SYSTEMS OF IMPRIMITIVITY FOR ERGODIC ACTIONS OF LOCALLY COMPACT ABELIAN GROUPS

By

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CONTENTS

INTRODUCTION

Mackey's theorem on induced representation gives all the systems of imprimitivity for a locally compact second countable group G, acting in a separable Hilbert space and based on a transitive G-space X. These systems of imprimitivity are obtained from the unitary representations of the closed subgroup H of G defining the transitive G-space X. The main tool for this study is a class of functions called cocycles. Mackey showed that the (G , X) systems of imprimitivity are connected in a one-one way to unitary operator valued cocycles on $G \times X$. This cocycle in turn gives a representation of the closed subgroup H, defining X. Systems of imprimitivity on non-transitive actions are not as well studied. In this thesis we study the systems of imprimitivity on some important cases of essentially non-transitive actions, and connect them to systems of imprimitivity on simpler spaces.

Using the notion of cocycles, Gamelin [7] showed that the systems of imprimitivity on the pair ($\mathbb{R}$, B) where $\mathbb{R}$ is the real line and B is the Bohr group are related in a one-one way to systems of imprimitivity on the pair (N , K) where N is the integer group and K is the annihilator of a cyclic subgroup of the dual group $\hat{B}$ of B. More precisely, he showed that every (N , K) cocycle extends to an ($\mathbb{R}$, B) cocycle and in the cohomology (an equivalence relation defined in the set of all cocycles) class of every ($\mathbb{R}$, B) cocycle, there is a cocycle which is extended from an (N , K) cocycle. Gamelin proved the above result for scalar valued cocycles and it was extended to the vector valued case by Muhly [20] and Bagchi [3]. The method Gamelin used is that of a flow built under a function. He views the action of $\mathbb{R}$ on B as a flow built under the constant function 1 with base space K. In chapter II of this thesis we show that Gamelin's method of extending cocycles can be used for a general flow built under a function.

To study the systems of imprimitivity on strictly ergodic actions, Mackey [19] introduced the notion of virtual subgroups. He showed, among other things, that using this notion one can generalize the notion of a flow built under a function. We show that Gamelin's method of extending cocycles is applicable to a generalized flow built under a constant function. The action of a locally compact second countable abelian subgroup H acting continuously on a locally compact second countable abelian group G by translation, can be viewed as a generalized flow built under a constant function. In chapter III we consider in detail, this particular case of group actions. The situation is similar to that of the pair $(\mathbb{R}, B)$.

This thesis is divided into three chapters. The first chapter is mainly introductory. We introduce the notion of a unitary operator valued cocycle and a system of imprimitivity. The definitions and results are taken from Varadarajan [24]. Strict cocycles are easier to handle and we mention some cases where we need to consider only strict cocycles. We show that we can take a unitary operator valued cocycle to satisfy a much stringent condition. This is obtained by generalizing a method of Doob's in obtaining a measurable stochastic process from a 'continuous in probability' stochastic process. This definition of a cocycle is needed for chapters II and III.

In chapter II we consider flows built under a function. We show that cocycles on the base space give rise to cocycles on the flow built under a function. Every strict cocycle on the flow built under a function is cohomologous to a cocycle extended from a cocycle on the base space. A proof of Gamelin's is modified to show that every cocycle on a flow built under a function has a strict version. Now appealing to a generalization due to S. G. Dani [5] of a deep theorem of Ambrose and Kakutani [2], we are able to show that every cocycle on a proper flow on a Lebesgue space has a strict version. Then we show that in most of the cases appearing in chapter III (our main chapter) cocycles can be taken to be strict. We also mention how the results go through for Mackey's generalization of a flow built under a function.

In chapter III we consider the main problem. Let R be a locally compact second countable abelian group and Γ a dense subgroup of R with another locally compact second countable topology such that the inclusion map of Γ into R is continuous. Such systems we call a 'pair', and denote it by (Γ, R). Let Γ_o be a closed subgroup of R such that $\Gamma_0 \subseteq \Gamma$. Then Γ_0 is closed in Γ as well. It is easy to see that $(\Gamma/\Gamma_0 , R/\Gamma_0)$ is also a pair. Let $B = \hat{\Gamma}$ and $S = \hat{R}$, and let K be the annihilator of Γ_0 in B. Then we can show that (S , B) is also a pair. (S , B) is called the dual pair of (Γ , R). The dual pair of $(\Gamma/\Gamma_0 , R/\Gamma_0)$ will be $(K \cap S , K)$. Given a system of imprimitivity on a pair, we get a system of imprimitivity on the dual pair by applying Stone's theorem. Let (V , E) be a $(K \cap S , K)$ system of imprimitivity. We show that (V, E) gives rise, in a natural fashion, to an (S , B) system of imprimitivity $(\bar{V} , \bar{E})$. Let (U , F) be the $(\Gamma/\Gamma_0 , R/\Gamma_0)$ system of imprimitivity which is the dual of (V , E) and let $(\bar{U} , \bar{F})$ be the (Γ , R) system dual to $(\bar{V} , \bar{E})$. We show that every $(\Gamma/\Gamma_0, R/\Gamma_0)$ system of imprimitivity (U , F) gives rise to a (Γ , R) system $(\breve{U} , \breve{F})$. Thus on (Γ , R) we have two systems of imprimitivity $(\bar{U} , \bar{F})$ and $(\breve{U} , \breve{F})$ starting from the same $(K \cap S , K)$ system (V , E). Under a mild assumption which is satisfied in many cases, and probably in all cases, we show that $(\bar{U} , \bar{F})$ and $(\breve{U} , \breve{F})$ are equivalent systems of imprimitivity. That is, we show that the following diagram commutes.

$$\begin{array}{ccc}
(K \cap S \, , \, K) & & (S \, , \, B) \\
(V \, , \, E) & \longrightarrow & (\bar{V} \, , \, \bar{E}) \\
\downarrow & & \downarrow \\
 & & (\bar{U} \, , \, \bar{F}) \\
(U \, , \, F) & \longrightarrow & (\widehat{U} \, , \, \widehat{F}) \\
(\Gamma/\Gamma_o \, , \, R/\Gamma_o) & & (\Gamma \, , \, R).
\end{array}$$

When $R = \mathbb{R}$, the additive group of real numbers, and Γ is a countable dense subgroup of $\mathbb{R}$, the assumption is satisfied. But we give a slightly different method so that the function which establishes the equivalence of $(\bar{U} , \bar{F})$ and $(\widehat{U} , \widehat{F})$ is a simple one.

Systems of imprimitivity for the real line $\mathbb{R}$ acting through translation on a compact group with Archemedean ordered dual were encountered by Helson and Lowdenslager [14] in their study of H^2 on Bohr group. In it they constructed a cocycle which is not a coboundary. Some of the subsequent papers are due to Helson [10], Helson and Kahane [13], Yale [25], Gámelin [7]. In all these, the authors work on the pair $(\mathbb{R}, B)$ where $\hat{B}$ is a discrete, dense subgroup of $\mathbb{R}$ and consider only scalar cocycles. In [7, 13, 14, 25] different methods of constructing non-tirvial scalar $(\mathbb{R}, B)$ cocycles are given whereas in [10, 11] deep analysis is made of the analytic structure of $(\mathbb{R}, B)$ cocycles. A connected account of these is given in Helson [12]. In this Helson considers cocycles on a flow also. Muhly [20] and later Bagchi [3] generalized some results due to Helson [10] using $(\mathbb{R}, B)$ systems of imprimitivity. The present work is at a more general level; it considers different pairs of groups, and ties up a general (R, B) system of imprimitivity with others which naturally arise from it.

Systems of imprimitivity in general set up was undertaken by G. W. Mackey in connection with the theory of group representations. A connected account of these is given in Varadarajan [24]. For all unexplained terminology we refer to Varadarajan [24].

CHAPTER I

COCYCLES AND SYSTEMS OF IMPRIMITIVITY

1. Basic Function Spaces:

Let X be a standard Borel space and $\mathbb{H}$ a separable complex Hilbert space. A function $F : X \longrightarrow \mathbb{H}$ is said to be (weakly) measurable if for every $\eta \in \mathbb{H}$, $x \longrightarrow \langle F(x), \eta \rangle$ is measurable. ($\langle \cdot , \cdot \rangle$ is the inner product in $\mathbb{H}$). Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be separable complex Hilbert spaces. A function A on X taking values in $\mathcal{B}(\mathbb{H}_1, \mathbb{H}_2)$, the set of all bounded linear operators from $\mathbb{H}_1$ to $\mathbb{H}_2$, is said to be (weakly) measurable if for any $\xi \in \mathbb{H}_1$ and $\eta \in \mathbb{H}_2$,

$$x \longrightarrow \langle A(x)\xi , \eta \rangle$$

is measurable.

Let X be a standard Borel space and α, a σ - finite measure on X and let $\mathbb{H}$ be a separable (complex) Hilbert space. By $L^2(X, \mathbb{H}, \alpha)$ we shall mean the set of all measurable functions (after identifying functions which agree α a. e.) F on X to $\mathbb{H}$ such that

$$\int_X \| F(x) \|^2 \alpha(dx) < \infty .$$

Equipped with the inner product,

$$(F_1 , F_2) = \int_X \langle F_1(x) , F_2(x) \rangle \ \alpha(dx) ,$$

$L^2(X, \mathbb{H}, \alpha)$ becomes a separable Hilbert space. Let α' be another σ - finite measure on X equivalent (in the sense of having the same null sets) to α. Let f be a version of $\frac{d\alpha'}{d\alpha}$. Then,

$$W : F \longrightarrow \frac{1}{\sqrt{f}} F$$

is an isometric isomorphism of $L^2(X, \mathbb{H}, \alpha)$ onto $L^2(X, \mathbb{H}, \alpha')$.

2. Borel G - spaces.

Let G be a separable Borel group. A standard Borel space X is said to be a G - space if for each $g \in G$, there is a Borel automorphism t_g of X such that,

(i) t_e is the identity automorphism, where e is the identity element of G.

(ii) $t_{g_1 g_2} = t_{g_1} \circ t_{g_2}$ for all $g_1 , g_2 \in G$.

We denote the value of t_g at x by gx. X is said to be a Borel G - space if the map $(g,x) \longrightarrow gx$ of $G \times X \longrightarrow X$ is measurable.

For any Borel G - space X and any $x \in X$,

$$G_x = \{ g \in G : gx = x \}$$

is a subgroup of G and is called the stability subgroup at x. The set,

$$Gx = \{ gx : g \in G \}$$

is called the orbit of x. A Borel G - space X is said to be transitive if there is $x \in X$ such that $Gx = X$. In this case, for all $y \in X$, $Gy = X$.

Let X be a standard Borel G - space and α, a σ-finite measure on X. For each $g \in G$, define the measure α_g on X by:

$$\alpha_g(E) = \alpha(gE), \ E \text{ a Borel subset of } X.$$

α is said to be quasi-invariant with respect to the action of G if, for each $g \in G$, α and α_g are mutually absolutely continuous. α is said to be invariant if $\alpha_g = \alpha$ for all $g \in G$. A quasi-invariant measure α on X is said to be ergodic if for each Borel subset E of X, $\alpha(E \triangle gE) = 0$ for each $g \in G$, if and only if

$\alpha(E) = 0$ or $\alpha(E^c) = 0$. Equivalently, any Borel function f on X such that

$$f(x) = f(gx) \quad \alpha \text{ a. e. , for each } g \in G,$$

is a constant α a.e. A quasi-invariant measure α on X is said to be <u>essentially transitive</u> if there is $x_0 \in X$ such that $\alpha(X - Gx_0) = 0$. An essentially transitive measure on X is ergodic. An ergodic measure which is not essentially transitive is said to be <u>strictly ergodic</u>.

<u>Notation</u>: For simplicity we shall write an ergodic (essentially transitive etc.) G - space X to mean a standard G - space X with a quasi-invariant measure α on it which is ergodic (essentially transitive etc.) with respect to the action of G.

<u>Example</u>: Let G be a locally compact second countable group and H a closed subgroup of G. Then $G/_H$ is, under the natural action of G on $G/_H$, a transitive G-space. Conversely, any transitive G-space X (where G is a locally compact second countable group) is isomorphic to the G-space $G/_H$ for some closed subgroup

H of G. Because of this simple nature of transitive G-spaces, cohomology classes of cocycles for them can be completely described. This in turn permits one to systematically study and apply systems of imprimitivity associated with transitive G-spaces. Situation with strictly ergodic actions is more complicated. (See for example pages 35 - 39 of [24]). Mackey [19] introduced the notion of virtual subgroups of groups to study strictly ergodic actions. We do not use this notion in this thesis, although some of the other notions such as that of a flow built under a function discussed in [19] play an important role in this work.

3. Borel sections:

Let H be a closed subgroup of a locally compact second countable group G. Then there is a Borel subset C of G which contains exactly one element from each right coset of H. We call such a Borel subset C of G, a Borel section of G with respect to H. Each element $g \in G$ can uniquely be written as $g = hc$ where $h \in H$ and $c \in C$. We shall denote by $[g]$ the part of g which belongs to H and by $\langle g \rangle$ the part of g which is in C. There is a one-one correspondence between C and G/H, the right coset space of H, viz. $c \longrightarrow Hc$. In fact, this is a Borel isomorphism between the two. Again, $g \longrightarrow (\langle g \rangle, [g])$ is a Borel isomorphism between G and $C \times H$.

Let $g_1, g_2, g_3 \in G$. Then,

(3.1) $[g_1 g_2 g_3] = [g_1 g_2][\langle g_1 g_2 \rangle g_3]$.

For, $g_1 g_2 g_3 = [g_1 g_2] \langle g_1 g_2 \rangle g_3$

$$= [g_1 g_2][\langle g_1 g_2 \rangle g_3] \langle \langle g_1 g_2 \rangle 3_3 \rangle .$$

Now, let G be an abelian group. Then G/H also is a group. The isomorphism $c \longrightarrow Hc$ makes C also a group, the group operation in C being the one carried over from G/H. It is easy to see that the group operation $\dot{+}$ thus defined in C is:

(3.2) $c_1 \dot{+} c_2 = \langle c_1 + c_2 \rangle$, $c_1, c_2 \in C$.

We will use the above two relations quite often in the sequel.

Let G be a locally compact second countable group and let H be a closed subgroup of G. Let C be a Borel section of G with respect to H. Let λ_0 be the Haar measure on H and let the Borel isomorphism between $C \times H$ and G be denoted by ξ :

$$(3.3) \qquad \xi(c,h) = hc .$$

If μ is any σ-finite measure on C, then

$$(3.4) \qquad \hat{\mu} = (\mu \times \lambda_0) \circ \xi^{-1}$$

is quasi-invariant with respect to translation by elements of H. The following lemma says that any σ-finite measure on G quasi-invariant with respect to translation by elements of H is equivalent to a measure of the above type.

3.1. Lemma. Let G be a locally compact second countable group and H a closed subgroup of G. Let ν be a σ-finite measure on G, quasi-invariant with respect to translation by elements of H. Let C be a Borel section of G with respect to H and ξ the Borel isomorphism between $C \times H$ and G defined by (3.3). Then ν is equivalent to $\hat{\mu}$ (as defined by (3.4)) for some measure μ on C.

Proof: We can assume, without loss of generality, that the identity e of G belongs to C.

We assume $\nu(G) < \infty$. Consider the measure $\sigma = \nu \circ \xi$ on $C \times H$. Let ν_1 and ν_2 be the marginals of σ on C and H respectively. ν_2 is quasi-invariant with respect to translation by elements of H. Therefore, ν_2 is equivalent to the Haar measure on H. Hence, it is enough to show that σ is equivalent to $\nu_1 \times \nu_2$.

Let $\sigma(A) = 0$, $A \subseteq C \times H$, a Borel set.

Therefore, $\sigma(A+(e,h)) = 0$, for all $h \in H$.

So, $\int_H \sigma(A+(e,h))\, d\nu_2(h) = 0$.

i.e., $\int_H \left(\int_{C\times H} I_{A+(e,h)}(c,h')\, d\sigma(c,h') \right) d\nu_2(h) = 0$

By Fubini's theorem,

$$\int_{C\times H} \left(\int_H I_A(c,h'-h)\, d\nu_2(h) \right) d\sigma(c,h') = 0.$$

So, for σ a.e. (c,h'), $\int_H I_A(c,h'-h)\, d\nu_2(h) = 0$.

Hence, for σ a.e. (c,h'), $\int_H I_A(c,h)\, d\nu_2(h) = 0$.

Therefore, for ν_1 a.e. $c \in C$, $\int_H I_A(c,h)\, d\nu_2(h) = 0$.

i.e., $(\nu_1 \times \nu_2)(A) = 0$.

Conversely, if $(\nu_1 \times \nu_2)(A) = 0$, then for ν_1 a.e. $c \in C$,

$$\int_H I_A(c,h)\, d\nu_2(h) = 0.$$

Therefore, for σ a.e. (c,h'), $\int_H I_A(c,h'-h)\, d\nu_2(h) = 0.$

Or, $$\int_{C\times H} \left(\int_H I_A(c,h'-h)\, d\nu_2(h) \right) d\sigma(c,h') = 0$$

Using Fubini's theorem, we get,

$$\int_H \left(\int_{C\times H} I_A(c,h'-h)\, d\sigma(c,h') \right) d\nu_2(h) = 0.$$

So, there exists $h_o \in H$ such that,

$$\int_{C\times H} I_A(c,h'-h_o) d\sigma(c,h') = 0.$$

i. e., $\sigma(A+(e,h_o)) = 0.$ Hence, $\sigma(A) = 0.$

Q. E. D.

The next lemma will be used in chapter II.

3.2. __Lemma:__ Let H be a closed subgroup of a locally compact second countable abelian group G. Let μ be a measure on a Borel section C of G with respect to H. Let $\hat{\mu}$ be defined as in (3.4). Let f be a Borel function on G such that

$$f(h+x) = f(x) \qquad \lambda_o \times \hat{\mu} \text{ a.e. } (h,x) \in H\times G.$$

Then there exists a Borel function $\hat{f}$ on G/H such that

$$f(x) = \tilde{f} \circ \pi(x) \quad \tilde{\mu} \text{ a.e. } x \in G,$$

where $\pi : G \dashrightarrow G/H$ is the natural homomorphism.

<u>Proof</u>: We have,

$$f(h+x) = f(x) \quad \lambda_o \times \tilde{\mu} \text{ a.e. } (h,x) \in H \times G.$$

That is, for λ_o a.e. $h \in H$,

$$f \circ \xi(c, h+h') = f \circ \xi(c, h') \quad \mu \times \lambda_o \text{ a.e. } (c, h').$$

Therefore,

$$f \circ \xi(c, h+h') = f \circ \xi(c, h') \quad (\mu \times \lambda_o \times \lambda_o) \text{ a.e. } (c, h, h').$$

Hence, by Fubini's theorem, there exists $h_o \in H$ such that,

$$f \circ \xi(c, h+h_o) = f \circ \xi(c, h_o) \quad (\mu \times \lambda_o) \text{ a.e. } (c, h).$$

Since λ_o is the Haar measure on H, this means that

(3.5) $$f \circ \xi(c,h) = f \circ \xi(c, h_o) \quad (\mu \times \lambda_o) \text{ a.e. } (c,h).$$

Define $\tilde{f}$ on G/H by:

$$f(Hc) = f \circ \xi(c, h_o), \quad c \in C.$$

$\tilde{f}$ is a Borel function on G/H and from (3.5) we have,

$$f(x) = \tilde{f} \circ \pi(x) \quad \tilde{\mu} \text{ a.e. } x \in G.$$

Q. E. D.

4. Unitary operator valued cocycles.

Let $\underline{\underline{U}}(\mathbb{H})$ stand for the group of unitary operators of a separable (complex) Hilbert space $\mathbb{H}$. Give $\underline{\underline{U}}(\mathbb{H})$ the smallest topology which makes all the maps $U \longrightarrow U\eta$, $\eta \in \mathbb{H}$, continuous. Then $\underline{\underline{U}}(\mathbb{H})$ is a metrizable topological group satisfying the second axiom of countability, and the Borel structure on $\underline{\underline{U}}(\mathbb{H})$ given by this topology is standard. Also the Borel structure on $\underline{\underline{U}}(\mathbb{H})$ given by this topology coincides with the smallest Borel structure on $\underline{\underline{U}}(\mathbb{H})$ which makes all the maps

$$U \longrightarrow \langle U\xi , \eta\rangle \quad , \quad \xi, \eta \in \mathbb{H} \text{ , measurable.}$$

Let G be a locally compact second countable group; λ the Haar measure on G. Let X be a standard Borel G-space and α a finite quasi-invariant measure on X. By a $(G,X,\underline{\underline{U}}(\mathbb{H}))$ cocycle relative to α we shall mean a Borel function $A : G \times X \longrightarrow \underline{\underline{U}}(\mathbb{H})$ such that,

(4.1) $\quad A(g_1 g_2 , x) = A(g_1 , x) A(g_2 , g_1 x)$

$$\text{for } (\lambda \times \lambda \times \alpha) \text{ a.e. } (g_1, g_2, x) \in G \times G \times X.$$

Two cocycles are identified if they agree $\lambda \times \alpha$ a.e.

Two $(G,X,\underline{\underline{U}}(\mathbb{H}))$ cocycles A_1 and A_2 (relative to α) are said to be <u>cohomologous</u> if there exists a Borel function $\rho : X \longrightarrow \underline{\underline{U}}(\mathbb{H})$ such that

$$(4.2) \quad A_1(g,x) = \rho(x)\, A_2(g,x)\, \rho^*(gx) \quad (\lambda \times \alpha) \text{ a.e.} (g,x).$$

When we want to emphasize the function ρ, we say that A_1 and A_2 are cohomologous (ρ). It is easy to see that the relation " A_1 and A_2 are cohomologous " is an equivalence relation. The equivalence classes are called <u>cohomology classes</u>. A cocycle A is said to be a <u>coboundary</u> if A is cohomologous to the cocycle which is everywhere equal to I, the identity in $\underline{\underline{U}}(\mathbb{H})$. Equivalently, A is a coboundary if there exists a Borel function $\rho : X \longrightarrow \underline{\underline{U}}(\mathbb{H})$ such that,

$$(4.3) \quad A(g,x) = \rho(x)\rho^*(gx) \quad (\lambda \times \alpha) \text{ a.e. } (g,x).$$

We give two instances where all cocycles are coboundaries.

4.1. <u>Lemma</u>: Let G_o and G be locally compact second countable groups, and G_o an algebraic subgroup of G. Let the inclusion map of G_o into G be continuous. G_o acts on G by translation. If α is a measure on G, quasi-invariant with respect to G_o such that G_o has full measure in G, then every (G_o, G) cocycle relative to α is a coboundary.

Proof: Let A be a (G_o, G) cocycle relative to α. G_o is a Borel subset of G. Let A_o be the restriction of A to $G_o \times G_o$. The measure α restricted to G_o is equivalent to the Haar measure λ on G_o. Hence, A_o is a (G_o, G_o) cocycle relative to λ and so A_o is a coboundary. Let ρ_o be the Borel function on G_o such that,

$$A_o(g,x) = \rho_o(x)\rho_o^*(gx) \quad \lambda \times \lambda \quad \text{a.e. } (g,x) \in G_o \times G_o.$$

Define ρ on G by:

$$\rho(x) = \rho_o(x) \quad \text{if } x \in G_o$$
$$= I \quad \text{if } x \notin G_o .$$

ρ is a Borel function on G. Define A_o' on $G_o \times G$ by:

$$A_o'(g,x) = A_o(g,x) \quad \text{if } x \in G_o$$
$$= I \quad \text{if } x \notin G_o .$$

A_o' is a (G_o, G) cocycle which is a coboundary (ρ). Also, $A(g,x) = A_o'(g,x)$ for $(\lambda \times \alpha)$ a.e. $(g,x) \in G_o \times G$. Hence, A is a coboundary.

Q. E. D.

4.2. Lemma: Let H be a closed subgroup of a locally compact second countable abelian group G and let H act on G by translation. Let μ be a measure on C, a Borel section of G with respect to H. Then every (H,G) cocycle relative to $\tilde{\mu}$ (as defined by (3.4)) is a coboundary.

Proof: Let λ_o be the Haar measure on H and let ξ be the Borel isomorphism from $C \times H$ onto G defined by (3.3). $\tilde{\mu} = (\mu \times \lambda_o) o \xi^{-1}$. Let A be an (H , G) cocycle relative to $\tilde{\mu}$. By the cocycle identity,

$$A(h_1+h_2 \, , g) = A(h_1,g) \, A(h_2,h_1+g) \qquad (\lambda_o \times \lambda_o \times \mu) \text{ a.e.}$$

That is,

$$A(h_1+h_2 \, , \xi(c,h)) = A(h_1, \xi(c,h)) \, A(h_2 \, , \xi(c,h_1+h))$$

$$(\lambda_o \times \lambda_o \times \lambda_o \times \mu) \text{ a.e. } (h_1,h_2,h,c).$$

So there is $h_o \in H$ such that,

$$A(h_1+h_2 \, , \xi(c,h_o)) = A(h_1 \, , \xi(c,h_o)) \, A(h_2 \, , \xi(c,h_1+h_o))$$

$$(\lambda_o \times \lambda_o \times \mu) \text{ a.e. } (h_1,h_2,c).$$

Let $\rho(g) = \rho(\xi(c,h)) = A(h \, , \xi(c,h_o))$.

ρ is a Borel function on G and

$$\rho(\xi(c,h_1+h_2)) = \rho(\xi(c,h_1)) \, A(h_2 \, , \xi(c,h_1+h_o))$$

$$(\lambda_o \times \lambda_o \times \mu) \text{ a.e. } (h_1,h_2,c).$$

Putting $\xi(c,h_1) = g_1$ we get,

$$A(h_2 \, , h_o+g_1) = \rho^*(g_1) \, \rho(g_1+h_2) \qquad (\lambda_o \times \tilde{\mu}) \text{ a.e. } (h_2,g_1).$$

That is, a translate of A is a coboundary. Hence, A itself is a coboundary.

Q. E. D.

A (G,X,$\underline{\underline{U}}$($\mathbb{H}$)) cocycle is said to be a <u>strict cocycle</u> if the cocycle identity (4.1) holds everywhere. We say that a cocycle has a strict version if A is almost everywhere equal to a strict cocycle. If a cocycle A is cohomologous to a strict cocycle, then A has a strict version. In particular, every coboundary has a strict version. Lemma 8.26 of [24] says that when the G-space X is transitive, then every (G , X) cocycle has a strict version. It is easier to handle strict cocycles. But we do not know whether every (G,X,$\underline{\underline{U}}$($\mathbb{H}$)) cocycle has a strict version. If G is a countable group, then every (G,X,$\underline{\underline{U}}$($\mathbb{H}$)) cocycle has a strict version. In chapter II we shall exhibit strict versions in some more special cases.

5. <u>Systems of Imprimitivity</u>.

Let $\mathbb{H}$ be a separable Hilbert space and X a standard Borel G-space, where G is a locally compact second countable group. By a <u>system of imprimitivity</u> based on (G , X) and acting in $\mathbb{H}$, we mean a pair (U , P) where

(i) U is a representation of G acting in $\mathbb{H}$

(ii) P is a spectral measure on the Borel subsets of X, acting in the same Hilbert space $\mathbb{H}$ such that for each Borel set $D \subseteq X$ and for each $g \in G$,

$$U_g^{-1} P(D) U_g = P(gD).$$

Two systems of imprimitivity (U,P) and (U', P') based on (G,X) and acting in $\mathbb{H}$ and $\mathbb{H}'$ respectively are said to be equivalent if there exists an isometric isomorphism S of $\mathbb{H}$ onto $\mathbb{H}'$ such that

$$SP(D)S^{-1} = P'(D) \quad \text{and}$$

$$SU_g S^{-1} = U'_g \quad ,$$

for each Borel set $D \subseteq X$ and each $g \in G$.

5.1. Example: Let μ be a measure on X, quasi-invariant under the action of G, and let A be a $(G,X,\underline{\underline{U}}(\mathbb{H}))$ cocycle relative to μ. We can define a system of imprimitivity (U^A, P) on (G,X) acting in $L^2(X, \mathbb{H}, \mu)$ by setting

$$(U^A_g f)(x) = \sqrt{\frac{d\mu_g}{d\mu}(x)}\ A(g,x)f(gx) \ , \quad x \in X, \ g \in G,$$

$$P(D)f = 1_D \cdot f$$

where 1_D is the characteristic function of D. (U^A, P) will be called a <u>concrete system of imprimitivity of multiplicity n</u>, where n $(= \infty, 1, 2, \cdots)$ is the dimension of the Hilbert space $\mathbb{H}$. If A is cohomologous to A', then (U^A, P) is equivalent to $(U^{A'}, P)$. More generally, let $\mu_\infty, \mu_1, \mu_2, \cdots$ be a sequence of mutually singular Borel measures on X, each μ_i quasi-invariant under G (some μ_i's may be zeros). For each n, let A_n be a $(G, X, \underline{\underline{U}}(\mathbb{H}_n))$ cocycle relative to μ_n, where $\mathbb{H}_n$ is a Hilbert space of dimension n. Then we can define a system of imprimitivity (U,P) on (G,X) acting in $\Sigma\ L^2(X, \mathbb{H}_n, \mu_n)$ by requiring that the restriction of (U,P) to $L^2(X, \mathbb{H}_n, \mu_n)$ be (U^{A_n}, P_n) where P_n is the spectral measure on X, acting in $L^2(X, \mathbb{H}_n, \mu_n)$ and consisting of multiplication by characteristic functions. Such a system of imprimitivity will be called a <u>concrete system of imprimitivity</u>. If (U', P') is another system of imprimitivity acting in $\Sigma\ L^2(X, \mathbb{H}_n, \mu_n')$ with associated cocycles A'_∞, $A'_1, A'_2, \cdots$, then (U , P) and (U' , P') are equivalent if and only if, for each n, μ_n and μ_n' are mutually absolutely continuous and A_n is cohomologous to A_n'.

We refer to Varadarajan [24] for proofs of the above stated results on systems of imprimitivity. In the sequel, we shall assume that $\mathbb{H}_n = \mathbb{C}^n$ if n is finite and $\mathbb{H}_\infty = \ell^2$. We will need the following theorem. This is proved in Varadarajan [24].

5.2. Theorem. Every (G,X) system of imprimitivity acting in a separable Hilbert space is equivalent to a concrete system of imprimitivity.

6. Another definition of a $(G,X,\underline{U}(\mathbb{H}))$ cocycle.

A is a $(G,X,\underline{U}(\mathbb{H}))$ cocycle relative to a quasi-invariant measure α on X, if A is $G \times X$ measurable and

$$A(g_1 g_2 , x) = A(g_1 , x) A(g_2 , g_1 x) \quad (\lambda \times \lambda \times \alpha) \text{ a.e.}$$

This is equation (4.1). However, for some purposes, it is convenient if (4.1) holds α a.e. for every pair $(g_1, g_2) \in G \times G$. The purpose of this section is to show that if we require the cocycle A to be only $\lambda \times \alpha$ measurable, then this can be accomplished. Further, if $H \subseteq G$ is a closed subgroup of G and λ_0 is the Haar measure on H, then $A_{|H \times X}$ is $\lambda_0 \times \alpha$ measurable.

6.1. Lemma. Let X be a Borel space, S a separable metric space and f a Borel map of X into S. Then there exists a sequence $\{f_n\}$ of Borel maps of X into S such that,

(i) each f_n takes only countably many values, and

(ii) $f_n(x) \longrightarrow f(x)$ uniformly for $x \in X$, as $n \longrightarrow \infty$.

This is lemma 8.3. of [24] and so the proof is omitted.

6.2. Theorem. Let G and X be standard Borel spaces with finite measures λ and α respectively. Let the function $w : G \times X \longrightarrow \mathbb{C}$ have the following properties:

(i) For each $g \in G$, $x \longrightarrow w(g,x)$ is Borel and belongs to $L^2(X,\alpha)$.

(ii) $g \longrightarrow w(g,\cdot)$ is a Borel function from G to $L^2(X,\alpha)$.

Then there exists a λ-null set $N \subseteq G$ and a function $\widetilde{w} : G \times X \longrightarrow \mathbb{C}$ such that,

(a) $\widetilde{w}$ is Borel on $(G-N) \times X$, and

(b) for every $g \in G$, $\alpha(\{x: \widetilde{w}(g,x) = w(g,x)\}) = \alpha(X)$.

If w is bounded by a constant K, then $\widetilde{w}$ can be so chosen that it is also bounded by the same constant K.

Proof: Since X is a standard Borel space, $L^2(X,\alpha)$ is a separable metric space. So by lemma 6.1, there exists a sequence $\{f_n\}$ of Borel maps of G into $L^2(X,\alpha)$ such that,

(1) each f_n takes only countably many values, and

(2) $f_n(g) \longrightarrow w_g$ uniformly for $g \in G$, as $n \to \infty$.

$(w_g(x) = w(g,x))$.

Define f_n^* on $G \times X$ as follows:

$$f_n^*(g,x) = f_n(g)(x).$$

Because of (1), f_n^* is $G \times X$ measurable. Since $f_n(g) \longrightarrow w_g$ in $L^2(X,\alpha)$, uniformly in g,

$$\lim_{m,n \to \infty} \int_X |f_n(g)(x) - f_m(g)(x)|^2 \, d\alpha(x) = 0,$$

uniformly in g.

So, by the definition of f_n^*,

$$\lim_{m,n \to \infty} \int_G \int_X |f_n^*(g,x) - f_m^*(g,x)|^2 \, d\alpha(x) \, d\lambda(x) = 0.$$

Thus, f_n^* converges in $\lambda \times \alpha$ measure, and therefore some subsequence $\{f_{n_j}^*\}$ of $\{f_n^*\}$ converges for a.e. (g,x) to a $G \times X$ measurable limit function f^*, defined at the points of convergence of this subsequence. By Fubini's theorem, there is a subset G_o of G, of full λ-measure such that the sequence of functions $\{f_{n_j}^*(g,\cdot)\}$ converges to $f^*(g,\cdot)$ α a.e. if $g \in G_o$.

Hence, if $g \in G_o$,

$$\alpha(\{x : f^*(g,x) = w(g,x)\}) = \alpha(X).$$

Define $\tilde{w}$ on $G \times X$ by:

$$\tilde{w}(g,x) = f^*(g,x) \quad \text{if } f^*(g,x) \text{ is defined and } g \in G_0 ,$$
$$= w(g,x) \quad \text{otherwise.}$$

$\tilde{w}$ satisfies all the requirements of the theorem.

Q. E. D.

6.3. Remarks and consequences.

(1) The modified function $\tilde{w}$ may not be Borel on $G \times X$, but it is $\lambda \times \alpha$ measurable.

(2) The theorem remains valid if the measure λ is σ-finite, for we need only replace it by an equivalent finite measure.

(3) If $g \longrightarrow w(g,\cdot)$ is continuous from G to $L^2(X,\alpha)$, then it is Borel and so the theorem 6.2 holds. This is the situation in the discussion that follows.

Suppose X is a Borel G-space where G is a locally compact second countable group. Let λ be the Haar measure on G and α a finite measure on X, quasi-invariant under the action of G. Then the Radon-Nikodym derivative,

$$\rho_g(x) = \frac{d\alpha_g}{d\alpha}(x)$$

has a version which is Borel in x, for each fixed g.

Further, from the theory of group representation, we know that

$$g \longrightarrow \sqrt{\frac{d\alpha_g}{d\alpha}\cdot 1} = \sqrt{\frac{d\alpha_g}{d\alpha}} = \sqrt{\rho_g}$$

is continuous from G to $L^2(X,\alpha)$. Hence, by theorem 6.2, we can get a $\lambda \times \alpha$ measurable version ρ' such that, for each $g \varepsilon G$, $\rho'_g(x) = \rho_g(x)$ α a.e. Also,

$$\rho'_{g_1 g_2}(x) = \rho'_{g_1}(x)\ \rho'_{g_2}(g_1 x) \quad \alpha \text{ a.e. } x \varepsilon X,$$

for each pair $g_1, g_2 \varepsilon G$.

Now, let $A : G \times X \longrightarrow \underline{\underline{U}}(\mathbb{H})$ be a function such that,

(i) for each g, $A(g,\cdot)$ is Borel on X, and

(ii) $g \longrightarrow \langle A(g,\cdot)e_i , e_j \rangle$ is continuous from G to $L^2(X,\alpha)$, where $\{e_i\}$ is a complete orthonormal set in $\mathbb{H}$.

Then, by theorem 6.2, for each $g \varepsilon G$, the function $a_{ij}(g,\cdot) = \langle A(g,\cdot)e_i , e_j \rangle$ can be modified on an α-null set so that $a_{ij}(\cdot,\cdot)$ is $\lambda \times \alpha$ measurable. From this one can obtain a function $A'(\cdot,\cdot)$ with values in $\underline{\underline{U}}(\mathbb{H})$ such that A' is $\lambda \times \alpha$ measurable and for every $g \varepsilon G$,

$$A'(g,x) = A(g,x) \quad \alpha \text{ a.e. } x \varepsilon X.$$

If A satisfies the cocycle identity α a.e. for each pair $g_1, g_2 \varepsilon G$, then so does A'.

Next, let $A : G \times X \longrightarrow \underline{\underline{U}}(\mathbb{H})$ be a function satisfying (i) and (ii) of the above paragraph. Let $H \subseteq G$ be a closed subgroup of G. $g \longrightarrow a_{ij}(g,\cdot)$ is continuous from H to $L^2(X,\alpha)$. Hence, by theorem 6.2, and the above discussion, for each $g \varepsilon H$, $A(g,\cdot)$ can be modified on an α-null set to get a function A' on $H \times X$ so that A' is $\lambda_o \times \alpha$ measurable, where λ_o is the Haar measure on H. By the same argument, A can be modified for each $g \varepsilon G - H$ on an α-null set so that the resulting function A' on $(G-H) \times X$ is $\lambda \times \alpha$ measurable. Thus A can be so modified to obtain a function A' such that A is $\lambda \times \alpha$ measurable and $A|_{H \times X}$ is $\lambda_o \times \alpha$ measurable. If A satisfies the cocycle identity α a.e. for each pair $g_1, g_2 \varepsilon G$, then so does A'.

Now suppose $A : G \times X \longrightarrow \underline{\underline{U}}(\mathbb{H})$ satisfies (4.1), viz. $A(g_1 g_2, x) = A(g_1, x)A(g_2, g_1 x)$ a.e. (g_1, g_2, x). Define the operator L_g on $L^2(X, \mathbb{H}, \alpha)$ by:

$$(L_g f)(x) = A(g,x)\sqrt{\frac{d\alpha_g}{d\alpha}(x)}\; f(gx) \;,\; g \varepsilon G,\; x \varepsilon X,\; f \varepsilon L^2(X, \mathbb{H}, \alpha).$$

From the theory of group representations (see pages 67,68 of [24]) we know that there is a representation U $(g \to U_g)$ of G such that,

$$L_g = U_g \quad \text{for a.e. } g \in G.$$

Also, there is a function $A' : G \times X \longrightarrow \underline{\underline{U}}(\mathbb{H})$ such that, for each $g \in G$, $A'(g,\cdot)$ is Borel on X and,

$$(U_g f)(x) = A'(g,x)\sqrt{\frac{d\alpha_g}{d\alpha}(x)}\, f(gx) \;, \quad x \in X,\ f \in L^2(X, \mathbb{H}, \alpha).$$

A' satisfies the cocycle identity (for α a.e.) for every pair $g_1, g_2 \in G$. If,

$$(T_g f)(\cdot) = \sqrt{\frac{d\alpha_g}{d\alpha}}\, f(g\cdot) \;, \quad f \in L^2(X, \mathbb{H}, \alpha), \text{ then}$$

$$g \longrightarrow \langle A'(g,\cdot)e_i \,,\, e_j \rangle = \langle U_g T_{-g} e_i \,,\, e_j \rangle$$

is a continuous function from G to $L^2(X,\alpha)$. So we can get a modification A'' of A' such that A'' is $\lambda \times \alpha$ measurable, $A'' = A'$ $(\lambda \times \alpha)$ a.e. and, for every $g_1, g_2 \in G$,

$$A''(g_1 g_2 \,,\, x) = A''(g_1 \,,\, x)\, A''(g_2 \,,\, g_1 x) \quad \alpha \text{ a.e. } x \in X.$$

Also, if H is a closed subgroup of G, then A'' can be so obtained that $A''|_{H \times X}$ is $\lambda_o \times \alpha$ measurable, where λ_o

is the Haar measure on H. Thus we have the following proposition.

6.4. <u>Proposition</u>. Let $A : G \times X \longrightarrow \underline{\underline{U}}(\mathbb{H})$ be a Borel function satisfying

$$A(g_1 g_2 , x) = A(g_1 , x)A(g_2 , g_1 x) \quad (\lambda \times \lambda \times \alpha) \text{ a.e.}$$

Let H be a closed subgroup of G and let λ_o be the Haar measure on H. Then we can get a function $A' : G \times X \longrightarrow \underline{\underline{U}}(\mathbb{H})$ such that,

(i) $A'(g,x) = A(g,x) \quad (\lambda \times \alpha)$ a.e. $(g,x) \in G \times X$,

(ii) A' is $\lambda \times \alpha$ measurable,

(iii) $A'|_{H \times X}$ is $\lambda_o \times \alpha$ measurable, and

(iv) for every $g_1, g_2 \in G$,

$$A'(g_1 g_2 , x) = A'(g_1 , x)A'(g_2 , g_1 x) \quad \alpha \text{ a.e. } x \in X.$$

Similarly, two $(G,X,\underline{\underline{U}}(\mathbb{H}))$ cocycles A_1 and A_2 relative to the quasi-invariant measure α on X, are cohomologous if and only if there exists a Borel function $\rho : X \longrightarrow \underline{\underline{U}}(\mathbb{H})$ such that for each $g \in G$,

$$A_1(g,x) = \rho(x) A_2(g,x) \rho^*(gx) \quad \alpha \text{ a.e. } x \in X.$$

CHAPTER II

COCYCLES ON A FLOW BUILT UNDER A FUNCTION

1. Flow built under a function.

Let $(\Omega, \mathcal{B}, \mu)$ be a finite measure space which is complete and let $S : \Omega \longrightarrow \Omega$ be a one-one, bimeasurable map, such that S preserves μ-null sets. S defines an action of N, the integer group, on Ω and μ is quasi-invariant under this action. Let F be a $\mathcal{B}$- measurable real-valued function on Ω such that $F \geq c > 0$ and $\int_\Omega F(w)\, d\mu(w) < \infty$, where c is a constant. Let $\bar{\Omega}$ denote the subset of $\Omega \times \mathbb{R}$ under the graph of F; i.e.,

$$\bar{\Omega} = \left\{(w,u) : w \in \Omega, 0 \leq u < F(w)\right\}$$

Equip $\mathbb{R}$ with the Lebesgue σ-algebra and the Lebesgue measure λ. $\bar{\Omega}$ is a measurable subset of the completed product σ-algebra on $\Omega \times \mathbb{R}$. Let $\bar{\mathcal{B}}$, $\bar{\mu}$ be the restrictions to $\bar{\Omega}$ of the completed product σ-algebra and the product measure $\mu \times \lambda$ on $\Omega \times \mathbb{R}$. Since $F \in L^1(\Omega, \mu)$, $(\bar{\Omega}, \bar{\mathcal{B}}, \bar{\mu})$ is again a finite measure space. On $\bar{\Omega}$ we define a one-parameter family $\{T_t : t \in \mathbb{R}\}$ of transformations, i.e., an action of the real line on $\bar{\Omega}$, as follows:

$$T_t(w,u) = (w, u+t), \quad \text{if} \quad 0 \leq u+t < F(w)$$

$$= \left(S^n w,\ u+t - \sum_{k=0}^{n-1} F(S^k w)\right),$$

(1.1) $$\text{if} \quad \sum_{k=0}^{n-1} F(S^k w) \leq u+t < \sum_{k=0}^{n} F(S^k w) \ , \ n=1,2,\cdots$$

$$= \left(S^{-n} w,\ u+t + \sum_{k=-1}^{-n} F(S^k w)\right),$$

$$\text{if} \ - \sum_{k=1}^{n} F(S^{-k} w) \leq u+t < - \sum_{k=1}^{n-1} F(S^{-k} w), \ n=1,2,\cdots \quad .$$

It can be shown that $\{T_t : t \in \mathbb{R}\}$ is a measurable flow on $\bar{\Omega}$ by which we mean that,

(i) $\{T_t : t \in \mathbb{R}\}$ makes $\bar{\Omega}$ a Borel $\mathbb{R}$-space, and

(ii) $\bar{\mu}$ is quasi-invariant under the action of $\mathbb{R}$.

We call $\{T_t : t \in \mathbb{R}\}$ the <u>flow built under the function</u> F, on the transformation S. Ω is called the <u>base space</u>, S the <u>base transformation</u> and F the <u>ceiling function</u>.

We note that,

(1.2) $$T_{F(w)}(w,0) = (Sw\ ,\ 0),$$

a relation which will be useful in the sequel.

Notation: For each $t \in \mathbb{R}$, and $w \in \Omega$ let,

$[t]_w$ = the integer n, such that

$$\sum_{k=0}^{n-1} F(S^k w) \leq t < \sum_{k=0}^{n} F(S^k w) \text{ , if } t \geq 0,$$

$$-\sum_{k=-1}^{n} F(S^k w) \leq t < -\sum_{k=-1}^{n+1} F(S^k w), \text{ if } t < 0.$$

$$\langle t \rangle_w = t - \sum_{k=0}^{[t]_w - 1} F(S^k w) \text{ , if } [t]_w \geq 0 \text{ ,}$$

$$= t + \sum_{k=-1}^{[t]_w} F(S^k w) \text{ , if } [t]_w < 0.$$

Remark: One can visualise translation on the real line as a flow built under the constant function 1 with base space Ω as the integers and S as addition by 1. If this is done, $[t]_w$ is independent of w and equals the integral part of t. For the integral part $[t]$ of t, the following relation is easily proved.

$$[t_1+t_2+t_3] = [t_1+t_2] + [t_3+\langle t_1+t_2 \rangle]$$

where $\langle t \rangle$ is the fractional part of t. (The above relation is (3.1) of chapter I). The following lemma is a generalization of this fact. This lemma was proved jointly with M. G. Nadkarni.

1.1. Lemma. For $t_1, t_2, t_3 \in R$, and $w \in \Omega$,

(1.3) $[t_1+t_2+t_3]_w = [t_1+t_2]_w + [t_3+\langle t_1+t_2\rangle_w]_{S^{[t_1+t_2]_w}(w)}$.

Proof: Let $n_1 = [t_1+t_2]_w$, $n_2 = [t_3+\langle t_1+t_2\rangle_w]_{S^{[t_1+t_2]_w}(w)}$ and $n = [t_1+t_2+t_3]_w$. Consider the case when $n_1, n_2 \geq 0$.

By definition,

$$\sum_{k=0}^{n_1-1} F(S^k w) \leq t_1+t_2 < \sum_{k=0}^{n_1} F(S^k w), \text{ and}$$

$$\sum_{k=0}^{n_2-1} F(S^k(S^{n_1} w)) \leq t_3+\langle t_1+t_2\rangle_w < \sum_{k=0}^{n_2} F(S^k(S^{n_1} w)).$$

i.e., $$\sum_{k=n_1}^{n_1+n_2-1} F(S^k w) \leq t_3+t_1+t_2-\sum_{k=0}^{n_1-1} F(S^k w) < \sum_{k=n_1}^{n_1+n_2} F(S^k w).$$

Therefore, $$\sum_{k=0}^{n_1+n_2-1} F(S^k w) \leq t_1+t_2+t_3 < \sum_{k=0}^{n_1+n_2} F(S^k w).$$

i.e., $n = [t_1+t_2+t_3]_w = n_1 + n_2$.

Similarly, we can consider the remaining cases and verify (1.3).

Q. E. D.

2. Cocycles on a flow built under a function.

Since $\mathbb{N}$ is countable, every $(\mathbb{N},\Omega)$ cocycle has a strict version. Let β be a measurable $\underline{\underline{U}}(\mathbb{H})$ valued function on Ω, and put

$$(2.1) \quad A_\beta(m,w) = \begin{cases} \beta(w)\beta(Sw)\cdots\beta(S^{m-1}w), & m > 0 \\ I, & m = 0 \\ \beta(S^{-1}w)^*\cdots\beta(S^m w)^*, & m < 0. \end{cases}$$

Then A_β is a strict $(\mathbb{N},\Omega)$ cocycle. Conversely, if A is an $(\mathbb{N},\Omega)$ cocycle and $\beta(w) = A(1,w)$, then $A = A_\beta$ a.e. Thus $(\mathbb{N},\Omega)$ cocycles are completely described in terms of a single $\underline{\underline{U}}(\mathbb{H})$ valued measurable function on Ω. In the rest of this section we establish a relation between strict $(\mathbb{N},\Omega)$ cocycles and strict $(\mathbb{R},\bar{\Omega})$ cocycles and show that every $(\mathbb{R},\bar{\Omega})$ cocycle has a strict version.

Let A be a strict $(\mathbb{N},\Omega)$ cocycle and define $\bar{A}$ on $\mathbb{R}\times\bar{\Omega}$ as follows:

$$(2.2) \quad \bar{A}(t, (w,u)) = A([t+u]_w, w).$$

Then $\bar{A}$ satisfies the cocycle identity (4.1) of chapter I. This can be verified by using lemma 1.1. Further, $\bar{A}$ is jointly measurable in view of the following lemma.

2.1. Lemma. $(t,(w,u)) \longrightarrow ([t+u]_w , w)$ is a measurable map of $\mathbb{R}\times\bar{\Omega}$ into $N\times\Omega$.

Proof: Since $(t,(w,u)) \longrightarrow w$ is measurable, it is enough to show that $(t,(w,u)) \longrightarrow [t+u]_w$ is measurable. Let n_o be a (positive) integer, and consider

$$E = \{(t,(w,u)) : [t+u]_w = n_o\} .$$

Now,

$\varphi_1 : (t,(w,u)) \longrightarrow (t+u, (w,u))$ is measurable from $\mathbb{R}\times\bar{\Omega}$ —— $\mathbb{R}\times\bar{\Omega}$ (Ambrose [1], p.731), and

$\varphi_2 : (t,(w,u)) \longrightarrow (t,w)$ is measurable from $\mathbb{R}\times\bar{\Omega}$ into $\mathbb{R}\times\Omega$. Therefore,

$\varphi_2 \circ \varphi_1 : (t,(w,u)) \longrightarrow (t+u , w)$ is measurable from $\mathbb{R}\times\bar{\Omega}$ into $\mathbb{R}\times\Omega$. Next,

$$V = \{(t,w) : [t]_w = n_o\}$$

$$= \{(t,w) : \sum_{k=0}^{n_o-1} F(S^k w) \leq t < \sum_{k=0}^{n_o} F(S^k w)\}$$

is a measurable subset of $\mathbb{R}\times\Omega$. Hence,

$$E = \{(t,(w,u)) : [t+u]_w = n_o\} = (\varphi_2 \circ \varphi_1)^{-1}(V)$$

is a measurable subset of $\mathbb{R}\times\bar{\Omega}$.

Q. E. D.

2.2. Theorem. Let B be a strict $(\mathbb{R}, \bar{\Omega})$ cocycle. Then B is strictly cohomologous to a cocycle $\bar{A}$ for some (N, Ω) cocycle A. Two (N, Ω) cocycles A_1 and A_2 are cohomologous if and only if the corresponding $(\mathbb{R}, \bar{\Omega})$ cocycles $\bar{A}_1$ and $\bar{A}_2$ are strictly cohomologous.

Proof: Define $\beta(w) = B(F(w), (w,0))$, and let A denote the (N, Ω) cocycle A_β giben by β according to (2.1). Define ρ on $\bar{\Omega}$ by:

$$\rho(w,u) = B(u, (w,0)).$$

ρ is measurable on $\bar{\Omega}$. For $t \in \mathbb{R}$, $(w,u) \in \bar{\Omega}$,

$$\rho(w,u)B(t,(w,u))\,\rho^*(T_t(w,u))$$

$$= B(u,(w,0))B(t,(w,u))B^*(\langle t+u\rangle_w, (S^{[t+u]_w}(w),0))$$

$$= B(u+t,(w,0))B^*(\langle t+u\rangle_w, (S^{[t+u]_w}(w), 0))$$

$$= B(F(w)+ \cdots + F(S^{[t+u]_w}(w)), (w,0))$$

$$= A([t+u]_w, w) = \bar{A}(t, (w,u))$$

So B and $\bar{A}$ are strictly cohomologous. If two (N, Ω) cocycles A_1 and A_2 are cohomologous (ρ_0), then $\bar{A}_1$ and $\bar{A}_2$ are cohomologous (ρ), where $\rho(w,u) = \rho_0(w)$. If $\bar{A}_1$ and $\bar{A}_2$ are strictly cohomologous (ρ), then A_1 and

A_2 are strictly cohomologous (ρ_o) where $\rho_o(w) = \rho(w,0)$.

Q. E. D.

2.3. Theorem. To every ($\mathbb{R}$, $\bar{\Omega}$) cocycle A, there corresponds a strict ($\mathbb{R}$, $\bar{\Omega}$) cocycle A' such that,

$$A(t, \bar{w}) = A'(t, \bar{w}) \text{ for a.e. } (t,\bar{w}) \in \mathbb{R} \times \bar{\Omega}.$$

Proof: Since A is an ($\mathbb{R}$, $\bar{\Omega}$) cocycle, we have:

$$(2.3) \quad A(s+t,(w,u)) = A(s,(w,u))A(t,T_s(w,u))$$

$$\text{a.e. } (s,t,(w,u)) \in \mathbb{R} \times \mathbb{R} \times \bar{\Omega}.$$

Since $[0,c) \times \Omega$ is of positive measure in $\bar{\Omega}$, there is u_o , $0 \le u_o < c$, such that,

$$A(s+t,(w,u_o)) = A(s,(w,u_o))A(t,T_s(w,u_o))$$

$$\text{a.e. } (s,t,w) \in \mathbb{R} \times \mathbb{R} \times \Omega.$$

Define A_1 on $\mathbb{R} \times \bar{\Omega}$ by:

$$A_1(t,(w,u)) = A(t,T_{u_o}(w,u)).$$

Then A_1 is an ($\mathbb{R}$, $\bar{\Omega}$) cocycle. Moreover,

$$(2.4) \quad A_1(s+t,(w,0)) = A_1(s,(w,0))A_1(t,T_s(w,0))$$

$$\text{a.e. } (s,t,w) \in \mathbb{R} \times \mathbb{R} \times \Omega.$$

Let J be a Borel subset of Ω of measure zero such

that, if $w \notin J$, (2.4) holds for a.e. $(s,t) \in \mathbb{R} \times \mathbb{R}$.
Replacing J by $\bigcup_{n=-\infty}^{\infty} S^n(J)$ we can assume $S(J) = J$ and $S^{-1}(J) = J$. Define A_2 on $\mathbb{R} \times \bar{\Omega}$ by,

$$A_2(t,(w,u)) = \begin{cases} I & \text{if } w \in J \\ A_1(t,(w,u)) & \text{if } w \notin J. \end{cases}$$

Then A_2 is an $(\mathbb{R}, \bar{\Omega})$ cocycle and $A_2 = A_1$ a.e. Also, for each $w \in \Omega$,

(2.5) $$A_2(s+t,(w,0)) = A_2(s,(w,0))A_2(t,T_s(w,0))$$

$$\text{a.e. } (s,t) \in \mathbb{R} \times \mathbb{R}.$$

Define the measurable function ρ on $\bar{\Omega}$ by:

$$\rho(w,u) = A_2(u,(w,0)).$$

Let A_3 be the $(\mathbb{R}, \bar{\Omega})$ cocycle defined by:

$$A_3(t,(w,u)) = \rho(w,u)A_2(t,(w,u))\rho^*(T_t(w,u)).$$

A_3 is cohomologous to A_2; for each $w \in \Omega$, A_3 satisfies

(2.6) $$A_3(s+t,(w,0)) = A_3(s,(w,0))A_3(t,T_s(w,0))$$

$$\text{a.e. } (s,t) \in \mathbb{R} \times \mathbb{R}.$$

Moreover,

$$A_3(u,(w,0)) = I, \quad 0 \leq u < F(w).$$

Fix $w \in \Omega$. For a.e. s, $F(w) \leq s < F(w) + F(Sw)$, we have,

(2.7) $A_3(s-F(w)+t,(Sw,0)) = A_3(s-F(w), (Sw,0)) \cdot$

$$A_3(t, T_s(w,0)) \text{ a.e. } t \in \mathbb{R}.$$

$$= A_3(t, T_s(w,0)) \quad \text{a.e.} \quad t \in \mathbb{R}.$$

(2.8) $A_3(s+t,(w,0)) = A_3(s,(w,0))\, A_3(t,T_s(w,0))$

$$\text{a.e. } t \in \mathbb{R}.$$

Fixing s for which (2.7) and (2.8) hold and putting $u = s+t - F(w)$, we have,

$$A_3(u+F(w), (w,0)) = A_3(s,(w,0))A_3(u,(Sw,0))$$

$$\text{a.e. } u \in \mathbb{R}.$$

Therefore, $A_3(s, (w,0))$ is constant for a.e. s in $F(w) \leq s < F(w)+F(Sw)$. Let $\beta(w)$ be this constant value. Then β is a $\underline{\underline{U}}(\mathbb{H})$ valued measurable function on Ω. For each $w \in \Omega$,

$$\beta(w) = A_3(s,(w,0)) \quad \text{a.e. } s,\ F(w) \leq s < F(w)+F(Sw).$$

Let A_β, be the (N,Ω) cocycle obtained from β by (2.1).

For each $w \in \Omega$,

$$(2.9)\begin{cases} \bar{A}_\beta(s,(w,0)) = I = A_3(s,(w,0)), \quad 0 \leq s < F(w) \\ \bar{A}_\beta(s,(w,0)) = \beta(w) = A_3(s,(w,0)), \\ \qquad\qquad\qquad \text{a.e. } s,\ F(w) \leq s < F(w)+F(Sw), \end{cases}$$

and,

$$(2.10)\begin{cases} A_3(s+t,(w,0)) = A_3(s,(w,0))\, A_3(t,(w,s)) \\ \qquad\qquad\qquad \text{a.e. } (s,t) \in [0,F(w))\times[0,c) \\ \bar{A}_\beta(s+t,(w,0)) = \bar{A}_\beta(s,(w,0))\, \bar{A}_\beta(t,(w,s)) \\ \qquad\qquad\qquad \text{a.e. } (s,t) \in [0,F(w))\times[0,c). \end{cases}$$

Applying (2.9) to (2.10) we have, for each $w \in \Omega$,

$$A_3(t,(w,s)) = \bar{A}_\beta(t,(w,s)) \text{ a.e. } (s,t) \in [0,F(w))\times[0,c).$$

Therefore,

$$\bar{A}_\beta(t,(w,s)) = A_3(t,(w,s)) \text{ a.e. } (t,(w,s)) \in [0,c)\times\bar{\Omega}\ .$$

Since A_3 and $\bar{A}_\beta$ are cocycles, the cocycle identity now implies that the equality $\bar{A}_\beta = A_3$ holds for a.e. $(t,\bar{w}) \in \mathbb{R}\times\bar{\Omega}$. Thus A_3 has a strict version and so A_2 has a strict version, since it is cohomologous to A_3. Since $A_1 = A_2$ a.e., A_1 has a strict version. But A_1 is a translate of A and so A has a strict version. This proves the theorem.

Q. E. D.

Remark: The above proof is an adaptation of the proof of lemma 12.3 of chapter VII of [8].

3. Ambrose-Kakutani Theorem.

Let $\{T_t : t \in \mathbb{R}\}$ and $\{T'_t : t \in \mathbb{R}\}$ be measurable flows on $(\Omega, \mathcal{B}, \mu)$ and $(\Omega', \mathcal{B}', \mu')$ respectively. We say that $\{T_t\}$ and $\{T'_t\}$ are isomorphic if there are invariant null sets N of Ω and N' of Ω' and a one-one, bimeasurable map φ of $\Omega - N$ onto $\Omega' - N'$ such that $\mu' = \mu\varphi^{-1}$ and $T'_t = \varphi T_t \varphi^{-1}$ (on $\Omega' - N'$). They are said to be quasi-isomorphic if $\mu' \equiv \mu\varphi^{-1}$ and $\varphi T_t \varphi^{-1} = T'_t$. Let the flow $\{T_t : t \in \mathbb{R}\}$ on $(\Omega, \mathcal{B}, \mu)$ be quasi-isomorphic to the flow $\{T'_t : t \in \mathbb{R}\}$ on $(\Omega', \mathcal{B}', \mu')$. Suppose every $(\mathbb{R}, \Omega)$ cocycle has a strict version. Then, it is easy to see that, every $(\mathbb{R}, \Omega')$ cocycle also has a strict version.

A measurable flow $\{T_t : t \in \mathbb{R}\}$ on $(\Omega, \mathcal{B}, \mu)$ is said to be proper if given a set A of positive μ-measure, there is a set $B \subseteq A$ of positive μ-measure and $t_0 \in \mathbb{R}$ such that $\mu(B - T_{t_0} B) > 0$. It is easy to see that if $\{T_t : t \in \mathbb{R}\}$ is ergodic, then it is proper. A deep theorem of Ambrose and Kakutani [2] says that if μ is invariant under $\{T_t : t \in \mathbb{R}\}$ and $\{T_t : t \in \mathbb{R}\}$ is proper, then Ω can be split up into countable number of invariant measurable sets $N, \Omega_1, \Omega_2, \ldots$ such that $\mu(N) = 0$ and restriction of $\{T_t : t \in \mathbb{R}\}$ to Ω_i is isomorphic to a flow built under a function. Thus, in view of the results of the last section, when μ is invariant under $\{T_t : t \in \mathbb{R}\}$ and the flow is proper, every $(\mathbb{R}, \Omega)$ cocycle has a strict version and we can describe the cohomology classes of $(\mathbb{R}, \Omega)$ cocycles in terms of (N, Ω_o) cocycles where Ω_o is a different measure space.

In [21] Rokhlin gives a new proof of Ambrose - Kakutani theorem when $(\Omega, \mathcal{B}, \mu)$ is a Lebesgue space and μ is invariant under the flow. In this case, the condition that $\{T_t : t \in \mathbb{R}\}$ is proper is equivalent to the condition that $\{w : T_t w = w \text{ for all } t \in \mathbb{R}\}$, the set of fixed points of the flow, has measure zero. In [5] Dani has modified Rokhlin's method to prove that if $\{T_t : t \in \mathbb{R}\}$ is a proper measurable flow on a Lebesgue space $(\Omega, \mathcal{B}, \mu)$ (μ is just quasi-invariant under $\{T_t : t \in \mathbb{R}\}$), then Ω can be split up into invariant measurable sets $N, \Omega_1, \Omega_2, \ldots$ such that

$\mu(N) = 0$ and $\{T_t : t \in \mathbb{R}\}$ restricted to Ω_i is quasi-isomorphic to a flow built under a function. Thus, cocycles associated with proper measurable flows on a Lebesgue space have strict versions and they can be described in terms of (N, Ω_0) cocycles; where Ω_0 is a different measure space.

4. Flows built under a constant function.

Let $\{T_t : t \in \mathbb{R}\}$ be a flow built under a constant function F, with base transformation S and base space $(\Omega, \mathcal{B}, \mu)$. We assume $F = 1$. In this case, $\bar{\Omega} = \Omega \times [0,1)$ and $\bar{\mu} = \mu \times \lambda$, where λ is the Lebesgue measure on $[0,1)$.

4.1. Lemma.

$$(4.1) \qquad \frac{d\bar{\mu}_t}{d\bar{\mu}}(w,u) = \frac{d\mu_{[t+u]}}{d\mu}(w), \quad \bar{\mu} \text{ a.e. } (w,u).$$

Proof: It is enough to prove that,

$$(4.2) \qquad \int_A \int_B \frac{d\bar{\mu}_t}{d\bar{\mu}}(w,u)\, d\mu(w)\, d\lambda(u) = \int_A \int_B \frac{d\mu_{[t+u]}}{d\mu}(w)\, d\mu(w) d\lambda(u)$$

for every Borel subset of $\Omega \times [0,1)$ of the form $B \times A$.

R.H.S. of (4.2) = $\int_A \mu_{[t+u]}(B)\, d\lambda(u)$

$$= \lambda(A_1)\mu(S^{[t]}B) + \lambda(A_2)\mu(S^{[t]+1}B),$$

where $A_1 = A \cap [0, 1-\langle t\rangle)$ and $A_2 = A \cap [1-\langle t\rangle, 1)$.

L.H.S. of (4.2) = $\bar{\mu}_t(B \times A) = \bar{\mu}(T_t(B \times A))$

$$= \bar{\mu}\big((S^{[t]}B \times (A_1+\langle t\rangle)) \cup (S^{[t]+1}B \times (A_2+\langle t\rangle-1)) \big)$$

$$= \lambda(A_1)\mu(S^{[t]}B) + \lambda(A_2)\mu(S^{[t]+1}B),$$

since the two sets occurring in the union are disjoint.

Q. E. D.

Let A be an (N, Ω) cocycle taking values in $\underline{\underline{U}}(\mathbb{H})$, the group of unitary operators of a separable Hilbert space $\mathbb{H}$. (The inner product in $\mathbb{H}$ is denoted by $\langle \cdot, \cdot \rangle$). Let $\bar{A}$ be the $(\mathbb{R}, \bar{\Omega})$ cocycle obtained from A. A defines a unitary operator U on $L^2(\Omega, \mathbb{H}, \mu)$ by

$$(Uf)(w) = A(1,w)\sqrt{\frac{d\mu_1}{d\mu}(w)}\; f(S^{-1}w), \quad f \in L^2(\Omega, \mathbb{H}, \mu).$$

Likewise, $\bar{A}$ defines a one-parameter group of unitary operators $\bar{U}_t$ on $L^2(\bar{\Omega}, \mathbb{H}, \bar{\mu})$ by:

$$(\bar{U}_t h)(w,u) = \bar{A}(t,(w,u)) \sqrt{\frac{d\bar{\mu}_t}{d\bar{\mu}}(w,u)}\ h(T_{-t}(w,u)) \ ,$$

$$h \in L^2(\bar{\Omega}, \mathbb{H}, \bar{\mu}).$$

Let E and $\bar{E}$ be the spectral measures associated (by the Stone's theorem) with U and $\{\bar{U}_t : t \in \mathbb{R}\}$ respectively. E is defined on the circle group and $\bar{E}$ on the real line. We prove the following:

4.2. Theorem. Suppose 1 is not an eigenvalue of U, and E is of uniform multiplicity n, $1 \leq n \leq \infty$. Then, $\bar{E}$ also is of uniform multiplicity n.

Proof: Since E is of uniform multiplicity n, there are functions $f_1, f_2, \cdots, f_n$ in $L^2(\Omega, H, \mu)$ such that,

(i) $(U^k f_i, f_j) = 0$ if $i \neq j$, $1 \leq i,j \leq n$,

(ii) the measures $\nu(\cdot) = (E(\cdot)f_i, f_i)$ are same for each $i = 1,2,\cdots,n$, and,

(iii) the closed linear span of $\{U^k f_i$: k integer, $i = 1,2,\cdots,n\}$ is $L^2(\Omega, H, \mu)$.

Define $\bar{f}_i$ on $\bar{\Omega}$ by:

$$\bar{f}_i(w,u) = f_i(w).$$

Then, $\bar{f}_i \in L^2(\bar{\Omega}, \mathbb{H}, \bar{\mu})$.

1) $$(\bar{U}_t \bar{f}_i, \bar{f}_j) = \int_{\bar{\Omega}} \left\langle \bar{A}(t,(w,u)) \sqrt{\frac{d\bar{\mu}_t}{d\bar{\mu}}(w,u)}\ \bar{f}_i(\bar{T}_{-t}(w,u)), \bar{f}_j(w,u) \right\rangle d\bar{\mu}(w,u)$$

$$= \int_0^1 \int_{\Omega} \left\langle A([t+u],w) \sqrt{\frac{d\mu_{[t+u]}}{d\mu}(w)} \cdot f_i(S^{-[t+u]}w), f_j(w) \right\rangle d\mu(w)\ d\lambda(u)$$

$$= \int_0^1 (U^{[t+u]} f_i, f_j)\ d\lambda(u)$$

$$= 0 \text{ if } i \neq j .$$

2) If the measures $\bar{\nu}_i$ on $\mathbb{R}$ are defined by

$$\bar{\nu}_i(\cdot) = (\bar{E}(\cdot)\bar{f}_i, \bar{f}_i),$$

then $\bar{\nu}_i = \bar{\nu}_j$, $1 \leq i,j \leq n$.

3) The closed linear span of $\{\bar{U}_t \bar{f}_i : t \in \mathbb{R}, i=1,2,\cdots,n\}$ is $L^2(\bar{\Omega}, \mathbb{H}, \bar{\mu})$. For, let $h \in L^2(\bar{\Omega}, \mathbb{H}, \bar{\mu})$ be such that $(\bar{U}_t \bar{f}_i, h) = 0$ for all $t \in \mathbb{R}$ and $i = 1,2,\cdots,n$. Let m be an integer; t_o, s_o real numbers such that $m \leq t_o < s_o \leq m+1$. Then,

$$0 = ((\overline{U}_{t_o} - \overline{U}_{s_o})\overline{f}_i , h)$$

$$= \int_{\Omega}\int_0^1 \left\{ \left\langle A([t_o+u],w)\sqrt{\frac{d\mu_{[t_o+u]}}{d\mu}}(w) f_i(S^{-[t_o+u]}w) , h(w,u)\right\rangle \right.$$

$$\left. - \left\langle A([s_o+u],w)\sqrt{\frac{d\mu_{[s_o+u]}}{d\mu}}(w) f_i(S^{-[s_o+u]}w) , h(w,u)\right\rangle \right\}$$

$$d\lambda(u)\ d\mu(w).$$

$$= \int_{\alpha}^{\beta} d\lambda(u)\left\{ \int_{\Omega}\left[\left\langle A(m+1,w)\sqrt{\frac{d\mu_{m+1}}{d\mu}}(w)\ f_i(S^{-(m+1)}w) , h(w,u)\right\rangle \right.\right.$$

$$\left.\left. - \left\langle A(m,w)\sqrt{\frac{d\mu_m}{d\mu}}(w)\ f_i(S^{-m}w) , h(w,u)\right\rangle\right] d\mu(w) \right\}$$

$$= \int_{\alpha}^{\beta} ((U^{m+1}f_i , h_u) - (U^m f_i , h_u))\ d\lambda(u) ,$$

where $\alpha = m+1 - s_o$, $\beta = m+1 - t_o$ and h_u is the u^{th} section of h. Varying t_o and s_o , this is true for all α, β with $0 \leq \alpha < \beta < 1$. Therefore,

$$(U^{m+1}f_i , h_u) = (U^m f_i , h_u) \qquad \lambda \text{ a.e. } u \in [0,1).$$

Or, $(U^m f_i , h_u) = (U^o f_i , h_u)$ λ a.e. $u \in [0,1)$.

This is true for all $i=1,2,\cdots,n$. Therefore,

$$h_u \in E(1)\,(L^2(\Omega, H, \mu)) \quad \lambda \text{ a.e. } u \in [0,1).$$

As 1 is not an eigenvalue of U, $E(1) = 0$. So $h_u = 0$ λ a.e. $u \in [0,1)$. That is, $h(w,u) = 0$ $\bar{\mu}$ a.e. $(w,u) \in \bar{\Omega}$. Therefore, the closed linear span of $\left\{ \bar{U}_t \bar{f}_i : t \in \mathbb{R}, i = 1,2,\cdots,n \right\}$ is $L^2(\bar{\Omega}, \mathbb{H}, \bar{\mu})$.

1), 2) and 3) prove that $\bar{E}$ is of uniform multiplicity n.

Q. E. D.

In [19] Mackey introduces a generalization of the concept of a flow built under a function. His generalization of a flow built under a constant function is as follows:

Let H be a closed subgroup of a locally compact second countable group G, and let S be an H-space. $S \times G$ can be made an $H \times G$-space by:

$$(h,g)(s,x) = (hs, g^{-1}xh), \quad (h,g) \in H \times G, \quad (s,x) \in S \times G.$$

The action of $\{e\} \times G$ on $S \times G$ commutes with the action of $H \times \{e\}$ on $S \times G$. Therefore, we get an action of G on the $H \times \{e\}$ orbit space $\widetilde{S \times G}^{H}$ of $S \times G$. This is

Mackey's generalization of a flow built under a constant function.

Let C be a Borel section of G with respect to the closed subgroup H. We assume, without loss of generality, that the identity $e \in C$. It can be shown that $S \overset{H}{\widetilde{\times}} G$ with the quotient Borel structure is Borel isomorphic to $\overline{S} = S \times C$. The action of G on $S \overset{H}{\widetilde{\times}} G$ when taken to $\overline{S}$ becomes:

$$(4.3)\quad g(s,c) = ([cg]s\ ,\ \langle cg\rangle). \qquad \text{(See section 3 of chapter I).}$$

Let μ be a finite measure on S, quasi-invariant under the action of H, and let ν be a finite measure on G/H, quasi-invariant under the natural action of G on G/H. There is a unique invariant (under this action of G on G/H) measure class on G/H). ν can be considered a measure on C, since there is a Borel isomorphism between C and G/H. Then $\bar{\mu} = \mu \times \nu$ on $\overline{S}$ is quasi-invariant under the action (4.3) of G on $\overline{S}$. An analogue of lemma 4.1 holds for $\bar{\mu}$:

$$(4.4)\quad \frac{d\bar{\mu}_g}{d\bar{\mu}}(s,c) = \frac{d\mu_{[cg]}}{d\mu}(s)\ ,\quad \bar{\mu}\ \text{a.e.}\ (s,c).$$

Let A be an (H,S) cocycle. Define $\overline{A}$ on $G \times \overline{S}$ by:

(4.5) $\overline{A}(g, (s,c)) = A([cg], s).$

$\overline{A}$ is a $(G, \overline{S})$ cocycle. This can be verified by using the fact that for $g_1, g_2, g_3 \varepsilon G$,

$$[g_1g_2g_3] = [g_1g_2][\langle g_1g_2\rangle g_3].$$

Also, any strict $(G, \overline{S})$ cocycle is cohomologous to a $(G, \overline{S})$ cocycle extended from an (H, S) cocycle. For, let B be a strict $(G,\overline{S})$ cocycle. Let A_o be the (H, S) cocycle defined by:

$$A_o(h,s) = B(h,(s,e)).$$

Define the Borel function ρ on $\overline{S}$ by:

$$\rho(s,c) = B(c, (s,e)).$$

Let A be the $(G, \overline{S})$ cocycle (cohomologous (ρ) to B) defined by:

$$A(g,(s,c)) = \rho(s,c)\ B(g,(s,c))\ \rho^*(g(s,c)).$$

Then,

$$\begin{aligned} A(g,(s,c)) &= B(c,(s,e))B(g,(s,c))B^*(\langle cg\rangle,([cg]s,e)) \\ &= B(cg,(s,e))B^*(\langle cg\rangle,([cg]s,e)) \end{aligned}$$

$$= B([cg] , (s,e)) = A_o([cg] , s)$$

$$= \overline{A}_o(g , (s,c)).$$

Thus, B is cohomologous to $\overline{A}_o$.

In chapter III, we shall consider in more detail, a particular case of a general flow built under a constant function.

5. Strict versions of cocycles.

Let N^n act on a space Ω . Since N^n is countable, every (N^n , Ω) cocycle has a strict version. Let A be an (N^n , Ω) strict cocycle. A is determined by n functions $f_1 , \cdots , f_n$ where

$$f_i(\cdot) = A(e_i , \cdot) = A((0,\cdots,0,1,0,\cdots,0) , \cdot)$$

where 1 occurs in the i^{th} position. Further f_i , f_j satisfy

(5.1) $f_i(w)f_j(e_iw) = f_j(w)f_i(e_jw).$

Conversely, if we are given n measurable $\underline{\underline{U}}(\mathbb{H})$ valued functions $f_1 , \cdots , f_n$ satisfying the equations (5.1), then we can get an (N^n,Ω) cocycle by setting

$$A(e_i , w) = f_i(w).$$

$[0,1)^n$ is a section of $\mathbb{R}^n$ with respect to N^n. So, by methods described at the end of section 4, an action of N^n on a space Ω will give rise to an action of $\mathbb{R}^n$ on $\bar{\Omega} = \Omega \times [0,1)^n$. An (N^n, Ω) cocycle A extends to an $(\mathbb{R}^n, \bar{\Omega})$ cocycle $\bar{A}$ (by formula (4.5)) and every strict $(\mathbb{R}^n, \bar{\Omega})$ cocycle is cohomologous to a cocycle extended from an (N^n, Ω) cocycle. Finally, by methods similar to that of theorem 2.3, one can prove that every $(\mathbb{R}^n, \bar{\Omega})$ cocycle has a strict version. Thus, all $(\mathbb{R}^n, \bar{\Omega})$ cocycles can be described in terms of (N^n, Ω) cocycles.

Now let us consider the special case when $\mathbb{R}^n$ with its usual topology is imbedded continuously and in a one-one way in a locally compact second countable abelian group B and that $\overline{\mathbb{R}^n} = B$. Then the dual $\hat{B}$ of B is dense in $\mathbb{R}^n$. We can assume, without loss of generality, that the subgroup Z^n of $\mathbb{R}^n$ generated by $\{(2\pi, 0, \cdots, 0), \cdots, (0, \cdots, 0, 2\pi)\}$ is contained in $\hat{B}$. Let K be the annihilator of the closed subgroup Z^n of $\mathbb{R}^n$. Then, $K \cap \mathbb{R}^n = N^n$ and $\overline{N^n} = K$. Also, one can show that the action of $\mathbb{R}^n$ on B is isomorphic to the action of $\mathbb{R}^n$ on $K \times [0,1)^n$ defined by (4.3);

viz.

$$(t_1,\cdots,t_n)(x,(u_1,\cdots,u_n)) = (([t_1+u_1],\cdots,[t_n+u_n])x\ ,\ (\langle t_1+u_1\rangle,\cdots,\langle t_n+u_n\rangle)\ ).$$

Thus, every $(\mathbb{R}^n, B)$ cocycle has a strict version and they can be described in terms of (N^n, K) cocycles. The next theorem says that in some more cases we can get strict versions of cocycles.

5.1. Theorem. Let H be a locally compact second countable abelian group of the form $\mathbb{R}^n \times K$ where K is a compact abelian group. Let H be imbedded continuously and in a one-one way in a locally compact second countable abelian group G and let the image of H be dense in G. Then every $(H,G,\underline{U}(\mathbb{H}))$ cocycle has a strict version.

Proof: We assume $H \subseteq G$.

Since K is compact in H, K is a compact subgroup of G and so closed in G. Hence by lemma 4.1, every (K, G) cocycle is a coboundary. Since H is dense in G, $H/K = \mathbb{R}^n$ is dense in G/K.

Let σ be a measure on G, quasi-invariant under the action of H. Obviously, σ is quasi-invariant under the action of K on G. Therefore, σ is equivalent to a measure of the form $(\lambda \times \mu)\circ \xi^{-1}$, where λ is the Haar measure on K, μ is some measure on G/K and ξ is the Borel isomorphism between G and $K \times G/K$ defined in section 3 of chapter I. Since σ is quasi-invariant under $\mathbb{R}^n$, the measure μ on G/K is quasi-invariant under the action of $\mathbb{R}^n$ on G/K.

Let A be a $(H,G,\underline{\underline{U}}(\mathbb{H}))$ cocycle relative to σ. We define a cocycle as in section 6 of chapter I. Then, $A|_{K \times G}$ is a (K , G) cocycle and so is a coboundary. Let this coboundary be given by the measurable function $\rho^* : G \longrightarrow \underline{\underline{U}}(\mathbb{H})$,

$$A(k,x) = \rho^*(x)\ \rho(x+k)\ , \quad k \in K\ , x \in G.$$

Define the (H , G) cocycle B as follows:

$$B((r,k)\ ,\ x) = \rho(x)\ A((r,k)\ ,\ x)\rho^*(x+(r,k))\ ,$$

$$r \in \mathbb{R}^n\ , k \in K\ ,\quad x \in G.$$

An easy computation shows that, for fixed $r \in \mathbb{R}^n$ and $j, k \in K$,

$$B((r,k)\ ,\ x+j) = B((r,e)\ ,\ x) = B(r,x) \quad \text{(say)}$$
$$\text{a.e. } x \in G.$$

Therefore, for fixed $r \in \mathbb{R}^n$, the function $B(r,\cdot)$ on G satisfies the conditions of lemma 3.1 of chapter I. Hence, there is a measurable function $B'(r,\cdot)$ on G/K such that,

$$B(r,x) = B'(r, \pi_K(x)) \quad \text{a. e. } x \in G$$

(where $\pi_K : G \longrightarrow G/K$ is the quotient map). From the relation: for every $r_1 , r_2 \in \mathbb{R}^n$,

$$B(r_1+r_2 , x) = B(r_1 , x)B(r_2 , x+r_1) \quad \text{a. e. } x \in G,$$

we get,

$$B'(r_1+r_2 , \hat{x}) = B'(r_1 , \hat{x})\, B'(r_2 , \hat{x}+r_1) \text{ a.e. } \hat{x} \in G/K .$$

Thus, we get an $(\mathbb{R}^n , G/K)$ cocycle B', and so it has a strict version, say B_o'. Define B_o on $H \times G$ by:

$$B_o((r,k) , x) = B_o'(r , \pi_K(x)).$$

Then B_o is a strict (H , G) cocycle which is almost everywhere equal to B. The given cocycle A is cohomologous to B and so A itself has a strict version.

Q. E. D.

CHAPTER III

SYSTEMS OF IMPRIMITIVITY ON LOCALLY COMPACT ABELIAN GROUPS WITH DENSE ACTIONS.

1. Pairs of groups.

By a pair (Γ, R) we will mean that

(i) Γ and R are locally compact second countable abelian groups, and

(ii) there exists a one-one, continuous homomorphism φ of Γ into R such that $\varphi(\Gamma)$ is dense in R.

Given a pair (Γ, R) there arises another pair in a natural way. Consider the dual groups $\hat{\Gamma}$ and $\hat{R}$ and the map $\hat{\varphi} : \hat{R} \longrightarrow \hat{\Gamma}$ defined by:

$$\langle x , \hat{\varphi}(\hat{y}) \rangle = \langle \varphi(x) , \hat{y} \rangle , \quad x \in \Gamma , \hat{y} \in \hat{R}.$$

It can be shown that $\hat{\varphi}$ is a one-one, continuous homomorphism of $\hat{R}$ into $\hat{\Gamma}$ and that $\hat{\varphi}(\hat{R})$ is dense in $\hat{\Gamma}$. The pair $(\hat{R}, \hat{\Gamma})$ is called the dual pair of (Γ, R).

Remark: It is convenient sometimes to identify Γ with $\varphi(\Gamma)$ and thus regard Γ as a dense subgroup of R. The topology of Γ is not the one induced from R, but it is such that the inclusion map of Γ into R is continuous.

1.1. <u>Example</u>: Take R to be the group $\mathbb{R}$ of real numbers with the usual topology, and Γ to be any countable dense subgroup of $\mathbb{R}$, with the discrete topology. Let φ be the inclusion map of Γ into $\mathbb{R}$. Then $(\Gamma, \mathbb{R})$ is a pair. $\mathbb{R}$ is densely imbedded in the compact group $\hat{\Gamma}$ and $(\mathbb{R}, \hat{\Gamma})$ is the dual pair.

Let (U, P) be a system of imprimitivity on the pair (Γ, R), (Γ acts on R through translation), acting in a separable Hilbert space $\mathbb{H}$. Apply Stone's theorem to U to yield a spectral measure Q on $\hat{\Gamma}$ and to P to yield a representation V of $\hat{R}$:

$$(1.1) \begin{cases} U_g = \int_{\hat{\Gamma}} \langle -y, g\rangle \, dQ(y) , & g \in \Gamma , \\ V_h = \int_R \langle x, h\rangle \, dP(x) , & h \in \hat{R} . \end{cases}$$

Since (U,P) is a (Γ, R) system of imprimitivity, we have

$$U_g^{-1} V_h U_g = \int_R \langle x,h\rangle \, d(U_g^{-1} P(x) \, U_g)$$

$$= \int_R \langle x,h\rangle \, dP(x+\varphi(g))$$

$$\langle -\varphi(g), h\rangle V_h$$

whence,

$$V_h^{-1} U_g^{-1} V_h = \langle -\varphi(g) , h \rangle U_g^{-1}$$

$$= \langle -g , \hat{\varphi}(h) \rangle U_g^{-1}$$

where $\hat{\varphi}$ is the dual map from $\hat{R}$ into $\hat{\Gamma}$. From (1.1) we have,

$$V_h^{-1} U_g^{-1} V_h = \int_{\hat{\Gamma}} \langle y , g \rangle \, d(V_h^{-1} Q(y) V_h)$$

$$= \int_{\hat{\Gamma}} \langle y-\hat{\varphi}(h) , g \rangle \, dQ(y)$$

$$= \int_{\hat{\Gamma}} \langle y , g \rangle \, dQ(y+\hat{\varphi}(h)) .$$

Therefore, $V_h^{-1} Q(D) V_h = Q(D+\hat{\varphi}(h))$ for each Borel set $D \subseteq \hat{\Gamma}$ and for each $h \in \hat{R}$. Hence (V,Q) is a system of imprimitivity of $(\hat{R} , \hat{\Gamma})$ acting in $\mathbb{H}$. We shall call (V,Q) the <u>dual system</u> of (U,P). We observe that a subspace of $\mathbb{H}$ reduces (U,P) if and only if it reduces (V,Q).

<u>Remarks</u>. The above definitions and results are taken from Bagchi [3]. Dual systems of imprimitivity first appear in the work of Stone [23] and von Neumann, and in general group context in the work of Mackey [17].

Pair, dual pair and dual systems of imprimitivity appear, somewhat implicitly, in the work of Helson and Lowdenslager [14]. Dual pairs were explicitly considered by de Leeuw and Glicksberg [6].

2. The pairs $(\Gamma/\Gamma_o\ ,\ R/\Gamma_o)$ and $(\Gamma\ ,\ R)$.

Let (Γ, R) be a pair where we regard Γ as a dense subgroup of R and the map φ is the inclusion map of Γ into R. Let $\Gamma_o \subseteq \Gamma$ be a closed subgroup of R. Then Γ_o is closed in Γ as well. Further, $(\Gamma/\Gamma_o\ ,\ R/\Gamma_o)$ is a pair which we call the <u>quotient pair</u>. We fix a Borel section Q of R with respect to Γ_o. Then every element $x \in R$ can uniquely be written in the form

$$x = \gamma_o + c \quad , \quad \gamma_o \in \Gamma_o \ , \ c \in Q.$$

We denote (see section 3 of chapter I) $\gamma_o = [x]$ and $c = \langle x \rangle$. Further, R is Borel isomorphic to $R/\Gamma_o \times \Gamma_o$ by the map $\xi : R/\Gamma_o \times \Gamma_o \longrightarrow R$ given by:

(2.1) $\xi(\bar{c}\ ,\ \gamma_o) = c + \gamma_o \quad , \quad c \in Q, \ \ \gamma_o \in \Gamma_o \ ,$

where $\bar{c}$ denotes the coset of Γ_o to which c belongs. As in (3.2) of chapter I, we define a group operation $\dotplus$ on Q by:

$$c_1 \dotplus c_2 = \langle c_1 + c_2 \rangle .$$

With this operation Q is group isomorphic to R/Γ_o, the isomorphism being $c \longrightarrow \bar{c}$.

2.1. Example. Let $R = \mathbb{R}$, Γ = a countable dense subgroup of $\mathbb{R}$, with discrete topology, and assume that $2\pi \in \Gamma$. Let Γ_o be the cyclic subgroup generated by 2π. Then $(\Gamma/\Gamma_o, \mathbb{R}/\Gamma_o)$ is a pair, $\mathbb{R}/\Gamma_o$ being the circle group and Γ/Γ_o a countable dense subgroup of it with discrete topology. The interval $[0, 2\pi)$ is a section of $\mathbb{R}$ with respect to Γ_o.

If ν is a measure on R/Γ_o and λ_o is the Haar measure on Γ_o, then $\check{\nu}$ shall denote the measure $(\nu \times \lambda_o) \circ \xi^{-1}$ on R. We will need the following lemmas.

2.2. Lemma. Let ν be a finite measure on R/Γ_o quasi-invariant under the action of Γ/Γ_o. Then $\check{\nu}$ is quasi-invariant under the action of Γ on R.

Proof: Let A be a measurable subset of R such that $\check{\nu}(A) = 0$. So, $(\nu \times \lambda_o)(\xi^{-1}(A)) = 0$. Therefore, for ν a.e. $\bar{c} \in R/\Gamma_o$ ($c \in Q$), $\lambda_o((\xi^{-1}(A))_{\bar{c}}) = 0$, where $(\xi^{-1}(A))_{\bar{c}}$ is the $\bar{c}$ th section of $\xi^{-1}(A)$.

Let $\gamma \in \Gamma$. Then $\gamma = [\gamma] + \langle\gamma\rangle$ where $[\gamma] \in \Gamma_o$ and $\langle\gamma\rangle \in Q$. Observe that $\langle\bar{\gamma}\rangle \in \Gamma/\Gamma_o$. For any $c \in Q$,

$$(\xi^{-1}(A+\gamma))_{\bar{c}} = \{\gamma_o \in \Gamma_o : (\bar{c},\gamma_o) \in \xi^{-1}(A+\gamma)\}$$

$$= \{\gamma_o \in \Gamma_o : c+\gamma_o \in A+\gamma\}$$

$$= \{\gamma_o \in \Gamma_o : \langle c - \langle\gamma\rangle\rangle + \gamma_o + [c-\gamma] \in A\}$$

$$= \{\gamma_o' - [c-\gamma] \in \Gamma_o : \langle c - \langle\gamma\rangle\rangle + \gamma_o' \in A\}$$

$$= (\xi^{-1}(A))_{\bar{c} - \langle\bar{\gamma}\rangle} - [c-\gamma].$$

Now,

$$\tilde{\nu}(A+\gamma) = (\nu \times \lambda_o)(\xi^{-1}(A+\gamma))$$

$$= \int_{R/\Gamma_o} \lambda_o((\xi^{-1}(A+\gamma))_{\bar{c}})\, \nu(d\bar{c})$$

$$= \int_{R/\Gamma_o} \lambda_o((\xi^{-1}(A))_{\bar{c}-\langle\bar{\gamma}\rangle} - [c-\gamma])\, \nu(d\bar{c})$$

$$= \int_{R/\Gamma_o} \lambda_o((\xi^{-1}(A))_{\bar{c}-\langle\bar{\gamma}\rangle})\,\nu(d\bar{c}),$$ since λ_o is the Haar measure on Γ_o.

$= 0$ since $\int_{R/\Gamma_o} \lambda_o((\xi^{-1}(A))_{\bar{c}})\,\nu(d\bar{c}) = 0$ and ν is quasi-invariant under the action of Γ/Γ_o.

Q. E. D.

2.3. Lemma. Any measure on R quasi-invariant under Γ is equivalent to a measure of the form $\tilde{\nu}$ for some measure ν on R/Γ_o quasi-invariant under Γ/Γ_o.

Proof: Let μ be a measure on R quasi-invariant under Γ. Let $\pi : R \longrightarrow R/\Gamma_o$ be the natural homomorphism and put $\nu = \mu o \pi^{-1}$. Then ν on R/Γ_o is quasi-invariant under the action of Γ/Γ_o and by lemma 3.1 of Chapter I, μ is equivalent to $\tilde{\nu} = (\nu \times \lambda_o) o \xi^{-1}$.

Q. E. D.

Let ν be a measure on R/Γ_o quasi-invariant under the action of Γ/Γ_o, and let A be a $(\Gamma/\Gamma_o, R/\Gamma_o, \underline{U}(\mathbb{H}))$ cocycle relative to ν. Define $\hat{A}$ on $\Gamma \times R$ by:

(2.2) $\tilde{A}(\gamma , x) = A(\gamma \Gamma_o , x \Gamma_o)$.

Then $\tilde{A}$ is a $(\Gamma , R , \underline{U}(\mathbb{H}))$ cocycle relative to $\tilde{\nu}$.

2.4. Theorem. Every $(\Gamma , R , \underline{U}(\mathbb{H}))$ cocycle relative to $\tilde{\nu}$ is cohomologous to a cocycle $\tilde{A}$ for some $(\Gamma/\Gamma_o , R/\Gamma_o , \underline{U}(\mathbb{H}))$ cocycle A relative to ν . Two $(\Gamma/\Gamma_o , R/\Gamma_o , \underline{U}(\mathbb{H}))$ cocycles A_1 and A_2 are cohomologous if and only if the corresponding (Γ , R) cocycles $\tilde{A}_1$ and $\tilde{A}_2$ are cohomologous.

Proof: Let B be a $(\Gamma , R , \underline{U}(\mathbb{H}))$ cocycle relative to $\tilde{\nu}$. By our definition of a cocycle, $B|_{\Gamma_o \times R}$ will be a (Γ_o , R) cocycle relative to $\tilde{\nu}$. Since Γ_o is closed in R, by lemma 4.2 of chapter I, $B|_{\Gamma_o \times R}$ is a coboundary. Hence $B|_{\Gamma_o \times R}$ can be taken to be strict; i.e.,

$$B(\gamma_o^1+\gamma_o^2 , x) = B(\gamma_o^1 , x)\, B(\gamma_o^2 , x+\gamma_o^1) \quad \text{for all}$$

$$x \in R, \ \gamma_o^1 , \gamma_o^2 \in \Gamma_o .$$

Define the $\tilde{\nu}$ - measurable map ρ on R by:

$$\rho(x) = B([x] , \langle x\rangle).$$

Let B_1 be the (Γ , R) cocycle defined by:

$$B_1(\gamma , x) = \rho(x)\, B(\gamma , x)\rho^*(\gamma+x).$$

B_1 is cohomologous (ρ) to B. If $h, h' \varepsilon \Gamma_o$ and $\gamma \varepsilon \Gamma$,

$$B_1(\gamma+h , x+h') = \rho(x+h')B(\gamma+h , x+h')\rho^*(\gamma+x+h+h')$$

$$= B([x]+h',\langle x\rangle)B(\gamma+h,x+h')B^*([\gamma+x]+h+h',\langle\gamma+x\rangle)$$

$$= B([x],\langle x\rangle)B(h',x)B(\gamma+h,x+h')B^*(h+h', \gamma+x)B^*([\gamma+x],\langle\gamma+x\rangle)$$

$$= B([x],\langle x\rangle)B(\gamma+h+h', x)B^*(h+h', \gamma+x)B^*([\gamma+x] ,\langle\gamma+x\rangle)$$

$$\tilde{\nu} \text{ a. e. } x \varepsilon R.$$

$$= B([x] ,\langle x\rangle) B(\gamma , x) B^*([\gamma+x] ,\langle\gamma+x\rangle) \quad \tilde{\nu} \text{ a.e. } x \varepsilon R.$$

$$= B_1(\gamma , x) \quad \tilde{\nu} \text{ a.e. } x \varepsilon R.$$

So there is a cocycle A on $(\Gamma/_{\Gamma_o} , R/_{\Gamma_o})$ relative to ν such that for each $\gamma \varepsilon \Gamma$,

$$B_1(\gamma , x) = \hat{A}(\gamma , x) \quad \tilde{\nu} \text{ a.e. } x \varepsilon R.$$

Hence, the given cocycle B and $\hat{A}$ are cohomologous.

Let two $(\Gamma/_{\Gamma_o} , R/_{\Gamma_o} , \underline{\underline{U}}(\mathbb{H}))$ cocycles A_1 and A_2 be cohomologous (ρ_o). Then $\hat{A}_1$ and $\hat{A}_2$ are cohomologous(ρ) where $\rho(x) = \rho_o(x\Gamma_o)$, $x \varepsilon R$. Now, let the $(\Gamma , R, \underline{\underline{U}}(\mathbb{H}))$ cocycles $\hat{A}_1$ and $\hat{A}_2$ be cohomologous (ρ). So, for each $\gamma \varepsilon \Gamma$,

$$\hat{A}_1(\gamma , x) = \rho(x) \hat{A}_2(\gamma , x)\rho^*(x+\gamma) \quad \tilde{\nu} \text{ a.e. } x \varepsilon R.$$

In particular, for $\gamma_o \in \Gamma_o$, we have,

$$I = \rho(x)\rho^*(x+\gamma_o) \quad \tilde{\nu} \text{ a.e. } x \in R.$$

Or, for each $\gamma_o \in \Gamma_o$, $\rho(x) = \rho^*(x+\gamma_o)$ $\tilde{\nu}$ a.e. $x \in R$.

So, by lemma 3.2 of chapter I, there is a measurable function ρ_o on R/Γ_o such that $\rho(x) = \rho_o(x\Gamma_o)$ $\tilde{\nu}$ a.e. $x \in R$. A_1 and A_2 are cohomologous (ρ_o).

Q. E. D.

Let (U,F) be a system of imprimitivity based on $(\Gamma/\Gamma_o, R/\Gamma_o)$ acting in a separable Hilbert space $\mathbb{H}$. Then there exists invariant measure classes $\mathcal{C}_\infty$, $\mathcal{C}_1$, $\mathcal{C}_2$, ⋯ and $\underline{\underline{U}}(\mathbb{H}_n)$ cocycles A_n relative to $\mathcal{C}_n$, $n = \infty, 1, 2, \cdots$, (where $\mathbb{H}_n = \mathbb{C}^n$ if $n = 1,2,\cdots$ and $\mathbb{H}_\infty = \ell^2$) such that (U,F) is equivalent to the concrete system of imprimitivity given by $\mathcal{C}_\infty$, $\mathcal{C}_1$, $\mathcal{C}_2$, ⋯ and cocycles A_∞, A_1, A_2, ⋯ . By $(\tilde{U},\tilde{F})$ we shall mean the (Γ, R) system of imprimitivity given by $\tilde{\nu}_\infty$, $\tilde{\nu}_1$, $\tilde{\nu}_2$, ⋯ ($\nu_n \in \mathcal{C}_n$, $n = \infty, 1, 2, \cdots$) and cocycles $\tilde{A}_\infty$, $\tilde{A}_1$, $\tilde{A}_2$, ⋯ . In view of theorem 5.2 of chapter I and theorem 2.4, we see that any (Γ, R) system of imprimitivity acting in a separable Hilbert space is equivalent to a (Γ, R) system of imprimitivity $(\tilde{U},\tilde{F})$ for some $(\Gamma/\Gamma_o, R/\Gamma_o)$ system of imprimitivity (U,F).

3. The pairs dual to $(\Gamma/\Gamma_o, R/\Gamma_o)$ and (Γ, R).

Now, let us consider the dual pairs $((R/\Gamma_o)^\wedge, (\Gamma/\Gamma_o)^\wedge)$ and $(\hat{R}, \hat{\Gamma})$. Let us put $\hat{R} = S$, and $\hat{\Gamma} = B$ and $(\Gamma/\Gamma_o)^\wedge = K$, the annihilator of Γ_o in B. Then $(R/\Gamma_o)^\wedge = K \cap S$, the annihilator of Γ_o in S. Thus the pair dual to $(\Gamma/\Gamma_o, R/\Gamma_o)$ is $(K \cap S, K)$ and the pair dual to (Γ, R) is (S, B).

3.1. Lemma. Any Borel section of S with respect to $K \cap S$ is also a Borel section of B with respect to K.

Proof: Let C be a Borel section of S with respect to $K \cap S$. By Kuratowski's theorem (p. 139 [18]) C is a Borel subset of B. Each coset of K can contain atmost one element from C, for if $c_1K = c_2K$, then $c_1c_2^{-1} \in K \cap S$, and so $c_1 = c_2$. Now, the relative topology of Γ_o in Γ is the same as the relative topology of Γ_o in R. So, $B/_K$ and $S/_{K \cap S}$ are topologically isomorphic, both being the dual of Γ_o. So, given $b \in B$, there is $c \in C$ such that

$$(bK, \gamma_o) = (c(K \cap S), \gamma_o) \quad \text{for all } \gamma_o \in \Gamma_o .$$

That is, $(b, \gamma_0) = (c, \gamma_0)$ or $(b^{-1}c, \gamma_0) = 1$ for all $\gamma_0 \varepsilon \Gamma_0$. Hence, $b^{-1}c \varepsilon K$ or $c \varepsilon bK$. So every coset of K contains an element of C. Therefore, C is a Borel section of B with respect to K.

Q. E. D.

Throughout the rest of this chapter, C will stand for a fixed Borel section of S with respect to $K \cap S$. The natural one-one correspondence $\alpha : c \longrightarrow \bar{c}$ between C and $S/_{K \cap S}$ is a Borel isomorphism, and a measure on $S/_{K \cap S}$ can be considered a measure on C and vice-versa. If μ is a finite measure on $S/_{K \cap S}$, its Fourier transform $\hat{\mu}$ is a function on $(S/_{K \cap S})\hat{} = \Gamma_0$. However, when considered as a measure on C, its Fourier transform is a function on R which when restricted to Γ_0 is $\hat{\mu}$, i.e., $\widehat{\mu \circ \alpha}\big|_{\Gamma_0} = \hat{\mu}$. Haar measure m on $S/_{K \cap S}$ is carried over into a measure on C which we denote by dc. This measure is invariant under the group operation on C defined by $c_1 \dot{+} c_2 = \langle c_1 + c_2 \rangle$. If $g \varepsilon L^2(S/_{K \cap S}, m)$, then g has a Fourier-Plancherel transform $\hat{g}$ defined on Γ_0. If we regard g as a function on C, i.e., if we look at $h = g \circ \alpha$, then h has a Fourier-Plancherel transform $\hat{h}$ defined on R, which when restricted to Γ_o coincides with $\hat{g}$ a.e. $\hat{h}$ is defined by:

$$(3.1) \quad \hat{h}(x+\gamma_o) = \int_C \langle x+\gamma_o , c\rangle h(c) \, dc , \quad x \in Q, \gamma_o \in \Gamma_o .$$

$$= \int_C \langle \gamma_o , c\rangle \langle x , c\rangle h(c) \, dc.$$

(Q is a section of R with respect to Γ_o).

Let ν be a complex-valued measure on R/Γ_o and let $\tilde{\nu} = (\nu \times \lambda_o) \circ \mathcal{E}^{-1}$ (as defined in section 2). Let $h \in L^2(C , dc)$ and put

$$(3.2) \quad f(x+\gamma_o) = \hat{\bar{h}}(x+\gamma_o) = \int_C \langle x+\gamma_o , c\rangle \bar{h}(c) \, dc,$$

$$x \in Q , \quad \gamma_o \in \Gamma_o .$$

Let $g \in \Gamma$, $g = u + [g]$ where $u = \langle g\rangle$. Define the function F_g on R by:

$$(3.3) \quad F_g(x+\gamma_o) = f(x+\gamma_o) \overline{f(x+\gamma_o - g)} .$$

Let us calculate the Fourier transform of the measure $F_g d\hat{\nu}$.

3.2. Lemma. For each $t \in S$,

$$\widehat{F_g d\tilde{\nu}}(t) = \int_C \hat{\nu}([t+c]) \langle g, \langle t+c\rangle\rangle h(\langle t+c\rangle) \bar{h}(c) \, dc.$$

<u>Proof</u>:

$$\int_C \hat{\nu}([t+c]) \langle g, \langle t+c \rangle\rangle h(\langle t+c \rangle) \bar{h}(c)dc$$

$$= \int_C \left\{ \int_{R/\Gamma_o} \langle x,[t+c]\rangle \nu(dx) \right\} \langle g,\langle t+c\rangle\rangle h(\langle t+c\rangle)\bar{h}(c)dc$$

$$= \int_{R/\Gamma_o} \langle x,t\rangle \int_C \langle x,c\rangle \bar{h}(c) \langle x, -\langle t+c\rangle\rangle \langle g,\langle t+c\rangle\rangle \cdot h(\langle t+c\rangle)\, dc\, \nu(dx)$$

$$= \int_{R/\Gamma_o} \langle x,t\rangle \int_C \langle x,c\rangle \bar{h}(c) \overline{\langle x-g , \langle t+c\rangle\rangle \bar{h}(\langle t+c\rangle)}\, dc\, \nu(dx)$$

$$= \int_{R/\Gamma_o} \langle x,t\rangle \int_{\Gamma_o} f(x+\gamma_o) \overline{\langle \gamma_o , -\langle t\rangle\rangle f(x+\gamma_o-g)}\, d\lambda_o(\gamma_o)\nu(dx)$$

(by Plancherel theorem)

$$= \int_{R/\Gamma_o} \int_{\Gamma_o} \langle x+\gamma_o , t\rangle f(x+\gamma_o) \overline{f(x+\gamma_o-g)}\, d\lambda_o(\gamma_o)\, \nu(dx)$$

$$= \int_R \langle x+\gamma_o , t\rangle \; F_g(x+\gamma_o)\, d\breve{\nu}(x+\gamma_o)$$

$$= \widehat{F_g\, d\breve{\nu}}\,(t).$$

Q. E. D.

<u>Remark:</u> The author is indebted to Professor M. G. Nadkarni for discussions in proving the above lemma. In particular, the idea of using Plancherel theorem is due to him.

Suppose $h \in L^2(C, dc)$ is such that $\hat{h}|_{\Gamma_0}$ is non-vanishing λ_0 a.e. Then by Wiener's theorem, $\{h(\langle \cdot + c \rangle) : c \in C\}$ spans $L^2(C, dc)$. Let ν be a measure on R/Γ_0. ν can be considered a measure on Q. Suppose $h \in L^2(C, dc)$ is such that the function $\hat{h}$ is $\tilde{\nu}$ a.e. non-zero on R. Then by Fubini's theorem, for ν a.e. $y \in Q$, $(\langle y, \cdot \rangle h(\cdot))\hat{}|_{\Gamma_0}$ is non-vanishing λ_0 a.e. on Γ_0. Hence, for ν a.e. $y \in Q$, $\{\langle y, \langle \cdot + s \rangle\rangle h(\langle \cdot + s \rangle) : s \in S\}$ spans $L^2(C, dc)$. Thus we have:

3.3. <u>Proposition</u>. Suppose $h \in L^2(C, dc)$ is such that $\hat{h}$ defined on R by,

$$\hat{h}(x+\gamma_0) = \int_C \langle x+\gamma_0 , c \rangle h(c)\, dc, \quad x \in Q, \quad \gamma_0 \in \Gamma_0 ,$$

is non-vanishing $\tilde{\nu}$ a.e. on R. Then for ν a.e. $y \in Q$, the collection $\{\langle y, \langle \cdot + s \rangle\rangle h(\langle \cdot + s \rangle) : s \in S\}$ spans $L^2(C, dc)$.

B is Borel isomorphic to $B/K \times K$, the isomorphism being $\eta : B/K \times K \longrightarrow B$ defined by:

(3.4) $\eta(\bar{c}\ , y) = c + y \quad , \quad c \in C\ , \quad y \in K.$

In the same way, S and ${}^{S}/_{K\cap S} \times (K \cap S)$ are Borel isomorphic.

Let μ be a complex-valued measure (finite) on K, and let λ be the Haar measure on ${}^{B}/_{K}$. Then by $\bar{\mu}$ we shall mean the measure $(\mu \times \lambda) \circ \eta^{-1}$ on B.

3.4. Lemma. Let μ be a finite complex-valued measure on K, quasi-invariant under the action of $K \cap S$. Then $\bar{\mu}$ is quasi-invariant under the action of S on B. Moreover,

(3.5) $$\frac{d\bar{\mu}_s}{d\bar{\mu}}(y+c) = \frac{d\mu_{[s+c]}}{d\mu}(y) \quad \text{for } (\mu \times \lambda) \text{ a.e. } (y,\bar{c}) \in K \times {}^{B}/_{K}.$$

Proof: Let D be a Borel subset of B and let $s \in S$.

$$\eta^{-1}(D+s) = \left\{ (y,\bar{c}) \in K \times {}^{B}/_{K} : \ y+c \in D+s \right\}$$

$$= \left\{ (y+[s+c]\ , \overline{\langle s+c \rangle}) : \ y+c \in D \right\}$$

$$= \left\{ (y+[s+c]\ , \overline{\langle s+c \rangle}) : \ (y,\bar{c}) \in \eta^{-1}(D) \right\}.$$

Therefore, $(\eta^{-1}(D+s))_{\overline{\langle s+c \rangle}} = (\eta^{-1}(D))_{\bar{c}} + [s+c]$.

Now,

$$\bar{\mu}_s(D) = \bar{\mu}(D+s) = (\mu \times \lambda)(\eta^{-1}(D+s))$$

$$= \int_{{}^{B}/_{K}} \mu((\eta^{-1}(D+s))_{\bar{c}})\, d\lambda(\bar{c})$$

$$= \int_{B/K} \mu((\eta^{-1}(D+s))_{\overline{\langle s+c\rangle}})\, d\lambda(\bar{c})$$, since λ is the Haar measure on B/K.

$$= \int_{B/K} \mu((\eta^{-1}(D))_{\bar{c}} + [s+c])\, d\lambda(\bar{c})$$

$$= \int_{B/K} \mu_{[s+c]}((\eta^{-1}(D))_{\bar{c}})\, d\lambda(\bar{c})$$

$$= \int_{B/K} \int_{(\eta^{-1}(D))_{\bar{c}}} \frac{d\mu_{[s+c]}}{d\mu}(y)\, d\mu(y)\, d\lambda(\bar{c})$$

$$= \int_{\eta^{-1}(D)} \varphi(y,\bar{c})\, d(\mu\times\lambda)(y,\bar{c})$$

where $\varphi(y,\bar{c}) = \dfrac{d\mu_{[s+c]}}{d\mu}(y)$

$$= \int_{D} \varphi \circ \eta^{-1}(x)\, d\bar{\mu}(x) \qquad (x = y+c).$$

Hence, $\dfrac{d\bar{\mu}_s}{d\bar{\mu}}(x) = \varphi \circ \eta^{-1}(x)$, $\bar{\mu}$ a.e. x.

Or, $\dfrac{d\bar{\mu}_s}{d\bar{\mu}}(y+c) = \dfrac{d\mu_{[s+c]}}{d\mu}(y)$, $(\mu\times\lambda)$ a.e. $(y,\bar{c})$.

Q. E. D.

Let A be a $(K \cap S\ ,\ K)$ cocycle relative to μ, a measure on K quasi-invariant under the action of $K \cap S$. Define $\bar{A}$ on $S \times B$ by:

$$(3.6)\quad \bar{A}(s\ ,\ y+c) = A([s+c]\ ,\ y), \quad s \in S,\quad y \in K,\quad c \in C.$$

Clearly, $\bar{A}$ is measurable on $S \times B$. Now, for fixed $s_1\ ,\ s_2 \in S$, and $c \in C$,

$$\bar{A}(s_1+s_2\ ,\ y+c) = A([s_1+s_2+c]\ ,\ y)$$

$$= A([s_1+c] + [s_2+\langle s_1+c\rangle]\ ,\ y) \quad \text{(by (3.1) of ch. I)}$$

$$= A([s_1+c], y)A([s_2+\langle s_1+c\rangle]\ ,\ y+[s_1+c]),\ \mu \text{ a.e. } y \in K.$$

$$= \bar{A}(s_1\ ,\ y+c)\ \bar{A}(s_2\ ,\ y+s_1+c), \quad \mu \text{ a.e. } y \in K.$$

Hence, by Fubini's theorem, for each fixed $s_1\ ,\ s_2 \in S$,

$$\bar{A}(s_1+s_2\ ,\ x) = \bar{A}(s_1\ ,\ x)\ \bar{A}(s_2\ ,\ x+s_1),\ \bar{\mu} \text{ a.e. } x \in B.$$

Therefore, $\bar{A}$ is an (S,B) cocycle relative to $\bar{\mu}$.

If two $(K \cap S\ ,\ K)$ cocycles A_1 and A_2 (relative to μ) are cohomologous (ρ_o), then the corresponding (S,B) cocycles $\bar{A}_1$ and $\bar{A}_2$ are cohomologous (ρ) where $\rho(x) = \rho_o([x])$.

<u>Remark</u>: For the special case when B is compact and $\Gamma = \hat{B}$ is a dense subgroup of $\mathbb{R}$, the method of obtaining $\bar{A}$ from a $(K \cap S\ ,\ K)$ cocycle A was given by Gamelin in [7].

He considers only scalar cocycles relative to the Haar measure on k.

<u>Notation</u>: Given a $(K \cap S, K)$ cocycle A we shall call $\bar{A}$ the <u>Gamelin cocycle</u> obtained from A.

Let (V,E) be a system of imprimitivity on $(K \cap S, K)$ acting in a separable Hilbert space $\mathbb{H}$. Then there exists invariant measure classes $\mathcal{C}_\infty, \mathcal{C}_1, \mathcal{C}_2, \cdots$ and $\underline{\underline{U}}(\mathbb{H}_n)$ valued cocycles A_n relative to μ_n ($\mu_n \in \mathcal{C}_n$, $n=\infty,1,2,\cdots$) such that (V,E) is equivalent to the concrete system of imprimitivity given by $\mathcal{C}_\infty, \mathcal{C}_1, \mathcal{C}_2, \cdots$ and $A_\infty, A_1, A_2, \cdots$. $(\bar{V}, \bar{E})$ will stand for the (S,B) system of imprimitivity given by $\bar{\mu}_\infty, \bar{\mu}_1, \bar{\mu}_2, \cdots$ and cocycles $\bar{A}_\infty, \bar{A}_1, \bar{A}_2, \cdots$. We call $(\bar{V}, \bar{E})$ the <u>Gamelin system of imprimitivity</u> obtained from (V, E).

4. <u>The Main Theorem</u>.

Let (V,E) be a system of imprimitivity on $(K \cap S, K)$; $(\bar{V}, \bar{E})$ be the Gamelin system of imprimitivity on (S,B) extended from (V,E). Let (U,F) and $(\bar{U},\bar{F})$ be the duals of (V,E) and $(\bar{V},\bar{E})$ respectively. Let $(\hat{U},\hat{F})$ be the (Γ, R) system of imprimitivity extended from the $(\Gamma/\Gamma_o, R/\Gamma_o)$ system (U,F).

$$\begin{array}{ccc}
(K \cap S\,,\,K) & & (S\,,\,B) \\
(V\,,\,E) & \longrightarrow & (\bar{V}\,,\,\bar{E}) \\
\big\downarrow & & \big\downarrow \\
 & & (\bar{U}\,,\,\bar{F}) \\
(U\,,\,F) & \longrightarrow & (\check{U}\,,\,\check{F}) \\
({}^{\Gamma}/_{\Gamma_o}\,,\,{}^{R}/_{\Gamma_o}) & & (\Gamma, R).
\end{array}$$

Thus on (Γ , R) we have obtained two systems of imprimitivity $(\bar{U},\bar{F})$ and $(\check{U},\check{F})$, starting from the same $(K \cap S\,,\,K)$ system (V,E). We now prove, under a mild assumption, that $(\bar{U},\bar{F})$ and $(\check{U},\check{F})$ are equivalent. The assumption is:

<u>Assumption (A)</u>: There is a function $h \in L^2(C,dc)$ such that for all $y \in Q$, $(\langle\cdot,y\rangle\, h(\cdot))^{\wedge}$ is non-vanishing λ_o a.e. on Γ_o. In many cases, we can choose the section C in such a way that assumption (A) is satisfied. In the appendix we show that if $S = \mathbb{R}^n \times L \times D$, where L is compact and D is discrete, and $K \cap S = H_1 \times L_1 \times D_1$, where H_1, L_1, D_1 are closed subgroups of $\mathbb{R}^n$, L, D respectively, then we can get a Borel section C of ${}^{S}/_{K \cap S}$ such that assumption (A) is satisfied.

We can assume, without loss of generality, that F is homogeneous of multiplicity n, $1 \leq n \leq \infty$. Let $\mathcal{C}_\nu$ be the measure class of F. Then (U,F) is equivalent to a concrete system of imprimitivity associated with ν and a $(\Gamma/\Gamma_o, R/\Gamma_o, \underline{U}(\mathbb{H}_n))$ cocycle D relative to ν. We will show that $\bar{F}$ is also homogeneous of multiplicity n, with associated measure class $\mathcal{C}_{\tilde{\nu}}$ and that the cocycle associated with $(\bar{U},\bar{F})$ is cohomologous to the (Γ, R) cocycle $\hat{D}$ extended from D. For simplicity, we assume that (V,E) is also homogeneous. (The proof for the general case is similar). Let $\mathcal{C}_\mu$ be the measure class associated with E and A be the $(K \cap S, K, \underline{U}(\mathbb{H}))$ cocycle corresponding to (V,E).

We fix $h \in L^2(C,dc)$ such that for each $y \in Q$ (Q is a section of R with respect to Γ_o), the Fourier coefficients of the function

$$c \longrightarrow \langle y, -c \rangle h(c)$$

is non-vanishing λ_o a.e. on Γ_o. For this h, we define the function f on R by (3.2) and for each $g \in \Gamma$, the functions F_g on R by (3.3):

$$f(y+\gamma_o) = \hat{h}(y+\gamma_o) = \int_C \langle y+\gamma_o , c\rangle \, \bar{h}(c) \, dc$$

$$F_g(y+\gamma_o) = f(y+\gamma_o) \, \overline{f(y+\gamma_o - g)} ,$$

$$y \in Q , \quad \gamma_o \in \Gamma_o .$$

Let $h_i \in L^2(K, \mathbb{H}, \mu)$, $i = 1,2$. Define $\bar{h}_i \in L^2(B, \mathbb{H}, \bar{\mu})$ by:

$$\bar{h}_i(x+c) = h_i(x) \, h(c) , \quad x \in K, \; c \in C, \; i = 1,2.$$

Let $g \in \Gamma$, $g = u + \gamma_o$, where $u = \langle g \rangle \in Q \cap \Gamma$ and $\gamma_o = [g] \in \Gamma_o$. Define the measures $\nu_{\bar{u}}^{1,2}$ on R/Γ_o and $\bar{\nu}_g^{1,2}$ on R as follows:

$$\nu_{\bar{u}}^{1,2}(X) = (F(X) \chi_{\bar{u}} h_1 , h_2) , \quad X \text{ a Borel subset of } R/\Gamma_o ,$$

$$\bar{\nu}_g^{1,2}(X') = (\bar{F}(X') \chi_g \bar{h}_1 , \bar{h}_2) , \quad X' \text{ a Borel subset of } R,$$

where $\chi_{\bar{u}}$, χ_g are the characters on K and B respectively defined by $\bar{u}$ and g.

Remarks: 1. The measures $\nu_{\bar{u}}^{1,2}$ and $\bar{\nu}_g^{1,2}$ are not the translates of the measures $\nu^{1,2}$ $(= \nu_o^{1,2})$ and $\bar{\nu}^{1,2}$ $(= \bar{\nu}_o^{1,2})$.

2. For measures appearing without superscripts, subscripts to them mean their translates.

4.1. Lemma: $\tilde{\nu}_g^{1,2}$ is absolutely continuous with respect to $\tilde{\nu}_{\bar{u}}^{1,2}$ and,

$$\frac{d\,\tilde{\nu}_g^{1,2}}{d\,\tilde{\nu}_{\bar{u}}^{1,2}} = F_g \quad . \tag{4.1}$$

(Here and in the sequel $\tilde{\nu}_{\bar{u}}^{i,j}$ means $\nu_{\bar{u}}^{i,j}$).

Proof: For $t \in S$,

$$\widehat{\tilde{\nu}_g^{1,2}}(t) = (\bar{V}_t \,\chi_g\, \bar{h}_1 \;,\; \bar{h}_2)$$

$$= \int_B \left\langle (\bar{V}_t \,\chi_g\, \bar{h}_1)(y) \;,\; \bar{h}_2(y) \right\rangle d\bar{\mu}(y) \;, \quad \text{where}$$

$\langle \cdot , \cdot \rangle$ is the inner product in $\mathbb{H}$.

$$= \int_B \left\langle \bar{A}(t,y) \sqrt{\frac{d\bar{\mu}_t}{d\bar{\mu}}(y)}\; \chi_g(y+t)\bar{h}_1(y+t) \;,\; \bar{h}_2(y) \right\rangle d\bar{\mu}(y)$$

$$= \int_C \int_K \Big\langle A([t+c],x) \sqrt{\frac{d\mu_{[t+c]}}{d\mu}(x)}\; \chi_g(x+[t+c])\cdot$$

$$\chi_g(\langle t+c \rangle)\, h_1(x+[t+c])\, h(\langle t+c \rangle),\; h_2(x)h(c) \Big\rangle \, d\mu(x)\; dc.$$

(by the definition of $\bar{A}$ and by lemma 3.4)

$$= \int_C (V_{[t+c]} \chi_{\bar{u}} h_1 , h_2) \chi_g(\langle t+c\rangle) h(\langle t+c\rangle) \overline{h(c)} \, dc$$

$$= \int_C \widehat{\nu_{\bar{u}}^{1,2}}([t+c]) \chi_g(\langle t+c\rangle) h(\langle t+c\rangle) \overline{h(c)} \, dc$$

$$= (F_g \, d\tilde{\nu}_{\bar{u}}^{1,2})^{\wedge}(t) \quad \text{(by lemma 3.2).}$$

Thus the two measures $d\bar{\nu}_g^{1,2}$ and $F_g d\tilde{\nu}_{\bar{u}}^{1,2}$ are same, because their Fourier coefficients are same. So

$$\frac{d\bar{\nu}_g^{1,2}}{d\tilde{\nu}_{\bar{u}}^{1,2}} = F_g \quad .$$

Q. E. D.

4.2. <u>Corollary</u>: Taking g to be the identity in Γ, we get

(4.2) $$\frac{d\bar{\nu}^{1,2}}{d\tilde{\nu}^{1,2}}(y+\gamma_o) = |f(y+\gamma_o)|^2 \quad , \quad y \in Q, \ \gamma_o \in \Gamma_o .$$

So by the choice of the function h in $L^2(C, dc)$ we get that the two measures $\bar{\nu}^{1,2}$ and $\tilde{\nu}^{1,2}$ are mutually absolutely continuous.

Since F is homogeneous of multiplicity n, there are n functions $h_1, h_2, \cdots, h_n \in L^2(K, \mathbb{H}, \mu)$ such that the following are satisfied.

(i) $(V_s h_i , h_j) = 0$ if $i \neq j$, $1 \leq i,j \leq n$, $s \in K \cap S$.

(ii) If $\nu_i(\cdot) = (F(\cdot)h_i , h_i)$, $1 \leq i \leq n$, then $\nu_i = \nu_j$ for all i,j, $1 \leq i,j \leq n$. Let $\nu_i = \nu$.

(iii) The closed linear span of $\{V_s h_i : s \in K \cap S, i=1,2,\cdots,n\}$ is $L^2(K , \mathbb{H}, \mu)$.

4.3. Theorem. $\overline{F}$ is also homogeneous of multiplicity n.

Proof: Consider $\overline{h}_1, \overline{h}_2, \cdots, \overline{h}_n \in L^2(B, \mathbb{H}, \overline{\mu})$. Then:

(1) $(\overline{V}_s \overline{h}_i , \overline{h}_j) = 0$ if $i \neq j$, $1 \leq i,j \leq n$; $s \in S$.

(2) If the measures $\overline{\nu}_i$ on R are defined by

$$\overline{\nu}_i(\cdot) = (\overline{F}(\cdot)\overline{h}_i , \overline{h}_i), \text{ then } \overline{\nu}_i = \overline{\nu}_j \text{ for } 1 \leq i,j \leq n.$$

(3) The closed linear span of $\{\overline{V}_s \overline{h}_i : s \in S, i=1,2,\cdots,n\}$ is $L^2(B, \mathbb{H}, \overline{\mu})$.

(1) and (2) are easily proved. To prove (3), let $\varphi \in L^2(B, \mathbb{H}, \overline{\mu})$ be such that,

$$(\overline{V}_s \overline{h}_i , \varphi) = 0 \text{ for all } s \in S \text{ and } i=1,2,\cdots,n.$$

We show that $\varphi = 0$. Now,

$$0 = \int_B \left\langle \overline{A}(s,x+c)\sqrt{\frac{d\overline{\mu}_s}{d\overline{\mu}}(x+c)}\ \overline{h}_i(x+c+s) , \varphi(x+c) \right\rangle d\overline{\mu}(x+c)$$

$$= \int_C \int_K \left\langle A([s+c],x)\sqrt{\frac{d\mu_{[s+c]}}{d\mu}(x)}\, h_i(x+[s+c])\, h(\langle s+c\rangle)\right., \left.\varphi(x+c)\right\rangle d\mu(x)\, dc.$$

$$= \int_C (V_{[s+c]}h_i\ ,\ \varphi_c)\ h(\langle s+c\rangle)\ dc, \quad \text{where } \varphi_c(x) = \varphi(x+c).$$

$$= \int_C \int_{R/\Gamma_o} \langle y,[s+c]\rangle\ \beta_i(y,c)h(\langle s+c\rangle)\ d\nu(y)dc,$$

where $\beta_i(\cdot,c)$ is the derivative of the measure $(F(\cdot)h_i,\varphi_c)$ with respect to ν. For each $c \in C$, the function $y \longrightarrow \beta_i(y,c)$ is in $L^1(R/\Gamma_o\ ,\nu)$ and for ν a.e. $y\varepsilon\ R/\Gamma_o$, the function $c \longrightarrow \beta_i(y,c)$ is in $L^2(C,dc)$. Further, the function $(y,c) \longrightarrow \beta_i(y,c)\ h(\langle s+c\rangle)$ is in $L^1(R/\Gamma_o \times C\ ,\ d\nu \times dc)$. Hence by Fubini's theorem,

$$0 = \int_{R/\Gamma_o} \langle y,[s]\rangle \int_C \langle y,[\langle s\rangle+ c]\rangle\ \beta_i(y,c)h(\langle s+c\rangle)\ dc\ d\nu(y).$$

This is true for all $s \varepsilon S$. So for ν a.e. $y \varepsilon\ R/\Gamma_o$,

$$\int_C \langle y\ ,\ [\langle s\rangle+c]\rangle\ \beta_i(y,c)h(\langle s+c\rangle)\ dc = 0.$$

For such a $y \varepsilon {}^{R}/_{\Gamma_o}$,

$$\int_C \left\{ \langle y,c\rangle \beta_i(y,c) \right\} . \left\{ \langle y, -\langle s+c\rangle\rangle h(\langle s+c\rangle) \right\} dc = 0.$$

By the choice of the function h in $L^2(C,dc)$ and by Proposition 3.3, we get

$$\langle y,c\rangle \beta_i(y,c) = 0 \quad \text{for a.e.} \quad c \varepsilon C.$$

Sine $\langle y,c\rangle \neq 0$, $\beta_i(y,c) = 0$ for a.e. $c \varepsilon C$.

Thus, $\beta_i(y,c) = 0$, for $(\nu \times dc)$ a.e. $(y,c) \varepsilon {}^{R}/_{\Gamma_o} \times C$, and this is true for all $i = 1,2,\cdots,n$. By property (iii) of $h_1, h_2, \cdots, h_n$, this means that $\varphi = 0$ a.e. $\bar{\mu}$.

(1), (2) and (3) imply that $\bar{F}$ is homogeneous of multiplicity n.

Q. E. D.

Let T be the isometric isomorphism between $L^2(K, H, \mu)$ and $L^2({}^{R}/_{\Gamma_o}, H_n, \nu)$ defined by:

$T F(X) h_i = (0,\cdots,0,1_X,0,\cdots,0)$, X a Borel subset of ${}^{R}/_{\Gamma_o}$ (1_X , the indicator function of X, is at the i^{th} position). For every Borel subset X of ${}^{R}/_{\Gamma_o}$ we have:

$$\nu_{\bar u}^{i,j}(X) = (F(X)\chi_{\bar u} h_i , h_j) , \quad u \varepsilon \Gamma \cap Q$$

$$= (F(X)\chi_{\bar u} h_i , F(X)h_j)$$

$$= (T F(X) T^{-1} T \chi_{\bar u} T^{-1} Th_i , T F(X) T^{-1} Th_j)$$

$$= \int_X \left\langle D(-\bar u,y)\sqrt{\frac{d\nu_{-\bar u}}{d\nu}(y)} (Th_i)(y-u) , (Th_j)(y)\right\rangle d\nu(y)$$

$$= \int_X d_{ij}(-\bar u,y)\sqrt{\frac{d\nu_{-\bar u}}{d\nu}(y)} d\nu(y)$$

where $D(\bar u,y) = (d_{ij}(\bar u,y))$.

So $$d_{ij}(\bar u,y) = \frac{d\nu_{-\bar u}^{i,j}}{d\nu}(y)\sqrt{\frac{d\nu}{d\nu_{\bar u}}(y)} \quad \nu \text{ a.e. } y \varepsilon R/\Gamma_o .$$

Similarly, proceeding with $\bar h_1, \bar h_2, \cdots, \bar h_n$ we get:

$$\bar d_{ij}(g,x) = \frac{d\bar\nu_{-g}^{i,j}}{d\bar\nu}(x)\sqrt{\frac{d\bar\nu}{d\bar\nu_g}(x)} \qquad \bar\nu \text{ a.e. } x \varepsilon R,$$

for each $g \varepsilon \Gamma$, where $\bar D(g,x) = (\bar d_{ij}(g,x))$ is the (Γ, R) cocycle corresponding to the (Γ, R) system of imprimivity $(\bar U,\bar F)$. Hence,

$$\bar{d}_{ij}(g,x) = \frac{d\bar{\nu}_{-g}^{\,i,j}}{d\breve{\nu}_{-\bar{u}}^{\,i,j}} \cdot \frac{d\breve{\nu}_{-\bar{u}}^{\,i,j}}{d\breve{\nu}} \cdot \frac{d\breve{\nu}}{d\bar{\nu}}(x) \cdot \sqrt{\frac{d\bar{\nu}}{d\breve{\nu}} \cdot \frac{d\breve{\nu}}{d\breve{\nu}_{\bar{u}}} \cdot \frac{d\breve{\nu}_{\bar{u}}}{d\bar{\nu}_g}(x)}$$

$$= \frac{d\bar{\nu}_{-g}^{\,i,j}}{d\breve{\nu}_{-\bar{u}}^{\,i,j}} \cdot \frac{d\breve{\nu}}{d\bar{\nu}}(x) \cdot \sqrt{\frac{d\bar{\nu}}{d\breve{\nu}} \cdot \frac{d\breve{\nu}_{\bar{u}}}{d\bar{\nu}_g}(x)} \cdot \breve{d}_{ij}(g,x),$$

where $\breve{D}(g,x) = (\breve{d}_{ij}(g,x))$ is the (Γ, R) cocycle extended from the $(\Gamma/\Gamma_o, R/\Gamma_o)$ cocycle D. Now,

$$\frac{d\bar{\nu}_g^{\,i,j}}{d\breve{\nu}_{\bar{u}}^{\,i,j}}(x) = F_g(x) = f(x)\cdot\overline{f(x-g)}\ , \quad x \in R,$$

(by Lemma 4.1)

and $\frac{d\bar{\nu}}{d\breve{\nu}}(x) = |f(x)|^2$ (by corollary 4.2.).

Therefore,

$$\bar{d}_{ij}(g,x) = \frac{\sqrt{\frac{d\bar{\nu}}{d\breve{\nu}}(x)}}{\bar{f}(x)} \cdot \frac{\bar{f}(g+x)}{\sqrt{\frac{d\bar{\nu}}{d\breve{\nu}}(g+x)}} \cdot \breve{d}_{ij}(g,x).$$

Or, $\bar{d}_{ij}(g,x) = \rho(x)\, \breve{d}_{ij}(g,x)\, \rho(g+x)^{-1}$, where

$$\rho(x) = \frac{\sqrt{\frac{d\bar{\nu}}{d\nu}(x)}}{\bar{f}(x)} = \frac{\sqrt{|f(x)|^2}}{\bar{f}(x)} = \frac{|\bar{f}(x)|}{\bar{f}(x)} ,$$

a function of modulus 1, on R.

Hence, $\bar{D}$ and $\breve{D}$ are cohomologous. Therefore, $(\bar{U},\bar{F})$ and $(\breve{U},\breve{F})$ are equivalent.

Thus, we have proved, under the assumption A, the following theorem.

4.4. Theorem. Let (V,E) be a $(K \cap S\ , K)$ system of imprimitivity; (U,F) its dual on $(\Gamma/\Gamma_o\ , R/\Gamma_o)$. Let $(\bar{V},\bar{E})$ be the Gamelin system of imprimitivity on (S, B) obtained from (V,E), and $(\bar{U},\bar{F})$ be its dual on (Γ, R). Let $(\breve{U},\breve{F})$ be the (Γ, R) system of imprimitivity obtained from the $(\Gamma/\Gamma_o\ , R/\Gamma_o)$ system (U,F). Then $(\breve{U},\breve{F})$ and $(\bar{U},\bar{F})$ are equivalent systems of imprimitivity.

4.5. Corollary. Every (S,B) system of imprimitivity is equivalent to a Gamelin system of imprimitivity obtained from a $(K \cap S\ , K)$ system of imprimitivity.

5. The Bohr groups.

In this section, we consider a special class of pairs which appear in the theory of harmonic analysis on compact groups with ordered duals. For these pairs, the assumption (A) is satisfied. We consider them separately because, for these pairs we can get a simple function which establishes the equivalence of the systems of imprimitivity.

A Bohr group B is a compact abelian group whose discrete dual $\Gamma = \hat{B}$ is a subgroup of the additive group $\mathbb{R}$ of real numbers, dense in the usual topology of $\mathbb{R}$. If Γ is countable, then B and Γ are second countable. We consider only those Bohr groups for which $\hat{B} = \Gamma$, is countable. The inclusion map from Γ into $\mathbb{R}$ is continuous and we get the pair $(\Gamma, \mathbb{R})$. Its dual pair is $(\mathbb{R}, B)$. The continuous homomorphism from $\mathbb{R}$ into B will be denoted by $t \longrightarrow e_t$. The elements e_t in B are identified by the relation

$$\langle e_t , \delta \rangle = \exp(it\delta), \quad t \in \mathbb{R}, \quad \delta \in \Gamma .$$

Let B be a Bohr group with Γ countable. Assume, without loss of generality, that $2\pi \in \Gamma$. Let $\Gamma_o = \{ 2\pi \cdot n : n \in N \}$ be the subgroup generated by 2π. Let K be the annihilator of Γ_o. K is a compact subgroup of B. An element e_t belongs to K if and only if t is an integer. Also $\{ e_t : 0 \leq t < 1 \}$ is a Borel section of B with respect to K.

$\mathbb{R}/\Gamma_o$ is the circle group T. We shall regard T as the interval $[0,2\pi)$ with addition modulo 2π. Γ/Γ_o is the dual $\hat{K}$ of K. Thus, the pair $(\Gamma/\Gamma_o , \mathbb{R}/\Gamma_o)$ is $(\hat{K} , T)$. Its dual pair is (N , K). Thus, the four pairs of groups are $(\hat{K} , T)$ and $(\Gamma , \mathbb{R})$, and, (N , K) and $(\mathbb{R} , B)$. A $(\hat{K} , T)$ cocycle extends to a $(\Gamma , \mathbb{R})$ cocycle and in the cohomology class of a $(\Gamma , \mathbb{R})$ cocycle, there is a cocycle which is extended from a $(\hat{K} , T)$ cocycle. Two $(\hat{K} , T)$ cocycles are cohomologous if and only if the extended $(\Gamma , \mathbb{R})$ cocycles are cohomologous. The action of $\mathbb{R}$ on B is quasi-isomorphic to the flow built on K under the constant function 1, with base transformation as translation by e_1 on K. So by theorem 2.3 of chapter II, every $(\mathbb{R} , B)$ cocycle has a strict version.

Every (N, K) cocycle extends to an $(\mathbb{R}, B)$ cocycle and in the cohomology class of an $(\mathbb{R}, B)$ cocycle, there is one which is extended from an (N, K) cocycle.

Let (V,E) be an (N,K) system of imprimitivity; $(\overline{V},\overline{E})$ the associated Gamelin system of imprimitivity on $(\mathbb{R}, B)$. (U,F) is the dual of (V,E) and $(\overline{U},\overline{F})$ is the dual of $(\overline{V},\overline{E})$. Let $(\breve{U},\breve{F})$ be the $(\Gamma, \mathbb{R})$ system of imprimitivity which is extended from the $(\hat{K}, T)$ system (U,F). We assume, as before, that F is homogeneous of multiplicity n, $1 \leq n \leq \infty$, with measure class $\mathcal{C}_\nu$. So (U,F) is given by a $(\hat{K},T,\underline{\underline{U}}(\mathbb{H}_n))$ cocycle D. We also assume that (V,E) is homogeneous. Let A be the associated cocycle and μ the measure corresponding to (V,E).

Instead of a function h in $L^2([0,1))$ satisfying assumption (A), we take the function $h = 1$ on $[0,1)$. Then the functions f and F_g $(g \in \Gamma)$ on $\mathbb{R}$, defined by (3.2) and (3.3) take the simple form

$$f(x) = \frac{1 - \exp(ix)}{x}$$

$$F_g(x) = \frac{(1 - \exp(ix))(1 - \exp(-i(x-g)))}{x(x-g)} .$$

When $g = 0$, we have

$$F_o(x) = \left(\frac{\sin x/2}{x/2} \right)^2 .$$

Theorem 4.2 of chapter II says that when $\nu(\{0\}) = 0$, then $\bar{F}$ is homogeneous of multiplicity n. So, when $\nu(\{0\}) = 0$, $(\bar{U},\bar{F})$ is given by a $(\Gamma, \mathbb{R}, \underline{U}(\mathbb{H}_n))$ cocycle $\bar{D}$. Proceeding as in section 4, we can show that $\bar{D}$ and $\tilde{D}$ are cohomologous (ρ) where ρ is the function on $\mathbb{R}$ defined by:

$$\rho(x) = \frac{|\bar{f}(x)|}{\bar{f}(x)} = \frac{|1 - \exp(-ix)|}{1 - \exp(-ix)} .$$

If $\nu(\{0\}) \neq 0$, then ν can be decomposed into two quasi-invariant measures ν_1 and ν_2 such that ν_1, ν_2 are mutually singular and ν_1 is concentrated on $\hat{K}$ and is equivalent to the Haar measure on $\hat{K}$. All $(\hat{K}, T)$ cocycles relative to ν_1 and all $(\Gamma, \mathbb{R})$ cocycles relative to $\tilde{\nu}_1$ are coboundaries. Observe that $F(\hat{K})$ reduces the system (U,F) and hence it reduces (V,E) also. Let (U', F') be the restriction of (U,F) to $F(\hat{K})$, and (V',E') that of (V,E) to $F(\hat{K})$. Then (U',F') and (V',E') are duals of each other. Clearly, F' is homogeneous of multiplicity n, with measure class $\mathcal{C}_{\nu_1}$ and hence cocycle of (U',F') is a coboundary. It is easy to see that E' is also homogeneous of multiplicity n, the measure class of E' is the Haar measure class on K and that the cocycle associated with (V',E') is a coboundary. Hence the multiplicity of $\overline{E}'$ is n with measure class same as the Haar measure class on B and the cocycle of $(\overline{V}',\overline{E}')$ is a coboundary. It follows that the dual system $(\overline{U}',\overline{F}')$ of $(\overline{V}',\overline{E}')$ is of uniform multiplicity n with associated measure class $\mathcal{C}_{\breve{\nu}_1}$. Hence the cocycle associated with $(\overline{U}',\overline{F}')$ is a coboundary. Thus $(\breve{U}',\breve{F}')$ and $(\overline{U}',\overline{F}')$ are equivalent.

Thus we have the theorem:

5.1. <u>Theorem</u>. Let (V,E) be an (N , K) system of imprimitivity; (U,F) its dual. Let $(\overline{V},\overline{E})$ be the Gamelin extension of (V,E) and $(\overline{U},\overline{F})$ be the dual of $(\overline{V},\overline{E})$. Let $(\breve{U},\breve{F})$ be the $(\Gamma, \mathbb{R})$ system of imprimitivity extended from the $(\hat{K}, T)$ system (U,F). Then $(\breve{U},\breve{F})$ and $(\overline{U},\overline{F})$ are equivalent systems of imprimitivity.

6. An example.

Let Γ be the following subgroup of $\mathbb{R}$, $\Gamma = \{ 2\pi m+n : m, n \text{ integers} \}$. Γ is dense in $\mathbb{R}$. Γ can be identified with $N \times N$ and its dual T^2 can be written as $T^2 = [0,1) \times [0,2\pi)$, where elements (x_1,y_1) and (x_2,y_2) are added according to

$$(x_1,y_1) + (x_2,y_2) = (x_1+x_2 (\text{mod } 1) \ , \ y_1+y_2 (\text{mod } 2\pi)).$$

We have:

$$\langle 2\pi m+n \ , \ (x,y) \rangle = \exp(2\pi i \cdot mx)\exp(iny),$$

for $2\pi m+n \ \varepsilon \ \Gamma$, and $(x,y) \ \varepsilon \ T^2$. For $t \ \varepsilon \ \mathbb{R}$,

$$\langle 2\pi m+n \ , \ e_t \rangle = \exp(it(2\pi m+n)).$$

This describes the pair ($\mathbb{R}$, T^2). The annihilator of the subgroup $\Gamma_o = \{ 2\pi m : m \ \varepsilon \ N \}$ is $K = \{0\} \times [0,2\pi)$, and the pair (N , K) is described by

$$n \longrightarrow e_n = (0 \ , \ n(\text{mod } 2\pi)).$$

Thus, in this case, (N , K) can be identified with (N , T). Hence, the dual pair ($\hat{K}$, T) can also be identified with (N , T).

Let q be a non-constant inner function on the circle. Let A_q be the cocycle defined by:

$$A_q(n, z) = \begin{cases} q(z)q(z+e_1) \cdots q(z+e_{n-1}) , & n > 0 \\ 1 , & n = 0 \\ q(z+e_{-1})^{-1} \cdots q(z+e_n)^{-1} , & n < 0. \end{cases}$$

Let (V,E) be the system of imprimitivity given by A_q. Let $H_2(T)$ be the Hardy space. Then,

$$V_1 H_2(T) = \{q(\cdot) f(\cdot+e_1) : f \varepsilon H_2(T)\} = q \cdot H_2(T) \subseteq H_2(T).$$

Let $(f_1 , f_2 , \cdots)$ (this set may be finite) be a complete orthonormal system of vectors in $H_2(T) \ominus q \cdot H_2(T)$. Then the cyclic subspaces $\{V_n f_i : n \varepsilon N\}$, $i=1,2,\cdots$ are mutually orthogonal, and together span $L^2(T)$. Also, $(V_n f_i , f_i) = \delta_{on}$ for each i. Thus, if F is the spectral measure corresponding to V, then F is homogeneous with multiplicity same as the dimension of $H_2(T) \ominus q \cdot H_2(T)$, and the measure class associated with F is the Haar measure class on T. Since (V,E) is irreducible, (U,F), the dual system of (V,E), is an irreducible system of imprimitivity based on (N , T).

1) If q has infinitely many zeros in the disc, then $H_2(T) \ominus q \cdot H_2(T)$ is infinite dimensional. So we have an irreducible system of imprimitivity based on (N , T) and acting in $L^2(T , \ell^2)$.

2) If $q(z) = \exp(ipz)$, p a positive integer, then $H_2(T) \ominus q\cdot H_2(T)$ is p-dimensional. In this case we shall calculate the cocycle D associated with (U,F).

For each k, $0 \leq k \leq p-1$, let h_k be the function $h_k(z) = \exp(ikz)$, $z \in T$. Then $h_0, h_1, \cdots, h_{p-1}$ have the properties:

(i) $(V_n h_i , h_j) = 0$, if $i \neq j$, $i,j = 0,1,2,\cdots,p-1$, $n \in N$.

(ii) For each $i = 0,1,\cdots,p-1$, $(F(\cdot)h_i , h_i)$ is the Haar measure on T.

(iii) The closed linear span of $\{V_n h_i : i=0,1,\cdots,p-1, n\in N\}$ is $L^2(T)$.

So, let S be the isometric isomorphism from $L^2(T)$ onto $L^2(T, \mathbb{C}^p)$ defined by:

$$S\, F(X)\, h_k = (0,0,\cdots,0,1_X,0,\cdots,0) ,$$

(1_X appears at the $(k+1)^{th}$ position), where X is a Borel subset of T; $k = 0,1,\cdots,p-1$. Then,

$$S(V_n h_k) = (0,\cdots,0,\exp(inz),0,\cdots,0),\ k=0,1,\cdots,p-1.$$

We have,

$$(U_n h)(z) = \exp(inz)h(z) ,\quad h \in L^2(T),\ \text{and}$$

$$(S\, U_n\, S^{-1}\, \hat{h})(z) = D(n,z)\, \hat{h}(z+e_n)\ , \quad \hat{h} \in L^2(T, \mathbb{C}^p).$$

Hence, taking $\hat{h}$ to be $(1,0,\cdots,0)$, $(0,1,0,\cdots,0)$, $\cdots$, $(0,0,\cdots,0,1)$ we get,

$$D(1,z) = \begin{bmatrix} 0, & 0, & \cdots & 0, & \exp(iz) \\ 1, & 0, & \cdots & 0, & 0 \\ \cdots & \cdots & & \cdots & \\ \cdots & \cdots & & \cdots & \\ 0, & 0, & \cdots & 1, & 0 \end{bmatrix} , \quad z \in T.$$

The cocycle D can be written in terms of $D(1,z)$. Thus, in this case we can calculate all the four cocycles.

<u>Remark</u>: The above example is the completion of an example given in Bagchi [3]. Bagchi uses the example to exhibit an $(\mathbb{N}\ ,\ T)$ irreducible system of imprimitivity acting in $L^2(T, \mathbb{C}^p)$ where p is any positive integer. However, he does not calculate the dual cocycle. [16] contains a construction, due to A. M. Gleason, of an $(\mathbb{N}\ ,\ T)$ cocycle having values in 2×2 unitary matrices giving rise to irreducible systems of imprimitivity of dimension 2. In [20] Muhly uses Gleason's example together with Gamelin's method of obtaining an $(\mathbb{R}\ ,\ B)$ cocycle, to exhibit an irreducible $(\mathbb{R}\ ,\ B)$ system of imprimitivity of multiplicity 2.

APPENDIX

In this appendix, we show that in many cases, assumption (A) is satisfied.

I. If $G = \mathbb{R}$, the real line, then any closed subgroup H of G is a cyclic group and so, without loss of generality, we can assume that $H = N$, the integer group. Take $C = [0,1)$ for a section of G with respect to H, and define f on C by:

$$f(x) = \exp(x) , \quad 0 \leq x < 1.$$

For any $t \in \mathbb{R}$,

$$\hat{f}(t) = \int_0^1 e^{itx} e^x \, dx = \frac{e^{it+1} - 1}{it+1} \neq 0.$$

So, the above choice of C and f satisfies assumption (A).

II. Suppose G is a locally compact second countable abelian group and H is an open subgroup of G. Then G/H is discrete. Let C be any section of G with respect to H such that C contains the identity e of G. Define f on C by:

$$f(e) = 1,$$
$$f(c) = 0, \quad \text{if } c \neq e, \quad c \in C.$$

Then, for $\gamma \varepsilon \hat{G}$,

$$\hat{f}(\gamma) = \int_C \langle \gamma , c \rangle \; f(c) \, dc = 1.$$

III. Let H_1 and H_2 be closed subgroups of locally compact second countable abelian groups G_1 and G_2 respectively. Suppose that we can get a Borel section C_1 of G_1 with respect to H_1, and a function f_1 on C_1 satisfying assumption (A). Similarly, suppose C_2, f_2 can be found satisfying assumption (A) for G_2 and H_2. Then $C = C_1 \times C_2$ is a Borel section of $G_1 \times G_2$ with respect to $H_1 \times H_2$, and the function f on C defined by:

$$f(c_1,c_2) = f_1(c_1) \cdot f_2(c_2) \; , \quad c_1 \varepsilon C_1, \quad c_2 \varepsilon C_2 \; ,$$

satisfies assumption (A).

IV. Take $G = \mathbb{R}^n$ and let H be a closed subgroup of G of the form $\mathbb{R}^m \times N^k$, $0 \leq m,k \leq n$, $m+k \leq n$. Then by I, II and III, we can get a Borel section C of G with respect to H and a Borel function f on C satisfying assumption (A). By theorem 9.11 of [15] we can, without loss of generality, take any closed subgroup H of $\mathbb{R}^n$ to be of the form $\mathbb{R}^m \times N^k$, $0 \leq m,k \leq n$, $m+k \leq n$. Thus, assumption (A) is valid if $G = \mathbb{R}^n$ and H is any closed subgroup of G.

V. Let G be a compact second countable abelian group and H a closed subgroup of G. Then the assumption (A) holds for G and H. This follows from the following more general lemma.

<u>Lemma</u>. Let ν be a finite measure on a compact group G. Then there is a continuous function f on G such that $(f\, d\nu)^{\wedge}$ is always non-zero.

<u>Proof</u>: Let $A = [0,1] \times [0, \frac{1}{2}] \times [0, \frac{1}{2^2}] \times \cdots$

and give A the product topology. Let $\hat{G} = \{\gamma_0, \gamma_1, \gamma_2, \cdots\}$, γ_0 being the identity in G. Put,

$$M_0 = \{ \tilde{a} = (a_0, a_1, a_2, \cdots) \in A : \sum_i a_i \hat{\nu}(\gamma_i) = 0 \}$$

$$M_1 = \{ \tilde{a} \in A : \sum_i a_i \hat{\nu}(\gamma_i + \gamma_1) = 0 \}$$

$$\cdots \quad \cdots \quad \cdots$$

$$M_n = \{ \tilde{a} \in A : \sum_i a_i \hat{\nu}(\gamma_i + \gamma_n) = 0 \}$$

$$\cdots \quad \cdots \quad \cdots$$

It is easy to see that each M_n is closed, nowhere dense subset of A. Hence, by Baire category theorem, $\bigcup_{n=0}^{\infty} M_n \neq A$. Let $\tilde{a} = (a_0, a_1, a_2, \cdots) \in A$ be such that $\tilde{a} \notin M_n$ for any n, and take $f(\cdot) = \sum_i a_i \langle \cdot, \gamma_i \rangle$. Then f is a continuous function on G and for each $\gamma \varepsilon G$,

$$(f\, d\nu)^{\wedge}(\gamma) = \sum_i a_i \hat{\nu}(\gamma_i + \gamma) \neq 0.$$

Q. E. D.

VI. Let $G = \mathbb{R}^n \times K \times D$, where K is a compact second countable abelian group and D is a discrete abelian group. Let H be a closed subgroup of G of the form $H_1 \times K_1 \times D_1$ where H_1, K_1, D_1 are closed subgroups of $\mathbb{R}^n$, K, D respectively. Then by the above steps, we can get a Borel section C of G with respect to H and a Borel function f on C such that,

$$\hat{f}(\gamma) = \int_C \langle \gamma, c \rangle f(c)\, dc \neq 0, \quad \text{for all } \gamma \varepsilon \hat{G}.$$

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