\therefore k_o \leq \bar{r} \quad \ldots \quad \ldots \quad \ldots \quad (6.2.6)$$

Again, the result (6.2.4) with $b = v$ implies

$$k_o \geq \bar{r} \quad \ldots \quad \ldots \quad \ldots \quad (6.2.7)$$

From (6.2.6) and (6.2.7), $k_o = \bar{r}$. This implies equality in (6.2.4). So, the system is a BIB Design by theorem 6.2.1 and it is symmetrical because $b = v$.

6.3 Non- Existence of Some PBIB Designs :

The following result has been proved in Mullin and Stanton(1966):

An (r, λ)- system with $b = v$ is always a symmetrical BIB Design ... ... (6.3.1)

(6.3.1) can be exploited to prove the non existence of certain PBIB Designs. The procedure is illustrated with a few examples.

Examples :

(i) GD Design with parameters $v = 16$, $b = 12$, $r = 6$, $k = 8$; $\lambda_1 = 2$, $\lambda_2 = 3$, $m = n = 4$ is non existent, because if it exists, then by considering each group as a block and adding 4 such blocks obtained from the 4 groups to the design, we would have got an (r, λ)- system with $r = 7$, $\lambda = 3$ and $b = v = 16$. By (6.3.1), the resulting design should be a BIB. But it is not so as there are two distinct block sizes, 8 and 4. Thus, we arrive at a contradiction and as such the GD Design with the given parameters cannot exist.

By similar arguments the non-existence of the GD Designs in (ii) and (iii) can be established.

(ii) GD Design with parameters $v = 12$, $b = 8$, $r = 4$, $k = 6$; $\lambda_1 = 1$, $\lambda_2 = 2$, $m = 4$, $n = 3$.

(iii) GD Design with parameters $v = 20$, $b = 16$, $r = 8$, $k = 10$; $\lambda_1 = 3$, $\lambda_2 = 4$, $m = 4$, $n = 5$.

(iv) L_2 Design with parameters $v = 25$, $b = 15$, $r = 6$, $k = 10$, $\lambda_1 = 3$, $\lambda_2 = 4$ does not exist, because if it exists then by considering each row as a block and each column as a block and adding these 10 blocks obtained from rows and columns to the L_2 Design, we would have got an (r, λ)- system with $r = 8$, $\lambda = 4$ and $b = v = 25$. Then, by (6.3.1), the resulting design should be a BIB. But it is not so, because there are two distinct block sizes, 10 and 5 and as such the L_2 Design with given parameters cannot exist.

Non-existence of several such PBIB Designs can thus be established with the help of (6.3.1).

6.4 Falsity of a Conjecture by Mullin and Stanton :

A counter - example is provided to conjecture 1 in Mullin and Stanton (1966), in this section. The conjecture states :

'For $\lambda \leq 2$ (and perhaps all λ), $\lambda(v-1) = r(r-1)$ implies $v = b$ if the corresponding design is irreducible'.

Here by 'design' is meant an (r, λ)- system. In Mullin and Stanton (1966), a design has been termed irreducible if it contains neither a complete block consisting of all v treatments nor a set of v single treatment blocks whose union is V.

The following counter example disproves the conjecture for $\lambda = 2$. The example gives an irreducible (r, λ)- system with $r = 4$, $\lambda = 2$ and $v = 7$, so that $\lambda(v-1) = r(r-1)$, but $b = 8$. Blocks in the system are :

$$\begin{array}{cccc} (1 & 2 & 3 & 4) \\ (1 & 2 & 5 & 6) \\ (1 & 3 & 5 & 7) \\ (1 & 4 & 6 & 7) \end{array}$$

(2 3 4 5 6 7), (2 7), (3 6), (4 5).

Similar counter examples can be provided for (r, λ)- systems with $\lambda > 2$. Hence, it can be asserted that the condition $\lambda(v-1) = r(r-1)$ is sufficient for an irreducible (r,λ)-system to be a symmetrical BIB Design only when $\lambda = 1$.

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7. ASSOCIATION MATRICES OF PBIB ASSOCIATION SCHEMES

7.1 Introduction :

The concept of using one or more known Incomplete Block Designs for the purpose of obtaining a solution to a new Incomplete Block Design is pretty old and in particular, the idea of constructing BIB Designs from Association Matrices of PBIB association schemes is originally due to Shrikhande and Singh (1962). Shrikhande and Singh (1962) considered the construction of some series of symmetrical BIB Designs and all their designs are obtained from association matrices of association schemes of two classes. Blackwelder (1969) considers a more general treatment of the problem and with the help of his theorem 3.1 which is same as theorem 1 of Shrikhande and Singh (1962) and theorem 4.1, gives a systematic procedure for constructing some more series of BIB Designs from association matrices for association schemes of two and three classes. In the present chapter, we prove a further generalised version of theorems 3.1 and 4.1 of Blackwelder (1969) and the generalised theorem is utilised for constructing some BIB Designs through association matrices of the available association schemes. It is shown that some more series can be added to the list of such designs provided by Blackwelder. The different two and higher associate association schemes exploited for the purpose have been listed and described in section 2 of the present chapter. To illustrate the potentiality of the general theorem proved in this chapter, a few BIB Designs are constructed from the higher associate cyclical association schemes of Adhikary (1967), which cover as particular cases the schemes proposed by Nandi and Adhikary (1965), and Raghava rao and Chandrasekhararao (1964). Of the designs so constructed three are unsolved cases as indicated in the lists of BIB Designs given by Rao (1961b)and Sprott (1962). Some of the series of BIB Designs constructed from association matrices have certain additional properties which can be exploited to construct suitable series of DBIB and GD Designs.

7.2 Definitions and Notations :

BIB, DBIB and General PBIB Designs along with some association schemes have already been introduced in previous chapters. The purpose of the present chapter is to give methods for constructing Incomplete Block Designs through association matrices. The association schemes exploited for the purpose are listed and described in the present section. The list is obviously not exhaustive.

7.2.1 Association Schemes of Two classes :

Association Schemes with two associate classes were studied extensively by Bose and Shimamoto (1952). Mesner's (1964) 'Negative Latin Square Design' and Adhikary's (1969b) 'Paracyclic Association Scheme' vindicate that the association schemes introduced by Bose and

Shimamoto do not exhaust all association schemes with two associate classes.

(a) Group Divisible (GD) Scheme :

Already explained in section 3.2 of chapter 3. The same notations will be used in the present chapter.

(b) Triangular Association Scheme :

Suppose, for some positive integer n, there is a set of $v = \binom{n}{2}$ elements. Arrange the v elements in an n x n array as follows : Leave the leading diagonal positions blank, and fill the $n(n-1)/2$ positions above the diagonal with the elements; now fill the remaining positions so as to make the array symmetric with respect to the diagonal. Define first associates as two elements which appear in the same row (equivalently, the same column) of the resulting array; if two elements do not appear in the same row, they are second associates.

Such an array is an association scheme, called a Triangular association scheme. The parameters of the association scheme are:

$$v = n(n-1)/2, \quad n_1 = 2n - 4, \quad n_2 = (n-2)(n-3)/2 ,$$

$$(p^1_{jk}) = \begin{pmatrix} n-2 & n-3 \\ n-3 & (n-3)(n-4)/2 \end{pmatrix}, \quad (p^2_{jk}) = \begin{pmatrix} 4 & 2n-8 \\ 2n-8 & (n-4)(n-5)/2 \end{pmatrix}$$

From the parametric values, we see that $n \geq 4$. The parameters of the triangular association scheme uniquely determine the association scheme when $n \neq 8$ (Connor, 1958 and Shrikhande, 1959 a).

(c) Singly Linked Block (SLB) Association Scheme :

Suppose N' is an incidence matrix for a BIB Design with b treatments, v blocks, k replications, r block size and $\lambda = 1$. Then, $bk = vr$ and $b-1 = k(r-1)$. This gives $v = k(rk - k + 1)/r$ and $b = rk - k + 1$.

It has been shown that in this case N is the incidence matrix of a PBIB Design with v treatments, b blocks, r replications, k plots per block, $\lambda_1 = 1$ and $\lambda_2 = 0$. Definiting, first associates as two treatments which appear together in some block of the derived PBIB Design, we have a two-class association scheme, called a singly linked block (SLB) association scheme, with the following parameters:

$$v = k(rk - k + 1)/r, \quad n_1 = r(k-1), \quad n_2 = (k-r)(r-1)(k-1)/r \ ,$$

$$(p^1_{jk}) = \begin{pmatrix} k - 2 + (r-1)^2 & (r-1)(k-r) \\ (r-1)(k-r) & (r-1)(k-r)(k-r-1)/r \end{pmatrix}$$

$$(p^2_{jk}) = \begin{pmatrix} r^2 & r(k - r - 1) \\ r(k - r - 1) & (k-r)^2 + 2(r-1) - k(k-1)/r \end{pmatrix}$$

For $r = 2$, the SLB scheme is the same as Triangular scheme with $n = k+1$.

(d) Latin Square and Pseudo - Latin Square ($L_g(n)$) Association Scheme:

Suppose, we have a set of $v = n^2$ elements, arranged in an $n \times n$ array. Letting two elements which appear in the same row or the same column be first associates and two elements which do not appear together in a row or column be second associates, we can define an L_2 (n) or in short L_2 association scheme.

For $3 \leq g \leq n+1$, if a set of (g-2) mutually orthogonal $n \times n$ Latin Squares exists, we can define a $L_g(n)$ association scheme from the $n \times n$ array of the v elements in the following manner. It two elements appear in the same row or column of the array, or if they correspond to the same symbol in one of the (g-2) Latin squares, they are first associates; otherwise the two elements are second associates.

For $2 \leq g \leq n+1$, the Latin square association scheme $L_g(n)$ has the following parameters.

$$v = n^2, \quad n_1 = g(n-1), \quad n_2 = (n-g+1)\,(n-1),$$

$$(p^1_{jk}) = \begin{pmatrix} (g-1)(g-2)+n-2 & (n-g+1)(g-1) \\ (n-g+1)(g-1) & (n-g+1)(n-g) \end{pmatrix}$$

$$(p^2_{jk}) = \begin{pmatrix} g\,(g-1) & g\,(n-g) \\ g\,(n-g) & (n-g)^2+g-2 \end{pmatrix}$$

Any association scheme with the parameters of an $L_g(n)$ association scheme will be called a Pseudo - Latin Square association scheme, whether or not it is obtained from a set of $(g-2)$ mutually orthogonal Latin Squares. Shrikhande (1959b) proved that the parameters of a L_2 association scheme uniquely determine the L_2 association scheme when $n \neq 4$.

(e) Negative Latin Square ($NL_g(n)$) Association Scheme :

It has been found that in many cases negative values of g and n will result in non-negative integers for the $L_g(n)$ parameters. Mesner (1964) first showed the existence of such association schemes. Substituting $-g$ for g and $-n$ for n in the parameters of $L_g(n)$ associa scheme, we obtain the set of parameters :

$$v = n^2 , \qquad n_1 = g(n+1) , \qquad n_2 = (n-g-1)\,(n+1),$$

$$(p^1_{jk}) = \begin{pmatrix} (g+1)(g+2) - n - 2 & (n-g-1)(g+1) \\ (n-g-1)(g+1) & (n-g-1)\,(n-g) \end{pmatrix}$$

$$(p^2_{jk}) = \begin{pmatrix} g\,(g+1) & g\,(n-g) \\ g\,(n-g) & (n-g)^2 - (g+2) \end{pmatrix}$$

Any association scheme with the above parameters is called Negative Latin Square ($NL_g(n)$) Association Scheme.

(f) Cyclic and Pseudo - Cyclic Association Scheme :

Consider a set of v elements, denoted by the integers 0,1,2,... ...., v-1. Suppose there is a set D of integers $(d_1,d_2,\ldots\ldots,d_{n_1})$ satisfying the following conditions :

(i) the d's are distinct and $0 < d_j < v$, $j = 1,2,\ldots\ldots, n_1$.

(ii) Among the $n_1(n_1-1)$ differences of the form d_i-d_j $(i \neq j ;$ $i,j = 1,2,\ldots\ldots, n_1)$ reduced (mod v), each of the numbers $d_1,d_2,\ldots,$ occurs α times and each of the numbers $e_1, e_2, \ldots\ldots, e_{n_2}$ occurs β times, where $d_1,d_2,\ldots\ldots, d_{n_1}$, $e_1, e_2, \ldots\ldots, e_{n_2}$ are all the integers 1,2,....., v-1. Clearly, $n_1\alpha + n_2\beta = n_1 (n_1-1)$.

(iii) The set $D = (d_1,d_2,\ldots\ldots, d_{n_1})$ is such that $D = (-d_1, -d_2 \ldots\ldots$ $\ldots, -d_{n_1})$ (mod v).

Given the element θ ($\theta = 0,1, \ldots\ldots, v-1$), define its first associates as the elements $\theta + d_1$, $\theta + d_2$, ... , $\theta + d_{n_1}$ (mod v); the remaining $(v- n_1 - 1)$ elements are the 2nd associates of θ. Then we have an association scheme, called Cyclic Association Scheme, with the following parameters :

$$v, \quad n_1, \quad n_2 = v - n_1 - 1 ,$$

$$(p^1_{jk}) = \begin{pmatrix} \alpha & n_1 - \alpha - 1 \\ n_1 - \alpha - 1 & n_2 - n_1 + \alpha + 1 \end{pmatrix} , \quad (p^2_{jk}) = \begin{pmatrix} \beta & n_1 - \beta \\ n_1 - \beta & n_2 - n_1 + \beta - 1 \end{pmatrix}$$

Conditions (i) and (ii) only were included in the definition given by Bose and Shimamoto (1952). Nandi and Adhikary (1966) realised the necessity of condition (iii) in order to make it an association scheme.

It can be shown that GD and L_2 association schemes are only particular cases of cyclical association scheme. Any association scheme with the parameters of the Cyclic association scheme will be called a Pseudo - Cyclic Association Scheme.

(g) Para Cyclic Association Scheme of Adhikary (1969b) :

The concept of Cyclic Scheme is generalised to paracyclic scheme as follows :

Consider the class of residues Mod v. Call this Abelian Group G. Let the v elements of G be divided into t disjoint sets $S_0, S_1, \ldots, S_{t-1}$ such that

$$\bigcup_{i=0}^{t-1} S_i = G$$

Let it be possible to select t sets of elements $A_0, A_1, \ldots A_{t-1}$ from 1, 2, ..., v-1 where

$$A_0 = (d_{01}, d_{02}, \ldots\ldots, d_{0n_1})$$
$$A_1 = (d_{11}, d_{12}, \ldots\ldots, d_{1n_1})$$
$$\vdots$$
$$A_{t-1} = (d_{t-1,1}, d_{t-1,2}, \ldots\ldots, d_{t-1,n_1}),$$

having the properties,

(i) d_{ij}'s $(j=1,2,\ldots\ldots n_1)$ are all distinct for a fixed i ; $i = 0,1, \ldots\ldots, t-1$.

(ii) If $\theta \in S_i$ and $\theta + d_{ik} \in S_j (j=0,1, \ldots\ldots, t-1)$, then $-d_{ik} \in A_j$.

(iii) If for any element $\theta \in S_i$ $(i = 0,1, \ldots\ldots, t-1)$, $\theta + d_{ik} \in S_j$, $(j=0,1, \ldots\ldots, t-1)$, then among the ordered differences that arise by taking the difference of all elements of A_j from every element of A_i (i.e. all possible $d_{il} - d_{jl'}$), the element d_{ik} should be repeated α times. If for any element $\theta \in S_i$, $\theta + e_{ik} \in S_j$, where e_{ik} is any non-zero element of $G - A_i$, then among the ordered differences that arise by taking the differences of all elements of A_j from every element of A_i, the element e_{ik} should be repeated β times.

(iv) α and β should be such that

$$n_1 \alpha + n_2 \beta = n_1 (n_1 - 1).$$

Then, an association scheme can be defined thus: if $\theta \in S_i$, the set of first associates of θ are $\theta + A_i$ and the remaining elements are second associates. Such an association scheme has been termed Paracyclic by Adhikary (1969b). Its parameters are

$$v, \quad n_1, \quad n_2$$

$$(p^1_{jk}) = \begin{pmatrix} \alpha & n_1-\alpha-1 \\ n_1-\alpha-1 & n_2-n_1+\alpha+1 \end{pmatrix}, \quad (p^2_{jk}) = \begin{pmatrix} \beta & n_1-\beta \\ n_1-\beta & n_2-n_1+\beta-1 \end{pmatrix}$$

The observation that any set of disjoint S_i's, $i = 0,1,\ldots\ldots, t-1$ with $\bigcup_{i=0}^{t-1} S_i = G$ serves the purpose of the association scheme has prompted us to make this slight modification in the definition given by Adhikary (1969b).

It has been shown by Adhikary that Cyclic and Triangular association schemes are particular cases of paracyclic association scheme.

.2 Association Schemes of Three or more Classes :

Rectangular Association Scheme of Vartak (1959) and Generalised Three Class Association Schemes of Adhikary (1966) have been introduced in section 3.2 of Chapter 3. The higher associate association schemes relevant for this chapter only are described below :

(a) Association Schemes from an Orthogonal Array :

Suppose we have an OA $[n^2, \sum_{i=1}^{m-1} \beta_i , n, 2]$, when n and β_i's ($i = 1,2, \ldots\ldots m-1$) are positive integers such that $n \geq 2$, $\sum_{i=1}^{m-1} \beta_i \leq n$, $m \geq 2$. Consider the n^2 columns of the array constituting

n^2 treatments. The n rows of the array are split into m-1 groups, the 1st group consisting of the first β_1 rows, 2nd group consisting of the next β_2 rows and so on, the (m-1)th group consisting of the last β_{m-1} rows. Now, define two treatments θ and φ as ith associates, if the columns corresponding to θ and φ coincide in exactly one position in the ith group of β_i rows, $i = 1,2,\ldots\ldots,m-1$. The two treatments are mth associates of each other, if the two columns do not coincide in any position in any of the (m-1) groups of rows. Since the array has strength 2 and index 1, we see that the definition of the association scheme is unambiguous, for two columns of the array can be alike in at most one position. Then, we have an m- class association scheme with $v = n^2$, $n_i = \beta_i (n-1)$, $i = 1,2, \ldots\ldots, m$, where $\beta_m = n + 1 - \sum_{i=1}^{m-1} \beta_i$.

$$p^i_{ii} = n - 2 + (\beta_i - 1)(\beta_i - 2), \quad i = 1,2, \ldots\ldots, m.$$

$$p^i_{jj} = \beta_j (\beta_j - 1), \quad i \neq j ; \; i,j = 1,2, \ldots\ldots\ldots, m.$$

$$p^i_{ij} = \beta_j (\beta_i - 1), \quad i \neq j ; \; i,j = 1,2, \ldots\ldots\ldots, m.$$

$$p^i_{jk} = \beta_j \beta_k , \quad i \neq j \neq k ; \; i,j,k = 1,2,\ldots\ldots\ldots, m.$$

This association scheme for three classes was given by singh and Shukla (1963). The extension to m- classes is obvious.

(b) Higher Associate Cyclical Association Scheme (Adhikary, 1967) :

Consider an Abelian Group, G. Let it be possible to decompose the non unit elements of G into m disjoint sets $A_1, A_2, \ldots\ldots, A_m$ such that

(i) $G - \{1\} = \bigcup_{i=1}^{m} A_i$, where $A_j^{-1} = A_j$, $\forall$ j.

(ii) A_i consists of n_i distinct elements, $i = 1,2, \ldots\ldots m$.

(iii) Among the $n_j(n_j-1)$ ratios arising out of the elements of A_j, each element of A_i occurs α_{ji} times; $i = 1,2,\ldots\ldots, m$; $j=1,2,\ldots,m-1$.

$$\text{So,} \quad \sum_{j=1}^{m} n_j \alpha_{ij} = n_i (n_i - 1)$$

(iv) Among the $n_j n_k$ elements of $A_j \times A_k$, the elements of A_i are repeated $\alpha_{jk}^{(i)}$ times each; $j \neq k$, $j,k = 1,2,\ldots, m-1$; $i=1,2,\ldots m$.

Identify the jth associates of any element θ as θA_j. Then, we shall have an association scheme of m classes having the parameters

$$v, \quad n_1, \quad n_2, \quad \ldots\ldots\ldots \quad n_m$$

$$(p_{jk}^{i}) = \begin{pmatrix} \alpha_{1i} & \alpha_{12}^{(i)} & \alpha_{13}^{(i)} & \ldots\ldots\ldots & \alpha_{1m}^{(i)} \\ \alpha_{12}^{(i)} & \alpha_{2i} & \alpha_{23}^{(i)} & \ldots\ldots\ldots & \alpha_{2m}^{(i)} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{1m}^{(i)} & \alpha_{2m}^{(i)} & \alpha_{3m}^{(i)} & \ldots\ldots\ldots & \alpha_{mi} \end{pmatrix}$$

$$i = 1,2, \ldots\ldots\ldots m.$$

The elements in the last row and column are obtained from the boundary conditions relating p^i_{jk} parameters.

7.3 Main Theorem :

The following theorem is a generalisation of theorems 3.1 and 4.1 of Blackwelder (1969).

Theorem 7.3.1 : Suppose we have an m-class association scheme with v treatments, with association matrices $B_o = I_v, B_1, B_2, \ldots, B_m$. Suppose, we have p non-identical sets of indices

$$S_i = \left\{ i_1, i_2, \ldots\ldots i_{t_i} \right\}, \qquad i = 1,2, \ldots\ldots p.,$$

where $i_1, i_2, \ldots\ldots i_{t_i}$ are t_i distinct integers such that

$i_j \in \left\{0,1,2, \ldots\ldots m\right\}$ for $j = 1,2, \ldots\ldots t_i \leq m$, $i=1,2,\ldots\ldots p$.

Then, the necessary and sufficient conditions for $C = [C_1 : C_2 : \ldots\ldots$ $\ldots\ldots : C_p]$ with $C_i = B_{i_1} + B_{i_2} \ldots\ldots + B_{i_{t_i}}$, $i= 1,2,\ldots\ldots p$, to be the incidence matrix of a BIB Design with parameters v, b, r, k, λ, where obviously $b = pv$ and hence $r = pk$, is that

$$\text{(i)} \quad \sum_{i=1}^{p} \left\{ \sum_{j=1}^{t_i} p^l_{i_j i_j} + 2 \sum_{\substack{j<j' \\ j,j'=1}}^{t_i} p^l_{i_j\, i_{j'}} \right\} = \lambda > 0$$

for $l = 1,2, \ldots\ldots\ldots m$.

and (ii) $\lambda (v - 1) = r (k - 1)$,

$$\text{where } r = \sum_{i=1}^{p} \sum_{j=1}^{t_i} n_{i_j}$$

Proof : C is a $v \times p v$ matrix with elements 0 and 1, where the columns correspond to blocks and rows to treatments.

Replication for a particular treatment

= The number of ones in the corresponding row of C.

Now, the number of one's in any given row of $C_i = n_{i_1} + n_{i_2} + \ldots$ $\ldots\ldots + n_{i_{t_i}} = n_i$, say, $i = 1, 2, \ldots\ldots p$.

So, the number of ones in any given row of $C = \sum_{i=1}^{p} n_i = r =$ the replication for each treatment.

Number of times two treatments occur together in the same block = The number of times the pair $\binom{1}{1}$ occur in the two corresponding rows of C. Suppose two treatments are lth associates of each other, $l = 1, 2, \ldots\ldots m$. Let us consider the two corresponding rows of C and obtain the two rowed submatrix of C.

Number of times the pair $\binom{1}{1}$ occurs in the part of the two rowed submatrix of C obtained from C_i is clearly ,

$$\sum_{j=1}^{t_i} p^{1}_{i_j i_j} + 2 \sum_{\substack{j<j' \\ j,j'=1}}^{t_i} p^{1}_{i_j i_{j'}} = \lambda_{i1} , \quad \text{say} , \quad i = 1,2,\ldots. p.$$

Hence the number of times the pair $\binom{1}{1}$ occurs in the full two rowed submatrix of C

$$= \sum_{i=1}^{p} \lambda_{i1} = \lambda_1 , \quad \text{say}.$$

Thus the necessary part of the theorem is obvious. For the sufficiency part

condition (i) implies $\lambda_1 = \lambda_2 \ldots\ldots = \lambda_m = \lambda$

= The number of blocks of C containing any pair of treatments together, as two treatments in any pair must be ith associates of each other for some i.

So, condition (i) implies C is the incidence matrix of an (r, λ)-system with v varieties and $b = p\,v$ blocks. Now, in addition, condition (ii) implies $b = \frac{v\,r}{k}$, where $k = \frac{\lambda(v-1)}{r} + 1$.

Hence, by theorem 6.2.1, the system is a BIB Design with parameters v, $b = pv$, $r = pk$, k, λ.

It may be noted that equality of block sizes has not been assumed in the conditions of the theorem and is implied automatically by the conditions (i) and (ii) of the theorem. So, the theorem establishes that in any PBIB Design with m associate classes the conditions (i) and (ii) as stated in Theorem 7.3.1 implies $\sum_{j=1}^{t_i} n_{i_j} = k$, same constant for $i = 1,2, \ldots\ldots p$.

In particular, if $t_1 = t_2 = \ldots\ldots = t_p = 1$, then $n_{1_1} = n_{2_1} = \ldots\ldots = n_{p_1}$ and as such the conditions (i) and (ii) of theorem 7.3.1 of this section implies conditions (i), (ii) and (iii) of theorem 4.1 of Blackwelder (1969). The other particular case considered in Theorem 3.1 of Blackwelder is $p = 1$. In this case C is $v \times v$ matrix. So, considering the sufficiency part of Theorem 7.3.1, condition (i) implies that C is the incidence matrix of an (r, λ)-system with $b = v$. Any such design has been proved to be a symmetrical BIB Design by Mullin and Stanton (1966). So, when $p = 1$, the condition (ii) of Theorem 7.3.1 is redundant and is implied by condition (i).

7.4 Construction of BIB Designs from Association Matrices :

The following BIB Designs which can be constructed with the help of Theorem 7.3.1 from association matrices of the schemes listed by Blackwelder are not included in the list of designs provided by him.

(i) Triangular Scheme : For $n=4$, $[B_0 + B_1]$ is an incidence matrix of BIB $(6, 6, 5, 5, 4)$.

(ii) Pseudo - Cyclic Scheme : For $\alpha = \beta$, B_1 is an incidence matrix of BIB (v, v, n_1, n_1, α).

(iii) Rectangular Scheme : For $l = n = 4$, $[B_1 + B_2 : B_3]$ is an incidence matrix of BIB $(9, 18, 8, 4, 3)$.

(iv) Three - class Scheme from an OA : For $\beta_1 = \beta_2 = \beta_3 = \frac{n+1}{3}$ where $n \equiv 2 \pmod 3$, $[B_0 + B_1 : B_0 + B_2 : B_0 + B_3]$ is an incidence matrix of BIB $\left(n^2, 3n^2, n^2+2, \frac{n^2+2}{3}, \frac{n^2+2}{3}\right)$.

And, in general for an m-class association scheme obtained from an OA, when, $\beta_1 = \beta_2 \; \ldots\ldots = \beta_m = \frac{n+1}{m}$ is an integer, say, β. $[B_1 : B_2 : \ldots\ldots : B_m]$ is an incidence matrix of BIB $(n^2, mn^2, n^2-1, \beta(n-1), \beta(n-1) - 1)$. Also, $[B_0 + B_1 : B_0 + B_2 : \ldots\ldots : B_0 + B_m]$ is an incidence matrix of BIB $(n^2, mn^2, n^2+m-1, \beta(n-1)+1, \beta(n-1) + 1)$.

(v) Equivalence of a Pseudo-Cyclic association scheme with parameters $v = 4u + 1$, $n_1 = n_2 = 2u$, $p^1_{11} = u-1$, $p^2_{11} = u$ and S_{4u+1} has been established by Shrikhande (1962). In fact, for any association scheme with two associate classes and $p^1_{12} = p^2_{12}$ it can be easily shown that writting $X = B_1 - B_2$, the inner product of any two rows of X is $v - 2 - 4\, p_{12}^{1}$. Now for an SLB association scheme with $k = 2r$ and consequently $v = (2r-1)^2 + 1$, the inner product of any two rows of $X = B_1 - B_2$ is zero. Hence X becomes a BOD. As X is symmetrical, by multiplying the rows and columns of X by -1 wherever necessary, it can be written as

$$X = \begin{pmatrix} 0 & E_{1,v-1} \\ E_{v-1,1} & Y \end{pmatrix}$$, where Y is a symmetrical

$(v-1) \times (v-1)$ matrix with diagonal elements 0 and off diagonal elements ± 1, satisfying the property

$$Y Y' = (v-1)\, I_{v-1} - E_{v-1,\, v-1} .$$

Hence, $Y = S_{(2r-1)^2}$. Now, the existence of $S_{(2r-1)^2}$ is equivalent to a Pseudo cyclic association scheme with parameters $v = (2r-1)^2$, $n_1 = n_2 = 2r(r-1)$, $p_{11}^{1} = r(r-1) - 1$, $p_{11}^{2} = r(r-1)$, which in its turn gives a BIB Design with parameters $v^* = (2r-1)^2$, $b^* = 2(2r-1)^2$, $r^* = 4r(r-1)$, $k^* = 2r(r-1)$, $\lambda^* = 2r(r-1) - 1$, as is evident from Blackwelder (1969). Thus, the existence of a Singly Linked Block association scheme with $k = 2r$ implies the existence of $S_{(2r-1)^2}$, implies the existence of BIB$((2r-1)^2, 2(2r-1)^2, 4r(r-1), 2r(r-1), 2r(r-1) - 1)$.

Now, we ~~prove~~ give the following Lemma to construct many more BIB Designs from the association matrices of the schemes listed by Blackwelder (1969).

Lemma 7.4.1 : If $[\, C_{(2k-1)\times(2k-1)} : D_{(2k-1)\times(2k-1)} \,]$ be the incidence matrix of an (r, λ)- system with $r = 2k-1$, where each row of C contains λ 1's, each column of C contains $(k-1)$ 1's and each column of D k 1's, then $\begin{bmatrix} C_{(2k-1)x(2k-1)} & : & D_{(2k-1)x(2k-1)} \\ E_{1,(2k-1)} & : & O_{1,(2k-1)} \end{bmatrix}$

is an incidence matrix of BIB $(2k, 2(2k-1), 2k-1, k, \lambda)$.

Making use of the Lemma 7.4.1 in addition to theorem 7.3.1, the following series of BIB Designs can be constructed through the association schemes listed by Blackwelder (1969).

(i) Pseudo-Cyclic Scheme : For $v = 4u+1$, $n_1 = 2u$, $p^1_{11} = u-1$, $p^2_{11} = u$, $\begin{bmatrix} B_1 & : & B_o + B_1 \\ E_{1,\,(4u+1)} & : & O_{1,\,(4u+1)} \end{bmatrix}$ is an incidence matrix of BIB $(4u+2,\ 8u+2,\ 4u+1,\ 2u+1,\ 2u)$.

(ii) $L_g(n)$ Scheme : For $n = 2g-1$,

$\begin{bmatrix} B_1 & : & B_o + B_1 \\ E_{1,\,n^2} & : & O_{1,\,n^2} \end{bmatrix}$ is an incidence matrix of

BIB $(\ (2g-1)^2+1,\ 2(2g-1)^2+1,\ (2g-1)^2,\ 2g(g-1)+1,\ 2g(g-1)\ )$.

(iii) $NL_g(n)$ Scheme : For $n = 2g+1$, $\begin{bmatrix} B_1 & : & B_o + B_1 \\ E_{1,n^2} & : & 0_{1,n^2} \end{bmatrix}$

is an incidence matrix of BIB ($(2g+1)^2 + 1$, $2(2g+1)^2$, $(2g+1)^2$, $2g(g+1)+1$, $2g(g+1)$).

(iv) Three class Association Scheme from an OA : For $\beta_1 = \frac{n+1}{2}$ when n is an odd integer, $\begin{bmatrix} B_1 & : & B_o + B_1 \\ E_{1,n^2} & : & 0_{1,n^2} \end{bmatrix}$ is an incidence matrix of BIB (n^2+1, $2n^2$, n^2, $\frac{n^2+1}{2}$, $\frac{n^2-1}{2}$). For $\beta_1 + \beta_2 = \frac{n+1}{2}$ when n is an odd integer $\begin{bmatrix} B_1 + B_2 & : & B_o + B_1 + B_2 \\ E_{1,n^2} & : & 0_{1,n^2} \end{bmatrix}$ is a solution of the same design.

It appears the method of constructing BIB Designs from Association matrices described in this section can also be successfully applied to the two associate paracyclic association schemes (1969b)and higher associate cyclical association schemes (1967) proposed by Adhikary. Adhikary's (1969b)example of paracyclic association scheme with $v = 16$, $n_1 = 6$, $p^1_{11} = p^2_{11} = 2$ readily provides a BIB (16, 16, 6, 6, 2) in the incidence matrix B_1. But no systematic attempt has been made in the present thesis to exploit this scheme for constructing new BIB Designs. To illustrate the further potentiality of the procedure suggested by theorem 7.3.1 and Lemma 7.4.1, some BIB Designs are constructed from higher associate cyclical association schemes of Adhikary (1967). Three of these Designs were unsolved as yet, as indicated in the lists of Rao (1961b) and Sprott (1962).

Example 7.4.1 : Consider the group formed by different powers of a when $a^{17} = 1$,

Let $A_1 : a, a^4, a^{13}, a^{16}$

$A_2 : a^3, a^5, a^{12}, a^{14}$

$A_3 : a^2, a^8, a^9, a^{15}$

$A_4 : a^6, a^7, a^{10}, a^{11}$.

Then the parameters of the 4 class association scheme are :

$$n_1 = n_2 = n_3 = n_4 = 4$$

$$(p^1_{jk}) = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}, \quad (p^2_{jk}) = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$(p^3_{jk}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}, \quad (p^4_{jk}) = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}$$

The distinct solutions obtained are :

(i) $[B_1 : B_2 : B_3 : B_4]$ is an incidence matrix of BIB (17, 68, 16, 4, 3).

(ii) $[B_1 + B_3 : B_2 + B_4]$ is an incidence matrix of BIB (17, 34, 16, 8, 7).

(iii) $[B_o + B_1 : B_o + B_2 : B_o + B_3 : B_o + B_4]$ is an incidence matrix of BIB (17, 68, 20, 5, 5).

(iv) $\begin{bmatrix} B_2 + B_4 & : & B_o + B_2 + B_4 \\ E_{1,\ 17} & : & O_{1,\ 17} \end{bmatrix}$ is an incidence matrix of BIB (18, 34, 17, 9, 8). This is indicated as an unsolved case in Sprott's list (1962).

Example 7.4.2 : Consider the group formed by different powers of a when $a^{13} = 1$,

Let A_1 : a , a^{12}

A_2 : a^2, a^{11}

A_3 : a^3, a^{10}

A_4 : a^4, a^9

A_5 : a^5, a^8

A_6 : a^6, a^7.

Then the parameters of the 6-class association scheme are

$$n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = 2.$$

$$(p^1_{jk}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad (p^2_{jk}) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

$$(p^3_{jk}) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad (p^4_{jk}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$(p^5_{jk}) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (p^6_{jk}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(i) $[B_1 + B_3 + B_4 \; : \; B_2 + B_5 + B_6]$ is an incidence matrix of BIB (13, 26, 12, 6, 5).

(ii) $\begin{bmatrix} B_2 + B_5 + B_6 & : & B_o + B_2 + B_5 + B_6 \\ E_{1;\ 13} & : & 0_{1,\ 13} \end{bmatrix}$ is an incidence matrix of BIB (14, 26, 13, 7, 6). This design is missing in Rao's list (1961b)of BIB Designs with $r = 11$ to 15.

Example 7.4.3 : Consider the group formed by different powers of a when $a^{11} = 1$.

Let
$A_1 : a, a^{10}$ $A_4 : a^4, a^7$
$A_2 : a^2, a^9$ $A_5 : a^5, a^6$
$A_3 : a^3, a^8$

Then the parameters of the 5 class association scheme are

$$n_1 = n_2 = n_3 = n_4 = n_5 = 2$$

$$(p^1_{jk}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad (p^2_{jk}) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix},$$

$$(p^3_{jk}) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad (p^4_{jk}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$(p^5_{jk}) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

(i) $[B_1 + B_3 \; : \; B_1 + B_4 \; : \; B_2 + B_3 \; : \; B_2 + B_5 \; : \; B_4 + B_5]$

is an incidence matrix of BIB (11, 55, 20, 4, 6). This design is indicated as an unsolved case in Sprott's list (1962).

(ii) $[B_0 + B_1 + B_3 \; : \; B_0 + B_1 + B_4 \; : \; B_0 + B_2 + B_3 \; :$

$B_0 + B_2 + B_5 \; : \; B_0 + B_4 + B_5]$ is an incidence matrix of BIB (11, 55, 25, 5, 10).

7.5 Construction of Further Series of BIB, DBIB and GD Designs :

The series of BIB Designs obtained with the help of theorem 7.3.1 (in some cases along with lemma 7.4.1) from association matrices have got certain properties which can be exploited to construct further series of BIB Designs, series of DBIB Designs and GD Designs. We first state and prove some lemmas from which series of BIB and DBIB's would follow. Of the lemmas that follow, 7.5.1 and 7.5.4 have already appeared as lemmas 2.6.1 and 2.6.2 respectively in Chapter 2. Here, they are restated along with the other related lemmas only for completeness.

Lemma 7.5.1 : If N is the incidence matrix of a BIB Design with parameters $v = b = 4t-1$, $r = k = 2t$, $\lambda = t$ and $\bar{N} = E_{vv} - N$ is the incidence matrix of its complement, then

$$M_1 = \begin{bmatrix} N & : & \bar{N} \\ 0_{1v} & : & E_{1v} \end{bmatrix}$$ is the incidence matrix of

a DBIB Design with parameters $v = 4t$, $b = 8t-2$, $r = 4t-1$, $k = 2t$, $\lambda = 2t-1$, $\mu = t-1$.

Proof : That M_1 is a BIB Design was proved by Bhat and Shrikhande (1970). That it is also a DBIB can be seen easily.

Lemma 7.5.2 : If N and $\bar{N}$ are as defined in lemma 7.5.1, then

$$M_2 = \begin{bmatrix} \bar{N} & : & \bar{N} & : & E_{v1} \\ \bar{N} & : & N & : & 0_{v1} \\ E_{1v} & : & 0_{1v} & : & 0 \end{bmatrix}$$ is the incidence matrix

of a symmetrical BIB Design with parameters $v = b = 8t-1$, $r = k = 4t-1$, $\lambda = 2t-1$. This result was proved by Bhat and Shrikhande (1970).

Arguing in a manner similar to that of Bhat and Shrikhande (1970), we have the following lemmas.

Lemma 7.5.3 : If N is the incidence matrix of a BIB Design with parameters $v = 2t+2$, $b = 4t+2$, $r = 2t+1$, $k = t+1$, $\lambda = t$, then

$$M_3 = \begin{bmatrix} N & : & N & : & E_{v1} & : & O_{v1} \\ N & : & \bar{N} & : & O_{v1} & : & E_{v1} \end{bmatrix}$$ is the incidence

matrix of a BIB Design with parameters $v = 2t'+2$, $b = 4t'+2$, $r = 2t'+1$, $k = t'+1$, $\lambda = t'$ where $t' = 2t+1$.

By repeated applications of the same lemma, the existence of a BIB $(2t+2,\ 4t+2,\ 2t+1,\ t+1,\ t)$, implies the existence of a series of BIB $(2t'+2,\ 4t'+2,\ 2t'+1,\ t'+1,\ t')$, where $t' = 2^p t+(2^p-1)$ for $p = 0$ or any positive integer.

Lemma 7.5.4 : If N is the incidence matrix of a BIB Design with parameters $v = 2t+2$, $b = 4t+2$, $r = 2t+1$, $k = t+1$, $\lambda = t$, then $M_4 = [N : \bar{N}]$ is the incidence matrix of a DBIB Design with parameters $v = 2t+2$, $b = 8t+4$, $r = 4t+2$, $k = t+1$, $\lambda = 2t$, $\mu = t-1$.

Hence by utilising lemma 7.5.3 along with lemma 7.5.4, the existence of a BIB $(2t+2,\ 4t+2,\ 2t+1,\ t+1,\ t)$, implies the existence of a series of DBIB $(2t'+2,\ 8t'+4,\ 4t'+2,\ t'+1,\ \lambda\ 2t',\ t'-1)$, where $t' = 2^p t + (2^p - 1)$ for $p = 0$ or any positive integer.

Lemma 7.5.5 : If N is the incidence matrix of a BIB Design with parameters $b = 2v = 2\ (2t+3)$, $r = 2k = 2\ (t+1)$, $\lambda = t$, then

$$M_5 = \begin{bmatrix} N & : & N & : & E_{v1} & : & E_{v1} \\ N & : & \bar{N} & : & 0_{v1} & : & 0_{v1} \\ E_{1b} & : & 0_{1b} & : & 0 & : & 0 \end{bmatrix}$$ is the

incidence matrix of a BIB Design with parameters $b = 2v = 2(2t'+3)$, $r = 2k = 2\ (t'+1)$, $\lambda = t'$ with $t' = 2t+2$.

By repeated applications of the same lemma, the existence of a BIB $(2t+3,\ 4t+6,\ 2t+2,\ t+1,\ t)$, implies the existence of a series of BIB $(2t'+3,\ 4t'+6,\ 2t'+2,\ t'+1,\ t')$, where $t' = 2^p t + 2(2^p-1)$ for $p = 0$ or any positive integer.

Lemma 7.5.6 : If N is the incidence matrix of a BIB Design with parameters $b = 2v = 2\ (2t+3)$, $r = 2k = 2\ (t+1)$, $\lambda = t$, then

$$M_6 = \begin{bmatrix} N & : & \bar{N} \\ E_{1b} & : & 0_{1b} \end{bmatrix}$$ is the incidence matrix

of a DBIB Design with parameters $v = 2t+4$, $b = 4(2t+3)$, $r = 2(2t+3)$, $k = t+2$, $\lambda = 2t+2$, $\mu = t$.

Hence by utilising Lemma 7.5.5 repeatedly along with Lemma 7.5.6, the existence of a BIB $(2t+3,\ 2(2t+3),\ 2(t+1),\ t+1,\ t)$, implies the existence of a series of DBIB $(2t'+4,\ 4(2t'+3),\ 2(2t'+3),\ t'+2,\ 2t'+2,\ t')$, where $t' = 2^p t + 2(2^p - 1)$ for $p = 0$ or any positive integer.

A. Series of BIB and DBIB Designs from Association Matrices :

(a) From Blackwelder (1969) and Shrikhande and Singh (1962), it is known that if a Singly Linked Block Scheme exists for $k = 2r+1$, the matrix B_1 for the scheme is an incidence matrix for a BIB $(4r^2 - 1,\ 4r^2 - 1,\ 2r^2,\ 2r^2,\ r^2)$. Hence, applying lemma 7.5.1 and 7.5.2 repeatedly, the existence of a linked block scheme for $k = 2r+1$, implies the existence of a series of DBIB $(2^{2+p}.\ r^2,\ 2^{3+p}.\ r^2 - 2,\ 2^{2+p}.\ r^2 - 1,\ 2^{1+p}.\ r^2,\ 2^{1+p}.\ r^2 - 1,\ 2^p.\ r^2 - 1)$ and a series of BIB $(2^{2+p}.\ r^2 - 1,\ 2^{2+p}.\ r^2 - 1,\ 2^{1+p}.\ r^2,\ 2^{1+p}.\ r^2,\ 2^p.\ r^2)$ for $p = 0$ or any positive integer. That these two series are implied by the existence of BIB $(4r^2 - 1,\ 4r^2 - 1,\ 2r^2,\ 2r^2,\ r^2)$ is quite well known from the property of H matrices and particularly from Bhat and Shrikhande (1970). The purpose of mentioning them here is only to indicate their construction from Association Scheme matrices.

(b) Blackwelder (1969) has shown that for a number of two-class association schemes under certain conditions, $[B_1 : B_2]$ is an incidence matrix of a BIB Design with parameters of the form

$$b = 2v = 2(2t+3), \quad r = 2k = 2(t+1), \quad \lambda = t.$$

The result also follows from the generalised theorem 7.3.1 of this chapter. From this result we get the following additional series, using lemmas 7.5.5 and 7.5.6. The existence of a series of BIB$(2t'+3, 4t'+6, 2t'+2, t'+1, t')$ and a series of DBIB $(2t'+4, 4(2t'+3), 2(2t'+3), t'+2, 2t'+2, t')$ with $t' = 2^p t + 2(2^p-1)$ for $p = 0$ or any positive integer, is implied by the existence of any of the following :

(i) a Pseudo-Cyclic scheme with parameters $v = 4u+1$, $n_1 = 2u$, $p_{11}^1 = u-1$, $p_{11}^2 = u$, for $t = 2u-1$.

(ii) $L_g(n)$ Scheme with $n = 2g-1$, for $t = 2g^2 - 2g - 1$.

(iii) $NL_g(n)$ Scheme with $n = 2g+1$, for $t = 2g^2 + 2g - 1$.

(c) By utilising Lemma 7.4.1, it has been shown in section 7.4 that for a number of two-class association schemes under certain conditions $\begin{bmatrix} B_1 & : & B_0 + B_1 \\ E_{1v} & : & 0_{1v} \end{bmatrix}$ is the incidence matrix of a BIB Design with parameters of the form $v = 2t+2$, $b = 4t+2$, $r = 2t+1$,

$k = t+1$, $\lambda = t$. From them we get the following series, using lemmas 7.5.3 and 7.5.4. The existence of a series of BIB $(2t'+2, 4t'+2, 2t'+1, t'+1, t')$ and a series of DBIB $(2t'+2, 8t'+4, 4t'+2, t'+1, 2t', t'-1)$ with $t' = 2^p t + (2^p-1)$ for $p = 0$ or any positive integer is implied by the existence of any of the following :

(i) Pseudo - Cyclic Scheme with parameters $v = 4u+1$, $n_1 = 2u$, $p^1_{11} = u-1$, $p^2_{11} = u$, for $t = 2u$.

(ii) $L_g(n)$ Scheme with $n = 2g-1$, for $t = 2g(g-1)$.

(iii) $NL_g(n)$ Scheme with $n = 2g+1$, for $t = 2g(g+1)$.

B. Series of GD Designs from Association Matrices :

Obviously by omitting some particular blocks in the series of BIB Designs obtained in the present section through lemmas 7.5.1 to 7.5.6, we get some series of corresponding GD Designs. But the procedure can be generalised and we can construct infinitely many GD Designs from association matrices of the known PBIB schemes. For constructing series of GD Designs from the same association matrices as used in the construction of the series of BIB and DBIB Designs already considered, the following lemmas may be made use of.

Lemma 7.5.7 : Let there exist OA $[p.\ 2^2,\ s,\ 2,\ 2]$ and let N be an incidence matrix of a BIB Design with parameters $v = b = 4t-1$, $r = k = 2t$, $\lambda = t$. Writing $N_1 = \begin{bmatrix} N \\ 0_{1v} \end{bmatrix}$ and M_7 as the matrix obtained from the OA be replacing each 1 in the array by the matrix N_1 and each 0 in the array by $\bar{N}_1$, M_7 is the incidence matrix of a GD Design with parameters $v = 4st$, $b = 4p(4t-1)$, $r = 2p(4t-1)$, $k = 2t\ s$; $\lambda_1 = 2p(2t-1)$, $\lambda_2 = p(4t-1)$, $m = s$, $n = 4t$.

Lemma 7.5.8 : Let there exist an OA $[p.\ 2^2,\ s,\ 2,\ 2]$ and let N be the incidence matrix of a BIB Design with parameters $b = 2v = 2(2t+3)$, $r = 2k = 2\ (t+1)$, $\lambda = t$. Writing $N_1 = \begin{bmatrix} N \\ E_{1b} \end{bmatrix}$ and M_8 as the matrix obtained from the OA by replacing each 1 in the array by the matrix N_1 and each 0 in the array by $\bar{N}_1$, M_8 is the incidence matrix of a GD Design with parameters $v = s(2t+4)$, $b = 8p\ (2t+3)$, $r = 4p(2t+3)$, $k = s(t+2)$, $\lambda_1 = 2p(2t+2)$, $\lambda_2 = 2p(2t+3)$, $m = s$, $n = 2t+4$.

The proofs of the lemmas are easy and hence omitted. These lemmas can be utilised for obtaining a large number of series of GD Designs from the same association matrices for the association schemes under the same conditions as utilised already in the construction of the series of BIB and DBIB Designs in this section.

Construction of series of GD Designs with the help of the incidence matrix of a BIB Design with parameters $v = 2t+2$, $b = 4t+2$, $r = 2t+1$, $k = t+1$, $\lambda = t$, has been considered extensively by Rao (1970) and hence is omitted here.

8. PROPERTY OF NEAR RESOLVABILITY

8.1 Introduction :

In this chapter, we give a method for constructing Incomplete Block Designs with the help of a known solution of some other Incomplete Block Design, which satisfies a certain property termed by us as 'near resolvability'. The method of construction given in this chapter is a generalisation of the procedures suggested by Shrikhande and Raghava rao (1963) in connection with resolvable Incomplete Block Designs. The required property of 'near resolvability' can be defined thus :

An equireplicate Incomplete Block Design with v treatments and b blocks will be called nearly resolvable, if the b blocks of the design can be divided into a number, say s of sets such that

(i) the treatments in any particular set of blocks are all distinct,

(ii) each set of blocks contains a constant number of treatments,

and (iii) for each pair of treatments, the number of sets of blocks in which at least one is absent is a constant, say p.

A near resolvable design becomes resolvable if $p = 0$. Thus, near-resolvability includes resolvability.

8.2 Main Results :

The main results of this chapter are contained in the following theorem and the subsequent corollaries.

Theorem 8.2.1 : Let there exist a 'near-resolvable' PBIB Design with l associate classes with parameters v, $b = ms$, r_1, k_1; $n_1, n_2, \ldots\ldots n_l$, $\lambda_1, \lambda_2, \ldots\ldots, \lambda_l$ and (p^i_{jk}), $i,j,k = 1,2,\ldots\ldots, l$ such that the b blocks can be divided into s sets, each set consisting of m blocks and containing mk_1 distinct treatments and the number of sets of blocks in which at least one of any pair of treatments is absent is p. Suppose, a GD Design also exists with parameters mk_1, b_2, r_2, k_2 ; λ_1', λ_2', m, k_1. Then, there always exists a PBIB Design with l associate classes with parameters v, $b = b_2 s$, $r = r_1 r_2$, $k = k_2$; n_i, $\lambda_i \lambda_1' + (s - \lambda_i - p)\lambda_2'$, $i = 1,2, \ldots\ldots l$ and (p^i_{jk}), $i,j,k = 1,2, \ldots\ldots l$.

Proof : From the conditions of the given PBIB Design, a set of m blocks out of the s sets as indicated in the theorem gives a division of mk_1 distinct treatments into m groups of k_1 treatments each. Hence, a GD Design with the given parameters can be obtained from any such set, where (i) if two treatments occur together within a block of the set, they will be occuring together in λ_1' blocks of the GD Design obtained from the set, (ii) if two treatments are both present in the set, but occur in different blocks, they will be occurring together in λ_2' blocks of the GD Design, and

(iii) if at least one of the two treatments is absent in the set, the pair does not occur together in any block of the GD Design obtained from the set.

Now, let such a GD Design with the given parameters be constructed for each of the s sets of the given PBIB Design. If two treatments are ith associates of each other in the given PBIB, they occur together in a block in λ_i of the s sets of blocks, in p sets at least one of the treatments is absent and in the remaining $(s - \lambda_i - p)$ sets, both the treatments are present, but in separate blocks. Thus, when all the s GD Designs obtained from the s sets of blocks are considered together, any two treatments which are ith associates of each other in the given PBIB, occur together in a block λ_1' times in λ_i of them, λ_2' times in $(s - \lambda_i - p)$ of them and in the remaining p GD Designs, they do not occur together at all. So, the two treatments occur together in $\lambda_i \lambda_1' + (s - \lambda_i - p). \lambda_2'$ blocks of the Incomplete Block Design obtained by the juxtaposition of s GD Designs already explained, $i = 1, 2, \ldots, l$.

Moreover, each treatment occurs in exactly r_1 sets of the s sets of blocks obtained from the given PBIB Design and hence in $r_1 r_2$ blocks of the resulting Incomplete Block Design, obtained by a juxtaposition of the s GD Designs as explained.

Hence, the resulting Incomplete Block Design is the required PBIB as described in the enunciation of the theorem.

The theorem 8.2.1 will remain true when $\lambda_1 = \lambda_2 \ldots\ldots = \lambda_1 = \lambda$ and any association scheme is redundant there. Thus, we have

Corollary 8.2.1 : Let there exist a near - resolvable BIB Design with parameters v, $b = ms$, r_1, k_1, λ such that the b blocks can be divided into s sets, each set consisting of m blocks and containing $m\,k_1$ distinct treatments and the number of sets of blocks in which at least one of any pair of treatments is absent is p. Suppose, a GD Design also exists with parameters mk_1, b_2, r_2, k_2 ; λ'_1 , λ'_2 , m, k_1 . Then there always exists a BIB Design with parameters v, $b = b_2 s$, $r = r_1 r_2$, $k = k_2$, $\lambda \lambda'_1 + (s - \lambda - p) \lambda'_2$.

Certainly the theorem 8.2.1 and Corollary 8.2.1 are true when the given PBIB or BIB is resolvable. In those cases $p = 0$ and $s = r_1$.

A very well known result in the construction of GD Designs is stated in the form of a lemma as follows.

Lemma 8.2.1 : If a BIB Design with parameters $v = m$, b_2, r_2, k'_2, λ exists, replacing each treatment in the Design by a group of k_1 treatments, a GD Design is obtained with parameters $v = mk_1$, b_2, r_2, $k_2 = k_1 k'_2$; $\lambda'_1 = r_2$, $\lambda'_2 = \lambda$, m, k_1.

Hence, by Lemma 8.2.1 and the fact that near resolvability includes resolvability, the two theorems of Shrikhande and Raghavarao (1963) can be obtained as particular cases of the theorem 8.2.1 and Corollary 8.2.1. But the GD Designs in theorem 8.2.1 and Corollary 8.2.1 can exist in various other ways than the one suggested by Lemma 8.2.1. Also, there exist near-resolvable designs which are not resolvable. So, the results of this section are more general than those of Shrikhande and Raghavarao (1963). To illustrate the use of theorem 8.2.1 and Corollary 8.2.1, the problem of constructing some series of BIB and GD Designs through them is taken up in the following section.

8.3 Construction of BIB and GD Designs :

8.3.1 BIB Designs :-

(i) The problem of construction of BIB Designs from Association Matrices of the known PBIB Association schemes has been extensively considered in chapter 7. For the t class association scheme obtained from an OA, it has been observed that if $\beta_1 = \beta_2 = \dots = \beta_t = \frac{n+1}{t}$, an integer, say β, $[B_1 : B_2 : \dots : B_t]$ is the incidence matrix of a BIB Design with parameters $v = n^2$, $b = t\,n^2$, $r = n^2 - 1$, $k = \beta(n-1)$, $\lambda = \beta(n-1) - 1$. The $t\,n^2$ blocks of the BIB so constructed can be divided into n^2 sets of t blocks each such that the $t\beta(n-1) = n^2 - 1$ treatments in any set of blocks are all distinct and for any two out of n^2 treatments, the number of sets in which at least one is absent is a constant, viz. 2. So, the BIB so constructed is 'nearly resolvable'.

Now, suppose a GD Design also exists with parameters $v = t.\beta(n-1) = n^2 - 1$, b_2, r_2, k_2 ; λ_1', λ_2' , t, β (n-1). By Corollary 8.2.1, we can then always construct a BIB Design with parameters $v = n^2$, $b = b_2 n^2$, $r = r_2 (n^2 - 1)$, $k = k_2$, $\lambda = (\beta(n-1) - 1)\lambda_1' + (n^2 - \beta(n-1) - 1)\lambda_2'$. In particular, the existence of a BIB Design with parameters $v = t$, b_2, r_2, k_2' , λ' along with the BIB Design given by $[B_1 : B_2 : \ldots\ldots : B_t]$ will in accordance with lemma 8.2.1 imply the existence of a BIB Design with parameters $v = n^2$, $b = b_2 n^2$, $r = r_2 (n^2 - 1)$, $k = k_2 = \beta(n-1).k_2'$, $\lambda = (\beta(n-1)-1)r_2 + (n^2 - \beta(n-1) - 1)\lambda'$.

Example : For $v = 9$, taking $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$ in 4-associate association scheme obtained from an OA, $[B_1 : B_2 : B_3 : B_4]$ is the incidence matrix of BIB Design with parameters $v = 9$, $b = 36$, $r = 8$, $k = 2$, $\lambda = 1$. Now, there exists a GD Design with parameters $v = 8$, $b = 6$, $r = 3$, $k = 4$, $\lambda_1 = 3$, $\lambda_2 = 1$, $m = 4$, $n = 2$ (Bose, Clatworthy and Shrikhande, 1954). Hence, we can construct a BIB Design with parameters $v = 9$, $b = 54$, $r = 24$, $k = 4$, $\lambda = 9$.

(ii) If $th+1$ is a prime power and x is a primitive root of $GF(th+1)$, then the t initial sets

$$(x^{i}, x^{i+t}, x^{i+2t}, \ldots\ldots\ldots, x^{i+(h-1)t}),$$

$$i = 0, 1, 2, \ldots\ldots, t-1,$$

form a difference set for a BIB Design with parameters

$$v = th+1, \quad b = t\,(th+1), \quad r = th, \quad k = h, \lambda = h-1.$$

From the construction method, it is clear the BIB Design is near resolvable. The $b = t\,(th+1)$ blocks of the Design can be divided into $(th+1)$ sets of t blocks each such that each set contains th distinct treatments and the number of sets of blocks in which at least one of any pair of treatments is absent is 2. So, if in addition to $th+1$ being a prime power, a GD Design with parameters $v = th$, b_2, r_2, k_2 ; λ_1' , λ_2' , t, h exists or in particular a BIB Design with parameters $v = t$, b_2, r_2, k_2', λ' exists, there will exist a BIB Design with parameters $v = th+1$, $b = b_2(th+1)$, $r = r_2\,th$, $k = k_2$, $\lambda = (h-1)\lambda_1' + h\,(t-1)\lambda_2'$ (in the second case $k_2 = k_2'\,h$, $\lambda_1' = r_2$ and $\lambda_2' = \lambda'$).

(iii) From Bose (1959), it is known that if s is a prime power, there always exists a resolvable BIB $(s^3+1, s+1, 1)$. Hence, by corollary 3.5.2b, if s and $s+1$ are both prime powers, there exist a series of resolvable BIB's ($(s^3+1)(s+1)^n$, s+1, 1), n = 0 or any positive integer. Then, series of BIB Designs can be obtained from these resolvable BIB's, exploiting Corollary 8.2.1.

(iv) From Bose and Shrikhande (1959), it is known that there always exists a resolvable BIB $(2^{m-1}(2^m-1), 2^{m-1}, 1)$, $m \geq 2$. Hence, by Corollary 3.5.2b, there always exists a series of resolvable BIB's $(2^{n(m-1)}(2^m-1), 2^{m-1}, 1)$, $m \geq 2$, $n \geq 1$. Now, series of BIB Designs can be obtained from these resolvable BIB's, exploiting corollary 8.2.1.

8.3.2 GD Designs :-

We give below methods of obtaining some series of GD Designs with the help of theorem 8.2.1. These Designs cannot be constructed by the method suggested in Shrikhande and Raghavarao (1963).

(i) A resolvable GD Design with parameters $v = s(s-1)$, $b = s^2$, $r = s$, $k = s-1$; $\lambda_1 = 0$, $\lambda_2 = 1$, $m = s-1$, $n = s$ always exists when s is a prime power. If, moreover (s-1) also is a prime power, there exists an OA $[(s-1)^2, s, s-1, 2]$ and hence a GD Design with parameters $v = s(s-1)$, $b = (s-1)^2$, $r = s-1$, $k = s$; $\lambda_1 = 0$, $\lambda_2 = 1$, $m = s$, $n = s-1$ exists by Corollary 3.5.3b. Hence, applying theorem 8.2.1, the GD Design with parameters $v = s(s-1)$, $b = s(s-1)^2$, $r = s(s-1)$, $k = s$; $\lambda_1 = s$, $\lambda_2 = s-1$, $m = s-1$, $n = s$ always exists when both s and (s-1) are prime powers.

(ii) If (s+1) is a prime power, there always exists a resolvable GD Design with parameters $v = (s^2-1)$, $b = (s+1)^2$, $r = s+1$, $k = s-1$; $\lambda_1 = 0$, $\lambda_2 = 1$, $m = s-1$, $n = s+1$ by Corollary 3.5.3b. Now, if s is a prime power, the series of GD Designs with parameters $v = b = s^2-1$, $r = k = s$; $\lambda_1 = 0$, $\lambda_2 = 1$, $m = s+1$, $n = s-1$ always exists (Raghavarao, 1971). Hence, a GD Design with parameters $v = s^2-1$, $b = (s+1)(s^2-1)$, $r = s(s+1)$, $k = s$; $\lambda_1 = s+1$, $\lambda_2 = s$, $m = s-1$, $n = s+1$ always exists when s and (s+1) are both prime powers, by theorem 8.2.1.

(iii) A GD Design with parameters $v = b = s^4-s$, $r = k = s^2$; $\lambda_1 = 0$, $\lambda_2 = 1$, $m = s^2+s+1$, $n = s^2-s$ exists for s a prime power (Raghavarao, 1971). Now, if (s^2+s+1) too is a prime power, there exists a resolvable GD Design with parameters $v = (s^2-s)(s^2+s+1) = s^4-s$, $b = (s^2+s+1)^2$, $r = s^2+s+1$, $k = s^2-s$; $\lambda_1 = 0$, $\lambda_2 = 1$, $m = s^2-s$, $n = s^2+s+1$ by Corollary 3.5.3b. Hence, by theorem 8.2.1, there always exists a GD Design with parameters $v = s^4-s$, $b = (s^4-s) \cdot (s^2+s+1)$, $r = s^2(s^2+s+1)$, $k = s^2$; $\lambda_1 = s^2+s+1$, $\lambda_2 = s^2+s$, $m = s^2-s$, $n = s^2+s+1$, when s and (s^2+s+1) are both prime powers.

(iv) A GD Design with parameters $v = b = s^2(s^2+s+1)$, $r = k = s(s+1)$; $\lambda_1 = s$, $\lambda_2 = 1$, $m = s^2+s+1$, $n = s^2$ exists for s, a prime power (Raghavarao, 1971). Now, if $(s^2 + s + 1)$ is a prime power, there exists a resolvable GD Design with parameters $v = s^2(s^2+s+1)$,

$b = (s^2 + s + 1)^2$, $r = (s^2 + s + 1)$, $k = s^2$; $\lambda_1 = 0$, $\lambda_2 = 1$, $m = s^2$, $n = s^2 + s + 1$ by Corollary 3.5.3b. Hence, a GD Design with parameters $v = s^2 (s^2 + s + 1)$, $b = s^2(s^2 + s + 1)^2$, $r = s (s+1) (s^2 + s + 1)$, $k = s (s + 1)$; $\lambda_1 = s^2 + s + 1$, $\lambda_2 = s^2 + 2s$, $m = s^2$, $n = s^2 + s + 1$ exists, when both s and $(s^2 + s + 1)$ are prime powers, by theorem 8.2.1.

9. MISCELLANEOUS METHODS FOR CONSTRUCTING INCOMPLETE BLOCK DESIGNS

9.1 Introduction :

In all the chapters so far, we have considered methods for constructing Incomplete Block Designs. In the present chapter we give some more methods. In the section 9.2, is given a method for constructing two series of PBIB Designs with (i) GD and (ii) Rectangular association scheme. In the section 9.3, is considered a method of constructing Incomplete Block Designs through S, Σ and allied matrices. In the section 9.4, the paracyclic association scheme of Adhikary (1969b) is generalised to three or more associate classes and a method is given for constructing PBIB Designs with higher associate paracyclic association schemes.

9.2 Two Series of PBIB Designs with (i) GD and (ii) Rectangular Association Schemes.

In this section, we prove the existence of

(i) a series of GD Designs with parameters $v = s(s-1)/2$, $b = s^2/2$, $r = s$, $k = s-1$; $\lambda_1 = 0$, $\lambda_2 = 2$, $m = s-1$, $n = s/2$, when $s = 2^p$, $p > 1$.

and (ii) a series of PBIB Designs with rectangular association scheme with parameters $v = b = s(s-1)/2$, $r = k = s-1$; $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 2$, $n_1 = s-1$, $n_2 = (s-3)/2$, $n_3 = (s-1)(s-3)/2$, when s is an odd prime power.

Some preliminary results which will be required in the proof of the main results are considered first.

Let s be a prime power and $\alpha_0 = 0$, $\alpha_1 = 1$, α_2 ....., α_{s-1} be the s elements of GF(s).

Let s(s-1)/2 treatments be denoted by the unordered pairs

$$\left.\begin{array}{l}(\alpha_x, \alpha_y), \quad x \neq y, \\ \qquad\qquad x,y = 0,1,2, \ldots\ldots, (s-1) \\ \text{and} \quad (\alpha_x, \alpha_y) = (\alpha_y, \alpha_x), \qquad x \neq y\end{array}\right\} \quad \ldots (9.2.1)$$

It is easy to see that the following lemmas hold for the elements $\alpha_0, \alpha_1, \ldots\ldots, \alpha_{s-1}$ of GF(s).

Lemma 9.2.1 : If for any $\alpha_i \neq 1$, $\alpha_x + \alpha_i \alpha_y = \alpha_{x'} + \alpha_i \alpha_{y'}$, then either $\alpha_x = \alpha_{x'}$ and $\alpha_y = \alpha_{y'}$ or $(\alpha_x, \alpha_y) \neq (\alpha_{x'}, \alpha_{y'})$ where the pairs represent the treatments defined in (9.2.1)

Lemma 9.2.2 : $\alpha_x + \alpha_i \alpha_y = \alpha_{x'} + \alpha_i \alpha_{y'}$ implies $\alpha_y + \alpha_i^{-1} \alpha_x = \alpha_{y'} + \alpha_i^{-1} \alpha_{x'}$ for all $\alpha_i \neq 0$.

Lemma 9.2.3 : For all $\alpha_i \neq \pm 1$, $\alpha_i^{-1} \neq \alpha_i \neq \pm 1$.

Lemma 9.2.4 : For any fixed $\alpha_i \neq \pm 1$ and α_j there exist (s-1) distinct treatments (α_x, α_y) as defined in (9.2.1) satisfying the equation $\alpha_x + \alpha_i \alpha_y = \alpha_j$.

Proof of Lemma 9.2.4 follows obviously from the lemma 9.2.1 and the fact $\alpha_i \neq -1$.

Case 1. $s = 2^p$, $p > 1$.

s elements of GF(s) can also be written as $\alpha_0 = 0$, $\alpha_1 = 1$, $\beta_1, \beta_2, \ldots\ldots, \beta_{(s-2)/2}, \beta_1^{-1}, \beta_2^{-1}, \ldots\ldots, \beta_{(s-2)/2}^{-1}$.

Let the treatments be as defined in (9.2.1). Let s/2 sets of s blocks each be constructed as follows :

(A) jth block in the ith set consists of all the treatments (α_x, α_y) satisfying the equation

$$\alpha_x + \beta_i \alpha_y = \alpha_j, \quad i = 1,2, \ldots\ldots, (s-2)/2, \quad j = 0,1, \ldots\ldots, (s-1) \qquad (9.2.2a)$$

(B) jth block in the s/2 th set consists of all the treatments (α_x, α_y) satisfying the equation

$$\alpha_x + \alpha_0 \alpha_y = \alpha_j, \quad j = 0,1, \ldots\ldots, (s-1) \qquad (9.2.2b)$$

Theorem 9.2.1 : The blocks in (9.2.2a and b) give a GD Design with parameters $v = s(s-1)/2$, $b = s^2/2$, $r = s$, $k = s-1$; $\lambda_1 = 0$, $\lambda_2 = 2$, $m = s-1$, $n = s/2$, when $s = 2^p$, $p > 1$.

Proof : By lemma 9.2.4, each block in (9.2.2) contains (s-1) treatments all distinct and by lemma 9.2.3 no block is repeated.

(i) Let us consider any two distinct treatments (α_x, α_y) and $(\alpha_{x'}, \alpha_{y'})$ with $\alpha_x + \alpha_y \neq \alpha_{x'} + \alpha_{y'}$, where $\alpha_x, \alpha_y, \alpha_{x'}$ and $\alpha_{y'}$ are all distinct elements of GF(s). Then, by lemma 9.2.2, there are two distinct β_i's and hence two sets in A such that the two treatments occur together in a block once in each of these two sets. These two treatments cannot obviously occur together in any block of the set in B.

(ii) Let us consider two distinct treatments (α_x, α_y) and $(\alpha_{x'}, \alpha_{y'})$ with $\alpha_x + \alpha_y \neq \alpha_{x'} + \alpha_{y'}$, where exactly one of α_x and α_y is identical with exactly one of $\alpha_{x'}$ and $\alpha_{y'}$. Suppose, without any loss of generality, $\alpha_x = \alpha_{x'}$ and $\alpha_y \neq \alpha_{y'}$, then because neither of the equalities $\alpha_x + \beta_i \alpha_y = \alpha_{x'} + \beta_i \alpha_{y'}$ and $\alpha_y + \beta_i \alpha_x = \alpha_{y'} + \beta_i \alpha_{x'}$ is possible for any β_i and only either of the equalities $\alpha_x + \beta_i \alpha_y = \alpha_{y'} + \beta_i \alpha_{x'}$ and $\alpha_y + \beta_i \alpha_x = \alpha_{x'} + \beta_i \alpha_{y'}$ is possible for exactly one β_i, it is proved that the treatment pair occurs exactly once together in a block in the sets of block contained in A. In the set of blocks contained in B, the treatment pair obviously occurs together exactly once.

Hence, when A and B are both included, any two treatments (α_x, α_y) and $(\alpha_{x'}, \alpha_{y'})$ with $\alpha_x + \alpha_y \neq \alpha_{x'} + \alpha_{y'}$ occur together exactly twice and if $\alpha_x + \alpha_y = \alpha_{x'} + \alpha_{y'}$, the two treatments occur together in neither the blocks of A nor the blocks of B.

So, the design is GD with (s-1) groups, ith group defined as consisting of all treatments (α_x, α_y) satisfying the equation $\alpha_x + \alpha_y = \alpha_i$, $i = 1, 2, \ldots\ldots, (s-1)$. Each group obviously consists of exactly s/2 distinct treatments.

Hence, the theorem follows.

Case 2 : s is an odd prime power. The s elements of GF(s) can be written as $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = -1$, $\beta_1, \beta_2 \ldots\ldots \beta_{(s-3)/2}$, $\beta_1^{-1}, \beta_2^{-1}, \ldots\ldots, \beta_{(s-3)/2}^{-1}$.

Let the treatments be written as in (9.2.1).

Let (s-1)/2 sets of s blocks each be constructed as follows :

(C) jth block in the ith set consists of all the treatments (α_x, α_y) satisfying the equation

$$\left.\begin{array}{l}\alpha_x + \beta_i \alpha_y = \alpha_j, \quad i = 1, 2, \ldots\ldots, (s-3)/2 \\ \qquad\qquad\qquad\quad j = 0, 1, \ldots\ldots, (s-1)\end{array}\right\} \quad (9.2.3a)$$

(D) jth block in the $(s-1)/2$ th set consists of all the treatments (α_x, α_y) satisfying the equation

$$\alpha_x + \alpha_0 \alpha_y = \alpha_j, \quad j = 0, 1, \ldots\ldots, (s-1) \qquad (9.2.3b)$$

Theorem 9.2.2 : The blocks in (9.2.3a & b) give a rectangular PBIB Design with parameters $v = b = s(s-1)/2$, $r = k = s-1$; $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 2$, $n_1 = s-1$, $n_2 = (s-3)/2$, $n_3 = (s-1)(s-3)/2$, when s is an odd prime power.

Proof : The proof in essence is similar to that of theorem 9.2.1.

There are two forms of grouping. In one form of grouping there are s groups of $(s-1)/2$ treatments each, ith group being given by all the treatments satisfying

$$\alpha_x + \alpha_y = \alpha_i, \quad i = 0, 1, 2, \ldots\ldots, (s-1).$$

The other form of grouping gives $(s-1)/2$ groups of s treatments each, jth group consisting of all the treatments satisfying

$$\alpha_x - \alpha_y = \gamma_j, \quad j = 1, 2, \ldots\ldots, (s-1)/2$$

where the non-null elements of GF(s) are written as γ_1, γ_2, ..., $\gamma_{(s-1)/2}$, $-\gamma_1$, $-\gamma_2$, ..., $-\gamma_{(s-1)/2}$.

Following the line of proof as for theorem 9.2.1, two distinct treatments (α_x , α_y) and ($\alpha_{x'}$, $\alpha_{y'}$), where $\alpha_x + \alpha_y \neq \alpha_{x'} + \alpha_{y'}$ and $\alpha_x \sim \alpha_y \neq \alpha_{x'} \sim \alpha_{y'}$ occur together exactly twice and if at least one of the equalities $\alpha_x + \alpha_y = \alpha_{x'} + \alpha_{y'}$ and $\alpha_x \sim \alpha_y = \alpha_{x'} \sim \alpha_{y'}$ holds, the two treatments do not occur together at all.

Now, the association scheme will be rectangular provided ith group in the first form of grouping has exactly one treatment common with jth group in the second form of grouping, $\forall$ i, j, i.e. there should be a unique treatment (α_x , α_y) satisfying the equations

$$\left.\begin{aligned} \alpha_x + \alpha_y &= \alpha_i \\ \alpha_x - \alpha_y &= \gamma_j \end{aligned}\right\} \qquad (10.2.4)$$

for all $i = 0, 1, 2, \ldots\ldots, s-1$

$j = 1, 2, \ldots\ldots\ldots, (s-1)/2$

That it is so can be seen easily. Hence, the theorem follows.

9.3 Incomplete Block Designs Through S, Σ and Allied Matrices :

Let X_{pxq} be a matrix with elements $\pm$ 1 and 0. Let N_1 and N_2 be incidence matrices of two Incomplete Block Designs with v treatments and b blocks. Let X (*) be the vp x bq matrix obtained from X, by replacing each 1 in X by N_1, each -1 by N_2 and each 0 by 0_{vb} .

Considering two different rows, say ith row and jth row of X, let us assume that the number of times the pairs $\binom{1}{1}$, $\binom{1}{-1}$, $\binom{-1}{1}$ and $\binom{-1}{-1}$ occur between them are respectively l_1, m_1, l_2 and m_2. Then, if X_i and X_j denote the submatrices of X (*) obtained from the ith and jth rows of X respectively,

$$X_i X_j' = l_1 . N_1 N_1' + m_1 . N_1 N_2' + l_2 . N_2 N_1' + m_2 . N_2 N_2' \qquad \ldots \quad (9.3.1)$$

and if the ith row of X contains l 1's and m -1's,

$$X_i X_i' = l . N_1 N_1' + m . N_2 N_2' \qquad \ldots \quad \ldots \quad (9.3.2)$$

Theorem 9.3.1 : The existence of Σ_{4t-1} and S_{4t+1}, implies the existence of a GD Design with parameters $v = b = 16 t^2 - 1$, $r = k = 4t (2t-1)$; $\lambda_1 = 4t (t-1)$, $\lambda_2 = (2t-1)^2$, $m = 4t+1$, $n = 4t-1$.

Proof : Let $\Sigma_{4t-1} = N_1 - N_2$, where N_1 and N_2 are incidence matrices (i.e. matrices with elements 0 and 1).

Then $N_1 N_1' = N_2 N_2' = t. I_{4t-1} + (t-1). E_{4t-1, 4t-1}$

and $N_1 N_2' + N_2 N_1' = -(2t-1). I_{4t-1} + (2t-1). E_{4t-1, 4t-1}$

Writing $X = S_{4t+1}$, the parameters used in (9.3.1) and (9.3.2) can be given as $l + m = 4t$ for any ith row

$l_1 + m_2 = 2t-1$, $l_2 = m_1 = t$ for any two rows i and j, $i \neq j$.

Hence, (9.3.1) and (9.3.2) give,

$$X_i X_j' = (2t-1)^2 \cdot E_{4t-1,\ 4t-1} \quad \forall\ i \neq j$$

$$X_i X_i' = 4t^2\ I_{4t-1} + 4t(t-1)\ E_{4t-1,\ 4t-1}, \quad \forall\ i .$$

$\therefore$ X (*) is the incidence matrix of the required GD Design.

Theorem 9.3.2 : The existence of S_{4t+1} and Σ_{4t+3}, implies the existence of a GD Design with parameters $v = b = (4t+1)(4t+3)$, $r = k = 4t(2t+1)$; $\lambda_1 = 4t^2 - 1$, $\lambda_2 = 4t^2$, $m = 4t+3$, $n = 4t+1$.

Proof : Writing $S_{4t+1} = N_1 - N_2$, where N_1 and N_2 are incidence matrices and $X = \Sigma_{4t+3}$, the proof follows immediately.

Theorem 9.3.3 : The existence of Σ_{4t-1}, implies the existence of a PBIB Design with L_2 association scheme with parameters $v = (4t-1)^2$, $b = 2\,(4t-1)^2$, $r = 4(2t-1)^2$, $k = 2(2t-1)^2$, $\lambda_1 = 4(t-1)(2t-1)$, $\lambda_2 = \lambda_1 + 1$.

Proof : Let $\Sigma_{4t-1} = N_1 - N_2$, where N_1 and N_2 are incidence matrices.

Writing $X = [\Sigma_{4t-1} : -\Sigma_{4t-1}]$, (9.3.1)and (9.3.2) give

$$X_i X_j' = -I_{4t-1} + (8t^2 - 12t + 5) \cdot E_{4t-1,4t-1}.$$

$$= -I_{4t-1} + (4(t-1)(2t-1) + 1) \cdot E_{4t-1,4t-1}\ ; \quad \forall\ i \neq j.$$

and $X_i X_i' = 4t\ (2t-1)\ I_{4t-1} + 4(t-1)(2t-1)\ E_{4t-1,4t-1}, \quad \forall\ i$

Hence, $X(*)$ is the incidence matrix of the required L_2 Design.

Theorem 9.3.4 : The existence of S_{4t+1}, implies the existence of a PBIB Design with L_2 association scheme with parameters $v = (4t+1)^2$, $b = 2(4t+1)^2$, $r = 8t^2$, $k = 4t^2$; $\lambda_1 = 4t(2t-1)$, $\lambda_2 = \lambda_1 + 1$.

Proof : Writing $S_{4t+1} = N_1 - N_2$, where N_1 and N_2 are incidence matrices

$$\text{and} \quad X = [\, S_{4t+1} \ : \ - S_{4t+1} \,],$$

The result follows immediately as in the case of theorem 9.3.3.

For the three class association scheme obtained from OA with $\beta_1 = \beta_2 = \beta_3 = \frac{n+1}{3}$, where $n \equiv 2 \pmod 3$, let us write

$$X = [\, B_1 - B_2 \ : \ B_2 - B_3 \ : \ B_3 - B_1 \,],$$

Then, the parameters defined for the relations (9.3.1) and (9.3.2) are as

$$l = m = n^2 - 1, \qquad \forall\ i$$

$$l_1 = m_2 = \frac{n^2 - 4}{3} \text{ and } l_2 = m_1 = \frac{n^2 - 1}{3}, \qquad \forall\ i \neq j.$$

Hence, we obtain the following theorem.

Theorem 9.3.5 : The existence of Σ_n or S_n , where $\frac{3n+5}{2} = p^2$, p a prime power of the form 2 (Mod 3), implies the existence of a GD Design with parameters $v = p^2 n$, $b = 3v$, $r = 3k$, $k = \frac{(p^2-1)(n-1)}{3}$;

$$\lambda_1 = \frac{3(n+1)(n-3)}{4}, \quad \lambda_2 = \frac{(n-1)^2}{2}, \quad m = p^2, \quad n$$

Proof : $\frac{3n+5}{2} = p^2$, p a prime power of the form 2(Mod 3), implies the existence of a three class association scheme obtained from OA with $\beta_1 = \beta_2 = \beta_3 = \frac{p+1}{3}$. Let

$$X = [\, B_1 - B_2 \; : \; B_2 - B_3 \; : \; B_3 - B_1 \,]$$

and Σ_n or S_n as the case may be can be written as $N_1 - N_2$, where N_1 and N_2 are incidence matrices.

Then, it can be proved easily that X (*) is the incidence matrix of the required design.

For the 4 class association scheme obtained from OA with $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \frac{n+1}{4}$, where $n \equiv 3$ (Mod 4), writing

$X = [\, B_1 - B_2 \; : \; B_3 - B_4 \,]$, the parameters in (9.3.1) and (9.3.2) can be given as

$$l = m = \frac{n^2-1}{2}, \quad \forall \; i$$

and $l_1 + m_2 = \frac{n^2-5}{4}$, $l_2 = m_1 = \frac{n^2-1}{8}$, $\forall \; i \neq j$.

Theorem 10.3.6 : The existence of Σ_{4t-1}, where $8t+1 = p^2$, p a prime power of the form 3 (Mod 4) implies the existence of GD Design with parameters $v = p^2(4t-1)$, $b = 2v$, $r = 2k$,

$$k = \frac{(p^2-1)(2t-1)}{2}\,; \quad \lambda_1 = 8t(t-1), \quad \lambda_2 = (2t-1)^2, \quad m = p^2, \quad n = 4t-1.$$

Proof : As p is a prime power of the form 3 (Mod 4), there exists a 4 class association scheme obtained from OA with $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \frac{p+1}{4}$. Writing

$$X = [\, B_1 - B_2 \; : \; B_3 - B_4 \,]$$

and $\Sigma_{4t-1} = N_1 - N_2$, where N_1 and N_2 are incidence matrices, X (*) is the incidence matrix of the required GD Design.

10.4 Higher Associate Para Cyclic Association Scheme :

Paracyclic Association Scheme of Adhikary (1969b) has been described in section 7.2 of chapter 7. In the present section we extend it to more than two associate classes and the procedure is similar to that employed by Adhikary (1967) in obtaining Higher associate cyclical association scheme. The procedure can be described as follows :

Let G be a module of v elements. Let us divide this set of v elements into t disjoint sets $S_0, S_1, \ldots S_{t-1}$, $\bigcup_{i=0}^{t-1} S_i = G$.

For each set S_i, the non-null elements of G are divided into m disjoint sets $A_{i1}, A_{i2}, \ldots, A_{im}$. $A_{ij} = \{ d_{ij_1}, d_{ij_2}, \ldots, d_{ij_{n_j}} \}$, $j = 1,2, \ldots m$ with $\bigcup_{j=1}^{m} A_{ij} = G - \{0\}$, $\forall i$, with the following properties :

9.4.1 If $\theta \in S_i$ and $\phi = \theta + d_{ij_l} \in S_k$, then $- d_{ij_l} \in A_{kj}$.

9.4.2 If $\theta \in S_i$ and $\phi = \theta + d_{ij_l} \in S_k$, then among the differences that arise by taking the differences of all elements of A_{kr} from all the elements of A_{is}, d_{ij_l} occurs $p^j_{rs} = p^j_{sr}$ times, for $r, s = 1,2, \ldots, m-1$ and $j = 1,2, \ldots m$. Then,

Theorem 9.4.1 : Defining the association scheme as : If $\theta \in S_i$, then the set of jth associates of θ is taken to be $\theta + A_{ij}$, $j = 1,2, \ldots m$, we have an m associate PBIB association scheme with parameters

$$v, \quad n_1, \quad n_2, \ldots \ldots \quad n_m,$$

$$(p^i_{jk}), \quad i, j, k = 1, 2, \ldots m.$$

The last row and column of any (p^i_{jk}) matrix for a fixed i are to be obtained from boundary conditions of the matrices.

Proof : Let $\theta \in S_i$. If ϕ is a jth associate of θ, there exists an element $d_{ij_1} \in A_{ij}$ such that $\phi = \theta + d_{ij_1}$. If $\phi \in S_k$, jth associates of ϕ will be $\phi + A_{kj} = \theta + d_{ij_1} + A_{kj}$. But $-d_{ij_1} \in A_{kj}$. $\therefore$ θ is the jth associate of ϕ.

Again, let $\theta \in S_i$ and $\phi = \theta + d_{ij_1} \in S_k$. Then, the set of sth associates of θ consists of elements $\theta + d_{is_1}$, $\theta + d_{is_2}$ ………… , $\theta + d_{is_{n_s}}$. Similarly, the set of rth associates of ϕ consists of elements $\phi + d_{kr_1}$, $\phi + d_{kr_2}$, ….., $\phi + d_{kr_{n_r}}$ i.e. $\theta + d_{ij_1} + d_{kr_1}$, $\theta + d_{ij_1} + d_{kr_2}$, ….., $\theta + d_{ij_1} + d_{kr_{n_r}}$.

Let $p^j_{sr} = \alpha$, say, elements be common between the two sets and let $(\theta + d_{is_{(1)}}, \theta + d_{is_{(2)}}, \ldots, \theta + d_{is_{(\alpha)}})$ be the same set as $(\theta + d_{ij_1} + d_{kr_{(1)}}, \theta + d_{ij_1} + d_{kr_{(2)}}, \ldots\ldots, \theta + d_{ij_1} + d_{kr_{(\alpha)}})$. Then,

$$\theta + d_{is_{(p)}} = \theta + d_{ij_1} + d_{kr_{(p)}}, \quad p = 1, 2, \ldots\ldots\ \alpha .$$

$$\text{or} \quad d_{is_{(p)}} - d_{kr_{(p)}} = d_{ij_1}, \quad p = 1, 2, \ldots\ldots\ \alpha .$$

i.e. the element d_{ij_1} is repeated $p^j_{sr} = \alpha$ times among the differences obtained by taking the differences of all the elements of A_{kr} from all the elements of A_{is}.

Hence, from the given conditions for any two treatments which are jth associates of each other, the number of elements common between the rth associates of one and sth associates of the other is a constant denoted by p^j_{rs}, $j = 1,2, \ldots\ldots m$

$r,s = 1,2, \ldots\ldots, m-1.$

$\therefore$ From the boundary conditions of (p^j_{rs}) matrices, the same is true for all $j, r, s = 1,2, \ldots\ldots m.$

Hence, the theorem is proved.

Examples of higher associate paracyclic association scheme :

Example 9.4.1 Association Scheme with three associate classes.

Consider the class of residues mod 10.

Let $S_0 = (0, 2, 4, 6, 8),$

$S_1 = (1, 3, 5, 7, 9),$

$A_{01} = (1, 3, 5, 7, 9),\quad A_{02} = (4, 6),\quad A_{03} = (2, 8)$

$A_{11} = (1, 3, 5, 7, 9),\quad A_{12} = (2, 8),\quad A_{13} = (4, 6).$

Then, we have the association scheme with parameters

$$v = 10, \quad n_1 = 5, \quad n_2 = 2, \quad n_3 = 2$$

$$(p^1_{jk}) = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad (p^2_{jk}) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad (p^3_{jk}) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Example 9.4.2 : Association Scheme with four associate classes.

Consider the class of residues mod 14.

Let $S_0 = (0, 2, 4, 6, 8, 10, 12)$

$S_1 = (1, 3, 5, 7, 9, 11, 13)$

$A_{01} = (1,3,5,7,9,11,13)$, $A_{02} = (2,12)$, $A_{03} = (4,10)$, $A_{04} = (6,8)$

$A_{11} = (1,3,5,7,9,11,13)$, $A_{12} = (4,10)$, $A_{13} = (6,8)$, $A_{14} = (2,12)$

Then, we have the association scheme with parameters

$$v = 14, \quad n_1 = 7, \quad n_2 = n_3 = n_4 = 2$$

$$(p^1_{jk}) = \begin{pmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \quad (p^2_{jk}) = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$(p^3_{jk}) = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad (p^4_{jk}) = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Higher associate cyclical association scheme of Adhikary (1967) is a particular case of higher associate paracyclic association scheme, viz., when $A_{0j} = A_{1j} = \ldots\ldots = A_{t-1,j}$, $\forall$ j , the paracyclic association scheme reduces to a cyclical association scheme.

Method of difference for constructing PBIB Designs with higher associate paracyclic association schemes :

The method is an obvious extension of the one given by Adhikary (1969 b) for PBIB Designs with two associate paracyclic association scheme -

Consider the module of residue classes mod v = ph. Let us divide the v elements into h disjoint sets as

$$S_0 = (0, h, 2h, \ldots\ldots , (p-1) h)$$

$$S_1 = (1, h+1, 2h+1, \ldots\ldots , (p-1)h + 1)$$

$$\vdots$$

$$S_{h-1} = (h-1, 2h-1, 3h-1, \ldots\ldots\ldots , ph-1).$$

Let it be possible to select A_{ij}'s, $i = 0,1,2,\ldots\ldots, h-1$, $j = 1,2,\ldots\ldots m$ satisfying the conditions of the theorem 9.4.1 and properties 9.4.1 and 9.4.2 preceding the theorem.

Let it be possible to select a set of t blocks satisfying the conditions :

(i) The blocks are of constant size, k.

(ii) Among the kt treatments occurring in the t blocks, the total number of elements appearing from S_j, $j = 0,1,2, \ldots\ldots h-1$ is a constant equal to r. So $kt = hr$.

(iii) All possible differences arising within blocks can be classified in the following manner :

Consider the elements within blocks which belong to S_j. Let s_{ju} be any such element occurring in the l-th block. Form the differences $a - s_{ju}$ where a is any element other than s_{ju} occurring in the l-th block. In this manner obtain differences from the t blocks for all s_{ju}'s belonging to S_j. Let the differences so obtained contain each element of A_{ji}, λ_i times, for $i = 1,2,\ldots\ldots$ $\ldots\ldots m$. The same is true for all j.

Obviously, $tk(k-1) = h \left\{ n_1 \lambda_1 + n_2 \lambda_2 \ldots\ldots + n_m \lambda_m \right\}$.

Next develop the initial blocks by adding $0, h, 2h, \ldots\ldots, (\alpha - 1)h$ in succession, when $\alpha = v/h$. The resulting design is a PBIB with m-class association scheme as given in theorem 9.4.1.

Example 9.4.3 : Consider the class of residues mod 10. Treating (1, 3, 5) and (0, 4, 8) as initial blocks, we get a solution of

PBIB Design with parameters

$$v = b = 10, \quad r = k = 3, \quad \lambda_1 = 0, \quad \lambda_2 = 2, \quad \lambda_3 = 1,$$

$n_1 = 5$, $n_2 = 2$, $n_3 = 2$ and association scheme as given in example 9.4.1

Example 9.4.4 : Consider the class of residues mod 14. Treating initial blocks as (1, 3, 5, 7) and (0, 4, 8, 12), we get a solution of PBIB Design with parameters

$$v = b = 14, \quad r = k = 4, \quad \lambda_1 = 0, \quad \lambda_2 = 3,$$

$\lambda_3 = 2$, $\lambda_4 = 1$, $n_1 = 7$, $n_2 = n_3 = n_4 = 2$ and the association scheme as given in the example 9.4.2.

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