SOME ASPECTS OF TOPOLOGICAL SEMIGROUP ACTS AND MACHINES

By

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A thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy

Calcutta

1974

ACKNOWLEDGEMENT

I thank the Research and Training School of the Indian Statistical Institute for the research facilities given to me.

I express my gratitude to Professor M.G.Nadkarni for supervising my research work. His comments, suggestions and advice have been of immense value to me.

I also thank Professor Ashok Maitra for his kind interest in my work and constant encouragement.

I am also indebted to Professor A.D.Wallace of University of Florida from whom I received much help and encouragement.

Professor Jane M. Day and Professor Keval P. Chinda of College of Notre Dame, Belmont, were very kind to go through the bulk of this work quite meticulously and give me their comments and suggestions which have proved very useful. I express my gratitude to both of them.

I also thank my teachers and colleagues, specially Dr. S.C.Bagchi, Dr. A.B.Raha and Shri K.K.Roy, at the Research and Training School who have been very helpful and encouraging to me.

Finally, I thank Mr. Gour Mohon Das for the careful typing of the material.

Kripasindhu Sikdar

CONTENTS

INTRODUCTION

The algebraic theory of sequential machines and automata is well known through the works of various authors, such as, Hartmanis and Stearns [23], Arbib [2], Booth [6] and Ginsburgh[21]. Generalising the concept of a complete sequential machine Ginsburg introduced the concepts of a quasi-machine and an abstract machine [20, 21] as abstract mathematical systems satisfying certain natural axioms and extended certain concepts and results of the classical theory of sequential machines. Fleck [18] considered automata in the generality of Ginsburg's quasi-machines except that he did not consider outputs and studied some algebraic properties of automata in relation to their structures. Ginsburg also suggested the possibility of introducing topology and defining the concept of a topological machine which would further generalise the concept of a quasi-machine or an abstract machine and, perhaps, could be an appropriate mathematical model for an analog or a continuous machine [25]. Subsequently, many authors mentioned about this possibility of topologizing machine theory. Wallace mentioned about topological machines which are topologized quasi-machines in a recent survey article [43] on binary topological algebras.

But no work about topological machines in the sense of Wallace [43] has been done so far. However, a good deal of work has been done about topological machines without outputs which are topologized automata of Fleck and, more commonly, referred to as semigroup acts or, simply, acts [7, 14, 15, 16] though some authors refer to them as topological automata [29] or topological machines [4, 5, 36] also. The study of topological automata or acts was initiated by Day and Wallace [15, 16] and, as remarked by Arbib in his editorial note in [1, p.270], their work opened the possibility of extending the concepts and results of the algebraic theory of machines to the topological case. As a matter of fact interests in the topological theory of automata and semigroups seem to be growing fast amongst mathematicians as evidenced by the volume of recent published works and several symposiums at the University of Florida and elsewhere. In her survey article on semigroup acts [14] Day also stressed the possibility of topologizing the algebraic theory of machines. Quoting the works of various authors, such as, Kalman [26], Wymore [45], Mesarovic' [31], and Balakrishnan [3], to name a few, she further remarked. 'Perhaps topology will play a larger role in system theory eventually'.

It is also interesting to note that the output function of a machine satisfies the same algebraic condition as the cocycle [24, 42] defined for a group act, in a measure theoretic set up, and the theory of cocycles play important roles in Harmonic Analysis, specially, in the study of induced representations of locally compact second countable groups [cf. Varadarajan [42]] and the invariant subspaces of $L^2(B)$, B being the Bohr group [cf. Helson [24]].

In view of all these observations we have ventured to study topological machines and write the present dissertation This dissertation is divided into three chapters and is based on the author's work during the period 1969-1973. Chapter I is devoted to some aspects of semigroup acts and Chapters II and III to machines.

Although an introduction and a summary are given in the beginning of each chapter we give below a brief summary of the problems considered and the results presented in this thesis.

In Chapter I, our main problem is to investigate the various kinds of partitioning of the state spaces of semigroup acts and a few related questions. Several results towards the characterisation of acts for which maximal orbits (or inverse-orbits i.e., subsets of the state spaces which are mapped onto a given point by one or more of the inputs) or orbits partition the state spaces are presented in Sections 2, 3, 4 and 5. Some remarks are also made in Section 6 concerning quotient acts induced by the above mentioned partitions of the state spaces. However, the study of these quotient acts is very much incomplete. In Section 7, we investigate how a product act inherits from the component acts certain property which may be the maximality of orbits (or inverse-orbits) or the partitioning of the state space by the maximal orbits (or inverse-orbits) or orbits. Finally, in Section 8, we investigate how a homomorphism carries over certain properties mentioned above from a semigroup act onto another.

In Chapter II, our main problem is to obtain structural characterisations of the output functions of topological machines. In Section 2, we obtain a few elementary results for a few special but fairly general situations. In Section 3, we obtain characterisations of output functions for machines whose input spaces are certain freely generated monoids or groups. Finally, in Section 4, we consider machines whose input semigroups act on themselves and are certain special classes of threads having identity and zero [11] and obtain results towards the structure of output functions of such machines.

In Chapter III, the primary objective is to extend certain concepts and results of the algebraic theory of machines to the topological case. First, in Section 2, a slightly general version of Kelemen's observations [28] concerning the existence of certain unique compatible topologies for recursions are given. In Sections 3 and 4, we obtain some sufficient conditions for the existence of a unique reduced form or a unique input-reduced form of a machine. Some results are also obtained concerning the topological version of a problem of Ginsburg [21] on the existence of an input-distinguished machine with a finite (compact) state space for any given input semigroup. In Section 5, topological versions of the concept of equivalence of machines and a few related results are presented. Finally, in Section 6, a few topological facts related to some problems of earlier sections are proved.

CHAPTER - 1

PARTITIONS OF THE STATE SPACES, QUOTIENTS, PRODUCTS AND HOMOMORPHISMS OF SEMIGROUP ACTS.

1. Introduction and Summary

In this introductory section we explain the basic concepts relevant to our discussion about semigroup acts and give a summary of the results obtained which are presented in the subsequent sections of this chapter.

1.1 Semigroup Acts. Let S be a topological semigroup and X a nonvoid Hausdorff space. An act, denoted by the pair (X, S), is a continuous (anonymous) function $X \times S \longrightarrow X$ such that, denoting the value of the (anonymous) function at the point (x, s) by xs, the associativity condition $x(s_1 s_2) = (xs_1)s_2$ holds for all $s_1, s_2 \in S$ and all $x \in X$. We shall refer to this situation as an action of S on X and say that S acts on X or use similar terminology. We shall often refer to X and S of an act (X, S) as the state space and the input semigroup respectively.

We have used juxtaposition to denote the semigroup operation as well as the action map and we shall continue to do do so unless the clarity of the situation dictates otherwise We shall mean by a semigroup a topological semigroup and by a space a nonvoid Hausdorff space throughout our discussion unless, of course, stated otherwise explicitly.

Of course what we have called an act should have been called a right act, to be more precise, while defining a left act, denoted by a pair (S, X), as a continuous function $S \times X \longrightarrow X$, where S is a semigroup and X is a space, such that $(s_1 s_2)x = s_1(s_2 x)$ holds for all $s_1, s_2 \in S$ and all $x \in X$. However, there is an obvious duality in these concepts. We formalize this briefly as given by Norris in his Ph.D. thesis [36]. For any semigroup $(S, \cdot)$, we define the dual semigroup to be $(S, *)$ where $s*t = t \cdot s$. Let S' denote the dual of S, suppressing mention of the operation on S. If now (X, S) is a (right) act, we define its dual to be $(X, S)' = (S', X)$ where $sx = xs$. It follows that (S', X) is a left act. If we make a similar definition for the dual of a left act then it follows that $(X, S)'' = (X, S)$ for any act (X, S). It can be easily seen that each theorem about (right) acts is logically equivalent to a 'dual' theorem for left acts. Because of this duality it is immaterial whether we study right or left acts. While in the literature it has become more or less standard to consider left acts we have deviated from this norm and, in this dissertation, we shall consider only right acts as we find it more convenient for writing, particularly so when we discuss about machines in the next two chapters.

We may also point out that by an act we are really meaning topological act and, if we do not consider any topology, then we may refer to an act by the term algebraic act. However, an algebraic act can also be regarded as a topological act if we think that both the input semigroup and the state space are given discrete topologies and, therefore, an algebraic act is also called a discrete act. By an act we shall mean, in the sequel, a topological act unless stated otherwise.

An act (X, S) can be viewed as a mathematical model of a physical system which can be at any moment in one of the several states (the elements of X) and changes from its present state x to the state xs upon receiving an input s (which is an element of S).

Before proceeding further we now list a few very standard examples of acts.

1.2. Examples. (1) The classical concept of a sequential machine or automaton without output [2, 6, 21, 23] provides examples of a very special class of algebraic acts. In this case, the input semigroup S is a free monoid generated by a finite input alphabet, the state space is a finite set X and the action map is such that $x\Lambda = x$ for all $x \in X$ where Λ is the identity element (null string) of S. The cocept of an algebraic act is more general where S may be any semigroup and X need not be finite.

(2) Any semigroup acts on itself via its multiplication.

(3) Any semigroup S acts on any space X via the identity $xs = x$ for all $x \in X$ and all $s \in S'$.

(4) If X is a locally compact space, then it is well known that the set $M(X)$ of all continuous functions from X into itself is a semigroup under functional composition in the compact-open topology and $M(X)$ acts on X via evaluation i.e., $xf = f(x)$ for each pair $(x, f) \in X \times M(X)$.

(5) If I is any right ideal of a semigroup S, then S acts on I by its multiplication.

(6) If S is a compact semigroup and C is a <u>right congruence on S</u> i.e., C is an equivalence relation on S and $(x, y) \in C$ implies that $(xs, ys) \in C$ for all $s \in S$, then $(S/C, S)$ is an act defined canonically by the identity $[x]s = [xs]$ where S/C is the quotient space and $[x]$ denotes the equivalence of x, for $x \in S$.

(7) Every topological transformation group [22] is an act.

In view of Example 1.2(1) an act is often referred to as an automaton [14, 18, 29] or a machine [5, 36] in both algebraic and topological literatures. However, we adopt the simpler term act and reserve the term machine for a more complex mathematical system (where we shall consider outputs) which we introduce and study in the next two chapters. A good guide to the literature on acts, both algebraic and topological, is the recent excellent survey article by Day [14].

We next introduce a few basic concepts concerning acts.

1.3. Definitions. Let (X, S) be an act and A and T be nonvoid subsets of X and S respectively. Then we denote by AT the subset of X which is the image of $A \times T$ under the action map i.e., $AT = \{y : y \in X \text{ and } y = xs \text{ for some } (x, s) \in A \times T\}$. If $A = \{x\}$, then the set xT will be referred to as T-orbit of $x \in X$. An S-orbit xS will be simply called an orbit. If an act is viewed as a model for a physical system, then the T-orbit xT of a point $x \in X$ is the set of all states of the system into which the system can go starting from x after receiving one or more inputs from T. We denote by $AT^{(-1)}$ the set of all points of X whose T-orbits intersect A i.e.,

$$AT^{(-1)} = \{y : y \in X \text{ and } yT \cap A \neq \emptyset\}$$

If $A = \{x\}$, then the set $xT^{(-1)}$ will be referred to as T-inverse-orbit of $x \in X$. An S-inverse-orbit will be simply called an inverse-orbit. An orbit is maximal if it is not properly contained in another orbit. An orbit is minimal if it does not properly contain another orbit. A maximal (respectively a minimal) inverse-orbit is similarly defined.

For a set X a family $\{X_t\}$ of subsets of X is called a cover of X if $\cup X_t = X$ and a cover is called a partition of X if any two distinct subsets belonging to it are disjoint.

An act (X, S) is called disjoint (respectively i-disjoint, or quasi-transitive) if the family of maximal orbits (respectively maximal inverse-orbits, or orbits) forms a partition of X.

A (continuous) homomorphism (respectively a topological isomorphism or, simply, an iseomorphism) from an act (X, S) onto an act (Y, T) is a pair (f, h) where f is a continuous map (respectively a homeomorphism) from X onto Y and h is a (continuous) homomorphism (respectively an iseomorphism) from S onto T satisfying for all $x \in X$ and all $s \in S$, $f(xs) = f(x)h(s)$. If $S = T$ and $h : S \to S$ is the identity map then the pair (f, h) defining a homomorphism (respectively, an iseomorphism) may be simply denoted by the single map f and we shall refer to f, in that case, as a homomorphism (respectively an iseomorphism). By a homomorphism we shall always mean continuous homomorphism unless stated otherwise.

Suppose $\{(X_i, S_i)\}$ and $\{(X_i, S)\}$ are two families of acts. If $\prod X_i$ is the product space of X_i's and $\prod S_i$ is the Cartesian product semigroup, then we can define the product act $(\prod X_i, \prod S_i)$ by $(x_i)(s_i) = (x_i s_i)$ and the product act $(\prod X_i, S)$ by $(x_i)s = (x_i s)$ for all $(x_i) \in \prod X_i$, $(s_i) \in \prod S_i$ and $s \in S$.

If (X, S) is an act, then an equivalence relation C on X is called a congruence if $(x, y) \in C$ implies that $(xs, ys) \in C$ for all $s \in S$. A congruence C on X is called a closed congruence on X if C is a closed subspace of $X \times X$. A continuous map f from a space X onto a space Y is called a quotient map if a subset A of Y is open iff $f^{-1}(A)$ is open in X. If C is such a congruence that the quotient space X/C is Hausdorff and the map $q \times i : X \times S \longrightarrow X/C \times S$, where $q : X \longrightarrow X/C$ is the canonical quotient map and $i : S \longrightarrow S$ is the identity map, is a quotient map, then the canonically induced action of S on X/C defined by $[x]s = [xs]$ for all $[x] \in X/C$ and $s \in S$, $[x]$ being the equivalence class containing x, defines an act $(X/C, S)$. Such an act $(X/C, S)$, whenever it is defined, will be called a quotient act of (X, S).

An act (X, S) is called a <u>compact act</u> if both X and S are compact and it is called an <u>onto act</u> if $XS = X$. (X, S) is called a unitary act or we shall say that <u>S acts unitarity on X</u> if $x \in xS$ for all $x \in X$. The properties of an act being unitary and onto are somewhat related which we point out in the following remark.

1.4. <u>Remark</u>. Let (X, S) be an act.

(1) If S has an identity 1 and (X, S) is onto, then $x1 = x$ for all $x \in X$, and hence, (X, S) is unitary. Conversely, every unitary act is onto.

(2) If S is compact and acts on X normally (i.e., $xtS = xSt$ for all $t \in S$), then (X, S) is unitary iff it is onto.

The proof of (1) is easy and (2) is a result due to Stadlander [40]

Several aspects of acts have been studied recently [cf. 14] and the situation when the input semigroup is a group is also well-known [cf. 9, 22]. For group actions the orbits form a partition of the state space but the situation is different in case of semigroup actions in which case all kinds of overlapping of orbits can take place. Our objective is to study acts from this point of view. Some works have been done by Stadlander [39, 40] and Borrego and De Vlun [8] which are somewhat similar to our theme of investigation.

Some of our results are purely algebraic and ~~some results are purely algebraic and~~ some results depend on topological theory of semigroups for which we refer to A.B.P. Miranda's book [37]. We present these results in the subsequent sections and give a summary in the following paragraph.

1.5. <u>Summary</u>. In Section 2, we obtain several results concerning maximal and minimal orbits (respectively inverse-orbits). We show that for a compact act every orbit (respectively inverse-orbit) is contained in a maximal orbit (respectively inverse-orbit) and, if the act is also onto, then the maximal orbits (respectively inverse-orbits) form a cover of the state space. We also obtain several results giving characterizations of maximal and minimal orbits (respectively inverse-orbits).

In Section 3, we first show that every compact onto act is a homomorphic image of a disjoint act and then obtain several results characterizing disjoint acts and i-disjoint acts.

Section 4 is primarily devoted to the study of quasi-transitive acts. Apart from the results about quasi-transitive acts we also make some remarks about point-transitive and transitive acts.

In Section 5, we make two remarks concerning partitions of a space induced by disjoint or quasi-transitive acts.

In Section 6, we make a few observations concerning quotient acts corresponding to disjoint or i-disjoint or quasi-transitive acts.

Section 7 is devoted to study how a product act inherits from the component acts a given property such as maximality of orbits (or inverse orbits) or disjointness (or i-disjointness or quasi-transitivity) of acts.

Finally, in Section 8, we study how a homomorphism from a compact unitary act onto another such an act carries a given property of act as mentioned above.

We give many illustrative examples and mention a few problems for further study.

2. Maximal and Minimal Orbits (respectively Inverse-Orbits)

In this section we present a few preliminary facts about orbits and inverse-orbits which will be useful in the sequel.

We start with some remarks about ideals in acts which are well known [cf. 40]. For an act (X, S) a nonvoid subset Y of X is called an *ideal* if $YS \subset Y$. Ideals in acts have properties similar to those of right ideals in semigroups. If (X, S) is a compact act, then every ideal A properly contained in X is contained in a maximal proper ideal and every ideal contains a minimal ideal. Further, if R is a minimal right ideal of S, then xR is a minimal ideal for any x in X. A minimal ideal is a minimal orbit.

Regarding orbits in an act, the following is a simple but useful result which also appears in Borrego and De Vun [8]. However, our proof is different and depends on the continuity of the act and an application of Zorn's lemma.

2.1. *Proposition.* For a compact act every orbit is contained in a maximal orbit.

Proof: Let (X, S) be a compact act and xS be an orbit. Let F be the collection of all orbits containing xS. Let $\leq$ be the partial order on F defined by, for yS and zS in F, $yS \leq zS$ if $yS \subset zS$. Let F_1 be a chain in F. F_1 can be taken as a directed set to define nets in X. As

X is compact a net $\{x_t\}$ with $x_tS \in F_1$ has a converging subnet $\{x_{t'}\}$ with $x_{t'}S \in F_1$. Let $\{x_{t'}\}$ converge to y. We shall show that yS is an upper bound of F_1. Let $x_tS \in F_1$ and $x_{t'}S$ be any element of F_1 corresponding to an element $x_{t'}$ of $\{x_{t'}\}$ such that $x_tS \subset x_{t'}S$. Then, for any $s \in S$, $x_ts = x_{t'}s_{t'}$ for some $s_{t'} \in S$. Since S is compact the net $\{s_{t'}\}$ has a converging subnet $\{s_{t''}\}$ converging to, say, s_1. As the subnet $\{x_{t''}\}$ corresponding to $\{s_{t''}\}$ also converge to y and the act is continuous it follows that $x_ts = x_{t'}s_{t'} = ys_1$. Therefore, $x_tS \subset yS$. Thus yS is an upper bound of F_1 and hence, by Zorn's lemma, the result follows.

The following simple example illustrates the fact that compactness is not necessary in the above proposition.

2.2. Example . Let $S = [0, \infty)$ act on $X = [x, \infty)$ for any real number x, by usual addition, both S and X being given the usual topology and S with usual addition as semigroup operation. Here, $X = [x, \infty) = x + S$ is the (unique) maximal orbit.

However, as the following example shows, in case of a non-compact act an orbit may not be contained in a maximal orbit.

2.3. <u>Example</u>. Let $S = [0, 1)$, with usual topology and usual multiplication, act on itself by its multiplication.

As a consequence of Proposition 2.1, the following is immediate.

2.4. <u>Remark</u>. If (X, S) is a compact onto act, then the family of all maximal orbits form a cover of X and, for any maximal orbit xS, $x \in xS$.

We next present a few facts characterizing maximal (minimal) orbits and inverse-orbits and showing a kind of 'dual' relations between orbits and inverse-orbits.

2.5. <u>Remark</u>. If (X, S) is a unitary act, then, for any $x, y \in X$,

(1) $xS \subset yS$ iff $yS^{(-1)} \subset xS^{(-1)}$, and

(2) $xS = yS$ iff $xS^{(-1)} = yS^{(-1)}$.

2.6. <u>Proposition</u>. If (X, S) is a unitary act, then the following statements are equivalent.

(1) xS is a maximal orbit

(2) $xS = yS$ iff $y \in xS^{(-1)}$.

(3) $xS^{(-1)} = yS^{(-1)}$ iff $y \in xS^{(-1)}$.

(4) $xS^{(-1)} \subset yS$ iff $y \in xS^{(-1)}$.

(5) $xS^{(-1)} \subset xS$.

(6) $(xS^{(-1)})S = xS$.

(7) $xS^{(-1)}$ is a minimal inverse-orbit.

Proof:

(1) => (2). If $y \in xS^{(-1)}$, then $x = ys$ for some $s \in S$ and so $xS \subset yS$. Hence, by (1), $xS = yS$. Again, as the act is unitary, $xS = yS$ implies that $y \in xS^{(-1)}$.

(2) => (3). Follows from Remark 2.5 (2).

(3) => (4). If $xS^{(-1)} \subset yS$, then, as the act is unitary, $x = ys$ for some $s \in S$ and so $y \in xS^{(-1)}$. Conversely, if $y \in xS^{(-1)}$ we shall show that $xS^{(-1)} \subset yS$. If $z \in xS^{(-1)}$, then, by (3), $xS^{(-1)} = zS^{(-1)}$ and hence, $z = yt$ for some $t \in S$.

(4) => (5) => (6). Easy.

(6) => (7). If, possible, let $yS^{(-1)} \subset xS^{(-1)}$ for some $y \in X$. Then, for some $s \in S$, $ys = x$ and hence, $xS \subset yS$. Again $yS \subset (xS^{(-1)})S = xS$. Hence $xS = yS$ or, equivalently, $xS^{(-1)} = yS^{(-1)}$.

(7) => (1). If $xS \subset yS$ for some $y \in X$, then $yS^{(-1)} \subset xS^{(-1)}$ and hence, $yS^{(-1)} = xS^{(-1)}$ or, equivalently, $xS = yS$.

The proof of the following proposition is also quite simple and similar. Therefore we state this without proof.

2.7. Proposition. If (X, S) is a unitary act, then the following statements are equivalent.

(1) xS is a minimal orbit

(2) $xS = yS$ iff $y \in xS$.

(3) $xS^{(-1)} = yS^{(-1)}$ iff $y \in xS$

(4) $xS \subseteq yS^{(-1)}$ iff $y \in xS$

(5) $xS \subseteq xS^{(-1)}$

(6) $(xS)S^{(-1)} = xS^{(-1)}$

(7) $xS^{(-1)}$ is a maximal inverse-orbit.

The following remark will also be of some use in the sequel.

2.8. Remark. If (X, S) is a compact unitary act, then the following statements are true.

(1) For any $x \in X$, $xS \subseteq \bigcup\{x_\alpha S^{(-1)} : x_\alpha S$ is a minimal orbit contained in $xS\}$.

(2) For any two minimal orbits xS and yS,

$$xS \cap yS^{(-1)} \neq \emptyset \quad \text{iff} \quad xS = yS .$$

(3) If xS is a maximal orbit and a union of maximal inverse-orbits $\{x_\alpha S^{(-1)}\}$, then $\{x_\alpha S\}$ are indeed all the minimal orbits contained in xS.

Proof: (1) If $y \in xS$, then $y = xs$ for some $s \in S$, and so, $yS \subseteq xS$. Therefore, if $x_\alpha S$ is a minimal orbit contained in yS, then $y \in yS^{(-1)} \subseteq x_\alpha S^{(-1)}$.

(2) If $xS \cap yS^{(-1)} \neq \emptyset$, then, for some $s,t \in S$, $xst = y$, and hence, $xS = yS$.

(3) If yS is any minimal orbit contained in xS, then $yS \cap x_\alpha S^{(-1)} \neq \emptyset$ for some α and so, by (2), $yS = x_\alpha S$.

We now give two examples of non-unitary acts for which some of the above results fail.

2.9. Example. Let $S = [a, \infty)$ be an additive semigroup of reals for some $a > 0$ and act on itself. Then the orbit of a is $[2a, \infty)$ and is the (unique) maximal orbit but the inverse-orbit of a is empty set. Therefore, (6) of Proposition 2.6 fails.

2.10. Example. Let X be any nonempty space. Define multiplication in X as follows. For any $x, y \in X$, $xy = x_0$ for some fixed $x_0 \in X$. Let G be any group and $S = G \times X$ be the Cartesian product semigroup with coordinatewise multiplication. If S acts on itself, then $G \times \{x_0\}$ is the only orbit which is, therefore, both maximal and minimal. Here (1) <=> (2) is not true in Proposition 2.7.

We conclude this section by recording two more facts about inverse-orbits. We omit the detailed proofs which are easy by our previous observations.

2.11. Proposition. If (X, S) is a compact act, then the following statements are true.

(1) Any inverse-orbit is contained in a maximal inverse-orbit.

(2) Any inverse-orbit contains a minimal inverse-orbit.

(3) If, further, (X, S) is onto, then the family of maximal inverse-orbits forms a cover of X and, if $xS^{(-1)}$ is a maximal inverse-orbit, then $x \in xS^{(-1)}$.

Proof: We prove only (1) and omit the proofs of (2) and (3) which are similar and quite easy. If $xS^{(-1)}$ is any inverse orbit, then, for a minimal orbit yS contained in xS, $yS^{(-1)}$ is a maximal inverse-orbit containing $xS^{(-1)}$.

Finally, we note the following which is similar to Remark 2.8 and omit the easy proof.

2.12. Remark. If (X, S) is a compact unitary act, then the following statements are true.

(1) For any $x \in X$, $xS^{(-1)} \subset \cup \{x_\alpha S : x_\alpha S^{(-1)}$ is a minimal inverse-orbit contained in $x\, S^{(-1)}\}$.

(2) For any two minimal inverse-orbits $xS^{(-1)}$ and $yS^{(-1)}$, $xS \cap yS^{(-1)} \neq \emptyset$ iff $xS^{(-1)} = yS^{(-1)}$.

(3) If $xS^{(-1)}$ is a maximal inverse-orbit and a union of maximal orbits $\{x_\alpha S\}$, then $\{x_\alpha S^{(-1)}\}$ are indeed all the minimal inverse-orbits contained in $xS^{(-1)}$.

3. Disjoint Acts and i-Disjoint Acts.

Though the family of maximal orbits (or inverse-orbits) of a compact onto act forms a cover of the state space it does not, in general, form a partition as illustrated below.

3.1. Example. Let $S = [0, 1]$, with usual topology and usual multiplication, act on $X = [-a, a]$, for some positive real number a and with usual topology, by usual multiplication. There are two maximal orbits viz., $-aS = [-a, 0]$ and $aS = [0, a]$ which intersect, and there is a unique minimal orbit viz., $0S = \{0\}$ and $0S^{(-1)} = X$ is the only maximal inverse-orbit.

3.2. Example. Let $S = \{(x, 0) : 0 \leq x \leq 1\} \cup \{(0, y): (0, y): 0 \leq y \leq 1\}$, considered as a subspace of the plane, and the multiplication in S be defined as $(x, y)(x', y') = (xx', xy' + y)$ for all $(x, y), (x', y') \in S$. Let $X = \{(x, 0) : -1 \leq x \leq 1\} \cup \{(0, y): -1 \leq y \leq 1\}$, considered as a subspace of the plane. Let the action of S on X be defined by, for $(x, y) \in X$ and $(x', y') \in S$, $(x, y)(x', y') = (xx', xy' + y)$. There are two maximal orbits, corresponding to the points $(-1, 0)$ and $(1, 0)$, which contain a common point $(0, 0)$. For any $-1 \leq y \leq 1$, $(0, y)$ is a minimal orbit as $(0, y)S = (0, y)$. The maximal inverse-orbits corresponding to the minimal orbits $(0, y)$ are $(0, y) \cup \{(x, 0) : x \geq y\}$ for $0 < y \leq 1$, $(0, y) \cup \{(x, 0): x \leq y\}$ for $-1 \leq y < 0$ and $[-1, 1]$ for $y = 0$. The maximal inverse-~~sets~~ orbits are also not disjoint.

However, we have the following result which also appears in Borrego and Devun [8].

3.3 <u>Proposition.</u> Let (X, S) be a compact onto act. There exists a disjoint act (X^*, S) such that (X, S) is a homomorphic image of (X^*, S). If the set $Y = \{x: xS$ is a maximal orbit of $(X, S)\}$ is closed, then X^* is compact. Also the action of S restricted to a maximal orbit of (X, S) is isomorphic to that on a maximal orbit of (X^*, S).

<u>Proof:</u> The proof involves a construction of a suitable act (X^*, S).

Let $X^* = \cup\{\{x\} \times xS : x \in Y\}$ be considered as a sub-space of $X \times X$. We first show that, if Y is closed, then X^* is closed, and hence, compact. Let $\{z_\alpha = (x_\alpha, x_\alpha s_\alpha)\}$ be a net in X^* converging to, say, $z = (x, y)$. We shall show that $(x, y) \in X^*$. Since Y is closed and S is compact, by the continuity of the act, it follows that $x \in Y$ and $y = xs$

for some $s \in S$, and so, $z \in X^*$.

Now define the action of S on X^* as follows: For any $(x, y) \in X^*$ and $s \in S$, $(x, y)s = (x, ys)$. It is clear that (X^*, S) is a disjoint act whose maximal orbits are $\{x\} \times xS$ for any $x \in Y$.

Finally, let $h : X^* \to X$ be defined by $h(x, y) = y$ for all $(x, y) \in X^*$ and $i : S \to S$ be the identity map on S. Then it is easily seen that (h, i) is a homomorphism from (X^*, S) onto (X, S) which restricted to any maximal orbit is an iseomorphism.

We shall now consider the conditions under which the maximal orbits (or inverse-orbits) form a partition of the state space of an act. Towards this our first result is as follows:

3.4. <u>Proposition.</u> Let (X, S) be a compact unitary act. Then the following statements are equivalent.

(1) (X, S) is disjoint.

(2) For any $x, y \in X$, $xS \cap yS \neq \emptyset$ implies that $xS^{(-1)} \cap yS^{(-1)} \neq \emptyset$.

(3) For any $\emptyset \neq A \neq B \subset X$, $AS \cap BS \neq \emptyset$ implies that $AS^{(-1)} \cap BS^{(-1)} \neq \emptyset$.

(4) Each inverse-orbit contains a unique minimal inverse-orbit.

(5) Each orbit is contained in a unique maximal orbit.

(6) Each maximal orbit is the union of the maximal inverse-orbits corresponding to the minimal orbits contained in it.

(7) Each maximal orbit is a union of maximal inverse-orbits.

(8) Each maximal orbit is a union of inverse-orbits.

(9) Each inverse-orbit is contained in an orbit.

(10) There exists a (unique) <u>closed congruence $C_{\hat{0}}$ on X</u> such that each equivalence class is an orbit.

<u>Proof:</u> (1) => (2). Suppose, for $x, y \in X$, $xS \cap yS \neq \emptyset$. Then the maximal orbits containing xS and yS intersect, and hence, are equal to, say, zS for some $z \in X$. But then $z \in xS^{(-1)} \cap yS^{(-1)}$.

(2) => (3). For $\emptyset \neq A \neq B \subset X$, let $z \in AS \cap BS$. Then for some $a \in A$, $b \in B$ and $s, t \in S$, we have $z = as = bt \in aS \cap bS$. Hence $aS^{(-1)} \cap bS^{(-1)} \neq \emptyset$ which implies that $AS^{(-1)} \cap BS^{(-1)} \neq \emptyset$.

(3) => (4). Let, if possible, $yS^{(-1)}$ and $zS^{(-1)}$ be two minimal inverse-orbits contained in some inverse-orbit $xS^{(-1)}$. Then $x \in yS \cap zS$, and hence, $yS^{(-1)} \cap zS^{(-1)} \neq \emptyset$ which implies, by Proposition 2.6, that $yS^{(-1)} = zS^{(-1)}$.

(4) => (5). Let, if possible, an orbit xS be contained in two maximal orbits, say, yS and zS. Then $yS^{(-1)} \cup zS^{(-1)} \subset xS^{(-1)}$. Now, by Proposition 2.6, $yS^{(-1)}$ and $zS^{(-1)}$ are two minimal inverse-orbits, and hence, are equal.

(5) => (6). Suppose xS is a maximal orbit and $\{x_\alpha S\}$ are all the minimal orbits contained in xS. We claim that $xS = \bigcup x_\alpha S^{(-1)}$. By Remark 2.8 (1), $xS \subset \bigcup x_\alpha S^{(-1)}$. Conversely, if $y \in x_\alpha S^{(-1)}$ for some α, then $x_\alpha S \subset yS \subset xS$. Therefore, $y \in xS$.

(6) => (7) => (8). Trivial.

(8) => (9). Let $yS^{(-1)}$ be any inverse-orbit and xS be a maximal orbit containing the point y. Then, by (8), $xS = \bigcup x_\alpha S^{(-1)}$, and so, $y \in x_\alpha S^{(-1)}$ for some α. Hence, $x_\alpha \in yS$ and $x_\alpha S \subset yS$. Thus $yS^{(-1)} \subset x_\alpha S^{(-1)} \subset xS$.

(9) => (6). Let xS be a maximal orbit and $\{x_\alpha S\}$ be the minimal orbits contained in xS. We have to show that $xS = \bigcup x_\alpha S^{(-1)}$. By virtue of Remark 2.8 (1), we need to show that $x_\alpha S^{(-1)} \subset xS$ for each α. Now, by (9), $x_\alpha S^{(-1)}$ is contained in a maximal orbit, say, yS. But $x \in x_\alpha S^{(-1)}$, since $x \in xS^{(-1)} \subset x_\alpha S^{(-1)}$, for each α. Therefore, $x \in yS$ which means that $xS \subset yS$, and so, $xS = yS$.

(6) => (1). Suppose $x_1 S$ and $x_2 S$ are two maximal orbits which intersect. Then there exists a minimal orbit $yS \subset x_1 S \cap x_2 S$. Let $zS^{(-1)}$ be a minimal inverse orbit contained in $x_1 S \cap x_2 S$ as $z \in zS^{(-1)}$ and, therefore, $zS = x_1 S = x_2 S$.

(1) => (10). Define $C_d \subset X \times X$ by including a point (x, y) in C_d if x and y are contained in the same maximal

orbit. It is clear that C_d is then a congruence on X such that each equivalence class is an orbit (infact, a maximal orbit). We now show that C_d is a closed subspace of $X \times X$. For that, let $\{(x_\alpha, y_\alpha)\}$ be a net in C_d converging to (x, y). Then, for each α, there exist $z_\alpha \in X$, and $s_\alpha, t_\alpha \in S$ such that $x_\alpha = z_\alpha s_\alpha$ and $y_\alpha = z_\alpha t_\alpha$. Now, by compactness of X and S and continuity of the act, we can conclude that there exist $z \in X$ and $s, t \in S$ such that $x = zs$ and $y = zt$, and so, $(x, y) \in C_d$. Therefore, C_d is closed.

(10) => (1). If C_d is such a closed congruence on X, then we claim that each equivalence class is indeed a maximal orbit. For, let $[x]$ be an equivalence class containing x and suppose $[x] = xS$. Let yS be a maximal orbit containing xS. Further, for any $z \in X$, $zS \subset [z]$; because, if $[z] = wS$, then $z \in wS$ implies that $zS \subset [z]$. Therefore, $x \in xS \subset yS \subset [y]$ which implies that $[x] = [y]$ and so $xS = yS$. Thus the maximal orbits form a partition of X.

An immediate result giving similar characterizations of i-disjoint acts is given below. We omit the details of the proof which are easy and follow the same pattern as that of Proposition 3.4. However, we include the proof of the part (1) => (10).

3.5. <u>Proposition.</u> Let (X, S) be a compact unitary act. Then the following statements are equivalent.

(1) (X, S) is i-disjoint

(2) For any $x, y \in X$, $xS^{(-1)} \cap yS^{(-1)} \neq \emptyset$ implies that $xS \cap yS \neq \emptyset$.

(3) For any $\emptyset \neq A \neq B \subseteq X$, $AS^{(-1)} \cap BS^{(-1)} \neq \emptyset$ implies that $AS \cap BS \neq \emptyset$.

(4) Each orbit contains a unique minimal orbit.

(5) Each inverse-orbit is contained in a unique maximal inverse-orbit.

(6) Each maximal inverse-orbit is the union of the maximal orbits corresponding to the minimal inverse-orbits contained in it.

(7) Each maximal inverse-orbit is a union of maximal orbits.

(8) Each maximal inverse-orbit is a union of orbits.

(9) Each orbit is contained in an inverse-orbit.

(10) There exists a (unique) closed congruence C_i on X such that each equivalence class is an inverse-orbit.

Proof: (1) => (10). Define $C_i \subseteq X \times X$ by including a point (x, y) in C_i if x and y are in the same maximal inverse-orbit. That C_i is a closed equivalence can be seen as in the proof of Proposition 3.4. We now show that C_i is indeed a congruence on X. Let $(x, y) \in C_i$. Then $x, y \in zS^{(-1)}$ for some maximal inverse-~~set~~ orbit $zS^{(-1)}$. That is, $zS \subseteq xS \cap yS$, and so, $xS^{(-1)} \cup yS^{(-1)} \subseteq zS^{(-1)}$. Now, for any $s \in S$, it is clear that $(xs)S \subset xS$ and $(ys)S \subset xS$ and $(ys)S \subset yS$, and so, $xS^{(-1)} \subset (xs)S^{(-1)}$ and $yS^{(-1)} \subset (ys)S^{(-1)}$. Now, by (5) which is equivalent to (1), both $(xs)S^{(-1)}$ and $(ys)S^{(-1)}$ are contained in the same maximal inverse-orbit $zS^{(-1)}$. Hence, $(xs, ys) \in C_i$. Thus C_i is a congruence on X. Further, each equivalence class is clearly an inverse-orbit (in fact, a maximal inverse-orbit).

We now give an example of a compact act which is both disjoint and i-disjoint.

3.6. Example. Let $I = [0, 1]$, the usual multiplicative semigroup, and $E = [0, 1]$ with min multiplication i.e., $xy = \min \{x, y\}$ for all $x, y \in E$. Then $X = E \times I$ is a acts on X by multiplication. The maximal orbits are arcs, $(e, 1)S$ which are pairwise disjoint. The minimal orbits are the points $(e, 0)$. Each maximal orbit $(e, 1)S$ is a maximal inverse-orbit corresponding to a minimal orbit $(e, 0)$. This act is, therefore, disjoint and i-disjoint.

However, if the ideal $E \times \{0\}$ is shrunk to a point to get X' and a quotient act (X', S), the maximal orbits are no longer disjoint although the quotient act is still i-disjoint as there is only one maximal inverse-orbit.

A suitable sub-act of the act given in Example 3.2 gives an example of an act which is disjoint (in fact, with only one maximal orbit) which is not i-disjoint as described below.

3.7. Example: [cf. Example 3.2]. Let S be as in Example 3.2 and $X = \{(x, 0): 0 \leq x \leq 1\} \cup \{(0, y): 0 \leq y \leq 1\}$.

Finally, Example 3.6 motivates us to state the following result characterising acts which are both disjoint and i-disjoint.

3.8. Proposition. Let (X, S) be a compact unitary act. The following two statements are equivalent.

(a) (X, S) is both disjoint and i-disjoint.

(b) Each maximal orbit is a maximal inverse-orbit and vice-versa.

Proof: (a) => (b). Let xS be a maximal orbit. Then as (X, S) is i-disjoint, by Proposition 3.5 (4), if yS is the unique minimal orbit contained in xS, we claim that $xS = yS^{(-1)}$. If $z \in xS$, then $zS \subseteq xS$ and zS contains a unique minimal orbit which must be yS, and hence, $z \in yS^{(-1)}$. Conversely, if $z \in yS^{(-1)}$, then $yS \subseteq zS$, and hence, as (X, S) is disjoint, by Proposition 3.4(5), the unique maximal orbit in which zS is contained in must be xS. Therefore, $z \in xS$.

To prove that each maximal inverse-orbit is a maximal orbit we can apply similar arguments.

(b) => (a). Suppose two maximal orbits $x_1 S$ and $x_2 S$ intersect and suppose $y_1 S^{(-1)}$ and $y_2 S^{(-1)}$ are two maximal inverse-orbits which equal $x_1 S$ and $x_2 S$ respectively. Then there exists a minimal orbit $zS \subset x_1 S \cap x_2 S$ so that $zS \subset y_1 S^{(-1)} \cap y_2 S^{(-1)}$ which implies that both y_1 and y_2 are in zS, and therefore, equivalently, $y_1 S = y_2 S = zS$ as zS is minimal. Therefore, by Proposition 2.6, $x_1 S = x_2 S$, and hence, (X, S) is disjoint.

Similarly, it can be shown that (X, S) is i-disjoimt.

4. Quasi-transitive Acts.

Let (X, S) be an act. We say that S acts on X point-transitively if $xS = X$ for some $x \in X$, transitively if $xS = X$ for all $x \in X$, and, quasi-transitively if $XS = X$ and, for any $x, y \in X$, either $xS = yS$ or $xS \cap yS = \emptyset$. A transitive act is clearly quasi-transitive and a quasi-transitive act is transitive iff it is point-transitive. In this section we shall present some results towards the characterization of quasi-transitive acts and mention some facts about point-transitive and transitive acts which will be of some use in the sequel.

To start with we mention a few examples of quasi-transitive acts.

4.1. Example. Let a topological group S act on a space X such that $XS = X$, or equivalently, [cf. Remark 1.4] $x1=x$ for all $x \in X$ where 1 is the identity of S. Such an act (X, S), called a topological transformation group [22], is always quasi-transitive.

4.2. Example: Let (X, S) be a compact unitary act. Then S acts point-transitively on each orbit, transitively on each minimal orbit and quasi-transitively on each ideal which is a union of minimal orbits.

4.3 Example. Let (X, S) be an onto act where S is a right simple semigroup. Then, as S is the only right ideal of S, every orbit is minimal, and hence, (X, S) is quasi-transitive.

Before proceeding further let us fix some notational conventions to be followed throughout the rest of this section as well as in the sequel.

4.4 Notations. Let (X, S) be an act. For an $x \in X$ we denote by ρ_x the map $\rho_x : S \longrightarrow X$ defined by $\rho_x(s) = xs$ for all $s \in S$. Similarly, for an $s \in S$ we denote by ρ_s the map $\rho_s : X \longrightarrow X$ defined by $\rho_s(x) = xs$ for all $x \in X$. Finally, we denote by C_x the right congruence on S defined by ρ_x i.e., $C_x = \{(s, t) : \rho_x(s) = \rho_x(t)\}$.

We now state a simple but useful characterization of point-transitive acts which is well-known [cf. 27, 29, 38]. We write the proof for completeness.

4.5 Proposition. Let a compact or discrete semigroup S act on a space X. Then (X, S) is point-transitive iff for some $x \in X$, the right congruence C_x on S induced by ρ_x is closed and satisfies (i) there exists an $e \in S$ such that $(es, s) \in C_x$ for all $s \in S$, and (ii) the canonical act $(S/C_x, S)$ is isomorphic to (X, S) through an isomorphism (h, i) where $h : S/C_x \to X$ is a homeomorphism and $i : S \to S$ is the identity map such that $h[e] = x$ where e is the element of S mentioned in (i), and $[e]$ denotes the equivalence class containing e.

Proof: Let $xS = X$. That C_x is closed can be shown by a net argument. If $e \in S$ be such that $xe = x$, then clearly $(es, s) \in C_x$ for all $s \in S$. Also, the map $h: S/C_x \to X$ defined by $h[s] = xs$ is clearly a homeomorphism, and hence, (h, i) is the required iseomorphism.

Conversely, if (i) and (ii) hold, then $[e]S = S/C_x$ and via the iseomorphism (h, i), $xS = X$.

The following remark is then immediate.

4.6 Remark. Let X be a nonvoid compact (or discrete) semigroup S acting on X point-transitively such that for some $x \in X$, $\rho_x : S \to X$ is a homeomorphism iff we can define a multiplication in X so as to make it a left unital semigroup iseomorphic to S.

We next present a few results concerning quasi-transitive acts. Our first proposition is very simple and we omit the proof.

4.7 Proposition. Let (X, S) be an act. Then the following statements are equivalent

(1) S acts quasitransitively on X.

(2) $XS = X$ and if, for any $x, y \in X$, $y \in xS$ then $x \in yS$.

(3) S acts unitarily on X and each orbit is minimal as well as mamimal.

(4) $xS = xS^{(-1)}$ for all $x \in X$.

(5) $AS = AS^{(-1)}$ for all $\emptyset \neq A \subset X$

(6) S acts unitarily on X and $A = AS^{(-1)}$ for any ideal $A \subset X$.

(7) If $A \subset X$ is any ideal, then $xS \cap A \neq \emptyset$ implies that $x \in A$ for any $x \in X$.

It is clear that a quasi-transitive act is both disjoint and i-disjoint; but the converse is not true as seen in Example 3.6.

If $A \subset X$ is an ideal, then let us call A a <u>prime ideal</u> if, for any $x \in X$, $xS \cap A \neq \emptyset$ implies that $x \in A$. Then Proposition 4.7 (7) says that an act is quasi-transitive iff every ideal is a prime ideal. The following remark characterizes all prime ideals.

4.8 <u>Remark</u>. Let (X, S) be a unitary act. Then an ideal $A \subset X$ is prime iff for any $x \in A$, xS is a minimal orbit, or in other words, S acts quasi-transitively on A.

For the next few results we shall need some results from the theory of compact semigroups, particularly the results concerning the structure of the minimal ideal of a compact semigroup. We refer to A. B. P. Miranda's book [37] for these results and follow the notations given there which we record below.

4.9. Notations. Let S be a compact semigroup. Then K is the minimal ideal of S, R stands for any minimal right ideal of S, E is the set of idempotents of S and $H(e)$ is the maximal subgroup of S containing $e \in E$. We also let $K' = K \cap E$ and $R' = R \cap E$. Further, we use the symbol TG for the term topological transformation group which will occur frequently.

Then our next result about quasi-transitive acts can be stated as follows.

4.10. Proposition. Let a compact semigroup S act on a space X. Then the following statements are equivalent.

(1) S acts on X quasi-transitively.

(2) R acts on X unitarily.

(3) For each $e \in K'$, $(Xe, H(e))$ is a TG and $\bigcup \{ Xe : e \in R' \} = X$.

(4) For each $x \in X$ there exists an $e \in R'$ such that $x = xe$.

(5) For each $x \in X$ there exists an $e \in K'$ such that $x = xe$.

(6) K acts on X unitarily.

(7) There exists a (unique) closed congruence C_0 on X such that each equivalence class is a minimal orbit.

Proof. (1) => (2). By Proposition 4.7 (3), for any $x \in X$, $x \in xS$ and xS is a minimal orbit. Now as R is a minimal right ideal, xR is a minimal orbit for any $x \in X$. Therefore, as $xR \subset xS$, it follows that $x \in xR = xS$.

(2) => (3). For any $e \in K'$, $XeH(e) = XeeSe = XeSe = XRe = Xe$. So $(Xe, H(e))$ is a TG. Also note that $XH(e) = Xe$. Now

$X = XR = X(\bigcup\{H(e): e \in R'\})$

$= \bigcup\{XH(e) : e \in R'\} = \bigcup\{Xe : e \in R'\}$.

(3) => (4). Since for any $x \in X$, $x \in Xe$ for some $e \in R'$ it follows that $x = xe$ for some $e \in R'$.

(4) => (5) => (6). Trivial.

(6) => (1). Since $K = \bigcup R$, for any $x \in X$, $x \in xK$ implies that $x \in xR$ for some R; and xR is a minimal orbit, and hence, $xR = xS$ since $x \in xR$ implies that $xS \subset xRS \subset xR$. Thus each orbit xS is minimal and S acts on X unitarily. Hence (1) follows.

(1) => (7). Define $C_o \subset X \times X$ by including a point (x, y) in C_o if $xS = yS$. Clearly, C_o is a congruence on X. To show that C_o is a closed subspace of $X \times X$, let $\{(x_\alpha, y_\alpha)\}$ be a net in C_o converging to (x, y). Then, by definition of C_o, there exist s_α and t_α in S such that $x_\alpha = y_\alpha s_\alpha$ and $y_\alpha = x_\alpha t_\alpha$ for each α. As $x_\alpha \to x$, $y_\alpha \to y$ and by compactness of S, we can assume $s_\alpha \to s$ and $t_\alpha \to t$ (otherwise, there exist converging subnets of s_α and t_α), by continuity of the act it follows that $x = ys$ and $y = xt$. Therefore, $xS = yS$ and $(x, y) \in C_o$. So C_o is a closed congruence such that each equivalence class is a minimal orbit.

(7) => (1). Trivial.

As an immediate corollary of the above result the following fact is true.

4.11. Remark. [cf. Lemma 7.2, [7]]. Let S be a right simple (or simple) compact semigroup acting on a space X. Then the following statements are equivalent.

(1) S acts on X quasi-transitively.

(2) S acts on X unitarily

(3) $XS = X$.

Note that $S = R$ or $S = K$ according as S is right simple or simple.

With some more restrictions on S or on the act we can have the following result some parts of which are similar to some results of Stadlander [cf. 38].

4.12. Proposition. Let a compact semigroup S act on a space X. If either S is left simple (i.e., $Ss = S$ for all $s \in S$) or the act is normal (i.e., $xSs = xsS$ for all $x \in X$ and all $s \in S$), then the following statements are equivalent.

(1) S acts on X quasi-transitively.

(2) For each $e \in K'$ and each $x \in X$, $(xS, H(e))$ is a TG and $XS = X$.

(3) For each $e \in K'$, $(X, H(e))$ is a TG.

(4) $\rho_s : X \to X$ is a homeomorphism for all $s \in K$.

(5) $x = xe$ for all $x \in X$ and all $e \in K'$.

(6) $\rho_s : X \to X$ is a homeomorphism for some $s \in K$.

(7) $x = xe$ for all $x \in X$ and for some $e \in K'$

Proof: (1) => (2). Let $x \in X$ and $e \in K'$. If S is left simple, then $xSe = xS$. Again, if the act is normal, then $xSe = xeS \subseteq xS$ which implies, by (1), that $xSe = xS$. Further, $xSH(e) = xSeSe = xSSe = xS$. Also, by (1), $XS = X$.

(2) => (3). Let $e \in K'$. Then $XH(e)$ $= (\cup \{ xS : x \in X \}) H(e) = \cup \{xS : x \in X\} = X$.

Therefore, $(X, H(e))$ is a TG.

(3) => (4). As $K = \cup \{H(e) : e \in K'\}$ and $(X, H(e))$ is a TG for all $e \in K'$, ρ_s is a homeomorphism for all $s \in K$.

(4) => (5). Let $e \in K'$ and $s \in H(e)$. Then ρ_s is a homeomorphism means that $(X, H(e))$ is a TG and so $x = xe$ for all $x \in X$.

(5) => (1). This follows from Proposition 4.10.

(4) => (6). Trivial

(6) => (7). Let $s \in H(e)$ for some $e \in K'$. Then ρ_s is a homeomorphism which implies that $x = xe$ for all $x \in X$.

(7) => (1). This follows from Proposition 4.10.

In connection with the above result we like to record the following remark which is clear from the above proof.

4.13. Remark. The statement 'ρ_s is a homeomorphism' can be replaced by 'ρ_s is onto' in (4) and (6) of Proposition 4.12. In case, S is left simple, then $S = K$. Further, to prove the equivalence of (4) and (6) the hypothesis S is left simple or the act is normal is superfluous as seen below. Also we note that any normal semigroup S acts on X normally and any commutative semigroup is normal.

4.14. Proposition. Let a compact semigroup S act on a space X. If $\rho_s : X \to X$ is onto for some $s \in K$, then ρ_s is onto for all $s \in S$ and ρ_s is, infact, a homeomorphism for all $s \in K$.

Proof: Let, for some $s_1 \in K$, ρ_{s_1} be onto. Then, if $s_1 \in H(e)$ for $e \in K'$, $Xs_1 = X$ implies that $(X, H(e))$ is a TG and so $x = xe$ for all $x \in X$. If $s \in S$, then $Xs = Xes = Xese = X$ as $ese \in H(e)$.

We next show that ρ_s is a homeomorphism for all $s \in K$. Note that, if $f \in K'$ and $f \neq e$, then there exists an iseomorphism $(i, \emptyset)$ from $(X, H(f))$ onto $(X, H(e))$, where $i : X \to X$ is the identity map and $\emptyset : H(f) \to H(e)$, defined by $\emptyset(s) = es$ for all $s \in H(f)$, is an isoomorphism [cf. Theorem 1.2.6 in [37]], because $xs = xes = x\emptyset(s)$ for all $x \in X$ and all $s \in H(f)$. Hence, as $(X, H(e))$ is a TG, $(X, H(f))$ is a TG for all $f \in K'$, and so, ρ_s is a homeomorphism for all $s \in K$.

In Proposition 4.12 we have proved equivalence of quasi-transitive acts and acts where each transition map $\rho_s : X \to X$ is onto (which is equivalent to saying that ρ_s is onto for some $s \in K$) under some bypothesis. The implication from the onto-ness of ρ_s to quasi-transitivity of the acts does not demand all these hypothesis. However, the assumption of onto-ness of some ρ_s is sufficiently strong and has some implication towards the algebraic structure of the input semigroup. This is the content of the following proposition which is a somewhat improved version of a result of Day [14].

4.15. Proposition. Let a compact semigroup S act on a space X effectively (i.e., for $s, t \in S$, $s \neq t$ implies that for some $x \in X$, $xs \neq xt$). If, for some $s \in K$, $\rho_s : X \to X$ is onto, then (X, S) is a TG.

Proof: By Proposition 4.14, we have $\rho_s : X \to X$ is onto for all $s \in S$. Now if $e \in E$, then $xe = X$ and so $x = xe$ for all $x \in X$. For, if $x \in X$, $x = ye$ for some $y \in X$ and so $xe = yee = ye = x$. So $xs = xese$ for all $x \in X$ and all $s \in S$ which implies, by the effectiveness of the act, that $ese = s$ for all $s \in S$. Therefore, e acts as the identity of S and, in fact, the only idempotent of S. For, if $f \in E$ and $f \neq e$, then $Xf = X$ implies that for any $x \in X$, $x = yf$ for some $y \in X$, and so, $xf = yff = yf = yfe = xe$. But the effectiveness of the act implies that $f = e$. As e is the only idempotent, which is the identity of S and S is compact, $K = H(e) = eSe = S$. Therefore, S is a group. Finally, ρ_e is onto implies that (X, S) is a TG.

Closely parallel to the above result we have the following proposition.

4.16. Proposition. Let a compact semigroup S act on a space X effectively. If for some $s \in K$, $\rho_s : X \to X$ is 1-1, then (X, S) is a TG.

Proof: Let, for $s \in K$, ρ_s be 1 - 1. Then, if $s \in H(e)$, $e \in K'$, as $es = s$ and for any $x \in X$, $xs = xes$, by the 1-1-ness of ρ_s, it follows that $x = xe$ for all $x \in X$. So ρ_e is onto, and hence, by Proposition 4.15, (X, S) is a TG.

If h is a homomorphism from a semigroup S onto a semigroup T and T acts on a space X, then we can extend this action of T on X to an action of S on X by letting $xs = xh(s)$ for all $x \in X$ and all $s \in S$ such that $xS = xT$ for all $x \in X$. Let us call the act (X, S) a homomorphic (more precisely, h-homomorphic) extension of the act (X, T). The following proposition says (among other things) that for a large class of acts every quasi-transitive act is a homomorphic extension of a TG.

4.17. Proposition. Let a compact semigroup S act on a space X. Then the following statements are true.

(1) If $XS = X$ and S is left simple, then S acts on X quasi-transitively and normally. But the converse is not true.

(2) If $XS = X$ and S acts on X normally, then S acts on X quasi-transitively.

(3) Let either S be left simple or S act on X normally.

If S acts on X quasi-transitively, then (X, S) is a homomorphic extension of a TG.

Proof: (1). As in the proof of (1) => (2) in Proposition 4.12 it is easily seen that $(xS, H(e))$ is a TG for all $x \in X$ and all $e \in K'$ and, therefore, S acts on X quasi-transitively. Further, as $xSs = xS$ and $xsS \subset xS$ for all $x \in X$ and all $s \in S$, by quasi-transitivity of the act, it follows that $xSs = xSs = xsS = xS$. So S acts on X normally.

To see that the converse is false we note that, if G is a compact group and $S = G \times G$ is given the multiplication $(s_1, s_2)(t_1, t_2) = (s_1 t_1, 1)$ for all $(s_1, s_2), (t_1, t_2) \in S$ where 1 is the identity of G, then S acts on G quasi-transitively and normally but S is not left simple.

(2) The proof of this is similar to that of (1).

(3) Let ξ be the congruence on S (i.e., ξ is an equivalence relation on S such that $(x, y) \in \xi$ implies $(xs, ys) \in \xi$ and $(sx, sy) \in \xi$ for all $s \in S$), the 'effectiveness congruence' [14], defined by $(s, t) \in \xi$ if $xs = xt$ for all $x \in X$. ξ is closed and, by compactness of S, the canonical quotient semigroup S/ξ is indeed a compact semigroup. Let the quotient semigroup S/ξ act canonically on X i.e., $x[s] = xs$ for all $x \in X$ and all $s \in S$, $[s]$ being the equivalence class containing s. Then, by Propositions 4.12 and 4.15, it follows that $(X, S/\xi)$ is a TG and (X, S) is clearly a homomorphic extension of $(X, S/\xi)$ via the homomorphism

$h: S \to S/\mathcal{E}$, $h(s) = [s]$ for all $s \in S$.

The following is yet another simple fact about quasi-transitive acts.

4.18. Proposition: Let a compact semigroup S act on a space X. If $\rho_x : S \to X$ is $1-1$ for all $x \in yS$, for some $y \in X$, then S acts on X effectively and S is a right simple semigroup. If, further, $XS = X$, then S acts on X quasi-transitively.

Proof: That S acts on X effectively is clear. To prove that S is right simple we show that S is left-cancellative. For any $s, t_1, t_2 \in S$, if $st_1 = st_2$, then $yst_1 = yst_2$. Now, by $1-1$-ness of ρ_{ys}, $t_1 = t_2$. Therefore, as S is compact, S is right simple.

We next record a few facts about transitive acts. The following is similar to Proposition 4.7 and the easy proof is omitted.

4.19. Proposition: Let (X, S) be an act. Then the following statements are equivalent.

(1) S acts on X transitively

(2) There is no ideal properly contained in X.

(3) $xS^{(-1)} = X$ for all $x \in X$

(4) $AS^{(-1)} = X$ for all $\emptyset \neq A \subset X$.

We also have the following some parts of which are, however, well-known [cf. 27, 38].

4.20. Proposition. Let a compact semigroup S act on X. Then the following statements are equivalent.

(1) S acts on X transitively

(2) R acts on X point-transitively.

(3) For each $e \in K'$, $(Xe, H(e))$ is a TG which is transitive, $xH(e) = Ke$ for all $x \in X$ and $\cup\{Xe : e \in R'\} = X$.

(4) K acts on X point-transitively.

(5) There exists an $x \in X$ such that the right congruence C_x on S induced by the map $\rho_x : S \Rightarrow X$ is closed and satisfies (i) and (ii) of Proposition 4.5 and (iii) for each pair $s, t \in S$ there exists an $f \in S$ so that $(sf, t) \in C_x$.

Proof.

(1) => (2). For any $x \in X$, xR is a minimal orbit and so, by (1), $xR = X$.

(2) => (1). If, for some $x \in X$, $xR = X$ which is a minimal orbit, then $yS = X$ for all $y \in X$.

(1) =>(3). By Proposition 4.10 (3), for any $e \in K'$, $(Xe, H(e))$ is a TG and $\cup\{Xe : e \in R'\} = X$. Now, if $y = xe \in Xe$, then $yH(e) = xeeSe = xRe = Xe$. Hence, $H(e)$ acts transitively on Xe such that $xH(e) = Xe$ for all $x \in X$.

(3) => (4). For any $x \in X$, $xK = \cup\{xH(e) : e \in K'\}$ $= \cup\{Xe : e \in K'\} = X$.

(4) => (2). Let, for some $x \in X$, $xK = X$. Then, as $K = \bigcup R$, $x \in xR$ for some R. But xR is a minimal orbit and so, $xR = xS = xK = X$.

(1) => (5). For any $x \in X$, $xS = X$ and so, by Proposition 4.5, the right congruence C_x on S induced by $\rho_x : S \to X$ satisfies (i) and (ii). We now show that C_x satisfies also (iii). By (1) and (ii), $[s]S = S/C$ for all $[s] \in S/C_x$. Therefore, for any $[t] \in S/C_x$, there exists $f \in S$ such that $[s]f = [t]$ i.e., $(sf, t) \in C_x$.

(5) => (1). We shall only show that $(S/C_x, S)$ is transitive. If $[s]S$ and $[t]S$ are any two orbits, then, by (iii), there exist f_1, $f_2 \in S$ such that $[s]f_1 = [t]$ and $[t]f_2 = [s]$, and so, $[s]S = [t]S$. Therefore, as $[e]S = S/C_x$ where e is an element of S satisfying (i), $(S/C_x, S)$ is transitive.

For an act (X, S), let ξ be the 'effectiveness congruence' on S [cf. 14] i.e., $(s, t) \in \xi$ if $xs = xt$ for all $x \in X$. We say that <u>(X, S) satisfies the property (P)</u> if there exists a point $y \in X$ such that for $s, t \in S$, $(s, t) \notin \xi$ implies that $ys \neq yt$. Then the following is a restatement of a result of Lin [29] which also appears in Day and Wallace [Corollary 1.31, 15].

4.21. <u>Proposition.</u> Let X be a nonvoid space. Then there exists a compact (or discrete) semigroup acting on X transitively such that (P) is satisfied iff a multiplication in X can be defined which makes X a compact (or discrete), left unital and right simple semigroup.

Furthermore, analogous to Proposition 4.12, we have the following.

4.22. Proposition. Let a compact semigroup S act on X such that $S/\mathcal{E}$ is isomorphic to $K/\mathcal{E}$. If S is left simple or S acts on X normally, then the following statements are equivalent.

(1) S acts on X transitively satisfying (P)

(2) For each $e \in K'$, $(X, H(e)/\mathcal{E})$ is a TG which is transitive and effective on X satisfying (P). Furthermore, X is a compact group isomorphic to $H(e)/\mathcal{E}$.

(3) For some $e \in K'$, the statement made in (2) holds.

We omit the easy proof. However, we remark that if S acts commutatively, then S acts normally and as in this case (P) is trivially satisfied we can omit the phrase 'satisfying (P)' in (1) - (3) above. Also the equivalence of (2) and (3) does not demand all the assumptions of the proposition.

5. Partition of a Space induced by a Disjoint or Quasi-transitive Semigroup Act.

When can we say that a given partition of a space X is induced by a disjoint or quasi-transitive action of some semigroup S ? Of course, the trivial partition of a space X formed by the points of X is induced by a disjoint or quasi-transitive action of a semigroup S iff S acts on X trivially i.e., $xs = x$ for all $x \in X$ and $s \in S$. We present below a few simple results concerning non-trivial partition that follow via Remark 4.6 and Proposition 4.21.

5.1. Proposition. Let X be a nonvoid space. If a compact (or discrete) semigroup S acts on X disjointly such that for each maximal orbit xS the map $\rho_x : S \to X$ is a homeomorphism, then X is partitioned by left-unital semigroups $\{X_t\}$ where each X_t is a maximal orbit isomorphic to S.

Conversely, if $\{X_t\}$ is a partition of X such that each X_t is clopen in X and a left-unital semigroup, then there exists a semigroup S (which is compact if each X_t is compact) acting on X disjointly such that each X_t is a maximal orbit.

Proof: The first part follows from Remark 4.6. Conversely, let $S = \prod X_t$, the Cartesian product of X_t's with coordinatewise multiplication. Let $f_t : X_t \times S \to X_t$ be defined as, for $(x_t) \in X_t$, $s \in S$, $f_t(x_t, s) = x_t P_t(s)$ where P_t is the projection from S onto X_t. Since each X_t is clopen in X and

$\{X_t\}$ forms a partition of X the map $f : X \times S \to X$ defined as, for $x \in X$, $s \in S$, $f(x, s) = f_t(x, s)$ if $x_t \in X_t$, is continuous by virtue of the continuity of f_t's. It is also clear that f is an action map as each f_t is so.

The hypothesis that $\{X_t\}$ forms a clopen partition of X made in the second part of Proposition 5.1 is not always satisfied. In Example 3.6 we have described a disjoint act (X, S) where X is the unit square and S is the usual unit interval semigroup, where maximal orbits do not form a clopen partition.

Analogous to Proposition 5.1 the following fact can be stated for quasi-transitive acts.

5.2. Proposition. Let a compact or discrete semigroup S act on space X quasi-transitively such that the action of S restricted to each orbit satisfies (P). Then each orbit is a left-unital, right simple semigroup iseomorphic to S/C for some closed congruence C on S.

Conversely, if $\{X_t\}$, where each X_t is a left-unital, right simple semigroup, is a clopen partition of a space X, then there exists a semigroup S (which is compact if each X_t is compact), namely $\prod X_t$, acting on X quasi-transitively such that each X_t is an orbit and the action of S on each X_t satisfies (P).

In this connection it may be worthwhile to mention the following problems. However, we do not know any answer.

5.3. Problems. (1) When can we say that a given partition of a space is induced by an i-disjoint action of a semigroup ?

(2) Let (X, S) be an act. Define the relation ∂ on X by $(x, y) \in \partial$ if $\{x\} \cup xS = \{y\} \cup yS$. ∂ induces a partition of X [cf. 39, 41]. When can we say that a given partition of a space X is induced by the ∂-relation on X for an action on X of a semigroup S ?

6. Quotient Acts of Disjoint (respectively, i-Disjoint or Quasi-transitive) Acts.

If C is a congruence on the state space X of an act (X, S), then we have seen in Section 1.3 how we can define canonically an act (X/C, S), called the quotient act of (X, S). In this section we make a few observations concerning the quotient act (X/C, S) where C is C_d (respectively C_i or C_o) which is the congruence on X induced by a disjoint (respectively an i-disjoint or a quasi-transitive) act (X, S) considered in Proposition 3.4 (10) (respectively Proposition 3.5(10) or Proposition 4.10 (7).

If (X, S) is a compact unitary act which is disjoint (respectively i-disjoint), then the congruence C_d (respectively C_i) on X is closed and hence, by the compactness of the act

(X, S), the quotient act $(X/C_d, S)$ (respectively $(X/C_i, S)$) is defined. We shall show that, if a compact semigroup S acts on a space X quasi-transitively, then also the quotient act $(X/C_0, S)$ is defined. We have already seen in Proposition 4.10(7) that C_0 is closed and we shall show in Proposition 6.2 that the quotient map $q_0 : X \to X/C_0$ is open. From these two facts it follows that X/C_0 is Hausdorff [cf. Proposition 8, p. 79 of Bourbaki [9]]. Further, as q_0 is open, the map $q_0 \times i : X \times S \to X/C_0 \times S$ is a quotient map, and so, the quotient act is defined.

Before proving that q_0 is open we state a result from Bourbaki [Proposition 6(c), p. 54, [9]] which will be useful in the sequel.

6.1. Proposition. Let R be an equivalence relation on a topological space X. Then the quotient map $q: X \to X/R$ is open iff the closure of each subset of X which is saturated with respect to R is saturated with respect to R.

A subset Y of X is *saturated with respect to R* if $Y = \bigcup \{[x] : x \in Y$, $[x]$ is the equivalence class with respect to R containing $x\}$.

Then we can prove the following.

6.2. Proposition. Let a compact semigroup S act on a space X quasi-transitively. Then the quotient map $q_0 : X \to X/C_0$ is open.

Proof: Let A be a subset of X saturated with respect to C_o. We shall show that the closure $\bar{A}$ of A is also saturated with respect to C_o whence, by Proposition 6.1, it will follow that q_o is open. So let A be such that $A = \cup\{xS: x \in A\}$. We shall show that $\bar{A} = \cup\{xS: x \in \bar{A}\}$. Since the act is unitary $\bar{A} \subset \cup\{xS : x \in \bar{A}\}$. Conversely, if $x \in \bar{A}$, then we show that $xs \in \bar{A}$ for any $s \in S$. Since $x \in \bar{A}$, there exists a net $\{x_\alpha\}$ in A such that $x_\alpha \to x$, and, since A is saturated with respect to C_o, it follows that for any $s \in S$, $\{x_\alpha s\}$ is also a net in A. But $x_\alpha \to x$ and the action map is continuous, and hence, $x_\alpha s \to xs$ which implies that $xs \in \bar{A}$.

However, for a compact unitary act which is disjoint (respectively i-disjoint), the quotient map $q_d : X \to X/C_d$ (respectively $q_i : X \to X/C_i$) is, in general, not open as seen in the following example.

6.3. Example. Let $X = \{(0, y): 0 \leq y \leq 1\} \cup \{(x, 0) : 0 \leq x \leq 1\}$ considered as a subspace of the plane and $S = [0, 1]$ be the usual unit interval semigroup acting on X via the identity $(x, y)s = (xs, y)$ for all $(x, y) \in X$ and all $s \in S$. The maximal orbits are $(0, y)$ for $0 \leq y < 1$ and $\{(x, 0): 0 \leq x \leq 1\}$ which are also the maximal inverse-orbits. (X, S) is both disjoint and i-disjoint. Here the set

$A = \{(0, y) : 0 \leq y < 1\}$ is saturated with respect to both C_d and C_i but the closure $\bar{A}$ of A is not saturated with respect either C_d or C_i. Hence, by Proposition 6.4, neither q_d nor q_i is open.

We next give a sufficient condition for q_d (respectively q_i) to be open.

6.4. Proposition. Let (X, S) be a compact unitary disjoint act. The quotient map $q_d : X \to X/C_d$ is open if, whenever $\{x_\alpha\}$ is a net in X such that $x_\alpha S$ is a maximal orbit for all α, and $x_\alpha \to x$, then xS is also a maximal orbit.

Proof. Let $A = \cup\{xS : x \in A$ and xS is a maximal orbit$\}$ be a subset of X saturated with respect to C_d. We shall show that the closure $\bar{A}$ of A is also saturated with respect to C_d i.e., $\bar{A} = \bigcup\{xS: x \in \bar{A}$ and xS is a maximal orbit$\} = B$, say.

Let $y \in \bar{A}$. Then there exists a net $\{y_\alpha\}$ in A such that $y_\alpha \to y$, $y_\alpha = x_\alpha s_\alpha$, for $x_\alpha \in A$ and $s_\alpha \in S$, and $x_\alpha S$ is a maximal orbit for all α. By the compactness of (X, S), we can assume $x_\alpha \to x$ and $s_\alpha \to s$ and then, by the continuity of the act, $y = xs$. Since, by our hypothesis, xS is a maximal orbit and $x \in \bar{A}$, it follows that $y \in B$. So $\bar{A} \subset B$.

Next we show that $B \subset \bar{A}$. First note that, if $x \in \bar{A}$, then, for any $s \in S$, $xs \in \bar{A}$ which follows by an argument similar to that in the proof of Proposition 6.2 and the fact that (X, S)

is disjoint [cf. Proposition 3.4(5)]. Hence, in particular, if $x \in \bar{A}$ and xS is a maximal orbit, then $xS \subseteq \bar{A}$, and hence, $B \subseteq \bar{A}$.

This completes the proof by virtue of Proposition 6.1.

6.5. Proposition. Let (X, S) be a compact unitary i-disjoint act. The quotient map $q_i : X \to X/C_i$ is open if, whenever $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x$ and $y \in xS^{(-1)}$, then there exists $y_\alpha \in x_\alpha S^{(-1)}$ for each α such that $y_\alpha \to y$.

Proof: As before, let $A = \bigcup \{xS^{(-1)}: x \in A$ and $xS^{(-1)}$ is a maximal inverse-orbit$\}$. We shall show that $\bar{A} = \bigcup \{xS^{(-1)}: x \in \bar{A}$ and $xS^{(-1)}$ is a maximal inverse-orbit$\} = B$, say.

We first show that $\bar{A} \subseteq B$. For this our first claim is that if $y \in A$, then $yS \subseteq A$. Since, if $y \in A$, then, for some $x \in A$ such that $xS^{(-1)}$ is a maximal inverse orbit, $y \in xS^{(-1)}$ and, therefore, $yS^{(-1)} \subseteq xS^{(-1)}$. Again, if $yS \subseteq y'S$, a maximal orbit so that $y'S^{(-1)}$ is a minimal inverse-orbit and $y'S^{(-1)} \subseteq yS^{(-1)} \subseteq xS^{(-1)}$. Therefore, in view of Proposition 3.5(6), $yS \subseteq y'S \subseteq y'S \subseteq xS^{(-1)} \subseteq A$. Now, if $y \in \bar{A}$, then there exists a net $\{y_\alpha\}$ in A such that $y_\alpha \to y$. If $y \in xS^{(-1)}$, a maximal inverse-orbit, then $ys = x$ for some $s \in S$. Now $y_\alpha s \in A$ for all α and $y_\alpha s \to ys = x$. Hence, $x \in \bar{A}$ and $\bar{A} \subseteq B$.

We next show that $B \subseteq \bar{A}$. For, if $y \in xS^{(-1)}$, a maximal inverse-orbit for $x \in \bar{A}$, then, for some $s \in S$, $ys = x$. There

There exists a net $\{x_\alpha\}$ in A such that $x_\alpha \to x$ and, by our hypothesis and the fact that (X, S) is i-disjoint [cf. Proposition 3.5(5)], there exists $y_\alpha \in x_\alpha S^{(-1)}$ for each α such that $y_\alpha \in A$ and $y_\alpha \to y$. Therefore, $y \in \bar{A}$ and $B \subset \bar{A}$.

This completes the proof by virtue of Proposition 6.1.

For a compact unitary disjoint (respectively i-disjoint) act (X, S), the quotient map q_d (respectively q_i) is a closed map. If a compact semigroup S acts on a space X quasi-transitively, then we de not know whether in general, (if X is not compact), the quotient map q_o will be a closed map. Of course, if S happens to be compact group, then it is well-known that the quotient map q_o is a closed map and, in fact, a proper map (i.e., a closed map such that $q_o^{(-1)}(y)$ is compact for all $y \in X/C_o$) [cf. Propositions 1 and 2, pp.251-252, Bourbaki [9]]. Therefore, if S is left simple or S acts normally on X, then since, by Proposition 4.17(3), (X, S) is a homomorphic extension of a quasi-transitive action on X by a compact group, q_o must be proper. The following remark gives a sufficient condition for q_o to be a closed map.

6.6. Remark. Let a compact semigroup S act on a space X quasi-transitively. If the quotient map q_o is such that, whenever for a net $\{x_\alpha\}$ in X which has no converging sub-net, the net $\{q(x_\alpha)\}$ has no converging sub-net, then q_o is a closed map.

Proof: It is necessary and sufficient for q_o to be a closed map that $q_o(\bar{A}) = \overline{q_o(A)}$ for any subset A of X [cf. Proposition 9, p. 56, [9]] where $\bar{A}$ denotes the closure of A. By the continuity of q_o, $q_o(\bar{A}) \subset \overline{q_o(A)}$. Conversely, if $x \in \overline{q_o(A)}$, then there exists a net $\{x_\alpha\}$ in $q_o(A)$ such that $x_\alpha \to x$. Then there exists a net $\{y_\alpha\}$ in A such that $q_o(y_\alpha) = x_\alpha$ for all α. By our assumption and the fact that $x_\alpha \to x$, $y_\alpha \to y$ such that $q_o(y) = x$, and then, since $y \in \bar{A}$, $x \in q_o(\bar{A})$.

The following remark connects topological structures of X and X/C_o for a quasi-transitive act (X, S).

6.7. Remark. Let a compact semigroup S act on a space X quasi-transitively.

(1) X/C_o is discrete iff each orbit xS is open (and hence clopen) in X.

(2) If the quotient map q_o is proper, then X is compact (respectively locally compact) iff X/C_o is compact (respectively locally compact).

Proof: (1) Trivial

(2) Follows from the corollary to Proposition 9, pp. 105-106, Bourbaki [9] [cf. Corollary 1 of Proposition 2, p. 252, Bourbaki [9]].

In the rest of this section we are concerned with the following problem.

6.8. Problem. Let a compact semigroup S act on a space X quasi-transitively. How are the dimensions of X, S and X/C_0 related ?

We do not attempt to solve this general problem in this dissertation. It may, however, be noted that the work of Stadlander [40] is closely related to this problem. We make some remarks giving some sufficient conditions for the equality of dimensions of X and X/C_0 when both are metric spaces and we consider Lebesgue covering dimension (dim) which is same thing as the strong inductive dimension (Ind) for metric spaces. We refer to the books of Nagata [35] and Nagami [34] for dimension theory.

First we quote a few facts from Nagata [35] for ready reference.

In what follows U and V are two metric spaces. Let f be a continuous map from U into V. A point q of $f(U)$ is called an unstable value of f if for every $\epsilon > 0$ there exists a continuous map g from U into V such that

$$\rho(f(p), g(p)) < \epsilon \quad \text{for every } p \in U,$$

$$g(U) \subset V - \{q\},$$

where we denote by ρ the metric of V.

Let I^{n+1} denote the $(n+1)$-dimensional unit cube i.e.,

$$I^{n+1} = \{ (x_1, \ldots, x_{n+1}) : |x_i| \leq 1, \quad i = 1, 2, \ldots, n+1 \}.$$

Then the following gives a characterization of dimension of a space.

6.9. Proposition [Theorem III.1, p. 52, Nagata [35]]. A space U has dimension $\leq n$ iff all values of every continuous map from U into I^{n+1} are unstable.

We shall also need the following result.

6.10. Proposition [Theorem III.6, p. 63, Nagata [35]]. Let f be a continuous closed map from U onto V such that $\dim f^{-1}(q) \leq k$ for every $q \in V$. Then $\dim U \leq \dim V + k$.

Now we can prove the following.

6.11. Proposition. Let f be a continuous closed map from U onto V. If $\dim f^{-1}(q) \leq 0$ for every $q \in V$ and there exists a continuous inverse $f^{(-1)}$ of f (i.e., $f^{(-1)}$ is a continuous map from V into U such that $f^{-1}(f(p)) = p$ for all $p \in U$), then $\dim U = \dim V$.

Proof: By virtue of Proposition 6.10, $\dim V \geq \dim U$. Now we show that $\dim V \leq \dim U$. Let $\dim U = n$. If f_1 is a continuous map from V into I^{n+1}, then $f_2 = f_1 \circ f$ is a continuous map from U into I^{n+1}. Let $q \in f_1(V) = f_2(U)$. Then, by Proposition 6.9, given $\epsilon > 0$, there exists a continuous map g from U into I^{n+1} such that

$$\rho(f_2(p),\ g(p)) < \epsilon \quad \text{for every} \quad p \in U,$$

$$g(U) \subset I^{n+1} - \{q\}$$

where ρ denotes the metric of I^{n+1}.

Now $h = g \circ f^{(-1)}$ is a continuous map from V into I^{n+1} such that

$$\rho(f_1(p),\ h(p)) = \rho(f_2(f^{(-1)}(p)),\ g(f^{(-1)}(p))) < \epsilon$$
$$\text{for every} \quad p \in V,$$

$$h(V) = g(U) \subset I^{n+1} - \{q\} .$$

Therefore, by Proposition 6.9, $\dim V \leq n = \dim U$.

Hence, $\dim U = \dim V$.

There are two more sufficient conditions for $\dim U = \dim V$ which we quote from Nagami [34].

6.12. Proposition. Let f be a continuous closed map from U **onto** V.

(1) If $f^{-1}(q)$ consists of k points $(k < \infty)$ for all $q \in V$, then $\dim U = \dim V$. [cf. Lemma 12-5 [Suzuki], p.73, [34]].

(2) If the boundary of $f^{-1}(q)$ is not dense-in-itself and $\dim f^{-1}(q) \leq 0$ for all $q \in V$, then $\dim U = \dim V$ [cf. Theorem 15-6, p. 97 [34]].

In view of the above results we can state some sufficient conditions for $\dim X = \dim X/C_0$ where (X, S) is a quasi-transitive act. For example, in view of Proposition 6.11, we can state the following.

6.13. Proposition. Let a compact semigroup S act on a metric space X quasi-transitively such that X/C_o is a metric space and the quotient map q_o $X \to X/C_o$ is closed. (If X is a compact metric space, then these conditions hold). If $\dim xS \leq 0$ for all $x \in X$ and q_o admits of a continuous inverse, then $\dim X = \dim X/C_o$.

Similar statements can be made concerning disjoint and i-disjoint acts.

In this section we have merely touched upon a general problem concerning quotient acts which we formulate below. This is analogous to a general problem concerning semigroups about which a considerable amount of work has been done and has been reviewed in a recent paper by Carruth [10].

6.14. Problem [cf. Problems 7 and 8 [10]].

Let (X, S) be an act and C a congruence on X (C may be any one of C_o, C_d or C_i) such that the canonical quotient act (X/C, S) is defined. When is it possible to make conclusions about the topological properties of X, topological and/or algebraic properties of S, or the action itself if the structure of X/C is known ?

7. Products of Disjoint (respectively i-Disjoint or Quasi-transitive) Acts.

Let $\{(X_i, S_i)\}$ and $\{(X_i, S)\}$ be two families of acts. In Section 1.3 we have defined the product acts $(\prod X_i, \prod S_i)$ and $(\prod X_i, S)$. In this section we examine how does a product act inherit a given property P from the component acts where P may be disjointness, i-disjointness, quasi-transitivity, etc. of acts.

We first study the product act $(\prod X_i, \prod S_i)$.

7.1. Lemma. Let $\{S_i\}$ be a family of compact semigroups. Then $\prod K_i$ is the minimal ideal of $\prod S_i$ iff K_i is the minimal ideal of S_i for each i.

Proof: This follows from the fact that K is the minimal ideal of a compact semigroup S iff K is an ideal of S such that $K = Kak$ for all $a \in K$ and the fact that, for arbitrary families of sets $\{A_r\}$ and $\{B_r\}$, $r \in \Gamma$, $\prod A_r = \prod B_r$ iff $A_r = B_r$ for all $r \in \Gamma$.

7.2. Lemma. $(\prod X_i, \prod S_i)$ is unitary (respectively point-transitive) iff (X_i, S_i) is unitary (respectively point-transitive) for each i.

Proof: Trivial.

Then we have the following result.

7.3. Proposition. Let $\{S_i\}$ be a family of compact semigroups. Then $(\prod X_i, \prod S_i)$ is quasi-transitive (respectively transitive) iff (X_i, S_i) is quasi-transitive (respectively transitive) for each i. Further, the quotient act $(\prod X_i/C_o, \prod S_i)$ is iseomorphic to the product $(\prod(X_i/C_o^i), \prod S_i)$ of the quotient acts $(X_i/C_o^i, S_i)$ where C_o and C_o^i are the closed congruences on $\prod X_i$ and X_i induced by the quasi-transitive actions of $\prod S_i$ and S_i respectively.

Proof: The first part for quasi-transitive (respectively transitive) case follows from Lemma 7.1 and 7.2 and Proposition 4.10(6) (respectively Proposition 4.20(4)).

The second part follows from Proposition 6.2 and a well-known fact which is the corollary to Proposition 8, p. 55 of Bourbaki [9].

7.4. Lemma. Let $\{(X_i, S_i)\}$ be a family of compact acts. Then, for $(x_i) \in \prod X_i$, $(x_i)\prod S_i$ is a maximal (respectively minimal) orbit of $(\prod X_i, \prod S_i)$ iff $x_i S_i$ is a maximal (respectively minimal) orbit of (X_i, S_i) for each i.

Proof: Follows from the definition of a maximal (respectively minimal) orbit and the facts that for arbitrary families of sets $\{A_r\}$, $\{B_r\}$, $r \in \Gamma$, (1) $\prod A_r \subset \prod B_r$ iff $A_r \subset B_r$ for each $r \in \Gamma$ and (2) $\prod A_r = \prod B_r$ iff $A_r = B_r$ for each $r \in \Gamma$.

Then we have the following result.

7.5. Proposition. Let $\{(X_j, S_j)\}$ be a family of compact unitary acts. Then $(\prod X_j, \prod S_j)$ is disjoint (respectively i-disjoint) iff (X_j, S_j) is disjoint (respectively i-disjoint) for each j. Further, if the quotient map $q_d^j : X_j \to X_j / C_d^j$ (respectively $q_i^j : X_j \to X_j / C_i^j$) is open for each j, then the quotient act $(\prod X_j / C_d, \prod S_j)$. (respectively $(\prod X_j / C_i, \prod S_j)$) is iseomorphic to the product $(\prod (X_j / C_d^j), \prod S_j)$ (respectively $(\prod (X_j / C_i^j), \prod S_j)$) of the quotient acts $(X_j / C_d^j, S_j)$ (respectively $(X_j / C_i^j, S_j)$)

where C_d and C_d^j (respectively C_i and C_i^j) are the closed congruences on $\prod X_j$ and X_j induced by the disjoint (respectively i-disjoint) actions of $\prod S_j$ and S_j respectively.

Proof: The first part for disjoint (respectively i-disjoint) case follows from Lemma 7.4 and Proposition 3.4(5) (respectively Proposition 3.5(4)).

The second part follows from the corollary to Proposition 8, p. 55 of Bourbaki [9].

While the product act $(\prod X_i, \prod S_i)$ inherits from the component acts (X_i, S_i) the properties mentioned in the beginning of this section, it is not so for the product act $(\prod X_i , S)$ as seen in the following examples.

7.6. Example. Let S be the usual unit interval semigroup and S act on itself by its multiplication. Then (S, S) is a compact unitary disjoint act (in fact, having only one maximal orbit). However, the product act $(S \times S, S)$ is not disjoint which can be easily seen.

7.7. Example. Let $S = [0, 1]$ be the usual unit interval with right-zero multiplication, i.e., $xy = y$ for all $x, y \in S$, and S act on itself by its multiplication. Then (S, S) is a transitive act and, hence, a quasi-transitive act. However, the product act $(S \times S, S)$ is not quasi-transitive and, hence, not transitive (in fact, not even an onto act) which can be easily verified.

In fact, without some restriction on the acts (X_i, S) we can not say anything about $(\prod X_i, S)$.

First we note the following fact about transformation group (or, in short, TG).

7.8. Proposition. Let $\{(X_i, S)\}$ be a family of acts. Then $(\prod X_i, S)$ is a TG iff (X_i, S) is a TG for all i.

Proof: Let, for any $s \in S$, $\rho_s : \prod X_i \to \prod X_i$ be defined by $\rho_s((x_i)) = (x_i s)$ for all $(x_i) \in \prod X_i$ and $\rho_s^i : X_i \to X_i$ by $\rho_s^i(x_i) = x_i s$ for all $x_i \in X_i$. Then, since $\rho_s((x_i)) = (\rho_s^i(x_i))$, the result follows from the fact that ρ_s is a homeomorphism iff ρ_s^i is a homeomorphism for all i.

As a corollary to Proposition 7.8 we can state the following which expresses an analogue of Proposition 7.3 and which holds for a large class of semigroup acts [cf. Proposition 4.17(3)].

7.9. Corollary. Let $\{(X_i , S)\}$ be a family of acts which are homomorphic extensions of group acts. Then $(\prod X_i, S)$ is quasi-transitive iff (X_i , S) is quasi-transitive for each i.

Proof: Let, for each i, (X_i, S) be a homomorphic extension of (X_i, G) where G is a group. Then $(\prod X_i, S)$ is a homomorphic extension of $(\prod X_i , G)$, and hence, the result follows from Proposition 7.8.

8. On Homomotphisms of Acts.

Throughout this section we let h to be a homomorphism from a compact unitary act (X, S) onto a compact unitary act (Y, S), that is, h is a map from X onto Y, which need not be continuous, such that $h(xs) = h(x)s$ for all $x \in X$ and all $s \in S$. Compactness is assumed to guarantee the existence of maximal and minimal orbits (and inverse-orbits). We investigate how h maps each maximal (minimal) orbits (inverse-orbits) or a disjoint (i-disjoint) act onto a maximal (minimal) orbit (inverse orbit) or a disjoint (i-disjoint) act respectively. This section is mainly algebraic.

Clearly, h maps an orbit onto an orbit. Regarding maximal orbits we have a few results.

8.1. Proposition. Every maximal orbit yS of (Y, S) is h-image of some maximal orbit xS of (X, S).

Proof. For any maximal orbit yS if $x \in h^{-1}(y)$ then $h(xS) = h(x)S = yS$. If $xS \subset x'S$, a maximal orbit, then $h(xS) = yS \subset h(x')S$. Now maximality of yS implies that $h(x')S = yS$.

8.2. Proposition. h maps each maximal orbit of (X, S) onto a maximal orbit of (Y, S) if, for any two maximal orbits x_1S and x_2S of (X, S), $x_1S \neq x_2S$ implies that neither $h(x_1)S \not\subset h(x_2)S$ nor $h(x_2)S \not\subset h(x_1)S$.

Proof. Let $x_1 S$ be a maximal orbit of (X, S). If $h(x_1)S$ is not a maximal orbit of (Y, S), then, by Proposition 8.1, there exists a maximal orbit $x_2 S$ of (X, S) such that $h(x_2)S$ is a maximal orbit of (Y, S) and $h(x_1)S \subset h(x_2)S$. But this implies that $x_1 S = x_2 S$, and hence, $h(x_1)S = h(x_2)S$. This completes the proof.

8.3. Proposition. h maps each maximal orbit of (X, S) onto a maximal orbit of (Y, S) if, for any $x_1, x_2 \in X$, $C = h(x_1)S \cap h(x_2)S \neq \emptyset$ implies that, if $C \subsetneq h(x_2)S$, then $C \subsetneq h(x_3)S$ for some $x_3 \in X$ such that $x_1 S \subset x_3 S$.

Proof. Suppose for some maximal orbit $x_1 S$ of (X, S) $h(x_1)S \subset h(x_2)S$, a maximal orbit of (Y, S) which corresponds to a maximal orbit $x_2 S$ of (X, S) by virtue of Proposition 8.1 So $C = h(x_1)S$ and either $C = h(x_2)S$ or $C \subsetneq h(x_2)S$. The latter case can not happen as then $C \subsetneq h(x_3)S$ for some $x_3 \in X$ such that $x_1 S \subset x_3 S$; but then $x_1 S = x_3 S$ and $C = h(x_1)S = h(x_3)S$.

So $h(x_1)S = h(x_2)S$.

8.4. Corollary. If h is 1-1, then h maps each maximal orbit of (X, S) onto a maximal orbit of (Y, S).

Proof. If h is 1-1, then we show that the hypothesis of Proposition 8.3 is satisfied. Let, for $x_1, x_2 \in X$, $C = h(x_1)S \cap h(x_2)S \neq \emptyset$. Then, if h is 1-1, $h^{-1}(C) = x_1 S \cap x_2 S$. If $C \subsetneq h(x_2)S$, then $h^{-1}(C) \subsetneq x_2 S$ and either (i) $h^{-1}(C) = x_1S$ or (ii) $h^{-1}(C) \subsetneq x_1S$. If (i) holds, then $x_1S \subset x_2S$ and we take x_3 of Proposition 8.3 as x_2. If (ii) holds, then we take x_3 of Proposition 7.3 as x_1. Thus the assertion of the corollary is proved.

Regarding disjoint acts we have the following results.

8.5. Proposition. h maps a disjoint act (X, S) onto a disjoint act (Y, S) if, for any $y \in Y$, $h^{-1}(y) = xA$ for some $x \in X$ and $\emptyset \neq A \subset S$.

Proof. Let, if possible, two maximal orbits y_1S and y_2S of (Y, S) interest. Then, by Proposition 8.1, suppose x_1S and x_2S are two maximal orbits of (X, S) such that $h(x_i)S = h(y_i)S$, $i = 1,2$. Now, for $y \in y_1S \cap y_2S \neq \emptyset$, $h^{-1}(y) \cap x_iS \neq \emptyset$, $i = 1,2$. Then as (X, S) is disjoint, $h^{-1}(y) = xA$ for some $x \in X$ and $\emptyset \neq A \subset S$ iff $h^{-1}(y)$ is contained in a unique maximal orbit and, so, $h^{-1}(y) \subset x_iS \cap x_2S$ which implies that $x_1S = x_2S$. Hence, $y_1S = y_2S$.

We now give an example which illustrates that, without the conditions assumed in Propositions 8.2 and 8.4, the conclusions of these propositions are, in general, not valid. This also illustrates that the converse of Proposition 8.5 is not true.

8.6. Example. Let $X = \{(0, y): 0 \leq y \leq 1\} \cup \{(1, y): 0 \leq y \leq \frac{1}{2}\}$ and $X' = \{(0, y): 0 \leq y \leq 1\}$ be considered as subspaces of the plane. Let $S = [0, 1]$, with usual multiplication, act on

X (and X') as follows: For $(x, y) \in X$ (or X') and $s \in S$, $(x, y)s = (x, ys)$ where ys denotes the usual product. Then the map $h : X \to X'$ defined by $h(x, y) = (0, y)$ for all $(x, y) \in X$ defines a homomorphism from (X, S) onto (X', S). It is easily seen that h does not map each maximal orbit of (X, S) onto a maximal orbit of (X', S) and the hypotheses of Propositions 8.2 and 8.4 are not true. It is also easy to see that the hypothesis of Proposition 8.5 is not true but even then h maps the disjoint act (X, S) onto the disjoint act (X', S).

The following gives a necessary and sufficient condition for a homomorphic image of an act to be disjoint.

8.7. Proposition. h maps an act (X, S) onto a disjoint act (Y, S) iff, for any $y \in Y$, there exists an orbit xS of (X, S) such that $h^{-1}(y) \cap xS \neq \emptyset$ and whenever $x_\alpha S$ is an orbit of (X, S) such that

$$h^{-1}(y) \cap x_\alpha S \neq \emptyset, \quad h(x_\alpha)S \subset h(x)S.$$

Proof: 'If part'. Let, if possible, y_1S and y_2S be two maximal orbits of (Y, S) which intersect. If $y \in y_1S \cap y_2S$, then, by Proposition 8.1, there exist maximal orbits x_1S and and x_2S of (X, S) such that $h(x_i)S = y_iS$, $i = 1,2$ and $h^{-1}(y) \cap x_1S \cap x_2S \neq \emptyset$. But then, by the hypothesis, we must have some orbit xS of (X, S) such that $h^{-1}(y) \cap xS \neq \emptyset$ and $y_iS \subset h(x)S$, $i = 1,2$ which implies, by the maximality of y_iS, $i = 1,2$, that $y_1S = y_2S$.

'Only if part'. Let $y \in Y$ and $F = \{x_\alpha S : h^{-1}(y) \cap x_\alpha S \neq \emptyset\}$. Let $F' = \{x'_\alpha S : x'_\alpha S$ is a maximal orbit of (X, S) such that $h(x'_\alpha)S$ is a maximal orbit containing $h(x_\alpha)S$ for each $x_\alpha S \in F\}$. Since $y \in h(x'_\alpha)S$, for all α, $h^{-1}(y) \cap x'_\alpha S \neq \emptyset$. But, for any $x'_{\alpha_i} S \in F'$, $i = 1, 2$, $h(x'_{\alpha_1})S = h(x'_{\alpha_2})S$ as (Y, S) is disjoint. Now, by definition of F and F', for any $x_\alpha S \in F$, the corresponding $x'_\alpha S \in F'$ which may be equal to $x_\alpha S$ is such that $h(x_\alpha)S \subset h(x'_\alpha)S$, and hence, as $h(x'_{\alpha_1})S = h(x'_{\alpha_2})S$ for any $x'_{\alpha_i} S \in F'$, $i = 1, 2$, the only if part follows.

8.8. Proposition. Let h be a homomorphism from (X, S) onto (Y, S). Then the following two statements are equivalent.

(1) (Y, S) is disjoint and h maps each maximal orbit of (X, S) onto a maximal orbit of (Y, S).

(2) For any two maximal orbits $x_i S$, $i = 1, 2$, of (X, S), $\bigcap_{i=1}^{2} h(x_i)S \neq \emptyset$ implies that $h(x_1)S = h(x_2)S$.

Proof. (1) => (2). Trivial.

(2) =>(1). Suppose $y_i S$, $i = 1, 2$, are any two maximal orbits of (Y, S) which intersect. By Proposition 8.1 suppose $x_i S$, $i = 1, 2$, are two maximal orbits of (X, S) such that $h(x_i)S = y_i S$, $i = 1, 2$. Then $\bigcap_{i=1}^{2} y_i S \neq \emptyset$ implies that $y_1 S = y_2 S$.

To show that h maps each maximal orbit onto a maximal orbit let xS be a maximal orbit of (X, S). Let $h(x)S \subset yS$, a maximal orbit of (Y, S), and let $x_1 S$ be a maximal orbit of

(X, S) such that $h(x_1)S = yS$. Now $h(x)S \cap h(x_1)S \neq \emptyset$ implies that $h(x)S = h(x_1)S = yS$.

Concerning minimal orbits we have the following two results.

8.9. Proposition.

(1) h maps each minimal orbit of (X, S) onto a minimal orbit of (Y, S).

(2) Each minimal orbit of (Y, S) is h-image of some minimal orbit of (X, S).

Proof. (1). Suppose xS is a minimal orbit of (X, S). Suppose $yS \subset h(x)S$ for some $y \in Y$. Then for any $s \in S$, there exists $s' \in S$ such that $ys = h(x)s' = h(xs')$. Now $z = xs'$ implies that $zS = xS$ as xS is minimal and so $h(z)S = h(x)S$. Also since $h(z) = ys$, $h(z)S \subset yS$ and so $h(x)S = h(z)S \subset yS \subset h(x)S$. So $h(x)S = yS$.

Proof. (2). Let yS be a minimal orbit of (Y, S). Let $x \in h^{-1}(y)$. So $h(x)S = yS$ and, if $x'S$ is a minimal orbit contained in xS, then $h(x'S) \subset h(x)S = yS$, which implies that $h(x'S) = yS$.

8.10. Corollary. A homomorphic image of a quasi-transitive (transitive) act is quasi-transitive (transitive).

We next consider maximal inverse-orbits and homomorphisms.

8.11. Proposition. Every maximal inverse-orbit $yS^{(-1)}$ of (Y, S) is h-image of a union of maximal inverse-orbits $\{x_\alpha S^{(-1)}\}$ of (X, S) such that $h(x_\alpha)S = yS$.

Proof. Notice that $yS^{(-1)}$ is a maximal inverse-orbit iff yS is a minimal orbit and, by Proposition 8.9(2), there exists a minimal orbit in (X, S) whose h-image is yS. So suppose $\{x_\alpha S\}$ are all the minimal orbits of (X, S) such that $h(x_\alpha)S = yS$. We claim that $yS^{(-1)} = \cup h(x_\alpha S^{(-1)})$. Note that $h(xS^{(-1)}) \subset h(x)S^{(-1)}$ for any $x \in X$ and $h(x_\alpha)S = yS$ iff $h(x_\alpha)S^{(-1)} = yS^{(-1)}$. Therefore, $h(x_\alpha S^{(-1)}) \subset yS^{(-1)}$, and hence, $\cup h(x_\alpha S^{(-1)}) \subset yS^{(-1)}$. Conversely, let $z \in yS^{(-1)}$. Then, for some $x \in X$, $h(x) = z$, as h is onto and there is $s \in S$ such that $h(x)s = y$ and $h(x)sS = yS$. There exists a minimal orbit $x'S \subset xsS$ so that $h(x'S) \subset h(x)sS = yS$. Now $x' = xst$ for some $t \in S$ and so $x \in x'S^{(-1)}$. So $h(x) = z \in h(x'S^{(-1)}) \subset \cup h(x_\alpha S^{(-1)})$.

8.12. Proposition. (Y, S) is i-disjoint iff for any two maximal inverse-orbits $x_i S^{(-1)}$, $i = 1, 2$, of (X, S) $\bigcap h(x_i S^{(-1)}) \neq \emptyset$ implies that $h(x_1)S = h(x_2)S$.

Proof. Suppose $y_i S^{(-1)}$, $i = 1, 2$, are any two maximal inverse-orbits of (Y, S) which intersect. Then, by Proposition 8.11, there exist maximal inverse-orbits $x_i S^{(-1)}$ of (X, S) such that $h(x_i)S = y_i S$, $i = 1, 2$ and $\bigcap h(x_i S^{(-1)}) \neq \emptyset$ when it follows that $y_1 S = y_2 S$.

Conversely, let, for any two maximal inverse-orbits $x_i S^{(-1)}$, $i = 1, 2$, of (X, S), $\cap h(x_i S^{(-1)}) \neq \emptyset$. Then $\cap h(x_i) S^{(-1)} \neq \emptyset$ as $h(xS^{(-1)}) \subset h(x)S^{(-1)}$ for any $x \in X$. Then as (Y, S) is i-disjoint, by Proposition 3.5(2), it follows that $\cap h(x_i) S \neq \emptyset$, and hence, by Proposition 8.9(1), $h(x_1)S = h(x_2)S$.

In general, $h(xS^{(-1)}) \subset h(x)S^{(-1)}$ for any $x \in X$ and $h(xS^{(-1)}) = h(x)S^{(-1)}$ iff for any $a \in h(x)S^{(-1)}$, $h^{-1}(a) \cap xS^{(-1)} \neq \emptyset$. The following gives a sufficient condition for the latter to happen in case of maximal inverse-orbits.

8.13. <u>Proposition.</u> h maps each maximal inverse-orbit of (X, S) onto a maximal inverse-orbit of (Y, S) if for any two maximal inverse-orbits $x_i S^{(-1)}$, $i = 1, 2$ of (X, S), $\cap h(x_i S^{(-1)}) \neq \emptyset$ implies that $h(x_1 S^{(-1)}) = h(x_2 S^{(-1)})$.

<u>Proof.</u> Let $xS^{(-1)}$ be a maximal inverse-orbit of (X, S). Then xS is a minimal orbit of (X, S), $h(x)S$ is a minimal orbit of (Y, S), by Proposition 8.9(1), and so $h(x)S^{(-1)}$ is a maximal inverse-orbit of (Y, S) such that $h(xS^{(-1)}) \subset h(x)S^{(-1)}$. Now, by Proposition 8.11, $h(x)S^{(-1)} = \cup\{h(x_\alpha S^{(-1)}) : x_\alpha S$ is a minimal orbit and $h(x_\alpha)S = h(x)S\}$ and, for any α, β such that $x_\alpha S$ and $x_\beta S$ are minimal orbits and $h(x_\alpha)S = h(x_\beta)S = h(x)S$, by Proposition 2.7(5), since $x_\alpha S \subset x_\alpha S^{(-1)}$, $x_\beta S \subset x_\beta S^{(-1)}$, $h(x)S \subset h(x_\alpha S^{(-1)}) \cap h(x_\beta S^{(-1)})$ which implies that

$h(x_\alpha S^{(-1)}) = h(x_\beta S^{(-1)})$. Therefore, $h(x S^{(-1)}) = h(x)S^{(-1)}$.

8.14. Proposition. Let (X, S) be disjoint. Then (Y, S) is i-disjoint if, for any two maximal inverse-orbits $x_1 S^{(-1)}$ of (X, S) that intersect, $h(x_1 S^{(-1)}) = h(x_2 S^{(-1)})$. If h maps each maximal inverse-orbit onto maximal inverse-orbit, then this condition is also necessary.

Proof. In view of Proposition 3.5(8), it is sufficient to show that any maximal inverse-orbit $yS^{(-1)}$ of (Y, S) is a union of orbits. By Proposition 8.11, $yS^{(-1)} = \bigcup h(x_\alpha S^{(-1)})$ where $x_\alpha S$ are all the minimal orbits of (X, S) such that $h(x_\alpha)S = yS$. Since (X, S) is disjoint, by Proposition 3.4(6), each maximal orbit xS is a union of maximal inverse-orbits corresponding to the minimal orbits contained in xS, and then, by the condition of the Proposition, if $xS = \bigcup x_\beta S^{(-1)}$ then, since $x \in \bigcap x_\beta S^{(-1)}$, it follows that $h(x S) = h(x_\beta S^{(-1)})$ for each β. This implies that for any maximal inverse orbit $x_\alpha S^{(-1)}$ there exist a maximal orbit $x^\alpha S$ such that $h(x^\alpha S) = h(x_\alpha S^{(-1)})$. So, if $yS^{(-1)} = \bigcup h(x_\alpha S^{(-1)})$, from the disjointness of (X, S) and the condition of the Proposition it follows that there exist maximal orbits $\{x^\alpha S\}$ such that $\bigcup h(x^\alpha S) = \bigcup h(x_\alpha S^{(-1)}) = y S^{(-1)}$ which is a union of orbits.

To prove the other way suppose (Y, S) is i-disjoint and h maps each maximal inverse-orbit onto a maximal inverse-orbit.

Each maximal inverse-orbit of (Y, S) is a union of maximal orbits by Proposition 3.5(7). Suppose two maximal inverse-orbits $x_i S^{(-1)}$, $i = 1, 2$, of (X, S) intersect. As (X, S) is disjoint, by Proposition 3.4(6), $\cup x_i S^{(-1)} \subset xS$, a maximal orbit. So $\cup h(x_i S^{(-1)}) \subset h(x) S \subset yS$, a maximal orbit. As (Y, S) is i-disjoint yS is contained in some maximal inverse orbit $y' S^{(-1)}$, and since, $h(x_i S^{(-1)}) = h(x_i) S^{(-1)}$, is a maximal inverse-orbit for $i = 1, 2$, it follows that $\cup h(x_i) S^{(-1)} \subset y' S^{(-1)}$, and hence, $h(x_i) S^{(-1)} = y' S^{(-1)}$. Thus $h(x_1 S^{(-1)}) = h(x_2 S^{(-1)})$.

CHAPTER II

ON SOME CLASSES OF TOPOLOGICAL MACHINES

1. Introduction and Summary

In this section we introduce the concept of a topological machine, give several examples and give a brief summary of the results presented in the subsequent sections.

1.1. Topological Machine. Let X be a nonvoid Hausdorff space and S and T be any two topological semigroups whose operations will be denoted by juxtaposition. A *topological machine* [43] M, denoted by a five-tuple $M = \langle X, S, T, f, g \rangle$, is defined by two functions f and g, $T \xleftarrow{g} X \times S \xrightarrow{f} X$, both continuous with respect to the product topology on $X \times S$, satisfying the following two axioms.

$$\text{A1. } f(x, s_1 s_2) = f(f(x, s_1), s_2)$$

$$\text{A2. } g(x, s_1 s_2) = g(x, s_1) g(f(x, s_1), s_2)$$

for all $x \in X$ and all $s_1, s_2 \in S$. X, S and T are referred to as the *state space*, the *input semigroup* and the *output semigroup* respectively. The two functions f and g are referred to as the *state-transition* (or *next state*) *function* and the *output function* respectively [cf. 2, 6, 21, 23]. The function f satisfying A1 was termed an *act* in Chapter I and we shall continue to do so referring to f as the action map. We shall also suppress the explicit mention of f and denote an act by the pair (X, S) using juxtaposition for the action map (as well as for semigroup operation) unless otherwise necessary to mention f explicitly. We shall also refer to an output function simply by the term op-function and, understanding that the underlying act (X, S) is given, a machine will be referred to only by an op-function $g : X \times S \to T$ for some semigroup T. By the term S-machine we shall mean a machine whose underlying act is such that the state space is same as the input semigroup which acts on itself by its multiplication. We shall use the term algebraic (or discrete) machine if no topology is considered and, in the sequel, the term machine will always refer to a topological machine unless stated otherwise explicitly. All spaces are assumed to be Hausdorff spaces and all semigroups (and groups) to be topological unless stated otherwise.

Returning back to the axioms postulated in the definition of a machine we would like to point out that often the following two additional axioms are also postulated though the axioms A1 and A2 are really the basic ones.

A3. There exists an identity u of S such that
$xu = x$ for all $x \in X$.

A4. If A3 holds, then g is constant restricted
to $X \times \{u\}$.

A machine (respectively an act) satisfying A3 and A4 (respectively A3) may be referred to as a machine (*respectively an act*) *with identity u*.

It is worthwhile to note the following simple consequences of A3 and A4.

1.2. Remark. (1) If A3 holds, then $g(x, u)$ is an idempotent of T for each $x \in X$.

(2) If A4 holds, then $g(X \times \{u\})$ **acts** as an identity of $g(X \times S)$.

Proof. (1) Since, by A2, $g(x, u) = g(x, uu) = g(x,u)g(x, u)$, (1) follows.

(2) Let $v = g(X \times \{u\})$ and $t \in g(X \times S)$.

Then $vt = vg(x, s)$, for some $(x, s) \in X \times S$,

$= g(x, u)g(x, s) = g(x, us) = g(x, s) = t$.

Also $tv = g(x, s)g(x, u)$

$= g(x, s)g(xs, u) = g(x, su) = g(x, s) = t$.

In view of the Remark 1.2, we may as well replace A4 by the following:

A4'. If A3 holds and v is the identity of T, then $g(x, u) = v$ for all $x \in X$.

A machine can be viewed as a mathematical model describing the external characteristics, namely the input-output **behaviour**, of physical systems like, for example, a computer, a vending machine or a tax display device etc. to name only a few, where the set X may correspond to the various internal states of the system, S to the set of possible inputs (programs, commands etc.) and T to the set of possible outputs (the results displayed). Then $g(x, s)$ is the output that we will have if the machine is in the state x and the input s is given; or we can think of $g(x, s)$ as the output resulting from the machine's transition from the state x to the state xs. We do not expect an output unless some action takes place and that means some input must be given. If the machine is idling in a state x, we do not expect any output. In the light of this discussion A2 can be interpreted as follows: if the machine is in the state x and the input s_1 is followed by the input s_2, then it is reasonable to expect the output $g(x, s_1)$ to be followed by the output $g(xs_1, s_2)$ i.e., $g(x, s_1 s_2) = g(x, s_1)g(xs_1, s_2)$ A3 means that there is one input u which leaves the machine in its current state x no matter what x is and A4 means that the output is the same, whenever u is the input, no matter what state x the machine is in. A3 does not seem to be unreasonable, and if A3 holds, it seems equally reasonable to assume A4.

We now list a few examples of machines.

1.3. Examples (1) Algebraic Machines. The classical concept of a sequential machine [2, 6, 21, 23] which models a sequential switching circuit, a basic component of all electronic digital machines, is a special type of algebraic machine where the state space X is a finite set, the input semigroup S and the output semigroup T are free monoids generated by some finite input alphabet and output alphabet respectively and the axioms A1, A2, A3 and A4' are satisfied. Ginsburg's *quasi-machines* and *abstract machines* [20, 21] are generalizations of complete sequential machines. A *quasi-machine* is an algebraic machine where the state space X need not be finite and the semigroups S and T may be any arbitrary ones. An *abstract machine* is a quasi-machine where the output semigroup T is left-cancellative.

(2) *Analog Computers*. Topological machines are not merely topologized quasi-machines but can be regarded as appropriate mathematical models for describing the behaviour of a large class of physical devices called *analog* or *continuous computers* [cf. Jackson [25]]. A simple example of an analog computer is that of an electric clock where the position of the hand changes continuously with time. The clock integrates with respect to time the angular velocity of the motor shaft and obtains a smooth, continuously changing angular displacement of the hands. The operation of this clock can be modelled by a machine as follows. Let X denote the set of possible angular displacements (states) of the hands from an initial position and so X can be taken to be $X = [0, \infty)$. Let S denote the time scale; again, $S = [0, \infty)$ with usual addition. The dial of the clock can be thought of as the unit circle and so, if we take T as the circumference of the unit circle with

right-zero multiplication i.e., $xy = y$ for all $x, y \in T$,
the machine model of the clock can be take to be the following:
$T \xleftarrow{g} X \times S \xrightarrow{f} X$ where $f(x, s) = x + vs$ where v is the (constant) angular velocity, x is the initial (angular) position of the hand and $f(x, s)$ is the position after an interval of time s. What we observe is the position of the tip of the hand on the dial and $g(x, s) = e^{i(x+vs)}$ gives the position of the tip of the hand on the dial when the angular position is $x + vs$.

Apart from these concrete examples, we list a few more mathematical ones which also can be conceived of as suitable models for some physical devices.

(3) Let (X, S) be an act, T a semigroup and v an idempotent of T. Then $g : X \times S \to T$, defined by, $g(x, s) = v$ for all $(x, s) \in X \times S$ is an op-function. Here the device just changes from the state x to the state xs but gives a constant output.

(4) Let (X, S) be an act, T a semigroup which may be the same as S, and $h : S \to T$ a homomorphism, then $g : X \times S \to T$, defined by, $g(x, s) = h(s)$ for all $(x, s) \in X \times S$, is an op-function.

A slight generalization of (4) is the following.

(5) Let (X, S) be an act such that $xS = x$ for all $x \in X$. Let T be a semigroup. Then a continuous map

$g : X \times S \to T$, is an op-function iff $g(x, s) = h_x(s)$ where $h_x : S \to T$ is a homomorphism for any $x \in X$. For example, if $S = T = [0, \infty)$, the usual additive semigroup and (X, S) is an above, then $g : X \times S \to S$ is a (continuous) op-function iff for each x there is a non-negative real number $\alpha(x)$ such that $g(x, s) = \alpha(x)s$.

Examples (4) and (5) are not unnatural and we can conceive of physical devices which conform to these models. For example, the Electricity Corporation may like to install a device in each house which will display the amount to be charged for the electricity consumed upto any moment. Here, the input to the device is the amount of electricity consumed and the output is the price of that. Here we can take X as the set of possible rates per unit of electricity and can be a subset of $[0, \infty)$, S and T can also be taken as the set $[0, \infty)$ with usual addition. Then our machine model will be

$$T \xleftarrow{g} X \times S \xrightarrow{f} S$$

where

$$f(x, s) = x$$

$$g(x, s) = xs$$

for all $(x, s) \in X \times S$. Note that no input changes the state x (the rate per unit) which is actually determined and fixed from time to time by the authority entirely from other considerations.

Finally, the following gives an example of S-machines.

(6) Let S be the usual multiplicative unit interval semigroup and S acts on itself by its multiplication. Let the function $g : S \times S \longrightarrow S$ be defined by

$$g(x, y) = \exp \left\{ - \int_y^1 \frac{m(tx)}{t} \, dt \right\}$$ for some continuous non-negative real valued function m on S. Then g is an op-function satisfying A2 and A4'.

We would also like to point outthat for group actions, in a measure theoretic set up, certain Borel functions called co-cycles satisfy the same algebraic conditions as the (continuous) op-functions in the present topological-algebraic set up. Cocycles play important roles in Harmonic Analysis and details of which are available in Varadarajan [42] and Helson [24]. In the following we merely mention what exactly cocyles are and what roles the play. (Here we deviate slightly from the standard notations in that we take actions on the right).

1.4. Cocycles. Let G be a locally compact second countable group with identity e and act on a standard Borel space X. Let μ be a measure on X which is quasi-invariant under the action of G, i.e., for each $g \in G$, μ and μ_g (where μ_g is defined by, for $A \subset X$, $\mu_g(A) = \mu(Ag)$) have the same null sets. Let M be a standard Borel group with identity 1. A Borel function $f : X \times G \to M$ is said to be a (X, G, M)-cocycle relative to μ if the following properties are satisfied.

(1) $f(x, e) = 1$, for μ-almost all $x \in X$

(2) $f(x, g_1 g_2) = f(x, g_1) f(xg_1, g_2)$ for $(\lambda \times \lambda \times \mu)$-almost all $(x, g_1, g_2) \in X \times G \times G$,

where λ is the Haar measure on G.

f is said to be a strict cocycle if it satisfies (1) and (2) everywhere. Functions satisfying (1) and (2) are called cocycles because these equations are generalizations of the identities which describe the cocycles in the cohomology theory of groups. It was G.W.Mackey who first studied the cocycles in the context of arbitrary transitive actions of locally compact second countable groups. It was the detailed study of these functions which enabled him to state and prove the generalizations of the classical work of Frobenius on induced representations of (finite) groups. Later H.Helson and D. Lowdenslager used these functions to study the invariant subspaces of $L^2(B)$, B the Bohr group. They discovered that these functions are in a one-one correspondence with the simply invariant subspaces of $L^2(B)$.

Following the above we could have described an op-function as a continuous cocycle defined on a semigroup act. However, since our motivation comes from algebraic theory of machines we shall stick to our terminology.

Various aspects of acts, both algebraic and topological, have been studied recently [cf. Day [14]] and abstract machines have been studied by Ginsburg [cf. 20, 21]. However, machines have not been studied in a topological-algebraic set up, though Ginsburg himself suggested such an undertaking [cf. [21]]. Similar views have also been expressed by several others; for example, see Arbib [p. 270, 1],Day [14], Wallace [43] and Wymore [45]. In the present chapter and the next we have initiated the study of topological machines which we believe to be a worthwhile beginning in view of our above discussions.

In this chapter our problem is to obtain results which characterise the op-functions when the underlying act is given. Some results characterising cocycles are known [cf. 42] but no such works concerning op-functions seem to be in print. It is difficult to obtain any result in a very general set up; but if we consider a special class of acts whose structures are well-understood, it is possible to describe op-functions for such acts. Indeed, this is our strategy which we will follow in the subsequent sections where we present our results a brief summary of which is given below.

1.5. Summary. In Section 2, we present a few elementary results characterizing the op-functions defined on a few special but fairly general classes of acts. In Section 3, we characterize op-functions for acts whose input spaces are freely generated monoids (or groups) in terms of continuous functions from the state space into the output space satisfying certain conditions. We also prove a few related facts. Finally, in Section 4, we consider op-functions on acts whose input space is a certain special type of thread with identity and interior zero and acts on itself. We are able to give a fairly complete picture of such op-functions. We also give many illustrative examples.

2. Some Elementary Results

In this section, we prove some elementary results concerning the structure of op-functions corresponding to some special classes of machines. We consider machines which satisfy A1 and A2 but need not satisfy A3 and A4 or A4'. Our first result is concerned with machines where the op-functions $g : X \times S \to T$ are of the form (*) $g(x, s) = h(xs)$ for some continuous function $h : X \to T$ i.e., the output depends on the state y into which the machine goes from the present state x and not what inputs bring the machine from the state x into the state y. Our Example 1.3(2) of the electric clock specifies such an op-function. Note that the condition (*) implies that $h(x) = g(x, u)$ if the underlying act (X, S) has an identity u. However, whether (X, S) has an identity u or not g defined via condition (*) is always an op-function.

2.1. Proposition. Let (X, S) be an act with identity u and T any semigroup. Let $g : X \times S \to T$ be a continuous function and let $h: X \to T$ be defined by $h(x) = g(x, u)$. Then the following statements are true.

(1) $h(xs) = g(x, s)$ iff $g(x, st) = g(xs, t)$ for all $x \in X$ and all $s, t \in S$.

(2) If T is a right zero semigroup, then g is an op-function iff $g(x, s) = h(xs)$ for all $(x, s) \in X \times S$.

(3) Suppose g is an op-function and h satisfies $h(xs) = g(x, s)$ for all $(x, s) \in X \times S$. Then $g(x, s)$ is an idempotent of T and a left identity for $g(x, st)$ for all $x \in X$ and all $s, t \in S$. If, in addition, $g(\{x\} \times S) = T$ for all $x \in X$, then T is a right zero semigroup.

Proof. (1) Suppose $h(xs) = g(x, s)$ for all $(x, s) \in X \times S$. Then, for any $t \in S$, $g(x, st) = h(x(st)) = h((xs)t) = g(xs, t)$. Conversely, suppose $g(x, st) = g(xs, t)$ for all $x \in S$ and all $s, t \in S$. Then note that

$$g(x, s) = g(x, su) = g(xs, u) = h((xs)u) = h(xs).$$

(2) If T is a right zero semigroup, then for any $x \in X$ and $s, t \in S$, g is an op-function iff $g(x, st) = g(x, s)g(xs, t) = g(xs, t)$ and hence, by (1), (2) follows.

(3) Since g is an op-function, for any $(x, s) \in X \times S$, $g(x, s) = g(x, su) = g(x, s)g(xs, u) = g(x, s)h(xsu) = g(x, s)h(xs) = g(x, s)g(x, s)$ and so, $g(x, s)$ is an idempotent of T. Also, for any $t \in S$, since $g(x, st) = g(x, s)g(xs, t) = g(x, s)g(x, st)$, by (1), it follows that $g(x, s)$ is a left identity of $g(x, st)$.

Further, if $a, b \in T$ and $g(x, s) = a$ and $g(xs, t) = b$ for some $x \in X$ and $s, t \in S$, then $ab = g(x, s)g(xs, t) = g(x, st) = h(xst) = h((xs)t) = g(xs, t) = b$. Hence, T is a right zero semigroup.

Let (X, S) be an act and T a semigroup (or a group). Then an op-function $g : X \times S \to T$ is called a <u>simple op-function</u> if there exists a continuous function $b : X \to T$ such that (*) $b(x)g(x, s) = b(xs)$, or equivalently, $g(x, s) = b(x)^{-1} b(x, s)$ if T is a group. The function b satisfying (*) is <u>said to define the op-function g</u>. We call such an op-function simple because if T is a group, then any simple op-function is completely defined by a continuous function $b : X \to T$.

We shall show that in many situations every op-function is simple. However, we give two examples below showing that there are examples of op-functions which are not simple.

2.2 <u>Example</u>. Let S be any commutative group acting on a space X and S^2, the Cartesian product group $S \times S$, act on X as $(x, (s_1, s_2)) \to xs_1s_2$. Let, for a commutative group H,

$h_i : S \to H$, $i = 1, 2$, be two distinct (continuous) homomorphisms. Then the (continuous) function $g : X \times S^2 \to H$ defined by $g(x,(s_1, s_2)) = h_1(s_1)h_2(s_2)$ is an op-function because the map $k : S^2 \to H$ defined by $k(s_1, s_2) = h_1(s_1)h_2(s_2)$ is a homomorphism. We show that g is not simple. For, if $h_1(s_1)h_2(s_2) = b(x)^{-1}b(xs_1s_2)$ for some continuous function $b : X \to H$, then note that for $s_1 = s_2^{-1}$, RHS = 1, but LHS $\neq 1$.

2.3. <u>Example.</u> Let S be any subgroup of the additive group R of real numbers such that S is dense in R with usual topology. Let S_d stand for S with discrete topology. If (X, S) is an act for which 0 is an identity, the action map $X \times S \to X$ will still be continuous if S is given discrete topology and so we also have an act (X, S_d). Then, for any non-continuous homomorphism h of S into a group H, $g : X \times S_d \to H$ defined by $g(x, s) = h(s)$ for all $(x, s) \in X \times S_d$, is an op-function which is not simple. For, if g is simple, then there exists a continuous map $b : X \to H$ such that $g(x, s) = h(s) = b(x)^{-1}b(xs)$ for all $(x, s) \in X \times S_d$. Now, by definition of h, there is a sequence $\{s_n\}$ in S such that $s_n \to 0$ in S but $h(s_n) \not\to 1$ in H where 1 is the identity of H. Then, because of the continuity of the act (X, S), if $x \in X$, $xs_n \to x0 = x$. Therefore, because $b : X \to H$ is assumed to be continuous, $b(x)^{-1}\, b(xs_n) \to 1$. But this is the same as $h(s_n) \to 1$, which is false. Hence no such continuous function b can exist.

That there exists a non-continuous homomorphism on S can be seen as follows. Let $H = R$. Then there exists a non-continuous homomorphism from R into R. This follows from a conside-ration of a Hamel basis for R and a cardinality argument. Then the restriction on S of any non-continuous homomorphism of R is non-continuous homomorphism of S as, if not, S being dense in R and a continuous homomorphism being automatically uniformly continuous there would arise a contradiction.

The following proposition is a slight generalization of a simple fact known in group theoretic set up [cf. 42].

2.4. Proposition. Let $g : S \times S \to H$ be an op-function for some semigroup S and a group H. If S has a left identity u (respectively a right zero z), then g is simple and the map b defining g such that $b(u) = 1$ (respectively $b(z) = 1$) is unique. (In case S has a left identity g is simple even if H is just a semigroup and not a group).

Proof. Let S have a left identity u. Then define $b : S \to H$ by $b(x) = g(u, s)$ for all $x \in S$. Since g is an op-function, for any $(x, y) \in S \times S$, $g(u, xy) = g(u, x)g(x, y)$

i.e., $b(xy) = b(x)\, g(x, y)$

i.e., $g(x, y) = b(x)^{-1}\, b(xy)$ and, therefore g is simple. Clearly, $b(u) = 1$. Now, if possible, let b_1 be another map defining g such that $b_1(u) = 1$. Then, for all $(x, y) \in S \times S$, $b_1(x)^{-1}\, b_1(xy) = b(x)^{-1}\, b(xy)$

$$\text{i.e., } b(x)\, b_1(x)^{-1} = b(xy)b_1(xy)$$

which implies that $b(u)b_1(u)^{-1} = b(y)b_1(y)^{-1} = 1$ for all $y \in S$ and that means $b = b_1$.

Next let S have a right zero z. Define $b : S \to H$ by $b(x) = g(x, z)^{-1}$ for all $x \in S$. Since g is an op-function, for all $(x, y) \in S \times S$,

$$g(x, z) = g(x, yz) = g(x, y)g(xy, z)$$

i.e., $g(x, y) = b(x)^{-1}\, b(xy)$ and so, g is simple. Clearly, $b(z) = 1$. Again, if possible, let b_1 be another map defining g such that $b_1(z) = 1$. Then, for all $(x, y) \in S \times S$, $b_1(x)^{-1}\, b_1(xy) = b(x)^{-1} b(xy)$ implies that $b(x)b_1(x)^{-1} = b(xz)b_1(xz)^{-1} = b(z)b_1(z)^{-1} = 1$, and so, $b = b_1$.

2.5. Remark. The uniqueness of the map b defining a simple op-function in Proposition 2.4 is subject to the condition that $b(u) = 1$ (or $b(z) = 1$). In general, however, if a simple op-function $g : X \times S \to H$ is defined by a map $b : X \to H$ and H is a group, then any translate $b_1 : S \to H$ of b (i.e., $b_1(x) = hb(x)$ for some $h \in H$ and all $x \in X$) also defines b.

2.6. Proposition. Let S be a commutative semigroup and H a group. They any op-function $g : S \times S \to H$ is simple and is defined, for any $a \in S$, by the map $b_a : S \to H$ (or any translate of b_a) where $b_a(x) = g(a, x)g(x, a)^{-1}$ for all $x \in S$. Further, any map $b : S \to H$ defining g is necessarily a translate of b_a for each $a \in S$.

<u>Proof.</u> For any $a \in S$, let $b_a : S \to H$ be defined as in the Proposition 2.6. Then, by the commutativity of S,
$g(x, ya) = g(x, ay)$ for all $(x, y) \in S \times S$, and so,
$g(x, y)g(xy, a) = g(x, a)g(xa, y)$. Therefore,

$$\begin{aligned} g(x, y) &= g(x, a)g(xa, y)(g(xy, a))^{-1} \\ &= g(x, a)g(a, x)^{-1}\, g(a, xy)g(xy, a)^{-1} \\ &= b(x)^{-1}\, b(xy). \end{aligned}$$

Hence the first part of the Proposition 2.6 follows.

Now suppose g is defined by a map $b : S \to H$ i.e., $g(x, y) = b(x)^{-1}\, b(xy)$ for all $(x, y) \in S \times S$. Then, for any $a \in S$, we have, for all $x \in S$,

$$b(x) = b(xa)g(x, a)^{-1}$$

and, by the commutativity of S,

$$b(xa) = b(ax) = b(a)g(a, x)$$

Therefore,

$$\begin{aligned} b(x) &= b(a)g(a, x)g(x, a)^{-1} \\ &= b(a)b_a(x) \end{aligned}$$

which proves the second part of the Proposition 2.6.

2.7. <u>Proposition.</u> Let a commutative semigroup S act on a space X such that the following two conditions are satisfied:

(C1) There exist $c \in X$ and $d \in S$ such that for each $x \in X$ there exists a unique $y \in S$ such that $xd = cy$.

(C2) If $\{s_\alpha\}$ is a net in S having no convergent subnet, then the net $\{cs_\alpha\}$, for $c \in X$, has no convergent subnet.

Then every op-function $g : X \times S \to H$, where H is a group, is simple.

Proof. Let $g : X \times S \to H$ be an op-function. Define, for fixed $c \in X$, $d \in S$ given by (C1), $b : X \to H$ by $b(x) = g(c, y)g(x, d)^{-1}$ for each $x \in X$, where y is the unique element in S such that $xd = cy$. We claim that the map b is continuous. If $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x$, then, by the continuity of the act, $x_\alpha d \to xd$ and then, since $x_\alpha d = cy_\alpha$, by (C2), we can assume $y_\alpha \to y$ such that $xd = cy$ which means that the correspondence $x \to y$ is continuous. Therefore, it follows that b is continuous being a composition of several continuous maps. Then, for any $(x, s) \in X \times S$, by the commutativity of S, $g(x, sd) = g(x, ds)$ which means that

$$g(x, s)g(xs, d) = g(x, d)g(xd, s).$$

Therefore,

$$\begin{aligned} g(x, s) &= g(x, d)g(xd, s)g(xs, d)^{-1} \\ &= g(x, d)g(cy, s)g(xs, d)^{-1} \\ &= g(x, d)g(c, y)^{-1}g(c, ys)g(xs, d)^{-1} \\ &= b(x)^{-1}\, b(xs), \text{ since } xd = cy \text{ implies, by the} \end{aligned}$$

commutativity of S, that $xsd = xds = cys$, and so, g is simple.

The following example illustrates the above proposition.

2.8. Example. Let $X = [c, \infty)$, $S = [d, \infty)$, for $-\infty < c < \infty$, $0 < d < \infty$ and $H = R$, the additive group of real numbers. Let the action map as well as the semigroup operation be usual addition of real numbers. The numbers c and d satisfy (C1) of Proposition 2.7. Also note that (C2) is satisfied in this case.

The above results can be slightly generalized as follows.

2.9. Remark. Let (X, S) be an act satisfying any one of the hypotheses of Propositions 2.4, 2.6 or 2.7. Then, for any space Y if S acts on the product space $Y \times X$ as follows: $(y, x)s = (y, xs)$ for all $(y, x, s) \in Y \times X \times S$, every op-function $g: (Y \times X) \times S \to H$, for a group H, is simple.

Proof. The proof is easy in all the cases and is illustrated for the case when S has a left identity e. Here S acts itself by its multiplication.

Define $b : Y \times S \to H$ as follows: $b(y, s) = g((y, e), s)$ for all $(y, s) \in Y \times S$. Then, from the identity

$$g((y, e), st) = g((y, e), s)g((y, s), t)$$

for any $t \in S$, it follows that g is simple and defined by b.

Likewise we can verify all other cases.

Our next result is concerned with extension of an op-function from a homomorphic image of an act. We recall that an act (X', S') is a homomorphic image of an act (X, S) if there exists a homomorphism from (X, S) onto (X', S') i.e., a pair (h, k) where $h : X \to X'$ is a continuous onto map and $k : S \to S'$ is a continuous onto homomorphism such that $h(xs) = h(x)k(s)$ for all $(x, s) \in X \times S$. Then we have the following proposition.

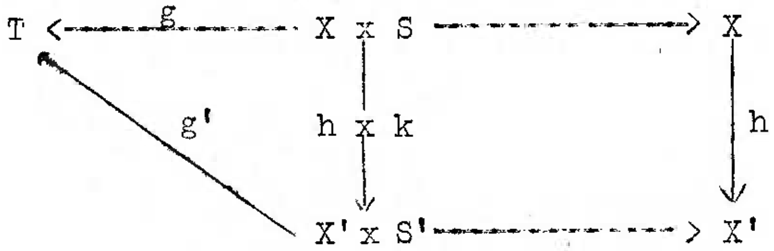
2.10. <u>Proposition.</u> Let (X', S') be a homomorphic image of an act (X, S) via the homomorphism (h, k) and T be a fixed semigroup.

If $g' : X' \times S' \to T$ is an op-function, then the function $g : X \times S \to T$, defined by $g(x, s) = g'(h(x), k(s))$ for all $(x, s) \in X \times S$, is an op-function satisfying the condition:

(C) : g is constant on $h^{-1}(h(x)) \times k^{-1}(k(s))$ for all $(x, s) \in X \times S$.

Conversely, if $g : X \times S \to T$ is an op-function satisfying (C), then the function $g' : X' \times S' \to T$, defined by $g'(x', s') = g(h^{-1}(x') \times k^{-1}(s'))$ for all $(x', s') \in X' \times S'$, is an op-function.

The proof of Proposition 2.10 is easy and omitted. The following diagram may be helpful in understanding this proposition



3. Machines with Freely Generated Commutative Monoids (or Groups) as Inputs.

Let $\{S_i : i \in I\}$ be a family of monoids (or groups). Then the Cartesian product monoid (or group), $X\{S_i : i \in I\}$, is the product space $X\{S_i : i \in I\}$ equipped with coordinatewise multiplication and the direct product of $\{S_i : i \in I\}$ is the submonoid (or subgroup) $(\overline{X})\{S_i : i \in I\}$ of $X\{S_i : i \in I\}$ consisting of those points which have all but a finitely many coordinates identities. For a finite family $\{S_1, \ldots, S_n\}$ of monoids (or groups) $X\{S_i : i = 1, \ldots, n\}$ and $(\overline{X})\{S_i : i = 1, \ldots, n\}$ are same and are often denoted by $S_1 (\overline{X}) \ldots (\overline{X}) S_n$. If additive notation is used, then the term direct sum is used instead of direct product. A monoid (or a group) S is the topological (respectively algebraic) direct product of its submonoids (or subgroups) $\{S_i : i \in I\}$ if S is topologically (respectively algebraically) isomorphic to the direct product $(\overline{X})\{S_i : i \in I\}$.

If a monoid (or a group) S is the direct product of its submonoids (or subgroups), then every element s of S different from the identity has a unique representation $s = s_1 s_2 \ldots s_n$ for some finitely many elements $s_1, \ldots, s_n$ which are not identities and come from some submonoids (or subgroups), say, $S_{m_1}, \ldots, S_{m_n}$ of S. A discrete commutative monoid (or a group) S is said to be freely generated by a set A of elements of S if S is the (algebraic) direct product of the monoids (or groups) $\{S_a : a \in A\}$ where each S_a is the infinite monogenic (or cyclic) monoid (or group) generated by $a \in A$ i.e., $S_a = \{ a^n : n \geq 0 \}$ (or $S_a = \{ a^n : n$ any integer$\}$).

In this section we obtain structural characterizations of op-functions (which are tacitly assumed to be continuous and satisfy A2 and A4') defined on an act whose input semigroup is a commutative monoid (or group) freely generated by a set of elements. Towards this we first prove the following result concerning op-functions on an act (X, S) where S is a commutative monoid and is the topological direct product of n submonoids.

3.1. Proposition. Let S be a commutative monoid which is the topological direct product of n submonoids $S_1, S_2, \ldots, S_n$. If S acts on a space X and T is any monoid, then a function $g : X \times S \to T$ is an op-function iff there exist (unique) op-functions $g_i : X \times S_i \to T$, $i = 1, \ldots, n$, such that

a) $g_i(x, s_i)g_j(xs_i, s_j) = g_j(x, s_j)g_i(xs_j, s_i)$ for all $x \in X$, $s_i \in S_i$ and $s_j \in S_j$ and $i, j = 1, 2, \ldots, n$ and $i \neq j$.

and

(b) $g(x, s) = g_1(x, s_1)g_2(xs_1, s_2) \ldots g_n(x \prod_{i=1}^{n-1} s_i, s_n)$ for all $x \in X$ and $s \in S$ where s has the (unique) representation $s = \prod_{i=1}^{n} s_i$, $s_i \in S_i$, $i = 1, \ldots, n$.

Proof. 'If part', First, for $n = 2$, it is shown that, if (a) and (b) hold, then g is an op-function. Let $S = S_1 \otimes S_2$ and

$s, s' \in S$ so that $s = s_1 s_2$ and $s' = s'_1 s'_2$ for $s_i, s'_i \in S_i$, $i = 1, 2$.

We shall show that, for any $x \in X$,

$$g(x, ss') = g(x, s)g(xs, s')$$

Now, $g(x, ss') = g(x, (s_1 s'_1)(s_2 s'_2))$

$$= g_1(x, s_1 s'_1) g_2(xs_1 s'_1, s_2 s'_2)$$

$$= g_1(x, s_1) g_1(xs_1, s'_1) g_2(xs_1 s'_1, s_2) g_2(xs_1 s'_1 s_2, s'_2)$$

$$= g_1(x, s_1) g_2(xs_1, s_2) g_1(xs_1 s_2, s'_1) g_2(xs_1 s_2 s'_1, s'_2)$$

$$= g(x, s)g(xs, s').$$

Next an induction is made on n. Suppose for $n = m$ the result is true. We shall show that the same holds for $n = m+1$. Let $S = S_1 \circledX \ldots \circledX S_{m+1}$, $S^* = S_1 \circledX \ldots \circledX S_m$ and so $S = S^* \circledX S_{m+1}$.

Let $s \in S$ and $s = \prod_{i=1}^{m+1} s_i$, $s_i \in S_i$

$$= t s_{m+1}, \quad t \in S^*.$$

Suppose g is defined by (a) and (b).

Now, $g(x, s) = g(x, ts_{m+1}) = g_*(x, t) g_{m+1}(xt, s_{m+1})$ where $g_* : X \times S^* \to T$ and $g_{m+1} : X \times S_{m+1} \to T$ are two op-functions and g_* is obtained via conditions (a) and (b). Induction is complete if g_* and g_{m+1} satisfy (a). That is to show that for all $x \in X$, $t \in S^*$ and $s_{m+1} \in S_{m+1}$

$$g_*(x, t) g_{m+1}(xt, s_{m+1}) = g_{m+1}(x, s_{m+1}) g_*(xs_{m+1}, t).$$

Assuming $t = \prod_{i=1}^{m} s_i$, then

$$\text{LHS} = g_1(x, s_1)g_2(xs_1, s_2)\ldots g_m(x\prod_{i=1}^{m-1} s_i, s_m)g_{m+1}(xt, s_{m+1}).$$

After repeated applications of (a) from the right one shows that LHS = RHS. Thus g satisfies A2 and, since each g_i satisfies A4', g satisfies A4' also.

Further, inview of (b) and that S is the topological direct product of $S_1, \ldots, S_n$, it can be easily seen that g is continuous.

'Only if'. If $g : X \times S \to T$ is an op-function then let $g_i : X \times S_i \to T$ be the restriction of g on $X \times S_i$, $i = 1, \ldots, n$. It is easy to see that g_i's satisfy (a) and (b) and g_i's, being the restrictions of g on $X \times S_i$, are unique.

If S is a commutative monoid which is the topological direct product of infinitely many submonoids of S, then the assertions of Propositions3.1 is false because the function g so defined via (b) may fail to be continuous. The following example illustrates this point.

3.2 Example. Let $S_i = R$, the usual additive group of reals, $i = 1, 2, \ldots$. Let S be the topological direct sum of S_i's. If R acts on a space X, then, taking R as the output semi-group, the function $g_i : X \times R \to R$, defined via $g_i(x, r_i) = r_i$ for all $x \in X$ and $r_i \in R$, is a (continuous) op-function for each i and the condition (a) of Proposition 3.1 is trivially satisfied. If $g: X \times S \to R$ is defined by (b) of

Proposition 3.1, then we shall show that g is not continuous.

If $\{s_n\}$ is a sequence in S,

where $s_n = (r_1, r_2, \ldots, r_m, \frac{1}{n}, \ldots, \frac{1}{n}, 0, 0 \ldots)$, $(m+1)$st to $(m+n)$th coordinates being equal to $\frac{1}{n}$ for all $n \geq 1$, then $\lim_{n \to \infty} s_n = (r_1, \ldots, r_m, 0, \ldots) = s$, say. But while $g(x, s_n) = r_1 + \ldots + r_m + 1$ for all $n \geq 1$, $g(x, s) = r_1 + \ldots + r_m$. So g is not continuous.

However, if $\{S_i : i \in I\}$ is an arbitrary family of submonoids of a discrete monoid S which is the (algebraic) direct product of $\{S_i : i \in I\}$, then we can state the following.

3.3 Proposition. Let S be as in the above paragraph and act on a space X. If T is a monoid, then a continuous function $g: X \times S \to T$ is an op-function iff there exist (unique) continuous op-functions $g_i : X \times S_i \to T$, $i \in I$, satisfying

(a) $g_i(x, s_i) g_j(xs_i, s_j) = g_j(x, s_j) g_i(xs_j, s_i)$

for all $x \in X$, $s_i \in S_i$, $s_j \in S_j$ and $i, j \in I$.

and (b) $g(x, s) = g_{i_1}(x, s_{i_1}) g_{i_2}(xs_{i_1}, s_{i_2}) \ldots$

$g_{i_n}(x \prod_{j=1}^{n-1} s_{i_{j-1}}, s_{i_{n-1}})$ for all $x \in X$ and $s \in S$ such that s has the (unique) representation

$$s = \prod_{j=1}^{n} s_{i_j}, \quad s_{i_j} \in S_{i_j}.$$

Proof. That g so defined satisfies the axiom A2 and A4' can be verified using Proposition 3.1 and that g is continuous follows from (b) and the fact that the continuity in the first coordinate only has to be established.

Now, in view of Proposition 3.3, if S is a free commutative monoid (or group) generated by a set $\{\lambda_i : i \in I\}$ of elements and S acts on a space X, then for any monoid (or group) T we can obtain structural description of any op-function $g : X \times S \to T$ in terms of functions $f_i : X \to T$, $i \in I$ satisfying certain condition similar to (a) of Proposition 3.3 While this is our objective in the rest of this section we shall state and prove our results only for the case when I is a finite set, the generalization to the case when I is an infinite set being quite easy.

Therefore our next proposition is the following.

3.4. Proposition. Suppose S is a discrete commutative monoid freely generated by the elements $\lambda_1, \lambda_2, \ldots, \lambda_n$, so that each element s of S has a unique representation

$$s = \prod_{i=1}^{n} \lambda_i^{m_i},$$

m_i is a non-negative integer, $i = 1, 2, \ldots, n$. If S acts on a space X and T is any monoid with identity 1, then any function $g : X \times S \to T$ is an op-function iff there exist (unique) continuous functions $f : X \to T$, $i = 1, 2, \ldots, n$ such that

(a) $f_i(x)f_j(x\lambda_i) = f_j(x)f_i(x\lambda_j)$ for all

$i, j = 1, 2, \ldots, n$; and

(b)(i) $$g_i(x, \lambda_i^{m_i}) = \begin{cases} \prod_{k=0}^{m_i-1} f_i(x\lambda_i^k) & \text{if } m_i > 0 \\ 1 & \text{if } m_i = 0 \end{cases}$$

for $i = 1, 2, \ldots, n$;

(ii) $$g(x, \prod_{i=1}^{n} \lambda_i^{m_i}) = g_1(x, \lambda_1^{m_1}) \ldots g_n(x \prod_{i=1}^{n-1} \lambda_i^{m_i}, \lambda_n^{m_n})$$

for all $x \in X$.

<u>Proof.</u> <u>'If part'</u>. Note that, if $S_i = \{\lambda_i^{m_i} : m_i \geq 0\}$, then $S = S_1 \otimes \ldots \otimes S_n$ and g_i defined by b(i) is an op-function on $X \times S_i$. In order that g defined by b(ii) be an op-function, it sufficies to show that the g_i's satisfy (a) of Proposition 3.1. Now, by repreated applications of (a), it can be shown that

$$g_i(x, \lambda_i^{m_i})\, g_j(x\lambda_i^{m_i}, \lambda_j^{m_j})$$

$$= f_i(x)f_i(x\lambda_i) \ldots f_i(x\lambda_i^{m_i-1})f_j(x\lambda_i^{m_i})f_j(x\lambda_i^{m_i}\lambda_j) \ldots$$
$$f_j(x\lambda_i^{m_i}\lambda_j^{m_j-1})$$

$$= f_j(x)f_j(x\lambda_j)\ldots f_j(x\lambda_j^{m_j-1})f_i(x\lambda_j^{m_j}).\ f_i(x\lambda_j^{m_j}\lambda_i)\ldots f_i(x\lambda_j^{m_j}\lambda_i^{m_i-1})$$

$= g_j(x, \lambda_j^{m_j})\, g_i(x\lambda_j^{m_j}, \lambda_i^{m_i})$. This completes the proof of 'if part'.

<u>'Only if part'</u>: Define $f_i(x) = g(x, \lambda_i)$ for all $x \in X$ and $g_i(x, \lambda_i^{m_i}) = g(x, \lambda_i^{m_i})$, $i = 1, 2, \ldots, n$. Then (a) and (b) are true. The uniqueness of f_i's follow from the fact that, by the condition b(i) and b(ii), $f_i(x) = g(x, \lambda_i)$ for all $x \in X$ and $i = 1, 2, \ldots, n$.

Next proposition is stated for the case when S is a commutative discrete group freely generated by finitely many elements.

3.5. <u>Proposition.</u> Let S be a commutative discrete groups freely generated by $\lambda_1, \ldots, \lambda_n$ so that each element s of S has a unique expression $s = \prod_{i=1}^{n} \lambda_i^{m_i}$, m_i is any integer, $i = 1, \ldots, n$. If S acts on a space X and T is a group with identity 1, then a function $g : X \times S \to T$ is an op-function iff there exist (unique)(continuous) functions $f_i : X \to T$, $i = 1, \ldots, n$, such that

a)(a) of Proposition 3.4 is satisfied and

$$\text{b)(i)}\quad g_i(x, \lambda_i^{m_i}) = \begin{cases} \prod_{k=0}^{m_i-1} f_i(x\lambda_i^k) & \text{if } m_i > 0 \\ 1 & \text{if } m_i = 0 \\ 1 \Big/ \prod_{k=0}^{-m_i-1} f_i(x\lambda_i^{m_i+k}) & \text{if } m_i < 0 \end{cases}$$

(b)(ii) $g(x, \prod_{i=1}^{n} \lambda_i^{m_i})$

$$= g_1(x, \lambda_1^{m_1})g_2(x\lambda_1^{m_1}, \lambda_2^{m_2})\dots g_n(x. \prod_{i=1}^{n-1} \lambda_i^{m_i}, \lambda_n^{m_n})$$

for all $x \in X$.

Proof. By virtue of Proposition 3.1, it is only necessary to verify that g_i's so defined satisfy (a) of Proposition 3.1, i.e., to show that, for $i \neq j$, $i, j = 1, \dots, n$, and m_i, m_j any integers.

$$(*)\quad g_i(x, \lambda_i^{m_i})g_j(x\lambda_i^{m_i}, \lambda_j^{m_j}) = g_j(x, \lambda_j^{m_j})g_i(x\lambda_j^{m_j}, \lambda_i^{m_i}).$$

Now, for the Case 1 when $m_i, m_j \geq 0$ (*) has been already verified in Proposition 3.4 and so we shall consider the remaining cases.

Case 2. $m_i = -\ell_i$, $\ell_i \geq 0$ and $m_j \geq 0$.

In this case, we can show that (*) is equivalent to

$$f_j(x')f_j(x'\lambda_j)\dots f_j(x'\lambda_j^{m_j-1})f_i(x'\lambda_j^{m_j})\dots f_i(x'\lambda_j^{m_j}\lambda_i^{\ell_i-1})$$

$$= f_i(x')f_i(x'\lambda_i)\dots f_i(x'\lambda_i^{\ell_i-1})f_j(x)\dots f_j(x\lambda_j^{m_j-1}).$$

where $x' = x\lambda_i^{m_i}$.

This can be easily verified by repeated applications of (a).

Case 3. $m_i \geq 0$ and $m_j = -\ell_j,\ \ell_j \geq 0$. This is similar to Case 2.

Case 4. $m_i = -\ell_i,\ \ell_i \geq 0$ and $m_j = -\ell_j,\ \ell_j \geq 0$.

Note that the condition (a):

$f_i(x)f_j(x\lambda_i) = f_j(x)f_i(x\lambda_j)$ is equivalent to (a'):

$$f_i(x\lambda_j)^{-1}\ f_j(x)^{-1} = f_j(x\lambda_i)^{-1}\ f_i(x)^{-1}$$

and (*) is equivalent to

$$f_i(x\lambda_i^{-1})^{-1}\ f_i(x\lambda_i^{-2})^{-1} \ldots f_i(x\lambda_i^{-\ell_i})^{-1}\ f_j(x\lambda_i^{-\ell_i}\lambda_j^{-1})^{-1} \ldots f_j(x\lambda_i^{-\ell_i}\lambda_j^{-\ell_j})^{-1}$$

$$= f_j(x\lambda_j^{-1})^{-1}\ f_j(x\lambda_j^{-2})^{-1} \ldots f_j(x\lambda_j^{-\ell_j})^{-1}\ f_i(x\lambda_j^{-\ell_j}\lambda_i^{-1})^{-1} \ldots f_i(x\lambda_j^{-\ell_j}\lambda_i^{-\ell_i})^{-1}$$

which is easily verified by repeated applications of (a').

This completes the proof of 'if part.'

'Only if' Define $f_i(x) = g(x, \lambda_i)$ for all $x \in X$ and $i = 1, \ldots, n$. Then f_i's satisfy (a); and, further, if $g_i(x, \lambda_i^{m_i}) = g(x, \lambda_i^{m_i})$, then (b) is also satisfied.

This completes the proof.

The following gives a condition when every op-function in the present set up is simple.

3.6. Proposition. Suppose S is a commutative discrete monoid freely generated by an arbitrary set $\{\lambda_i : i \in I\}$ of generators. If T is a group and S acts on a space X, then any op-function $g: X \times S \to T$ is simple iff there exists a continuous function $b: X \to T$ such that $f_i(x) = g(x, \lambda_i) = b(x)^{-1} b(x\lambda_i)$ for all $x \in X$ and all λ_i, $i \in E$

The proof is trivial.

The following gives a situation when every op-function is simple.

3.7. Proposition. Let S be a sub-semigroup (or subgroup) of the additive real line X generated by a single element λ. Then for any group T, every op-function $g: X \times S \to T$ is simple.

Proof: Let $g: X \times S \to T$ be any op-function. Let $f(x) = g(x, \lambda)$ for all $x \in X$. Because of Proposition 3.6, we need to show that

(*) $f(x) = b(x)^{-1} b(x+\lambda)$ for all $x \in X$, for some continuous function $b : X \to T$. Now note the following property (P) of real numbers.

(P) : Every real number has a unique representation $y = x + n\lambda$ for $0 < x \leq \lambda$ and n an integer. Now take any continuous function $b : [0, \lambda] \to T$ such that $b(\lambda) = b(0)f(0)$. Then for any $y > \lambda$, if $y = x + n\lambda$, $n \geq 1$, define $b(y)$ so as to satisfy (*) i.e., set

$$b(y) = b(x + n\lambda) = b(x + \overline{n-1}\,\lambda) f(x + \overline{n-1}\,\lambda) \ldots$$

$$= b(x) f(x) f(x + \lambda) \ldots f(x + \overline{n-1}\,\lambda).$$

and for $y \leq 0$, if $y = x - n\lambda$, $n \geq 1$,

$$b(y) = b(x - n\lambda) = b(x - \overline{n-1}\,\lambda)\ f(x - n\lambda)^{-1}$$
$$= b(x) f(x - \lambda)^{-1}\ f(x - 2\lambda)^{-1} \ldots f(x - n\lambda)^{-1}$$

Then b is a well-defined continuous map from X into T and, by the very construction, satisfies (*) for all $x \in X$. Hence g is simple.

Next we give an example of op-function which is not simple.

3.8. Example. Let S be the discrete subgroup of the additive real line X generated by 1 and an irrational number λ. Let T be the circle group and f_1 and f_2 be two functions from X into T defined by

$$f_1(x) = \exp\,(ix) \quad \text{and} \quad f_2(x) = \exp\,(i\lambda x)$$

for all $x \in X$. It can be easily seen that f_1 and f_2 satisfy the condition (a) of Proposition 3.5.

Then, via Proposition 3.5 and after some simplifications, the op-function $g : X \times S \to T$ constructed from f_1 and f_2 is defined as:

$$g(x, m) = \begin{cases} \exp\left[i\left(mx + \frac{m(m-1)}{2}\right)\right] & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ \exp\left[-i\left(-mx - \frac{-m(-m+1)}{2}\right)\right] & \text{if } m < 0 \end{cases}$$

$$g(x, n\lambda) = \begin{cases} \exp\left[i\lambda\left(nx + \frac{n(n-1)\lambda}{2}\right)\right] & \text{if } m > 0 \\ 1 & \text{if } n = 0 \\ \exp\left[-i\lambda\left(-nx - \frac{-n(-n+1)}{2}\lambda\right)\right], & \text{if } n < 0 \end{cases}$$

and $g(x, m+n\lambda) = g(x, m)g(x+m, n\lambda)$.

Then, for $m < 0$ and $n > 0$, it can be shown that

$$(*) \quad g(x, m+n\lambda) = \exp\left[i(m+n\lambda)x\right]. \ \exp\left[\tfrac{i}{2}(m+n\lambda)^2\right].\exp\left[-\tfrac{i}{2}(m+n\lambda^2)\right]$$

If g is simple there should exist a continuous $b : X \to T$ such that

$$g(x, m+n\lambda) = b(x)^{-1}\, b(x+m+n\lambda)$$

for all $x \in X$ and $m+n\lambda \in S$.

Now consider a sequence $m_k + n_k\lambda \to 0$. It can be assumed that $n_k > 0$, $n_k \to \infty$ and $m_k < 0$ for all $k \geq 1$. So if g is trivial one should have

$$\lim_{m_k + n_k\lambda \to 0} g(x, m_k + n_k\lambda) = 1$$

whence from (*)

$$\lim_{m_k + n_k\lambda \to 0} \exp\left[-\tfrac{i}{2}(m_k + n_k\lambda^2)\right] = 1$$

That is, $m_k + n_k\lambda^2 \to 2\ell\pi$ for some constant ℓ and that means $\frac{m_k}{n_k} \to \lambda^2$. But, as $m_k + n_k\lambda \to 0$, $\frac{m_k}{n_k} \to -\lambda$.

This is a contradiction, and so, g is not simple.

In concluding this section we give a characterization of op-functions for the action of discrete group of rationals Q

on the set of reals R. Let G_n be the group generated by $\frac{1}{n}$, $n \geq 1$. Then $Q = \bigcup_{n=1}^{\infty} G_n$. For a group H with identity 1 an op-function $g_n : R \times G_n \to H$ is described as

$$(\alpha)\quad g_n\left(x, \frac{m}{n}\right) = \begin{cases} \prod_{i=0}^{m-1} f_n\left(x + \frac{i}{n}\right), & \text{if } m > 0 \\ 1 & \text{if } m = 0 \\ 1/\prod_{i=0}^{-m-1} f_n\left(x + \frac{m}{n} + \frac{i}{n}\right), & \text{if } m < 0 \end{cases}$$

for all $x \in R$ and all m, where $f_n : R \to H$ is a continuous function and is defined by $f_n(n) = g_n(x, \frac{1}{n})$. The following is then a description of op-functions $g : R \times Q \to H$.

3.9. Proposition. A function $g : R \times Q \to H$ is an op-function iff there exists a sequence of continuous functions $f_n : R \to H$, $n \geq 1$ satisfying

$$(\beta)\qquad f_i(x) = f_{ij}(x)\, f_{ij}\left(x + \frac{1}{ij}\right) \cdots f_{ij}\left(x + \frac{j-1}{ij}\right)$$

for all $x \in R$ and $i, j \geq 1$

and $g(x, \frac{m}{n}) = g_n(x, \frac{m}{n})$ as given by (α).

Proof. If f_n's satisfy (β) and g is defined via (α) it is easy to see that g satisfies the condition A2 of an op-function. For, if $\frac{m}{n}, \frac{m'}{n'} \in Q$, then $\frac{m}{n} = \frac{mn'}{nn'}$ and $\frac{m'}{n'} = \frac{m'n}{nn'}$ and so $\frac{m}{n'}, \frac{m'}{n'} \in G_{nn'}$.

Therefore, $g(x, \frac{m}{n} + \frac{m'}{n'}) = g_{nn'}(x, \frac{m}{n} + \frac{m'}{n'})$ will satisfy A2. Only thing that is necessary to verify is that g defined via (α) is unambiguous.

That is, $g_n(x, \frac{m}{n}) = g_{nn'}(x, \frac{mn'}{nn'})$ for all integers m, n, n', with $n, n' \geq 1$.

Expanding the RHS and using (β) for the two cases when $m > 0$ and $m < 0$ the above equality can be easily established.

Conversely, if $g : R \times Q \to H$ is an op-function, then define $f_n(x) = g(x, \frac{1}{n})$ for all $x \in X$ and $n \geq 1$. It is easy to see that f_n's satisfy (α) and (β).

A final remark is worth making in this context. Examples of op-functions which are not simple are given in both Sections 2 and 3 for actions of discrete subgroups of additive real line R which are dense in R with usual topology. But what can be said about op-functions on $R \times S$ where S is a dense subgroup, the topology on S being the induced topology from R. If H is complete metric, then every uniformly continuous op-function $R \times S$ into H has a unique uniformly continuous extension to $R \times R$, and hence, must be simple. What can be said about the structures of continuous op-functions ? More generally, suppose S is a dense submonoid (or subgroup) of a group H acting on a space X and T is a monoid (or group). Can every op-function $g : X \times S \to T$ be extended to an op-function $g' : X \times H \to T$? We do not know any answer.

4. S-Machines whose Input Semigroups are certain special types of Threads having identity and zero and Output Semigroups contain zero.

We have seen in Section 2 (cf. Proposition 2.6) that if S is a commutative semigroup and H a group, then every op-function $g : S \times S \to H$ is simple. However, if H is a group with zero (i.e., H is a semigroup with zero 0 such that $H \setminus 0$ is a group, for example, H can be the multiplicative semigroup R^+ of nonnegative real numbers), then this may not be the case. For instance, if $S = [0, 1]$ with usual multiplication, then S is a subsemigroup of R^+ and not every op-function $g : S \times S \to S$ is simple. In fact, if $S = [0, 1]$ with usual multiplication, then we shall prove in the sequel the following proposition which completely characterizes all op-functions $g : S \times S \to S$.

4.1. Proposition. Let $S = [0, 1]$ with usual multiplication and $g : S \times S \to S$ be any op-function. Let (C_0) denote the condition that : $g(0, x) = 0$ for some $x \in S$.

Then :

1) If (C_0) holds, then either

(a) $g(x, y) = 0$ for all $(x, y) \in S \times S$,

or (b)(i) $g(x, 0) = 0$ for all $x \in S$ and

(ii) $g(x, y) \neq 0$ for all $x \in S$ and $y > 0$.

2) if (C_0) does not hold, then $g(x, y) \neq 0$ for all $(x, y) \in S \times S$, and, hence, g must be simple.

The arguments required to prove the Proposition 4.1 are quite elementary. However, similar arguments can be made use of to study op-functions when S is a more general interval semigroup such as a standard thread or a thread with identity and interior zero [11]. This motivates the discussion of this section and our discussion is carried on for a certain special class of threads with identity and interior zero. From this discussion results for the case of a standard thread and, in particular, Proposition 4.1 will follow as special cases.

Towards this we first describe the structure of threads with identity and interior zero. We refer to Clifford [11], Day [13] and Paalman-de Miranda [37] for this material. However, we shall mainly follow the notations and terminologies of Clifford [11].

By a *thread* we shall mean a compact connected linearly ordered semigroup with both end points as idempotents. A *unit thread* is a semigroup topologically isomorphic (or, simply, iseomorphic) to $[0, 1]$ with usual real multiplication and a *nil thread* is a semigroup iseomorphic to the semigroup $[\frac{1}{2}, 1]$ with multiplication defined by $xy = \max\{\frac{1}{2}$, usual real product of x and $y\}$. By a *ligament* we shall mean either a unit thread or a nil thread. A standard thread is a thread with one end point as zero and the other end point as identity.

The following result describes the structure of a standard thread.

Theorem [cf. Clifford [11], Day [13]]. Let S be a standard thread with E as the set of idempotents. Then E is a closed subset of S, and, if $x, y \in E$, $xy = \min\{x, y\}$; the complement of E is the union of disjoint open intervals, and, if P is one of these, then the closure of P is a subsemigroup of S which is a ligament; and, finally, if $x \in P$ and $y \notin P$, then $xy = \min\{x, y\}$. In particular, S is Abelian.

The next result describes the structure of a thread with identity and interior zero.

Theorem [cf. Clifford [11]]. Let $T = [f, u]$ be a thread with u as identity and having interior zero 0 such that $f < 0 < u$ (if necessary taking the order dual). Let $S = [0, u]$ and $S' = [f, 0]$. Then S is a standard thread, S' is an order dual of a standard thread (i.e., S' is obtained from a standard thread by reversing the order) and the multiplication $*$ in T is defined via a continuous onto homomorphism $\emptyset : S \to S'$ as follows: For $x, y \in S$ and $x', y' \in S'$,

$$x * y = xy, \quad x' * y = x' \emptyset(y)$$
$$x * y' = \emptyset(x) y', \quad x' * y' = x' y',$$

where the multiplication in S (and S') is denoted by juxtaposition. Further, $\emptyset(x) = f * x = x * f$ for all $x \in S$.

However, in the following discussion we shall consider a thread T with identity and interior zero such that the the map $\emptyset : S \to S'$ mentioned in the description of the structure of T is actually an iseomorphism i.e, we consider a T where S' is an order dual of S. Let E and E' denote *the set of idempotents of S and S' respectively. Then, for* every $e \in E$, $\emptyset(e) \in E'$.

Let T_1 be a semigroup with zero 0 such that for $x, y \in T_1$, $x \neq 0$, $y \neq 0$ implies that $xy \neq 0$ and E_1, the set of idempotents of T_1, is totally disconnected.

For the rest of this section we **assume** that we are given an S-machine defined by a (continuous) op-function $g : T \times T \to T_1$ satisfying A2, where T and T_1 are as described above. We now proceed to describe the structure of g for which we shall need a series of intermediate results of which the first is the following.

4.2. Proposition. Let $g : T \times T \to T_1$ be an op-function.

Then :

(a) For any $e \in E$ and for all $x \in [\emptyset(e), e]$,

i) $g(0, e) = g(x, e) \in E_1$, and,

ii) if $g(0, e) = 0$, then $g(e, x) = 0$

(b) For any $e' \in E'$ and for all $x \in [e', 0]$,

i) $g(0, e') = g(x', e') \in E_1$, and,

ii) if $g(0, e') = 0$, then $g(e', x') = 0$.

(c) For any $x \in T$, the following statements are true.

i) If $g(x, y) = 0$ for some $y \in S$, then $g(x, y') = 0$ for all $y' \in [\emptyset(y), y]$.

ii) If $g(x, y) = 0$ for some $y \in S'$, then

$g(x, y') = 0$ for all $y' \in [y, 0]$.

Proof. a(i). For any $e \in E$ and any $x \in [\emptyset(e), e]$, it is clear that $x*e = e*x = x$. Therefore, by A2, $g(x, e) = g(x, e*e) = g(x, e)g(x*e, e) = g(x, e)g(x, e) \in E_1$ for all $x \in [\emptyset(e), e]$. Now, since $[\emptyset(e), e]$ is connected, E_1 is totally disconnected and g is continuous, it follows that $g(0, e) = g(x, e) \in E_1$ for all $x \in [\emptyset(e), e]$.

Proof. a(ii): For any $e \in E$ and any $x \in [\emptyset(e), e]$,

$g(e, x) = g(e, x*e) = g(e, x)g(x, e) = 0$

since $g(x, e) = g(0, e) = 0$ by a(i).

Proof. b(i): For any $e' \in E'$ and any $x' \in [e', 0]$,

$x'*e' = e'*x' = x'$, and so, by A2,

$g(x', e') = g(x', e'*e') = g(x', e')\, g(x', e') \in E_1$.

Therefore, since $[e', 0]$ is connected, E_1 is totally disconnected and g is continuous, it follows that $g(0, e') = g(x', e') \in E_1$.

Proof. b(ii): Follows from b(i) in the same way as a(ii) follows from a(i).

Proof. c(i): Let, for some $y \in S$, $g(x, y) = 0$. We consider two cases:

Case 1. Let $y' \in [0, y]$.

If y and y' belong to the same ligament $U = [e_1, e_2]$ of S, then there exists a $z \in U$, (in fact, $y' \leq z \leq y$) such that $y' = y*z$. Therefore,
$g(x, y') = g(x, y*z) = g(x, y)g(x*y, z)$, by A2, and then since $g(x, y) = 0$, it follows that $g(x, y') = 0$.

If y and y' do not belong to the same ligament, then, since $y' < y$, $y*y' = y'$ and so, $g(x, y') = g(x, y*y') = g(x, y)g(x*y, y') = 0$ because $g(x, y) = 0$.

Case 2. Let $y' \in [\emptyset(y), 0]$.

The proof is exactly similar to that of Case 1 whether y and y' belong to the same ligament of S' or not.

Proof. c(ii): Similar to the case 2 of c(i).

In view of Proposition 4.2(c), if the condition (C_o): $g(0, x) = 0$ for some $x \in T$ is satisfied by g let us define the two elements $x_o \in S$ and $x'_o \in S'$ as follows:

$$x_o = \sup \{x \in S : g(0, x) = 0\}$$

and

$$x'_o = \inf \{x' \in S : g(0, x') = 0\}.$$

Then we can prove the following result which will be very useful in the sequel.

4.3. Proposition. Let $g : T \times T \to T_1$ be an op-function satisfying the condition (C_o) so that x_o, and x'_o exist. Then

i) $x_o \in E$ and $g(0, x_o) = 0$

ii) $x'_o \in E'$ and $g(0, x'_o) = 0$

iii) $x'_o \neq f$ iff $x_o \neq u$ and $\phi(x_o) = x'_o$.

Proof. (i): Let, if possible, $x_o \notin E$ and the ligament of S containing x_o be $U = [e_1, e_2]$. Then there exist $x_1, x_2 \in U$ such that $x_o < x_1$, $x_o < x_2$ and $x_1 * x_2 < x_o$, and so, by **A2**, $g(0, x_1 * x_2) = g(0, x_1) g(0, x_2)$. But, in view of Proposition 4.2 c(i), since both $g(0, x_1) \neq 0$ and $g(0, x_2) \neq 0$ implies that $g(0, x_1 * x_2) \neq 0$ we arrive at a contradiction to the fact that $g(0, x_1 * x_2) = 0$ for $x_1 * x_2 < x_o$. This contradiction shows that $x_o \in E$.

Again, by Proposition 4.2 c(i) and the definition of x_o, since $g(0, y) = 0$ for all $y \in [0, x_o)$, by the continuity of g, it follows that $g(0, x_o) = 0$.

Proof. (ii): Similar to the proof of (i).

Proof. (iii): If $x'_o \neq f$, then $x_o \neq u$. For, otherwise, $g(0, x_o) = g(0, u) = 0$ implies, by Proposition 4.2 c(i), that $g(0, y) = 0$ for all $y \in [\phi(u), u] = [f, u]$, since ϕ is an iscomorphism and $\phi(u) = f$, and so, $g(0, f) = 0$ which is a contradiction to the definition of x'_o . Therefore, $x'_o \neq f$ implies that $x_o \neq u$, $g(0, f) \neq 0$ and, for all $x > x_o$, $g(0, x) \neq 0$. Hence, if $x' = \phi(x)$ for some $x > x_o$, then

$g(0, x') = g(0, f* x') = g(0, f* x) = g(0, f)\, g(0, x) \neq 0$ since both $g(0, f) \neq 0$ and $g(0, x) \neq 0$. But, since $\emptyset$ is an iseomorphism, $x > x_o$ iff $\emptyset(x) < \emptyset(x_o)$, and hence, for all $x' < \emptyset(x_o)$, $g(0, x') \neq 0$. On the other hand, since $g(0, x_o) = 0$ implies that $g(0, x) = 0$ for all $x \in [\emptyset(x_o), x_o]$, we conclude that $\emptyset(x_o) = x'_o$.

The converse case of (iii) is obvious.

At this point we like to remark that in proving Proposition 4.2 we do not require that (a) the map $\emptyset: S \to S'$ is an iseomorphism and (b) T_1 satisfies: for $x, y \in T_1$, $x \neq 0$, $y \neq 0$ implies $x\, y \neq 0$. However, we have used both (a) and (b) in the proof of Proposition 4.3, and, as the following examples show, these conditions can not be dropped.

4.4 <u>Example.</u> Let T be the usual unit thread $[0, 1]$ and T_1 be the nil thread $T / [0, \frac{1}{2}]$. Let $q: T \to T_1$ be the natural homomorphism. Define $g : T \times T \to T_1$ by $g(x, y) = q(y)$. Then g is an op-function and $g(0, y) = \bar{0}$ iff $y \leq \frac{1}{2}$ where $\bar{0}$ denotes the zero of T_1. Here $x_o = \frac{1}{2} \notin E$. In this example the condition (b) is not satisfied and the question of (a), of course, does not arise.

4.5. <u>Example.</u> Let $T = [-1, 1]$ with multiplication defined by letting $[0, 1]$ be the usual unit interval, $[-1, 0]$ the order dual of it, and $\emptyset : [0, 1] \to [-1, 0]$ be defined by $\emptyset(x) = -x$. Then subintervals like $[-\frac{1}{2}, 0]$, $[-\frac{1}{2}, \frac{1}{2}]$ and $[-\frac{3}{4}, \frac{1}{2}]$ are ideals of T. Let $T_1 = T/[-\frac{3}{4}, \frac{1}{2}]$ and define $g : T \times T \to T_1$ by $g(x, y) = q(y)$ where $q : T \to T_1$ is the canonical homomorphism. Here, $x_o = \frac{1}{2} \neq 1$, $x'_o = -\frac{3}{4} \notin E'$ and $\phi(x_o) = -\frac{1}{2} \neq x'_o$. In this example, though the condition (a) is satisfied, the condition (b) is not true.

4.6. Example. Let $T = [-\frac{1}{2}, 1]$ with multiplication defined as follows. Let $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ both be iseomorphic to the usal unit interval, so that $\frac{1}{2}$ is the identity for $[0, \frac{1}{2}]$ and zero for $[\frac{1}{2}, 1]$. If $x \in [0, \frac{1}{2}]$ and $y \in [\frac{1}{2}, 1]$, let $xy = yx = x$. Finally, let $\phi : [0, 1] \to [-\frac{1}{2}, 0]$ be defined by

$$\phi(x) = \begin{cases} -x & \text{if} \quad x \leq \frac{1}{2} \\ -\frac{1}{2} & \text{if} \quad x > \frac{1}{2} \end{cases}$$

ϕ is a homomorphism, because $[\frac{1}{2}, 1]$ is a subsemigroup of $[0, 1]$. (This ϕ would not be a homomorphism if $[0, 1]$ had usual multiplication). This ϕ defines a multiplication on T in the way we have indicated in describing a thread having identity and interior zero in the beginning of this section. Note that $[-\frac{1}{2}, \frac{1}{2}]$ is an ideal of T. Let $T_1 = T/[-\frac{1}{2}, \frac{1}{2}]$ and define $g : T \times T \to T_1$ by $g(x, y) = q(y)$ where $q : T \to T_1$ is the canonical homomorphism. In this case $x_o = \frac{1}{2} \neq u$, $\phi(x_o) = -\frac{1}{2} = x'_o = f$. In this example both the conditions (a) and (b) are violated.

Our next result is the following:

4.7. Proposition. Let $g' : T \times T \to T_1$ be an op-function satisfying the condition (C_0). Then $g(x, y) = 0$ for all $x \in T$ and all $y \in [x'_0, x_0]$.

Proof. From Propositions 4.2(a)(i) and 4.3(i) and the relation $g(x, y) = g(x, x_0 * y) = g(x, x_0)g(x * x_0, y)$ it follows that $g(x, y) = 0$ for all $x, y \in [\emptyset(x_0), x_0]$.

Now, if $x \notin [\emptyset(x_0), x_0]$, then $x * x_0 = x_0$ or $\emptyset(x_0)$, and in any case, $g(x * x_0, y) = 0$ for all $y \in [\emptyset(x_0), x_0]$ and, hence, $g(x, y) = g(x, x_0 * y) = g(x, x_0)g(x * x_0, y) = 0$.

Thus, $g(x, y) = 0$ for all $x \in T$ and all $y \in [\emptyset(x_0), x_0]$ and so, by virtue of Proposition 4.3(iii), if $x'_0 \neq f$, then Proposition 4.7 is proved. But, if $x'_0 = f$, then $g(0, f) = 0$, and so, $g(x', f) = 0$ for all $x' \in S'$. Therefore, $g(x, y') = g(x, y' * f) = g(x, y')g(x * y', f) = 0$ for all $x \in T$ and all $y' \in S'$ since $x * y' \in S'$.

Thus, Proposition 4.7 is proved.

From Proposition 4.7 it is clear that for all $x \notin [x'_0, x_0]$, $g(0, x) \neq 0$ and $g(0, x) \neq 0$ for $x \in S$ (or $x \in S'$) implies that $g(0, y) \neq 0$ for all $y \geq x$ (or $y \leq x$). Therefore, unless $g(x, y) = 0$ for all $x, y \in T$, $g(0, x) \neq 0$ for some $x \in T$. Let us now define the two elements $y_0 \in S$ and $y'_0 \in S'$ as:

$$y_0 = \inf \{y \in S : g(0, y) \neq 0\}$$

and

$$y'_0 = \sup \{y' \in S' : g(0, y') \neq 0\}.$$

The following remarks will be useful in the sequel.

4.8. Remarks: Let $g: T \times T \to T_1$ be an op-function and x_o, x'_o, y_o, y'_o be defined as above.

i) If both x_o and y_o exist, then $x_o = y_o$ and similarly, if both x'_o and y'_o exist, then $x'_o = y'_o$. Further, if $x'_o = y'_o$, then $\emptyset(x_o) = x'_o$.

ii) If y_o exists, then $g(0, y_o) \neq 0$ implies that $y_o = y'_o = 0$.

Proof. (i): Since T is connected, the existence of both x_o and y_o implies that $x_o = y_o$ and, similarly, the existence of both x'_o and y'_o implies that $x'_o = y'_o$. Further, if $x'_o = y'_o$, then $x'_o \neq f$, and so by Proposition 4.3(iii), $\emptyset(x_o) = x'_o$.

Proof.(ii): If $g(0, y_o) \neq 0$, then $y_o = 0$ since otherwise there exists $0 < x < y_o$ such that $g(0, x) = 0$ and so x_o exists and $x_o = y_o$ but then $g(0, x_o) = g(0, y_o) = 0$ which is a contradiction.

Further, there is no $x' \in X'$ such that $g(0, x') = 0$ since otherwise $g(0, 0) = 0$ which is a contradiction. Therefore, y'_o exists and $y'_o = y_o = 0$.

Then we have the following.

4.9. Proposition. Let $g: T \times T \to T_1$ be an op-function.

(a) If y_o and y'_o exist, then the following statements are true.

(i) $g(x, e) \neq 0$ for all $x \in T$ and all $e \in E$ such that $g(0, e) \neq 0$

(ii) $g(e, x) \neq 0$ for all $x > y_o$ and all $e \in E$.

(iii) $g(x', e') \neq 0$ for all $x' \in S'$ and all $e' \in E'$ such that $g(0, e') \neq 0$

(iv) $g(e', x') \neq 0$ for all $x' < y_o'$ and all $e' \in E'$.

(v) If $g(0, y_o) \neq 0$, then $g(x, e) \neq 0$ and $g(e, x) \neq 0$ for all $x \in T$ and all $e \in E \cup E'$.

(b) If y_o exists but y_o' does not (which means that $x_o' = f$), then the following hold.

(i) For any $e \in E$ such that $g(0, e) \neq 0$, $g(x, e) \neq 0$ for all $x \geq \emptyset(e)$.

(ii) $g(e, x) \neq 0$ for all $x > y_o$ and all $e \in E$.

Proof. a(i): Let $e \in E$ such that $g(0, e) \neq 0$. Then, for all $x \in [\emptyset(e), e]$, $g(x, e) \neq 0$, by Proposition 4.2 a(i). So let us consider an $x \notin [\emptyset(e), e]$ and, if possible, let $g(x, e) = 0$. We shall show that this leads to contradictions proving that $g(x, e) \neq 0$. We shall distinguish two cases:

Case 1. $x > e$.

If $x \notin E$, let $x \in [e_1, e_2]$, a ligament of S. We first claim that $g(x, e_2) \neq 0$. For, if $g(x, e_2) = 0$, then, by Proposition 4.2 a(i), $g(0, e_2) = g(x, e_2) = 0$ since $x \in [e_1, e_2]$ (and hence, $x \in [\emptyset(e_2), e_2]$). Hence, $g(0, y) = 0$ for all

$y \in [\emptyset(e_2), e_2]$, by Proposition 4.2(c)(i), which implies that $g(0, e) = 0$ since $e < e_2$ and so $e \in [\emptyset(e_2), e_2]$. This is a contradiction to our basic assumption that $g(0, e) \neq 0$, and hence, our claim that $g(x, e_2') \neq 0$ is established.

Now, since $g(x, e_2) \neq 0$, there exists a θ, $e_1 < \theta < e_2$, such that $g(x, y) \neq 0$ for all $y \in [\theta, e_2]$, since, otherwise, by the continuity of g, it will follow that $g(x, e_2) = 0$. Then note that for any $y \in [\theta, e_2]$, as $e \leq e_1 < e_2$, $e * y = e = y * e$, and so, $g(x, e) = g(x, y * e) = g(x, y)g(x * y, e)$, and since $g(x, y) \neq 0$, $g(x, e) = 0$ iff $g(x * y, e) = 0$. That is, there exists an x', namely,

$$x' = x * \theta, \text{ such that } g(x, e) = 0 \text{ iff}$$
$$g(y, e) = 0 \quad \text{for all } y \in [x', x].$$

Again, if $x \in E$, then, as $x > e$, arguing as above, since $g(x, x) \neq 0$, there exists a θ, $e < \theta < x$, such that $g(x, y) \neq 0$ for all $y \in [\theta, x]$ from which it will follow that there exists an x', namely $x' = x * \theta = \theta$, such that $g(x, e) = 0$ iff $g(y, e) = 0$ for all $y \in [x', x]$. Let us now <u>define an element</u> $x_1 \in S$ as:

$$x_1 = \inf \{ x' \in S : g(x, e) = 0 \text{ iff } g(y, e) = 0 \text{ for all } y \in [x', x] \}.$$

Note that, by the continuity of g, $g(x_1, e) = 0$ if $g(x, e) = 0$.

Now we <u>claim that $x_1 \leq e$</u>. For, if possible, let $x_1 > e$.

Then, since $g(x_1, c) = 0$, arguing as before, there exists an $x' < x_1$ such that $g(y, c) = 0$ iff $g(x_1, e) = 0$ for all $y \in [x', x_1]$ which is a contradiction to the definition of x_1, and hence, $x_1 \leq c$. But $x_1 \nleq c$, since $g(x_1, c) = 0$ and we have already seen that $g(y, e) \neq 0$ for all $y \in [\emptyset(e), e]$.

This contradiction arises from our assumption that $g(x, c) = 0$ for some $x > c$ and, therefore, $g(x, c) \neq 0$ for all $x > c$.

Case 2: $x < \emptyset(c)$.

If $x \notin E'$, let $x \in [c_1', c_2']$, a ligament of S'. Again, we first claim that $g(x, c_1') \neq 0$. For, if $g(x, c_1') = 0$, then, by Proposition 4.2(b)(i), $g(0, c_1') = g(x, c_1') = 0$ since $x \in [c_1', 0]$. Hence, $g(0, y) = 0$ for all $y \in [c_1', 0]$, by Proposition 4.2(c)(ii), which implies that $g(0, \emptyset(c)) = 0$ since $c_1' < x < c_2' \leq \emptyset(c)$. But $g(0, \emptyset(c)) \neq 0$ since $g(0, c) \neq 0$ which follows from the facts, since $\emptyset$ is an iso-morphism and $y_0' = \emptyset(y_0)$, by Remark 4.8(i), that $c > y_0$ iff $\emptyset(c) < y_0'$ and $g(0, c') \neq 0$ for all $x' < y_0'$. Therefore, our claim that $g(x, c_1') \neq 0$ is established.

Now, since $g(x, c_1') \neq 0$, there exists a θ, $c_1' < \theta < c_2'$, such that $g(x, y) \neq 0$ for all $y \in [c_1', \theta]$ since, otherwise, by the continuity of g, $g(x, c_1') = 0$. Then note that, for any $y' \in [c_1', \theta]$, $y' * \emptyset(c) = \emptyset(c) = \emptyset(c) * y'$, since $c_1' < \emptyset(c)$. Therefore, $g(x', \emptyset(c)) = g(x, y' * \emptyset(c)) = g(x', y')g(x * y', \emptyset(e))$

implies, as $g(x, y') \neq 0$, $g(x, \emptyset(e)) = 0$ iff $g(x * y', \emptyset(e)) = 0$. That is, there exists an x', namely $x' = x * \theta$, such that $g(x, \emptyset(e)) = 0$ iff $g(y', \emptyset(e)) = 0$ for all $y' \in [x, x']$.

If $x \in E'$, since $x < \emptyset(e) < y_o' = \emptyset(y_o)$, arguing as before, $g(x, x) \neq 0$, and so, there exists a θ, $x < \theta < \emptyset(e)$, such that $g(x, y) \neq 0$ for all $y \in [x, \theta]$. Therefore, arguing as before, there exists an x', $x < x' < \emptyset(e)$, such that $g(x, \emptyset(e)) = 0$ iff $g(y, \emptyset(e)) = 0$ for all $y \in [x, x']$.

Now, let us define an element $x_1 \in S'$ as:

$$x_1 = \sup \{x' \in S' : g(x, \emptyset(e)) = 0 \text{ iff } g(y, \emptyset(e)) = 0 \text{ for } y \in [e, x']\}.$$

Note that, by the continuity of g, $g(x_1, \emptyset(e)) = 0$ if $g(x, \emptyset(e)) = 0$.

Now we first claim that $x_1 \geq \emptyset(e)$. For, if $x_1 < \emptyset(e)$, arguing as before, we can have an x', $x_1 < x_1' < \emptyset(e)$, such that $g(x_1, \emptyset(e)) = 0$ (which is implied by $g(x, \emptyset(e)) = 0$) implies that $g(y, \emptyset(e)) = 0$ for all $y \in [x_1, x']$ contradicting the definition of x_1. But, again, $x_1 \not> \emptyset(e)$, since $g(x_1, \emptyset(e)) = 0$ and $g(y, \emptyset(e)) \neq 0$ for all $y \in [\emptyset(e), 0]$, by virtue of Proposition 4.2(b)(i) and the fact $g(0, e) \neq 0$ which implies that $g(0, \emptyset(e)) \neq 0$ as $\emptyset$ is an isomorphism. This proves that for all $x < \emptyset(e)$, $g(x, e) \neq 0$.

Thus, $g(x, e) \neq 0$ for all $x \in T$ and all $e \in E$ such that $g(0, e) \neq 0$.

for all $x' \in S'$ and, since $\{(x', e_1') : x' \in S'\}$ is a compact set, there exists a θ, $e_1' < \theta < y_1'$, such that $g(x', y') \neq 0$ for all $x' \in S'$ and $y' \in [e_1', \theta]$. Then, by choosing y' such that $e_1' < \theta < y' < y_1'$ and $y' * \theta > y_1'$ we arrive at a contradiction from the relation $g(e', y' * \theta) = g(e', y')g(e' * y', \theta)$ and the facts that $g(e', y') \neq 0$ and $g(e' * y', \theta) \neq 0$.

Thus, $g(e', x') \neq 0$ for all $x' < y_0'$ and all $e' \in E'$.

Proof. a(v): By Remark 4.8(ii), $y_0 = y_0' = 0$, and hence, by a(i) and a(iii), $g(x, e) \neq 0$ for all $x \in T$ and all $e \in E$ and $g(x', e') \neq 0$ for all $x' \in S'$ and all $e' \in E'$. Now we show that $g(x, e') \neq 0$ for all $x \in S$ and $e' \in E'$. For, if $g(x, e') = 0$ for some $x \in S$ and $e' \in E'$, then, by Proposition 4.2(c)(ii), $g(x, y') = 0$ for all $y' \in [e', 0]$, and thus, $g(x, 0) = 0$ which is a contradiction since $g(x, 0) \neq 0$. Therefore, $g(x, e) \neq 0$ for all $x \in T$ and all $e \in E \cup E'$.

Again, by a(ii) and a(iv), $g(e, x) \neq 0$ for all $x \in S$ and all $e \in E$ and $g(e', x') \neq 0$ for all $x' \in S'$ and all $e' \in E'$. Next we show that $g(e, x') \neq 0$ for all $e \in E$ and $x' \in S'$. For, if $g(e, x') = 0$ for some $e \in E$ and $x' \in S'$, then $g(e, 0) = 0$, by Proposition 4.2(c)(ii), which is a contradiction since $g(e, 0) \neq 0$. Similarly, $g(e', x) \neq 0$ for all $e' \in E'$ and $x \in S$. Thus $g(e, x) \neq 0$ for all $x \in T$ and all $e \in E \cup E'$.

Proof.a)(iii): We give an outline of the proof omitting the details as it is very similar to the previous ones.

Let for $e' \in E'$, $g(0, e') \neq 0$ Then, by Proposition 4.2.b(i) $g(x', e') \neq 0$ for all $x' \in [e', 0]$. So let $x' < e'$. If $x' \notin E'$, let $x' \in [e_1', e_2']$, a ligament of S'. Then we can show that $g(x', e_1') \neq 0$, by using Propositions 4.2 (b)(i) and 4.2 (c)(ii), which will imply that there exists $\theta > x'$ such that $g(x', e') = 0$ iff $g(y', e') = 0$ for all $y' \in [x', \theta]$. If $x' \in E'$, then, since $g(x', x') \neq 0$, there exists a $\theta > x'$ such that $g(x', e') = 0$ iff $g(y', e') = 0$ for all $y' \in [x', \theta]$. Now, if we define $x_1' = \sup\{\theta \in S' : g(x', e') = 0$ iff $g(y', e') = 0$ for all $y' \in [x', \theta]\}$ we can show that x_1' is neither $> e'$ nor $\leq e'$. Therefore, $g(x', e') \neq 0$ for all $x' \in S'$.

Proof.a(iv): We again give only an outline of the proof.

Let $e' \in E'$ and $e_1' \in E'$ such that $e_1' < y_0'$. Then, since $g(0, e_1') \neq 0$, by a(iii), $g(e', e_1') \neq 0$ and hence, $g(e', x') \neq 0$ for all $x' \leq e_1'$ which follows via Proposition 4.2.(c)(ii).

Therefore, $g(e', x') \neq 0$ for all $x' \leq e_1'$ and all $e' \in E'$ where $[e_1', y_0']$ is a ligament of S'.

Now, if possible, let, for some $x' \in (e_1', y_0')$, $g(e', x') = 0$ for some $e' \in E'$ and define $y_1' = \inf\{x' \in (e_1', y_0') : g(e', x') = 0\}$. Note that, by the continuity of g, $g(e', y_1') = 0$ and so $y_1' > e_1'$. Now, $g(0, e_1') \neq 0$ implies, by a(iii), that $g(x', e_1') \neq 0$

for all $x' \in S'$ and, since $\{(x', e_1') : x' \in S'\}$ is a compact set, there exists a θ, $e_1' < \theta < y_1'$, such that $g(x', y') \neq 0$ for all $x' \in S'$ and $y' \in [e_1', \theta]$. Then, by choosing y' such that $e_1' < \theta < y' < y_1'$ and $y' * \theta > y_1'$ we arrive at a contradiction from the relation $g(e', y' * \theta) = g(e', y')g(e' * y', \theta)$ and the facts that $g(e', y') \neq 0$ and $g(e' * y', \theta) \neq 0$.

Thus, $g(e', x') \neq 0$ for all $x' < y_0'$ and all $e' \in E'$.

Proof. a(v): By Remark 4.8(ii), $y_0 = y_0' = 0$, and hence, by a(i) and a(iii), $g(x, e) \neq 0$ for all $x \in T$ and all $e \in E$ and $g(x', e') \neq 0$ for all $x' \in S'$ and all $e' \in E'$. Now we show that $g(x, e') \neq 0$ for all $x \in S$ and $e' \in E'$. For, if $g(x, e') = 0$ for some $x \in S$ and $e' \in E'$, then, by Proposition 4.2(c)(ii), $g(x, y') = 0$ for all $y' \in [e', 0]$, and thus, $g(x, 0) = 0$ which is a contradiction since $g(x, 0) \neq 0$. Therefore, $g(x, e) \neq 0$ for all $x \in T$ and all $e \in E \cup E'$.

Again, by a(ii) and a(iv), $g(e, x) \neq 0$ for all $x \in S$ and all $e \in E$ and $g(e', x') \neq 0$ for all $x' \in S'$ and all $e' \in E'$. Next we show that $g(e, x') \neq 0$ for all $e \in E$ and $x' \in S'$. For, if $g(e, x') = 0$ for some $e \in E$ and $x' \in S'$, then $g(e, 0) = 0$, by Proposition 4.2(c)(ii), which is a contradiction since $g(e, 0) \neq 0$. Similarly, $g(e', x) \neq 0$ for all $e' \in E'$ and $x \in S$. Thus $g(e, x) \neq 0$ for all $x \in T$ and all $e \in E \cup E'$.

Proof.b(i) and (ii): It is clear from the proof of a(i) and a(ii).

Then we have the following important corollary:

4.10. Corollary: Let $g: T \times T \to T_1$ be an op-function

(a) If y_o and y'_o exist, then the following statements are true.

i) $g(x, y) \neq 0$ for all $x \in T$ and $y > y_o$.

ii) $g(x', y') \neq 0$ for all $x' \in S'$ and $y' < y'_o$.

iii) If, further, the condition (c_1): $g(x, f) \neq 0$ for all $x \in S$ holds, then $g(x, y') \neq 0$ for all $x \in S$ and $y' < y'_o$.

iv) If $g(0, y_o) \neq 0$, then $g(x, y) \neq 0$ for all $x, y \in T$.

(b) Let $x'_o = f$(i.e., y'_o does not exist) and y_o exist. If, further, the condition (c_2): $g(x', e) \neq 0$ for all $x' \in S'$ and $e \in E$ such that $g(0, e) \neq 0$ holds, then $g(x, y) \neq 0$ for all $x \in T$ and $y > y_o$.

Proof.a(i): We first claim that (A): $g(x, y) \neq 0$ for all $x \in T$ and $y \geq e > y_o$ where $e \in E$. First note that, for any $e \in E$ such that $e > y_o$, since $g(0, e) \neq 0$, by Proposition 4.9.a(i), $g(e, e) \neq 0$ for all $x \in T$. Now, if possible, let for some $y_1 \in [e_1, e_2]$, a ligament of S such that $e_1 > y_o$, $g(x, y_1) = 0$ for some $x \in T$. Then it follows, by Proposition 4.2.c(i), that $g(x, y) = 0$ for all $y \in [\emptyset(y_1), y_1]$, and hence, $g(x, e_1) = 0$ which is a contradiction as $e_1 > y_o$. Therefore, our claim (A) is established.

Now, since $g(x, e_1) \neq 0$ for all $x \in T$ where e_1 corresponds to the right end point of the ligament $[y_o, e_1]$ and the set $\{(x, e_1) : x \in T\}$ is compact, it follows that there exists a θ, $y_o < \theta < e_1$, such that $g(x, y) \neq 0$ for all $x \in T$ and $y \in [\theta, e_1]$. Let $y' = \inf \{\theta \in [y_o, e_1]: g(x, y) \neq 0$ for all $x \in T$ and $y \in [\theta, e_1]\}$

We claim that $y' = y_o$. Clearly $y' \geq y_o$. So, if possible, let $y' > y_o$. Now $g(x, y) \neq 0$ for all $x \in T$ and $y \in [y', e_1]$.

We can choose $y_1, y_2 > y'$ such that $y_o < y_1 * y_2 < y'$, and then,

$g(x, y_1 * y_2) = g(x, y_1)g(x * y_1, y_2)$ implies that $g(x, y_1 * y_2) \neq 0$ as both $g(x, y_1) \neq 0$ and $g(x * y_1, y_2) \neq 0$. But $y_1 * y_2 < y'$ which is a contradiction to the definition of y'. Therefore, $y' = y_o$.

Thus, $g(x, y) \neq 0$ for all $x \in T$ and $y > y_o$.

<u>Proof.a(ii)</u>: Again we can easily show that $g(x', y') \neq 0$ for all $x' \in S'$ and $y' \leq e' < y_o'$ where $e' \in E'$ by using Proposition 4.9.a(iii) and arguments similar to those in the proof of a(i).

Then, as before, from the facts that $g(x', e_1') \neq 0$ for all $x' \in S'$ where e_1' corresponds to the left end point of the ligament $[e_1', y_o']$ of S' and that $\{(x', e_1'): x' \in S'\}$ is a compact set, there exists a θ', $e_1' < \theta < y_o'$, such that $g(x', y') \neq 0$ for all $x' \in S'$ and $y' \in [e_1', \theta']$. Now, if we define y'' as

$$y'' = \sup\left\{ \theta' \in [e_1', y_0'] : g(x', y') \neq 0 \text{ for all } x' \in S' \text{ and } y' \in [e_1', \theta'] \right\}$$

then we can show that $y'' = y_0'$.

This proves that $g(x', y') \neq 0$ for all $x' \in S'$ and $y' < y_0'$.

Proof.a(iii): For any $x \in S$ and $y' < y_0'$, since $x * f = \emptyset(x) \in S'$ and $f * y' = y'$, $g(x, y') = g(x, f*y') = g(x, f)g(\emptyset(x), y') \neq 0$, because $g(x, f) \neq 0$, by (C_1), and $g(\emptyset(x), y') \neq 0$, by a(ii).

Proof.a(iv): By Remark 4.8(ii), $y_0 = y_0' = 0$. Now $g(0, 0) \neq 0$ implies, by Proposition 4.9.a(i), that $g(x, 0) \neq 0$ for all $x \in T$ which, in turn, implies that $g(x, f) \neq 0$ for all $x \in T$ because, by Proposition 4.2.c(ii), $g(x, f) = 0$ will imply $g(x, 0) = 0$. Thus, the condition (C_1) is satisfied. Now a(iv) follows from a(i) — a(iii).

Proof.(b): Because of the condition (C_2), and Proposition 4.9.b(i), Proposition 4.9.a(i) is true. Now if we look at the proof of a(i) we see that Proposition 4.9.a(i) implies that $g(x, y) \neq 0$ for all $x \in T$ and $y > y_0$.

From the above discussion it is clear that, if we had considered an op-function $g: S \times S \to T_1$, where S is a standard thread instead of a thread T we considered above, then we could have obtained by somewhat less efforts the following.

4.11. Remark: Let, for a standard thread S, $g: S \times S \to T_1$ be an op-function where T_1 is the same as before such that y_o exists. Then, for all $x \in S$ and $y > y_o$, $g(x, y) \neq 0$ and $g(x, y) = 0$ for all $x \in S$ and $y \leq y_o$. Further, if $g(0, y_o) \neq 0$, then $g(x, y) \neq 0$ for all $x, y \in S$. In particular, if $S = [0, 1]$, the unit thread, then, if y_o exists, $y_o = 0$, and, hence, Proposition 4.1 is obtained as a very special case.

For the rest of this section let us assume that T_1 is a group with zero 0 (i.e., T_1 is a semigroup with zero 0 such that $T_1 \setminus 0$ is a group). Then towards the structure of an op-function $g: T \times T \to T_1$ we have the following results.

4.12 Proposition. Let $g: T \times T \to T_1$ be a function. Then the following statements are equivalent.

i) g is an op-function such that $g(x, y) \neq 0$ for all $x, y \in T$.

ii) g is an op-function such that $g(0, 0) \neq 0$.

iii) There exists a continuous function $b : T \to T_1$ such that $b(x) \neq 0$ for all $x \in T$ and $g(x, y) = b(x)^{-1} b(x * y)$ for all $x, y \in T$.

Proof. Follows from Remark 4.8 (ii), Corollary 4.10.a(iv) and Proposition 2.4 or Proposition 2.6.

The next few results are concerned with op-functions $g : T \times T \to T_1$ such that neither $g(x, y) = 0$ for all $x, y \in T$ nor $g(x, y) \neq 0$ for all $x, y \in T$. However, for this case, the description of the structure of g is not complete.

4.13. Proposition. Let $g : T \times T \to T_1$ be a function. Then the following two statements are equivalent.

1) g is an op-function such that both y_o and y_o' exist, $g(0, y_o) = 0$ and the condition (C_1), i.e., $g(x, f) \neq 0$ for all $x \in S$, is satisfied.

2) There exists an $e_o \in E$ such that, for any idempotent $e > e_o$, there are three continuous functions $h_i : T \to T_1$ $i = 1,2,3$, satisfying

(i)(a) $h_1(x) \neq 0$ for all $x \in T$, $i = 1, 2$, and $h_3(x) \neq 0$ iff $x \notin [\phi(e_o), e_o]$, and

(b) there exist two constants $d_1, d_2 \in T_1$ such that $d_1 \neq 0$, $d_2 \neq 0$ and

$$h_1(x) = h_3(x)^{-1} d_1 \qquad \text{for all } x \geq e,$$

$$h_1(x) = h_3(x)^{-1} d_2 \qquad \text{for all } x \leq \phi(e),$$

and $$h_2(x) = h_3(x)^{-1} d_2 \qquad \text{for all } x \notin (\phi(e), e);$$

and (ii)(a) $g(x, y) = 0$ iff $x \in T$ and $y \in [\phi(e_o), e_o]$,

(b) $$g(x, y) = \begin{cases} h_1(x)\, h_1(x * y)^{-1} & \text{if } y \geq e \\ h_2(x)\, h_2(x * y)^{-1} & \text{if } y \leq \phi(e) \end{cases}$$

for all $x \in T$,

(c) $g(x, y) = h_3(x)^{-1} h_3(x * y)$ for all $x \notin [\phi(e_o), e_o]$ and $y \in T$,

(d) $g(x, \cdot)$ is a homomorphism from $[e_o, e]$ (and from $[\phi(e), \phi(e_o)]$) into T_1 for all $x \in [\phi(e_o), e_o]$ (and for all $x \in [\phi(e_o), 0]$).

(e) $g(x, y') = h_2(x)g(\emptyset(x), y')$ for all $x \in [0, e_o]$ and $y' \in [\emptyset(e), \emptyset(e_o)]$, and, finally,

(f) g, defined via (ii)(a) — (ii)(e), is continuous.

Proof: 1) => 2). Let $e_o = y_o$. Then, by Remark 4.8(i), $\emptyset(e_o) = y_o'$. Now define, for any idempotent $e > e_o$, $h_i : T \to T_1$, $i = 1,2$, by $h_1(x) = g(x, e)$ and $h_2(x) = g(x, \emptyset(e))$ for all $x \in T$. Let $h_3 : T \to T_1$ be defined by $h_3(x) = g(u, x)$ for all $x \in T$. Clearly, h_i, $i = 1,2,3$, is a continuous function.

Now, by virtue of Proposition 4.7 and Corollary 4.10(a), $g(x, y) = 0$ iff $x \in T$ and $y \in [\emptyset(e_o), e_o]$, and hence, (i) and (ii)(a) are satisfied.

We shall next verify (i)(b). For that, let $d_1 = g(u, e)$ and $d_2 = g(u, \emptyset(e))$. Then, for any $x \geq e$, $h_3(x)\, h_1(x) = g(u, x)g(x, e) = g(u, x * e) = g(u, e) = d_1$; for any $x \leq \emptyset(e)$, $h_3(x)h_1(x) = g(u, x * \emptyset(e)) = g(u, \emptyset(e)) = d_2$; and finally, for any $x \notin [\emptyset(e), e]$, $h_3(x)h_2(x) = g(u, x * \emptyset(e)) = g(u, \emptyset(e)) = d_2$, and, therefore, (i)(b) is verified.

Now, for any $x \in T$ and $y \geq e$, $g(x, e) = g(x, y * e) = g(x, y)g(x * y, e)$ so that $g(x, y) = h_1(x)h_1(x * y)^{-1}$, and, for any $x \in T$ and $y \leq \emptyset(e)$, $g(x, \emptyset(e)) = g(x, y)g(x * y, \emptyset(e))$ so that $g(x, y) = h_2(x)h_2(x * y)^{-1}$. Therefore, (ii)(b) is satisfied.

Again, for any $x \in [\emptyset(e_o), e_o]$ and $y \in T$, $g(u, x * y) = g(u, x)g(x, y)$, and so, by (i)(a), $g(x, y) = h_3(x)^{-1} h_3(x * y)$. Therefore, (ii)(c) is satisfied.

Let $x \in [\emptyset(e_o), e_o]$ and $y_1, y_2 \in [e_o, e]$. Then

$$g(x, y_1 * y_2) = g(x, y_1)g(x * y_1, y_2) = g(x, y_1)g(x, y_2)$$

and, therefore, $g(x, \cdot)$ is a homomorphism from $[e_o, e]$ into T_1 for any $x \in [\emptyset(e_o), e_o]$. Similarly, $g(x, \cdot)$ is a homomorphism from $[\emptyset(e), \emptyset(e_o)]$ into T_1 for any $x \in [\emptyset(e_o), r]$. Therefore, (ii)(d) is satisfied.

Next, for any $x \in [0, e_o]$ and $g' \in [\emptyset(e), \emptyset(e_o)]$,
$g(x, y') = g(x, \emptyset(e) * y') = g(x, \emptyset(e))g(x * \emptyset(e), y') = h_2(x)g(\emptyset(x), y')$, and so, (ii)(e) is also satisfied.

Finally, g, being given to be an op-function, is continuous.

2) => 1). We shall show that g, defined by (ii), is well defined by virtue of (i), and is an op-function satisfying the conditions of 1)

To show that g is well-defined via (ii)(a) - (ii)(e), we shall have to only check that the values of $g(x, y)$ for those $x, y \in T$ for which g is defined, in (ii)(b) and (ii)(c), in-terms of both h_1 and h_3 (or h_2 and h_3) are same whether g is defined by h_1 or h_3 (or by h_2 or h_3) and this can be easily done by virtue of (i)(b). For example, for any $y \geq e$ and $x > e_o$, $g(x, y) = h_1(x)h_1(x * y)^{-1}$, by (ii)(b), and $g(x, y) =$ $h_3(x)^{-1} h_3(x * y)$, by (ii)(c). But, since, for $y \geq e$ and $x \geq e$, $x * y \geq e$, and, for $y \geq e$ and $e_o < x < e$, $x * y = x$, we see that, if $x * y \geq e$, by virtue of the relation $h_1(x) =$ $h_3(x)^{-1} d_1$ for all $x \geq e$, $g(x, y) = h_1(x)h_1(x * y)^{-1} =$ $h_3(x)^{-1} d_1 d_1^{-1} h_3(x * y) = h_3(x)^{-1} h_3(x * y)$ and, if $x * y = x$, $g(x, y) = h_1(x)\, h_1(x)^{-1} = h_3(x)^{-1} h_3(x) = 1$.

Similarly, we can verify all other cases and show that g is well-defined and, by (ii)(f), g is continuous. So only things that remain to be shown are that g satisfies axiom A2 and the conditions of 1) are satisfied.

Now, by (ii)(a), for all $x \in T$ and $y \in [\emptyset(e_o), e_o]$, g trivially satisfies A2, and, if $y \geq e > e_o$ (or $y \leq \emptyset(e) < \emptyset(e_o)$) g satisfies A2 by virtue of (ii)(b). Again, if $x \notin [\emptyset(e_o), e_o]$ and $y \in (e_o, e)$ [or $y \in (\emptyset(e), \emptyset(e_o))$], g satisfies A2 by virtue of (ii)(c) and, finally, if $x \in [\emptyset(e_o), e_o]$ and $y \in (e_o, e)$ [or $y \in (\emptyset(e), \emptyset(e_o))$], g satisfies A2 by virtue of (ii)(d) and (ii)(e).

Finally, by (ii)(a) it follows that $g(0, y_o) = 0$ in view of Remark 4.8(ii), and, as $e_o \neq u$, (ii)(a) further implies that $g(x, f) \neq 0$ for all $x \in T$ and hence, the condition (C_1) is satisfied as well as both y_o and y_o' exist, since $x_o' \neq f$.

If instead of a thread T we consider a standard thread (or a unit thread) S, then concerning op-functions $g : S \times S \to T_1$, the Proposition 4.13 takes the following special form.

4.14. Proposition. Let $g : S \times S \to T_1$ be a function then the following two statements are equivalent.

1) g is an op-function such that y_o exists and $g(0, y_o) = 0$.

2) There exists an $e_o \in E$ such that, for any idempotent $e > e_o$, there are two continuous functions $h_i : S \to T_1$, $i = 1, 2$, satisfying

   i)(a) $h_1(x) \neq 0$ for all $x \in S$ and $h_2(x) \neq 0$ for all $x > e_o$, and

   (b) there exists a constant $d \in T_1$ such that $d \neq 0$ and $h_1(x) = h_2(x)^{-1} d$ for all $x \geq e$.

   ii)(a) $g(x, y) = 0$ iff $x \in S$ and $y \in [0, e_o]$,

   (b) $g(x, y) = h_1(x) h_1(xy)^{-1}$ for all $x \in S$ and $y \geq e$,

   (c) $g(x, y) = h_2(x)^{-1} h_2(xy)$ for all $x > e_o$ and $y \in S$.

   (d) $g(x, \cdot)$ is a homomorphism from $[e_o, e]$ into T_1 for all $x \in [0, e_o]$, and, finally,

   (e) g, defined via (ii)(a) - (ii)(d), is continuous

In case S is the unit thread, then 2) can be replaced by the following.

2)' $g(x, 0) = 0$ for all $x \in S$ and $g(x, y) = h(x)^{-1}h(xy)$ for all $x \in S$ and $y > 0$ where h is a continuous function $h : S \to T_1$ such that $h(x) \neq 0$ iff $x > 0$.

Finally, we have the following.

4.15. Proposition. Let $g : T \times T \to T_1$ be a function. Then the following two statements are equivalent.

1) g is an op-function such that y_0 exists, x_0' exists, and $x_0' = f$, and the condition (C_2), i.e., $g(x', e) \neq 0$ for all $x' \in S'$ and $e \in E$ such that $g(0, e) \neq 0$ holds.

2) There exists an $e_0 \in E$ such that for any idempotent $e > e_0$, there are three continuous functions $h_i : T \to T_1$, $i = 1, 2, 3$, satisfying.

i)(a) $h_1(x) \neq 0$ for all $x \in T$,

$h_2(x) \neq 0$ iff $x > e_0$, and

$h_3(x) \neq 0$ iff $x \notin [\emptyset(e_0), e_0]$, and

(b) there exist two non-zero constants, $d_1, d_2 \in T_1$ such that

$h_1(x) = h_2(x)^{-1} d_1$ for all $x \geq e$,

$h_1(x) = h_3(x)^{-1} d_2$ for all $x \notin (\emptyset(e), e)$;

and iii)(a) $g(x, y) = 0$ iff $x \in T$ and $y \leq e_0$,

(b) $g(x, y) = h_1(x)h_1(x * y)^{-1}$ for all $x \in T$ and $y \geq e$,

(c) $g(x, y) = h_2(x)^{-1} h_2(x * y)$ for all $x > e_0$ and $y \in T$,

(d) $g(x, y) = h_3(x)^{-1} h_3(x * y)$ for all $x < \emptyset(e_0)$ and $y \in T$,

(e) $g(x, \cdot)$ is a homomorphism from $[e_o, e]$ into T_1 for all $x \in [\emptyset(e_o), e_o]$, and finally,

(f) g, defined via (ii)(a) - (ii)(e), is continuous.

Proof. The proof is very similar to that of Proposition 4.13 and so we give only an outline.

1) => 2). Let $e_o = y_o$. Define, for any idempotent $e > e_o$, $h_i : T \to T_1$, $i = 1,2,3$ as, for all $x \in T$,

$$h_1(x) = g(x, e), \; h_2(x) = g(u, x) \text{ and}$$

$$h_3(x) = \begin{cases} g(f, x) & \text{if } x \geq 0 \\ g(f, \emptyset^{-1}(x)) & \text{if } x < 0 \end{cases}$$

Now, by Proposition 4.7 and Corollary 4.10(b), (i)(a) and (ii)(a) are satisfied. If we set $d_1 = g(u, e)$ and $d_2 = g(f, e)$, then (i)(b) can be easily verified. The verifications of (ii)(b) - (ii)(f) are routine and omitted.

2) => 1). As in the proof of Proposition 4.13, it is easy to show that, by virtue of (i)(b), the definition of g, via (ii)(a) - (ii)(e), is unambiguous and g is a continuous op-function. Further, (ii)(a) guarantees the conditions of 1) to be satisfied by g.

In this section we have studied op-functions $g : T \times T \to T_1$ for a special class of threads amongst those with idempotent end points having identity and interior zero. There are other types of threads with identity and with or without (interior) zero but not having both end points as idempotents, e.g., the interval [-1, 1] with usual real multiplication and many types of interval semigroups [12, 37]. While it is of interest to study op-functions in case of other types of threads and interval semigroups we do not make an attempt to do so in this dissertation.

CHAPTER III

ON SOME PROPERTIES OF TOPOLOGICAL MACHINES

1. Introduction and Summary

1.1. Introduction

Let $M = \langle X, S, T, f, g \rangle$ be an algebraic machine. A set of Hausdorff topologies on X, S and T for which M becomes a topological machine, i.e., S and T become topological semigroups and f and g become continuous, will be referred to as a set of compatible topologies on M. There may be several sets of compatible topologies on M and two topological machines corresponding to two different sets of compatible topologies on M will be referred to as two topological variants of M. While it will be of some interest to us to obtain conditions that guarantee the uniqueness of one or more of a set of compatible topologies on M, our main objective in this chapter is to generalize certain basic concepts and results of conventional algebraic machines to the topological case. But because of the topological structures endowed in the input, output and state spaces of a topological machine it is not possible to obtain immediate generalizations of the results of the algebraic theory to the topological set up. In fact, we shall show that with natural generalizations of the concepts of algebraic theory to topological case we need to put much topological restrictions in order to obtain results for topological machines which are generalizations of the corresponding results for algebraic machines. However, this is quite a common feature in many parts of topological algebra. For example, if S is an algebraic semigroup and C is a congruence on S, then it is well known that the multiplication in S induces canonically a multiplication in S/C, the set of all equivalence classes with respect to C, so as to make S/C an algebraic semigroup, called the quotient semigroup. But, if S is a topological semigroup and C is any congruence on S, then this canonically defined multiplication in S/C may not make S/C a topological semigroup, and, in fact, S/C may not be even Hausdorff or, even if S/C is Hausdorff, the canonical multiplication in S/C may fail to be continuous. Of course, if S is compact, then it is well known that S/C will be a topological semigroup and this harsh condition of compactness of S which is, of course, not necessary is quite a standard hypothesis. In a recent paper [30], however, B Madison discussed this problem and obtained several other sufficient conditions for which S/C becomes a topological semigroup. Similar problems arise in the case of acts or machines too and some of the results of Madison which we shall mention in the sequel will be of much relevance to our discussion in this chapter. In the next few sections we shall generalize the concepts of state equivalence, input equivalance, machine equivalence, reduced and input-reduced forms, etc., and the basic results related to these concepts from the algebraic theory to the topological case. For the algebraic theory of machines we refer to Ginsburg [21], Hartmanis and Stearns [23] and Arbib [2].

We shall follow our earlier conventions in using the term machine for a topological machine, a space for a Hausdorff space, a semigroup for a topological semigroup and that all topologies to be Hausdorff topologies unless stated otherwise. We shall also assume that the output semigroups of all machines considered in this chapter are left cancellative. The letter M (with or without subscript or superscript) shall be used to denote a machine $M = \langle X, S, T, f, g \rangle$ (with same subscript or superscript on X, S, T, f, and g). We also assume that all machines in this chapter satisfy A1 and A2 but need not satisfy A3 and A4 or A4'.

We conclude this introductory section by giving a brief summary of the contents of the subsequent sections of this chapter.

1.2. Summary. In Section 2 certain results of Kelemen [28] for recursions concerning uniqueness of compatible topologies are presented in a slightly general set up which are applicable in the sequel. In Section 3 the concepts of state equivalence, iseomorphism of machines and reduced form of a machine are introduced and certain sufficient conditions are obtained for the existence and uniqueness upto iseomorphism of the reduced form of a machine. In Section 4 the concepts of input equivalence, input iseomorphism of machines and input-reduced form of a machine are introduced and certain sufficient conditions are obtained for the existence and uniqueness upto input-iseomorphism of the input reduced form of a machine. In this section certain results are also obtained concerning the topological version of a problem of Ginsburg on the existence of a input-distinguished machine with a compact state space for any given input semigroup. In Section 5 the concepts of machine equivalences are introduced and certain results analogous to algebraic theory are proved. Finally, in Section 6 a few relevant topological facts are proved.

2. Uniqueness of Certain Compatible Topologies

For each set of compatible topologies for a machine M we get a topological variant of M. Under what conditions are one or more of these compatible topologies uniquely determined ? This question for recursions was discussed by Keleman [28] We can state his results in a slightly general set up from which similar results can be directly read off for topological machines. The purpose of this section is to mention these briefly.

Let X, Y, Z be any three spaces. For a net $\{x_\alpha\}$ in X $\lim x_\alpha = \infty$ if $\{x_\alpha\}$ does not have a converging subnet.

A continuous function $\sigma : X \to Y$ is said to be IP (infinity preserving), if whenever $\{x_\alpha\}$ is a net in X such that $\lim x_\alpha = \infty$, then $\lim \sigma(x_\alpha) = \infty$. A continuous function $\mu : X \times Y \to Z$ is said to be IP (or weakly IP or W IP) on X if the continuous partial map $\mu_x : Y \to Z$, $\mu_x(y) = \mu(x, y)$, is IP for all (or some) $x \in X$.

Then the results of Kelemen can be stated in a slightly general form as follows. The proofs are essentially the same as those of Kelemen and we include them for the sake of completeness.

2.1. Proposition. Let, for any two spaces X and Z, and any non-empty set Y, $\mu : X \times Y \to Z$ be a function.

Let μ be effective on X (i.e. $\mu(x, y_1) = \mu(x, y_2)$ for all $x \in X$ implies $y_1 = y_2$).

i) Let T_1 and T_2 be two topologies on Y such that under each of T_1 and T_2, μ is continuous with respect to product topology on $X \times Y$ and is WIP on X. Then $T_1 = T_2$.

ii) Let T_1 and T_2 be two compact topologies on Y such that under each of T_1 and T_2, μ is continuous with respect to product topology on $X \times Y$. Then $T_1 = T_2$.

Proposition 2.1 follows immediately from the following.

2.2. Proposition. Let, for any three spaces X, Y and Z, $\mu : X \times Y \to Z$ be a continuous function which is effective and WIP on **X**. Then Y is homeomorphic to the subspace $\{\mu_y : X \to Z : y \in Y\}$ of $C(X, Z)$, the set of all continuous maps from X to Z with compact-open topology, where $\mu_y(x) = \mu(x, y)$ for all $x \in X$.

Proof. Let $W = \{\mu_y : y \in Y\}$. Since $\mu_y = \mu | X \times \{y\}$, μ_y is continuous and thus $W \subset C(X, Z)$. Let $h : Y \to W$ be defined by $h(y) = \mu_y$ for all $y \in S$ We will show that h is a homeomorphism. h is clearly onto and if $h(y_1) = h(y_2)$, then $\mu_{y_1}(x) = \mu_{y_2}(x)$ for all $x \in X$ which implies that $y_1 = y_2$ since μ is effective. Thus h is a bijection.

The notation $(K, V) = \{\sigma \in C(X, Z): \sigma(K) \subset V\}$, where K is compact and V is open, is used to denote a subbasic open set of the compact-open topology. We next show that h is continuous. Let $h(y) \in (K, V) \cap W$, a subbasic open set in W. Then $\mu_y(K) \subset V$. Choose U_o in Y and V_o open in X such that $y \in U_o$, $K \subset V_o$ and $\mu(V_o \times U_o) \subset V$. This can be done since K is compact and μ is continuous. Let $t \in U_o$, then $\mu_t(K) \subset \mu(V_o \times U_o) \subset V$ implies that $\mu_t \in (K, V)$ which in turn implies that $h(U_o) \subset (K, V) \cap W$ and, since $y \in U_o$, this means that h is continuous.

To complete the proof, we now show that h is open. Let $O \subset Y$ be open and let $\mu_y \in h(O)$. If we can find (A)

$K_1, K_2, \ldots K_n$ compact in X and $U_1, U_2, \ldots, U_n$ open in Z such that $\mu_y \in \bigcap\{(K_i, U_i);\ i = 1, \ldots, n\} \cap W \subset h(O)$ then we are finished. Suppose the desired sets do not exist. Let $\mathcal{F}$ be the family of all finite intersections of subbasic open sets of $C(X, Z)$ that contain μ_y. Thus if $F \in \mathcal{F}$, then $\mu_y \in F$ and $F = \bigcap\{(K_i, U_i);\ i = 1, \ldots, n\}$ for some n where each K_i is compact in X and each U_i is open in Z. Let D be an index set for $\mathcal{F}$ and if $\alpha, \beta \in D$, define $\alpha < \beta$ if $F_\beta < F_\alpha$. Since $F_\alpha, F_\beta \in \mathcal{F}$ implies that $F_\alpha \cap F_\beta \in \mathcal{F}$, it follows that $(D, <)$ is a directed set. For each $\alpha \in D$, choose y_α such that $\mu_{y_\alpha} \in F_\alpha$ but $y_\sigma \notin O$. Since (A) does not occur, we can always make this choice.

Now $\{y_\alpha\}$ is a net in Y. We first show that $\lim y_\alpha = \infty$. Suppose $\{y_\beta\}$ **were** a subnet of $\{y_\alpha\}$ which converged to y_o. Then $y_o \in Y \setminus O$ since $\{y_\alpha\} \subset Y \setminus O$ which is closed. Thus $y_o \neq y$ since $y \in O$. By the effectiveness of μ, there exist $x \in X$ such that $\mu(x, y_o) \neq \mu(x, y)$. Choose U, V open in Z such that $\mu(x, y_o) \in U$, $\mu(x, y) \in V$ and $U \cap V = \emptyset$ and then select U' open in Y such that $y_o \in U'$ and $\mu(\{x\} \times U') \subset U$. Finally, let $\partial \in D$ be such that $F_\partial = (x, V)$. Then $\alpha > \partial$ implies $\mu_{y_\alpha} \in F_\partial$ which means $\mu(x, y_\alpha) \in V$ **and** thus $y_\alpha \notin U'$. But this contradicts the fact that a subnet of $\{y_\alpha\}$ converges to $y_o \in U'$. Therefore, $\{y_\alpha\}$ has no convergent subnets, i.e. $\lim y_\alpha = \infty$.

We now show that $\lim y_\alpha = \infty$ contradicts the fact that μ is WIP on X. Let for any given $x \in X$ $z_x \in Z$ be such that $\mu(x, y) = z_x$, and V be any open set in Z with $z_x \in V$. Choose ∂ so that $F_\partial = (x, V)$. Then, for $\alpha > \partial$, $\mu_{y_\alpha} \in F_\partial$ which implies that $\mu(x, y_\alpha) \in V$. Thus $\lim \mu(x, y_\alpha) = z_x$ for all $x \in X$ which contradicts the fact that μ is WIP on X. Therefore, (A) may not be denied which means that h is an open map, and hence, is a homeomorphism.

2.3 Proposition. Let, for any two spaces X and Y and any non-empty set $Z, \mu : X \times Y \to Z$ be a function. Let, for some $x_o \in X$, $\mu_{x_o}(Y) = \mu(x_o, Y) = Z$.

i) Let T_1 and T_2 be two topologies on Z such that under each of T_1 and T_2, the partial map μ_{x_o} is IP and continuous. The $T_1 = T_2$.

ii) Let T_1 and T_2 be two compact topologies on Z such that under each of T_1 and T_2, μ is continuous. Then $T_1 = T_2$.

Proof. We use nets and the notations $_Y\lim$, $_1\lim$, $_2\lim$ to indicate limits taken in Y, (Z, T_1) and (Z, T_2) respectively. Suppose that the set $F \subset Z$ is closed in T_1 but not in T_2. Then there is a net $\{z_\alpha\} \subset F$ such that $_2\lim z_\alpha = z_1 \in Z \setminus F$. For each α, choose $y_\alpha \in Y$ such that $\mu(x_o, y_\alpha) = z_\alpha$ and note that $_2\lim(\mu(x_o, y_\alpha)) = z_1$, which implies that $_Y\lim y_\alpha \neq \infty$

because μ_{x_o} is IP. Thus, there is a convergent subnet $\{y_\beta\}$ of $\{y_\alpha\}$. Let ${}_Y\lim y_\beta = y_1$ and observe that $\mu(x_o, y_1) = \mu(x_o, {}_Y\lim y_\beta) = {}_2\lim \mu(x_o, y_\beta) = {}_2\lim \mu(x_o, y_\alpha) = {}_2\lim z_\alpha = z_1$ since $\{\mu(x_o, y_\beta)\}$ is a subnet of $\{\mu(x_o, y_\alpha)\}$ and $\mu_{x_o} : \{x_o\} \times Y \to (Z, T_2)$ is continuous. Let $\mu(x_o, y_\beta) = z_\beta$ for each β. Then the continuity of $\mu_{x_o} : \{x_o\} \times Y \to (Z, T_1)$ implies that $z_1 = \mu(x_o, y_1) = \mu(x_o, {}_Y\lim y_\beta) = {}_1\lim \mu(x_o, y_\beta) = {}_1\lim z_\beta$ and $\{z_\beta\} \subset F$ converges to z_1 in T_1. But F is closed in T_1 and $z_1 \in Z$ F is a contradiction. Thus, every set that is closed in T_1 is closed in T_2. Similarly, every set closed in T_2 is closed in T_1 so that $T_1 = T_2$.

The roles of X and Y can be interchanged in the above propositions.

Kelemen's results stated above can be used to state various conditions on f and g that guarantee uniqueness of one or more compatible topologies on a machine. We do not state them explicitly here.

3. On the Reduced Form of a Machine

All machines considered in this section are assumed to have the same input and output semigroups S and T respectively.

Two machines M_1 and M_2 are said to be topologically isomorphic or, simply, iseomorphic, written $M_1 \cong M_2$, if there exists a homeomorphism $h : X_1 \to X_2$, satisfying, for each $x \in X_1$ and each $s \in S$, the following conditions of algebraic isomorphism [21],

1) $g_1(x, s) = g_2(h(x), s))$, and

2) $h(f_1(x, s)) = f_2(h(x), s))$.

A state x_1 of M_1 is said to be equivalent to a state x_2 of M_2, written $x_1 \sim x_2$, if $g_1(x_1, s) = g_2(x_2, s)$ for each $s \in S$. A machine M is in reduced form or distinguished if for $x, y \in X$ $x \sim y$ implies that $x = y$. A machine M' is a reduced form of M if there exists a continuous onto map $h: X \to X'$ such that $x \sim h(x)$ for all $x \in X$, and M' is distinguished.

We now proceed to investigate whether for a machine there exists a reduced form, and if so, whether a reduced form is unique upto iseomorphism.

The following lemma is well known [cf. Lemma 3.1 [21]] and follows from the fact that the output semigroup is left cancellative.

3.1 Lemma. Let M_1 and M_2 be two machines. For $x_1 \in X_1$ and $x_2 \in X_2$, if $x_1 \sim x_2$, then, for any $s \in S$, $f_1(x_1, s) \sim f_2(x_2, s)$.

We shall also need the following topological fact.

3.2. Lemma. Let X be any arbitrary topological space (X need not satisfy any separation axiom), Y be any T_2 space and D be any non-empty set. Let $\{h_k, k \in D\}$ be a family of continuous maps from X into Y and let R be the equivalence relation on X defined by xRy iff $h_k(x) = h_k(y)$ for all $k \in D$. Then the quotient space X/R is a Hausdorff space.

Proof. Note that the product space Y^D is a T_2 space and the map $h : X \to Y^D$, defined by $h(x) = (h_k(x))$, $k \in D$, is continuous. Then the lemma follows from a known fact [cf. Proposition 9, p. 79, [9]].

It is well known [cf. Theorem 3.2, [21]] that if M is an algebraic machine, then there exists a unique (upto isomorphism) reduced form M' of M. M' is defined by taking the state space X' as the quotient set $X/\sim$, the set of all equivalence classes with respect to the equivalence relation $\sim$ on X, and the functions f' and g' are canonically defined via Lemma 3.1 so that the Figure 1 becomes commutative. In this figure $q : X \to X'$ is the canonical map defined by $q(x)$ = equivalence class of x with respect to $\sim$ and

$i : S \to S$ is the identity map.

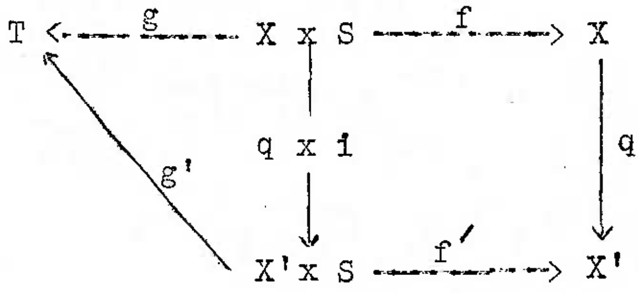


Figure 1

For a topological machine M, if there exists a Hausdorff topology on X' that makes M' a topological machine and q a continuous map, then we get a reduced form of M. The quotient topology on X' is Hausdorff by Lemma 3.2, if we take the set D of the lemma as the set S so that $x \sim y$ if $g_s(x) = g_s(y)$ for all $s \in S$, $g_s : X \to T$ being defined by $g_s(x) = g(x, s)$. Therefore, a Hausdorff topology on X' that makes M' a reduced form of M must be weaker than or equal to the quotient topology on X'. Moreover, after a moment's reflection it would be clear that any reduced form M' of M must be obtained (upto isomorphism) by giving a Hausdorff topology on X' that makes the maps q, f' and g' of Figure 1 continuous. For, if M'' is a reduced form of M and $p: X \to X''$ is the continuous map such that $x \sim p(x)$ for all $x \in X$, then we can establish a one-one correspondence between the state spaces X' of M' and X'' of M', namely the map, $h : X' \to X''$ defined by $h(x') = p \circ q^{-1}(x)$ for all $x \in X'$, by virtue of the fact that M'' is distinguished, such that the topology of X'' can be carried over to X' and that will make M' a topological machine iseomorphic to M'' and a reduced form of M. We do not give the details of the arguments but it is clear from the following commutative diagrams given in Figure 2

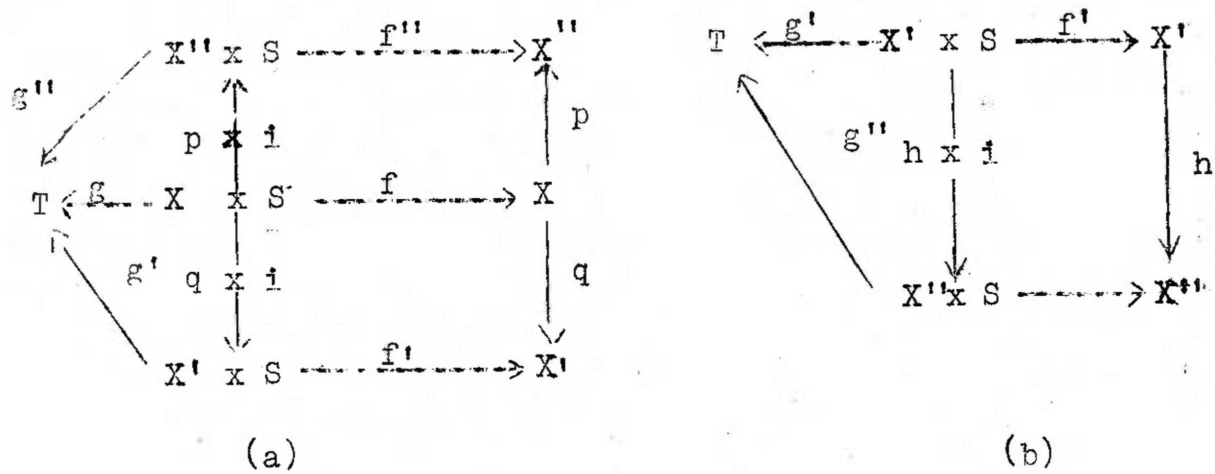


Figure 2.

Therefore for a topological machine M, a reduced form M' exists iff these exists a Hausdorff topology on X' which is weaker than or equal to the quotient topology on X', that makes M' a topological machine, and, if M' is a reduced form of M, then M' is the unique (upto iseomorphism) reduced form of M iff the compatible Hausdorff topology on X' is the unique Hausdorff topology that makes the maps f', g' and q of Figure 1 continuous. If the quotient topology on X', which is Hausdorff by Lemma 3.2, makes f' and g' continuous we shall refer to this reduced form of M as the <u>quotient machine of M.</u>

The rest of this section is primarily concerned with machines for which the quotient machine is defined and is the unique (upto iseomorphism) reduced form. Incidentally, if M' is a reduced form of M, then under some conditions there exists a topological variant M'' of M such that M' is the quotient machine of M''. For obtaining such conditions we need to solve the following topological problem. Suppose Y is any non-empty Hausdorff space, X is any non-empty set and $f : X \to Y$ is an onto map. Under what conditions can we give a Hausdorff topology on X such that f becomes continuous open (or Y becomes the quotient space X/f) ? A sufficient condition for this is the following:

3.3. Lemma. Let X, Y and f be as in above. If, for any $y_1, y_2 \in Y$, there exists a 1-1 correspondence between $f^{-1}(y_1)$ and $f^{-1}(y_2)$, then X can be given a Hausdorff topology such that f becomes continuous open.

Proof. Let $A_y = f^{-1}(y)$, $y \in Y$ and $A = A_{y_0}$ for a fixed $y_0 \in Y$. There exists a 1-1 onto map $h_y : A \to A_y$ for each $y \in Y$. Let $B_a = \{h_y(a) : y \in Y\}$. Then $\{B_a\}$ is a partition of X and, for each $a \in A$, there exists a 1-1 onto map $h_a : B_a \to Y$ defined as : $h_a(h_y(a)) = y$. Note that $h_a = f|B_a$. Give B_a the T_2 topology making h_a a homeomorphism and then to X give the union topology [9] which is the required Hausdorff topology on X making f continuous open.

In view of the above lemma we can now state the following for machine.

3.4. Proposition. Suppose for a machine M there exists a reduced form M' and the canonical map $q : X \rightarrow X'$ is such that $q^{-1}(x_1)$ and $q^{-1}(x_2)$ are in 1-1 correspondence for every pair $x_1, x_2 \in X'$, then M' is the quotient machine for a topological variant of M.

Proof. Since the new topology on X obtained from X' via Lemma 3.3 makes q continuous open, by considering the Figure 1, it is easy to show that this new topology on X is indeed a compatible topology defining a topological machine which is a topological variant of M.

Towards the existence and uniqueness of the quotient machine for a machine we have some sufficient conditions only. We first note some such conditions in the following remark. We may recall here that a continuous map f from a space X onto a space Y is a quotient map if $A \subset Y$ is open iff $f^{-1}(A)$ is open in X.

3.5 Remark. If $q : X \rightarrow X'$ is the canonical quotient map and $q \times i : X \times S \rightarrow X' \times S$ is a quotient map [c.f. Figure 1], then for a machine M the quotient machine M_q is defined. If M_q is defined and the quotient topology on X' is minimal Hausdorff [44], then it is the unique (upto iseomorphism) reduced form. It is known that a compact Hausdorff space is minimal Hausdorff [44].

Incidentally, we quote in the following some results from Madison [30] which give several sufficient conditions for the map $q \times i$ of Remark 3.5 to be open.

3.6. Remark. [c.f. 30]. The map $q \times i$ of Remark 3.5 is a quotient map if any one of the following holds.

1) S is locally compact.

2) $X' \times S$ is a k-space. (A space X is a k-space if a subset A of X is open (closed) in X whenever $A \cap K$ is open (closed) in K for each compact subset K of X. X is a k-space iff X is a quotient space of a locally compact space).

3) q is a bi-quotient map. (A map $f : X \to Y$ is bi-quotient if, whenever $y \in Y$ and $\mathcal{U}$ is a covering of $f^{-1}(y)$ by open sets of X, then finitely many $f(U)$, $U \in \mathcal{U}$, cover some neighbourhood of $y \in Y$. A bi-quotient map is a quotient map and q is a bi-quotient map if q is either open or proper.)

We do not make an attempt to reproduce the proofs of Madison of Remarks 3.6 but our point is only to record the existence of such results which are relevant to our present discussion.

The following example illustrates Remark 3.5.

3.7. Example. Let R be the usual real line, T the circle group and S a sub-semigroup (without identity) of additive group R generated by 1 and λ, an irrational number. Let S act on R by usual addition. Let f_1 and f_2 be two functions from R into T defined by

$$f_1(x) = \exp(ix) \quad \text{and}$$

$$f_2(x) = \exp(i\lambda x) \quad \text{for all } x \in R$$

Then, as seen in Section 3 of Chapter II, the function g defined on $R \times S$ with values in T as

$$g(x, m+n\lambda) = \prod_{j=o}^{m-1} f_1(x+j) \prod_{j=o}^{n-1} f_2(x+m+j\lambda)$$

$$= \exp\left[i\left\{mx+\frac{m(m-1)}{2}+n(x+m)+\frac{n(n-1)}{2}\lambda\right\}\right]$$

for all $m, n \geq 1$, is an output function. It can be seen easily that $x_1 \sim x_2$ iff $x_1 \equiv x_2 \pmod{2\pi}$ whence it follows that $R/\sim = T$ which is compact and the quotient map $q : R \to R/\sim$ is open. So the quotient machine is defined and its the unique reduced form.

The following gives another sufficient condition for the existence and uniqueness of a reduced form of a machine.

3.8. Proposition. Suppose, for a machine M, there is some $s \in S$ such that $x \not\sim y$ implies that $g(x, s) \neq g(y, s)$, and $g_s : X \to T$, $g_s(x) = g(x, s)$, is a continuous open map. Then the quotient machine is defined and is the unique (upto iseomorphism) reduced form.

Proof. From the given conditions it follows that the quotient space $X/\sim = X'$ is homeomorphic to $g_s(X)$ and the canonical map $q: X \to X'$ is open. So the quotient machine is defined and g'_s is a homeomorphism between X' and $g'_s(X') = g_s(X)$ [c.f. Figure 1], and hence, there is no weaker T_2 topology on X' making g'_s (and hence g') continuous. Therefore, the quotient machine is the unique reduced form.

The following example illustrates the above proposition.

3.9. Example. Let a compact semigroup S with identity 1 act quasi-transitively on a space X (i.e., the orbits form a decomposition of X) [c.f. Sections 4 and 6 of Chapter 1]. Let X' be the orbit space i.e., the quotient space obtained from X by coalescing the orbits, and $q : X \to X'$ the quotient map which is known to be open. Let $T = X'$ be equipped with right zero multiplication. Define the output function $g : X \times S \to T$ by $g(x, s) = q(xs)$ for all $(x, s) \in X \times S$. Then the partial function $g_1(x) = g(x, 1)$ is a map from X onto X' which is continuous open and $g_1(x) = g_1(y)$ implies that $g_s(x) = g_s(y)$ for all $s \in S$.

In the light of our discussion of Kelemen's results in Section 2 we state the following proposition giving some sufficient conditions for the uniqueness of a reduced form of a machine, if it is defined.

3.10. Proposition. Let $M = \langle X, S, T, f, g \rangle$ be a machine and $M' = \langle X', S, T, f', g' \rangle$ be a reduced form of M. Let $q : X \to X'$ be the quotient map. Then M' is unique upto iseomorphism if any one of the following three conditions hold.

1) g is WIP on S.

2)(a) f is WIP on S and q is IP, and

(b) $x_1 \sim x_2$ iff $f(x_1, s) \sim f(x_2, s)$ for all $s \in S$.

(Note that, if M satisfies A3, then (b) is automatically satisfied).

3)(a) For some $x_o \in X$, $f(x_o, S) = X$, and

(b) the partial map f_{x_o} and q are IP.

Proof. 1) g is WIP on S implies g' is WIP on S. For, if, for a net $\{x'_\alpha\}$ in X', $\lim x'_\alpha = \infty$, then, if $x_\alpha \in q^{-1}(x'_\alpha)$, we see that $\lim x_\alpha = \infty$ and so there is some $s \in S$ such that $\lim g(x_\alpha, s) = \lim g'(x'_\alpha, s) = \infty$.

Further, g' is always effective on S. For, if $g'(x'_1, s) = g'(x'_2, s)$ for all $s \in S$, then, if $x_i \in q^{-1}(x'_i)$, $i = 1, 2$, $g(x_1, s) = g(x_2, s)$ for all $s \in S$ and so $x_1 \sim x_2$ and hence, $x'_1 = q(x_1) = q(x_2) = x'_2$.

Therefore, Proposition 2.1(a) can be applied.

2). Again it is easy to see that (a) implies that f' is

WIP on S and (b) implies that f' is effective on S.

3) Note that (a) implies that $f'(q(x_o), S) = X'$ and (b) implies that the partial map $f'_q(x_o)$ is IP. Hence, the Proposition 2.3(a) can be applied.

The following example illustrates the above proposition where, however, all the three conditions are satisfied.

3.11. <u>Example</u>. Let M be a machine defined by :

$$R \xleftarrow{\;g\;} R^2 \times R \xrightarrow{\;f\;} R^2$$

where R is the usual real line, R^2, the Cartesian (additive) product group, and f and g are defined as:

$$f((r_1, r_2), r) = (r_1 + r, r_2 + r)$$

$$g((r_1, r_2), r) = b(r_1 + r, r_2 + r) - b(r_1, r_2)$$

for all $(r_1, r_2, r) \in R^2 \times R$, and b is a continuous map for $R^2 \longrightarrow R$ defined by $b(r_1, r_2) = (r_1 + r_2)^2$.

Note that

$$g((r_1, r_2), r) = 4\{r^2 + r(r_1 + r_2)\}.$$

and $(r_1, r_2) \sim (r_1', r_2')$ iff $r_1 + r_2 = r_1' + r_2'$ so that $R^2/\sim$ is R.

Therefore, the quotient machine M' which is defined is

$$R \xleftarrow{g'} R \times R \xrightarrow{f'} R$$

where $f'(r, s) = r + s$

and $g'(r, s) = 4(s^2 + rs)$

for all $(r, s) \in R \times R$.

It is easy to see that all the three conditions of Proposition 3.10 hold.

4. On Input-distinguished Machines

In this section all machines are taken to have the same output semigroup.

For a machine M two inputs s_1 and s_2 are input-equivalent, written $s_1 \approx s_2$, if $g(x, s_1) = g(x, s_2)$ and $g(x, s_1 s) = f(x, s_2 s)$ for each $x \in X$ and each $s \in S$. M is called input-distinguished if no two distinct inputs are input-equivalent. M' (with $X' = X$) is an input-reduced form of M if there exists a continuous onto homomorphism $h : S \to S'$ such that $s \approx h(s)$ for all $s \in S$ and M' is input-distinguished. Two machines M_1 and M_2 are input-iseomorphic if there exists an iseomorphism $h : S_1 \longrightarrow S_2$ and a homeomorphism $k : X_1 \longrightarrow X_2$ such that :

(1) $k(f_1(x, s)) = f_2(k(x), h(s))$, and

(2) $g_1(x, s) = g_2(k(x), h(s))$ for all $x \in X_1$ and all $s \in S_1$.

We first study whether for a machine an input-reduced ~~form~~ form exists and, if so, whether an input reduced form is unique upto input-iseomorphism.

The following algebraic fact is well known and so we state this without giving any proof.

4.1. Lemma. [cf. Lemma 3.2 of [21]]. Let M be a machine. Then:

(1) If, for $s_1, s_2 \in S$ and $x, y \in X$, $s_1 \approx s_2$ and $x \sim y$, then $f(x, s_1) \sim f(y, s_2)$.

In particular, if M is distinguished and, for $s_1, s_2 \in S$, $s_1 \approx s_2$, then $f(x, s_1) = f(x, s_2)$ for all $x \in X$.

(2) If, for $s_1, s_2, s_3, s_4 \in S$, $s_1 \approx s_2$ and $s_3 \approx s_4$, then $s_1 s_3 \approx s_2 s_4$. It follows that $\approx$ is a congruence relation on S.

We shall also need the following fact.

4.2. Lemma. The quotient topology on $S/\approx$ is Hausdorff.

Proof. Let $D = X \cup X \times S$. Now $s_1 \approx s_2$ if $g_x(s_1) = g_x(s_2)$ and $g_{(x,s)}(s_1) = g_{(x,s)}(s_2)$ for all $x \in X$ and $s \in S$ where $g_x : S \longrightarrow T$ (respectively $g_{(x,s)} : S \longrightarrow T$) is defined by

$g_x(s_1) = g(x, s_1)$ (respectively $g_{(x,s)}(s_1) = g(x, s_1 s))$.
Hence, by Lemma 3.2, the result follows.

For algebraic machines the following result is well known [cf. Theorem 3.3 of [21]].

4.3 Proposition. For any algebraic machine M these exists an input-reduced form M' such that:

(1) $X = X'$

(2) There exists a homomorphism $h : S \to S'$ satisfying $g(x, s) = g'(x, h(s))$ for all $s \in S$ and each $x \in X$.

(3) If M is distinguished, then any input-distinguished machine M'' satisfying (1) and (2) above is input-isomorphic to M'.

M' is defined by taking $X' = X$, $S' = S/\approx$, which is the well-defined canonical quotient semigroup via Lemma 4.1(2) and f' and g' are defined via Lemma 4.1(1) so that the Figure 3 becomes commutative.

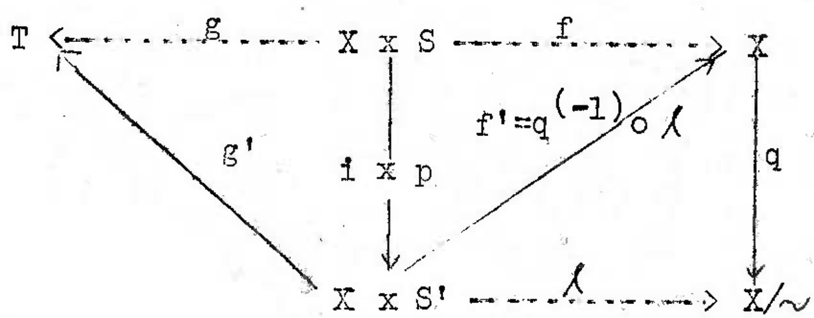


Figure 3

In Figure.3, $p : S \to S' = S/\approx$, is the canonical map defined by $p(s)$ = the equivalence class containing s with respect to $\approx$, $q : X \to X/\sim$ is the canonical map defined by $q(x)$ = the equivalence class containing x with respect to $\sim$, $i : X \to X$ is the identity map, $\lambda : X \times S' \to X/\sim$ is defined by $\lambda(x, s')$ = the equivalence class with respect to $\sim$ containing $f(x, s)$, for any $s \in p^{-1}(s')$ and $q^{(-1)}: X/\sim \to X$ is a map which selects one point of X from each equivalence class with respect to $\sim$.

For the topological case, however, several problems arise as in the discussion of the previous section. Firstly, though, by Lemma 4.1.(2), $S/\approx$ inherits canonically a semigroup operation from that of S, there may not exist a topology on $S/\approx$ making it a topological semigroup; in fact, even the quotient topology on $S/\approx$, which is Hausdorff by Lemma 4.2, may not make $S/\approx$ a topological semigroup. The quotient topoloty on $S/\approx$ makes it a topological semigroup if the map ~~p x pi~~ $p \times p : S \times S \to S/\approx \times S/\approx$ is a quotient map where $p : S \to S/\approx$ is the canonical quotient map. Several sufficient conditions for $p \times p$ to be a quotient map similar to those stated in Remarks 3.6 are available in Madison [30]; but we do not state them here. Secondly, there should exist not only a topology on $S/\approx$ making it a topological semigroup, but also a topology on $X/\sim$ so that p, g', ℓ, q and $q^{(-1)}$ of Figure 3 become continuous.

Let, for topological spaces X and Y, $\sigma : X \to Y$ be a continuous map. Then a map $\sigma^{(-1)} : Y \to X$ is a <u>continuous inverse</u> of σ if $\sigma^{(-1)}$ is continuous and $\sigma(\sigma^{(-1)}(y)) = y$ for all $y \in Y$.

Then for a topological machine M there exists an input reduced form satisfying (1) and (2) of Proposition 4.3 if there exists a topology on the quotient set $S/\approx$ and a topology on $X/\sim$ such that $S/\approx$ becomes a topological semigroup, the maps p, g', ℓ, q of Figure 3 continuous and q admits of a continuous inverse $q^{(-1)}$. Further, if M is distinguished then, arguing as in the case of existence of a reduced form of a machine, there exists an input-reduced form iff there exists a topology on $S/\approx$ making it a topological semigroup and the maps p, g', f' of Figure 3 continuous and a unique (upto input-iseomorphism) input-reduced form iff such a topology on $S/\approx$ is unique.

Now we shall state two results giving sufficient conditions for the existence and uniqueness (upto input-iseomorphism) of an input-reduced form of a distinguished machine analogous to results of Section 3.

Analogous to Proposition 3.8 we can state the following.

4.4. Proposition. Let M be a distinguished machine. Let there exist an $x_o \in X$ such that the partial map $g_{x_o} : S \to T$, $g_{x_o}(s) = g(x_o, s)$, is continuous open and $s \not\approx t$ implies $g_{x_o}(s) \neq g_{x_o}(t)$. Then the quotient topology is the unique T_2-topology on $S/\approx$ making $M' = \langle X, S/\approx, T, f', g' \rangle$ [c.f. Figure 3] the unique input-reduced form of M.

Proof: Similar to that of Proposition 3.8.

Next we give an example to illustrate the above.

4.5. Example. Let $R^+ = [0, \infty)$ with usual addition, and R^{+2} the usual Cartesian product of R^+ with itself. Let M be defined by

$$R^+ \xleftarrow{g} R^+ \times R^{+2} \xrightarrow{f} R^+$$

$$f(r, (r_1, r_2)) = r + r_2$$

and $g(r, (r_1, r_2)) = 2r\, r_2 + r_2^2$ for all $(r, (r_1, r_2)) \in R^+ \times R^{+2}$.

Note that since $r \neq r'$ implies $g(r,(0,r)) \neq g(r',(0,1))$ Hence, $r \not\sim r'$ and so M is distinguished. Further, note that $g_o : R^{+2} \to R^{++}$ is continuous open and $g_o(r_1, r_2) = r_2^2 = r_2'^2 = g_o(r_1', r_2')$ iff $r_2 = r_2'$ whence $f(r,(r_1, r_2)) = f(r,(r_1', r_2'))$ for all $r \in R^+$ and so $(r_1, r_2) \approx (r_1', r_2')$.

Therefore, all the assumptions of Proposition 4.4 hold good.

As in Section 3 we state a proposition below giving some sufficient conditions for the uniqueness of an input-reduced form, if it exists, in view of Kelemen's results of Section 2.

4.6. Proposition Let $M = \langle X, S, T, F, g \rangle$ be a distinguished machine and $M' = \langle X, S', T, f', g' \rangle$ be an input-reduced form of M. M' is unique (upto input-iseomorphism) if any one of the following conditions hold good.

1)(a) g is WIP on X, and

(b) if $g(x, s_1) = g(x, s_2)$ for all $x \in X$, then

$g(x, s_1 s) = g(x, s_2 s)$ for all $x \in X$ and all $s \in S$.

2)(a) f is WIP on X, and

(b) if $f(x, s_1) = f(x, s_2)$ for all $x \in X$, then

$g(x, s_1) = g(x, s_2)$ for all $x \in X$.

Proof: 1) Follows from Proposition 2.1(a) if we note that (a) implies that g' is WIP on X and (b) implies that g'

is effective on X.

2) Similar argument is needed.

We now give two examples.

4.7. Example. Let R and R^2 be as in Example 3.11. Define a machine M as:

$$R \xrightarrow{g} R \times R^2 \xrightarrow{f} R$$

where $f(r, (r_1, r_2)) = r + r_1 + r_2$

and $g(r, (r_1, r_2)) = (r_1 + r_2)^2 + 2r(r_1 + r_2)$

for all $(r, (r_1, r_2)) \in R \times R^2$.

Note that M is disginguished and

$(r_1, r_2) \approx (r_1', r_2')$ iff $r_1 + r_2 = r_1' + r_2'$, and so, $R^2/\approx = R$.

Note that g satisfies 1(a) and (b). So

$M' : R \xleftarrow{g'} R \times R \xrightarrow{f'} R$, where $f'(r, s) = r + s$ and $g'(r, s) = s^2 + 2rs$ for all $(r, s) \in R \times R$, is the unique input reduced form of M.

4.8. Example. Let everything be as in the above Example 4.7 except that $g(r, (r_1, r_2)) = (r - r_1 + r_2)^2$ for all $(r_1 (r_1, r_2)) \in R \times R^2$. Note that f satisfies 2(a) and (b) and M' is samething except $g'(r, s) = (r + s)^2$ for all

$(r, s) \in R \times R$ and g' is not effective.

Next we discuss the topological version of a problem of Ginsburg concerning input-distinguished machines. The problem is to find conditions on a semigroup S which guarantee the existence of an input-distinguished machine $M = \langle X, S, T, f, g \rangle$ with a compact state space X [cf. 21]. As noted by Ginsburg [21], for each semigroup S there exists an input-distinguished machine $M = \langle X, S, T, f, g \rangle$. For, without any loss of generality, we can assume that S has an identity and then define M as follows. Let T be the semigroup obtained by defining a right zero multiplication on S i.e. $s_1 s_2 = s_2$ for all $s_1, s_2 \in S$. Then, taking $X = S$, define $T \xleftarrow{g} X \times S \xrightarrow{f} X$ by $f(s_1, s_2) = s_1 s_2$ and $g(s_1, s_2) = s_1 s_2$ for all $(s_1, s_2) \in X \times S$. But, in general, X need not be compact if S is not. Ginsburg provided with examples of infinite semigroups [21] for which these exists no finite-state input-distinguished machine. In the sequel, we make some observations towards the existence of an input-distinguished machine with compact state-space for any given input semigroup.

4.9. <u>Remark.</u> If a semigroup S admits of a compactification S^* of which S is a sub-semigroup, then there exists an input-distinguished machine with a compact state space, namely, S^* and S as input semigroup.

In the following we make some observations where given an input semigroup we obtain conditions under which there exists an input-distinguished machine with a compact state space satisfying some additional hypotheses.

4.10. Proposition. Let S be a semigroup with identity Then there exists an input-distinguished machine with a compact state space and an output semigroup having right zero multiplication if there exists a compact space X on which S acts effectively and there exists a 1-1 continuous map from X into S.

Conversely, if, for a semigroup S with identity, there exists an input-distinguished machine with a compact state space X and an output semigroup with right zero multiplication, then S must act on X effectively.

Proof: Suppose S acts effectively on a compact space X. Suppose T is the semigroup obtained by defining right zero multiplication on S. Then define the machine

$M = \langle X, S, T, f, g \rangle$ as : f is the given action of S on X and $g(x, s) = h(f(x, s))$ for some 1-1 continuous map $h : X \to T$ [cf. Proposition 2.1 of Chapter II]. Now M is input-distinguished since, for each pair $s_1, s_2 \in S$, $s_1 \neq s_2$, there exists $x \in X$ such that $f(x, s_1) \neq f(x, s_2)$, and hence, $g(x, s_1) \neq g(x, s_2)$.

Conversely, suppose $M = \langle X, S, T, f, g \rangle$ is an input distinguished machine with S having identity, X compact and T having right zero multiplication. Then, by Proposition 2.1 of Chapter II, there exists a continuous map $h : X \to T$ such that $g(x, s) = h(f(x, s))$ for all $x \in X$ and $s \in S$. Since, for each pair $s_1, s_2 \in S$, $s_1 \neq s_2$, there exists $x \in X$ such that either $f(x, s_1) \neq f(x, s_2)$ or $g(x, s_1) \neq g(x, s_2)$ (equivalently, $h(f(x, s_1)) \neq h(f(x, s_2))$), it follows that $f(x, s_1) \neq f(x, s_2)$.

4.11. Corollary. If S is any infinite semigroup having identity, then there exists no finite-state input-distinguished machine with the output semigroup having right zero multiplication.

A closely related result on effective acts, which may have some independent interest, is as follows.

4.12. Proposition. A semigroup S acts effectively on a locally compact (compact) space iff there exists a semigroup S^* such that

(1) there exists a locally compact (compact) right ideal X of S^* on which S^* acts effectively, and,

(2) there exists a continuous 1-1 homomorphism h from S into S^*.

In (1) the statement S^* acts effectively on X can be replaced by saying that $h(S)$ acts effectively on X.

Proof. 'If'. Define the act $f : X \times S \to$ as : $f(x, s) = x.h(s)$ for all $x \in S$.

'Only if'. Suppose $f : X \times S \to X$ is an effective act with X locally compact (compact). Let S^* be the semigroup of all continuous maps from X into itself under the operation of composition of maps and compact-open topology. Then the map $h : S \to S^*$, $h(s) = f_s : X \to X$, $f_s(x) = f(x, s)$, is 1-1 continuous homomorphism and the map $\Phi(x) = k_x$, $k_x(X) = x$, for all $x \in X$, is a homeomorphism and $\Phi(X)$ is a locally compact (compact) ideal of S^* such that S^* acts (canonically and) effectively on $\Phi(X)$.

If S acts quasi-transitively on a space X, the equivalence relation on X defined by identifying the orbits, is referred to as the orbit equivalence relation on X. Let R_1 and R_2 be two equivalences on a set X. R_1 is said to be weaker than R_2 if each R_1-equivalence class is contained in some R_2-equivalence class. Then the following is another observation concerning Ginsburg's problem.

4.13. Proposition Given a semigroup S there exists an input distinguished machine with a compact state-space on which S acts quasi-transitively such that the orbital equivalence relation is weaker than the state equivalence relation ($\sim$) iff there exists a compact space Y and a semigroup T such that a continuous map $g : Y \times S \to T$ exists for which $g_y : S \to T$,

$g_y(s) = g(y, s)$, is a (continuous) 1-1 homomorphism for all $y \in Y$.

<u>Proof</u>. '<u>Only if</u>'. Let $M = < X, S, T, f, g >$ be a machine of the type described. Let Y be the (compact) quotient space of X obtained by coalescing the orbits under the action of S on X. Then there exists a machine $M' = < Y, S, T, f', g' >$ defined canonically so as to make the Figure 4 commutative as follows:

$f'(x', s) = q(f(x, s))$, and $g'(x', s) = g(x, s)$ for some $x \in q^{-1}(x')$, $x' \in X$ and $s \in S$. Then g' satisfies the requirements.

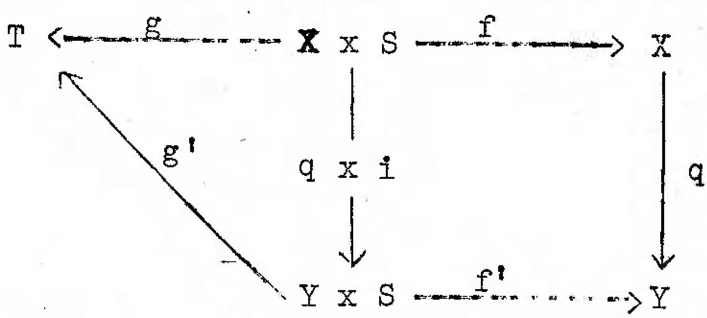


Figure 4

'<u>If</u>' . Define, taking $X = Y$, $M = < X, S, T, f, g >$ as : $f(x, s) = x$ for all $x \in X$ and all $s \in S$ and g as given. Then M is a desired machine.

Ginsburg's problem is, however, not yet satisfactorily and completely solved.

5. On Equivalence of Machines

All machines of this section have same input and output semigroups. For a machine M let $X' = X/\sim$, the quotient set and $\Phi = \{g_x : S \to T : g_x(s) = g(x, s), x \in X$ and $s \in S\}$. Two algebraic machines M_1 and M_2 are said to be (behaviourally) equivalent if there exist two maps $h : X_1 \to X_2$ and $k : X_2 \to X_1$ such that $x_1 \sim h(x_1)$ and $x_2 \sim k(x_2)$ for all $x_1 \in X_1$ and $x_2 \in X_2$ [21] or, equivalently, if $\Phi_1 = \Phi_2$. Then, via a 1-1 correspondence between X' and Φ, two (algebraic) machines are (behaviourally) equivalent iff their reduced forms [which are unique (upto isomorphism) and (behaviourally) equivalent to the original forms] are isomorphic [21, 46]. The purpose of this section is to discuss the topological version of the above concept and result.

Two (topological) machines M_1 and M_2 are said to be (behaviourally) equivalent, written $M_1 \approx M_2$, if there exist two continuous maps $h : X_1 \to X_2$ and $k : X_2 \to X_1$ such that $x_1 \sim h(x_1)$ and $x_2 \sim k(x_2)$ for all $x_1 \in X_1$ and $x_2 \in X_2$. However, the topological version of the equivalent form of this concept in the algebraic setting is not equivalent to this but is somewhat weaker. Accordingly, we say that M_1 and M_2 are weakly (behaviourally) equivalent, written $M_1 \sim M_2$ if $\Phi_1 = \Phi_2$ and the resultant 1-1 correspondence between X_1' and X_2', both being given quotient topologies is a homeomorphism. The concept of iseomorphism ($\cong$) of machines signifies that of structural equivalence and is a stronger concept than those above. These remarks are justified by the following:

5.1. Proposition. Let M_1 and M_2 be two (topological) machines.

(a) $M_1 \cong M_2 \Rightarrow M_1 \approx M_2$

(b) If the (canonical) quotient maps $q_i : X_i \to X_i'$, $i = 1, 2$, are open (or closed), then $M_1 \approx M_2 \Rightarrow M_1 \sim M_2$.

(c) For algebraic machines, $M_1 \approx M_2$ iff $M_1 \sim M_2$.

Proof. (a) is obvious and (c) is well-known [21]. For (b) look at the commutative Figure 5 where h_i are the maps establishing $\approx$ between M_1 and M_2 and h is defined by $h(x') = q_2 o h_1 o q_1^{-1}(x')$ for $x' \in X_1'$. Note that h is a homeomorphism.

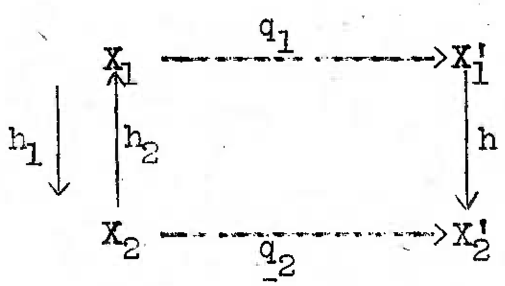


Figure 5.

In the rest of this section we obtain some conditions under which for topological machines part (c) of Proposition 5.1 holds.

5.2. Proposition. Let M_1, M_2 be two machines for which the quotient maps q_i's are open (or closed) and admit of continuous inverses. Then $M_1 \approx M_2$ iff $M_1 \sim M_2$.

Proof. $M_1 \approx M_2 \Rightarrow M_1 \sim M_2$ by (b) of Proposition 5.1. To prove the other way, let $h : X_1' \to X_2'$ be the desired homeomorphism and $k_i : X_i' \to X_i$ be continuous inverses of q_i, $i = 1, 2$. Define $h_1 : X_1 \to X_2$ and $h_2 : X_2 \to X_1$ by

$$h_1(x) = k_2 o h o q_1(x), \quad \text{for } x \in X_1$$

and

$$h_2(x) = k_1 o h^{-1} o q_2(x), \quad \text{for } x \in X_2 .$$

Then h_1 and h_2 are two required continuous maps.

While the existence of continuous inverse of a map demands much topological restrictions, which we discuss subsequently, the following observation is worth recording.

5.3. Proposition. Let M_1 and M_2 be two machines such that q_i's are open (or closed) and $M_1 \approx M_2$. Then q_i's have continuous inverses iff there txist two continuous maps $h_1 : X_1 \to X_2$ and $h_2 : X_2 \to X_1$ such that $x_i \sim y_i$ implies that $h_i(x_i) = h_i(y_i) \sim x_i$ for $i = 1, 2$.

Proof. 'If part'. Look at the commutative Figure 5 in connection with proof of Proposition 5.1. Define $k_i : X_i' \to X_i$ by

$$k_1(x_1') = h_2 o q_2^{-1} o h(x_1'),$$

and $$k_2(x_2') = h_1 o q_1^{-1} o h(x_2') \quad \text{for all } x_i' \in X_i',\ i = 1,2.$$

It is easy to see that k_i is a continuous inverse of q_i, $i = 1, 2$.

'Only if part'. The proof of this is contained in the proof of part (b) of Proposition 5.1 and that of Proposition 5.2.

A final remark given below contains an analogue of a result for abstract machines [cf. 21, 46].

5.4. Remark.

(1) Let M be a (topological) machine such that the quotient machine M' is defined. Then $M' \approx M$ iff there exists a continuous inverse of q.

(2) Let M_1 and M_2 be two machines for which the quotient machines M_1' and M_2' are defined and $M_i \approx M_i'$, $i = 1,2$. Then $M_1 \approx M_2$ iff $M_1' \cong M_2'$.

We now give an example to illustrate some of the above discussions.

5.5 **Example.** Let R and R^2 be as in Example 3.11. Define a machine $M : R \xleftarrow{g} R^2 \times R^2 \xrightarrow{f} R^2$ as follows.

$$f((r_1, r_2), (r_1', r_2')) = (r_1 + r_1', r_2 + r_2')$$

and $$g((r_1, r_2), (r_1', r_2')) = r_1' + 2r_2 r_2' + r_2'^2$$

for all $((r_1, r_2), (r_1', r_2')) \in R^2 \times R^2$.

The output function g is simple [cf. Section 2 of Chapter II] and is obtained via the defining map $b : R^2 \to R$, $b(r_1, r_2) = r_1 + r_2^2$.

Note that $(r_1, r_2) \sim (r_1', r_2')$ iff $r_2 = r_2'$. Therefore $R^2/\sim = R$ and there exists a continuous inverse of the quotient map $q : R^2 \to R^2/\sim = R$. Hence the (unique) quotient machine is defined and is equivalent to M.

6. Miscellaneous Topological Results

In this section we present some topological results which may not have any direct bearing on the material of this chapter but are somewhat related to a few problems treated in earlier sections.

6.1. Compactness of the Range of a Continuous Open Map

In view of Remark 3.5 we would like to obtain necessary and sufficient conditions for the compactness of the range of a continuous open map. Towards this we have two results which we present below

We shall first prove a lemma.

6.1.1. Lemma. Let X be any Hausdorff space, Y be any T_1-space and h be any continuous map from X into Y. Let F be a family of nonvoid compact subsets of X linearly ordered under inclusion i.e., for A, B in F, $A \leq B$ if $B \subseteq A$. Then $h(\bigcap_{A \in F} A) = \bigcap_{A \in F} h(A)$.

Proof. We need only to show that for any $y \in \bigcap_{A \in F} h(A)$ there exists an $x \in {}_{A \in F} A$ such that $h(x) = y$. Let $A_o \in F$ and

$F_1 = \{ A_1 = A_o \cap A: A \in F \}$. Then $\bigcap_{A_1 \in F_1} A_1 = \bigcap_{A \in F} A$ and

$\bigcap_{A_1 \in F_1} h(A_1) = \bigcap_{A \in F} h(A)$. Now $F_2 = \{ A_2 = h^{-1}(y) \cap A_1 : A_1 \in F_1 \}$

is a collection of closed sets of the compact space $h^{-1}(y) \cap A_o$ having finite intersection property and, hence, F_2 has a non-void intersection which proves the lemma.

We then have

6.1.2. Proposition. Let X be any locally compact T_2-space, Y be any T_1-space and h be any continuous open map from X onto Y. Then Y is compact iff there exists a compact subset C of X such that $h(C) = Y$ and h is one-to-one on C^o, the interior of C.

Proof. We need to verify 'only if' part. Let Y be compact and $F = \{A_x : A_x$ is a compact neighbourhood of $x \in X\}$. Then $F_o = \{A_x^o : x \in X\}$ is an open cover for X and $h(\bigcup_{x \in X} A_x^o) = \bigcup_{x \in X} h(A_x^o) = Y$. Since $h(A_x^o)$ is open for all $x \in X$, $\{h(A_x^o) : x \in X\}$ is an open cover for Y and as Y is compact there is a finite sub-cover, say, $\{h(A_{x_1}^o), \ldots h(A_{x_n}^o)\}$ for Y. So $h(\bigcup_{i=1}^{n} A_{x_i}) = Y$ and thus there is a compact set $A = \bigcup_{i=1}^{n} A_{x_i}^- \subset X$ such that $h(A) = Y$. Let $F_1 = \{A : A$ is a compact subset of X such that $h(A) = Y\}$ be partially ordered under set inclusion as in Lemma 6.1.1. Then any chain in F_1 has an upper bound, namely the intersection, by virtue of Lemma 6.1.1 and, hence, by Zorn's Lemma there exists a maximal element C in F_1 and $h(C) = Y$. We show that h is one-to-one on C^o. If h is not one-to-one on C^o there exist two distinct points x_1 and x_2 in C^o such that $h(x_1) = h(x_2)$. Since X is a T_2-space and h is open there exist two disjoint open neighbourhoods N_{x_1} and N_{x_2} of x_1 and x_2 respectively which are completely contained in C^o such that $h(N_{x_1})$ and $h(N_{x_2})$ are two open neighbourhoods of $h(x_1) = h(x_2) = y$ say. Then $V = f(N_{x_1}) \cap f(N_{x_2})$ is an open neighbourhood of y. Consider $U_1 = h^{-1}(V) \cap N_{x_1}$ and $U_2 = h^{-1}(V) \cap N_{x_2}$. $h(U_1) = h(U_2)$ and $h(C^o \setminus U_1) = h(C^o)$. So $h(C \setminus U_1) = Y$. But $C \setminus U_1$ is a compact proper subset of C which is a contradiction. This proves the result.

The phrase 'h is one-to-one on C^o' in Proposition 6.1.2 can not be replaced by 'h is one-to-one on C' as shown by the following counter-example.

6.1.3. Example. Let X be the real line and h be the map given by $h(x) = e^{ix}$. Then h is a continuous open map from X onto $h(X)$, the unit circle. Obviously, there is no compact subset C of X such that $h(C) = h(X)$ and h is one-to-one on C.

However, when X is any connected subset of the real line with usual topology and h is a real valued continuous open map then h is one-to-one on a minimal compact set $C \subset X$.

More generally, we have the following results the proof of which is easy and is omitted.

Let X and Y be any two connected linearly ordered spaces equipped with respective order topoligies. Let h be any non-constant continuous map from X into Y. let

$h_s = \sup_{x \in X} h(x)$, $h_i = \inf_{x \in X} h(x)$ and $E = h^{-1}\{h_i, h_s\}$ which may be empty. Then E is a closed subset of X and E^c, the complement of E, is nonvoid and is a disjoint union of open intervals, the connected components of E^c.

Then we have

6.1.4. <u>Proposition</u>. h is open (with respect to the range h(X)) iff h is one-to-one (or, equivalently, strictly monotone) on each component of E^c.

Further, we have two important corollaries.

6.1.5. <u>Corollary</u>. If h is open then h(X) is compact iff there exists a compact subset of X h-homeomorphic to h(X).

6.1.6. <u>Corollary</u>. Suppose X is as above and X has a first element, Y is any T_2-space and h is a continuous open map from X onto Y. Then the assertion of Corollary 6.1.5 holds.

6.2. <u>A Result on the Existence of a Continuous Inverse of a Map</u>.

The discussion of Section 5 shows the relevance of the problem of the existence of continuous inverses of maps. The problem of existence of a continuous inverse of a map f is also related to the problem of continuous selections as formulated and studied by Michael [cf. 17, 32, 33] and, in fact, they are same if f is open and closed or f is open and domain of f is compact Hausdorff [cf. 32]. As noted by Michael, the continuous selection problem has a solution only for very much restricted spaces [33] e.g., o-dimensional complete metric spaces or when the domain space is o-dimensional paracompact and the range space is complete metric etc. In the following we make an observation on the existence of a continuous inverse of a (continuous) map which is modelled on the Example 6.2.3 given in the sequel and seems to be new.

Suppose f is a (continuous) map from a topological space (X, T) onto a topological space (Y, T') and the following conditions are satisfied.

(1) There exists a linear order $<$ on X such that $<$-order topology on X is weaker than T.

(2) There exist $<$-order-preserving bijections $h_{xy}: f^{-1}(x) \to f^{-1}(y)$ satisfying:

(a) $h_{xy} = h_{yx}^{-1}$ and $h_{xy} = h_{zy} \circ h_{xz}$ for all $x, y, z \in Y$.

(b) for any $z \in f^{-1}(x)$, $z < h_{xy}(z)$ iff $w < h_{xy}(w)$ for all $w \in f^{-1}(x)$.

(c) for any $w, z \in f^{-1}(x)$, $z < w$ implies that $h_{xy}(z) < w$ for all $y \in Y$, and

(d) for any $x \in Y$ and $z \in f^{-1}(x)$ and for every T-open set $A \subseteq X$, there exists a $<$-open set $B \subseteq X$ such that $\{y \in Y : h_{xy}(z) \in A\} = \{y \in Y : h_{xy}(z) \in B\}$.

Define $\leq_*$ on Y as $x \leq_* y$ if, for all $z \in f^{-1}(x)$, $z < h_{xy}(z)$. $\leq_*$ is a linear order on Y induced by the linear order $<$ on X via the map f. An equivalent definition of $\leq_*$ is : $x \leq_* y$ if, for all $w \in f^{-1}(y)$, $h_{yx}(w) < w$.

6.2.1. Remarks. (a) From 2(a) it follows that for any $w \in f^{-1}(x)$, $w = h_{xy}(w)$ iff $x = y$.

(b) From 2(a) and 2(b) it follows that

2(b') : for any $z \in f^{-1}(x)$, $z > h_{xy}(z)$ iff $w > h_{xy}(w)$ for all $w \in f^{-1}(x)$.

(c) Similarly if follows that

2(c'): for $w, z \in f^{-1}(x)$, $z > w$ implies that $h_{xy}(z) > w$ for all $y \in Y$.

Then we have:

6.2.2. Proposition. Suppose f, X, Y are as in above satisfying the conditions (1) and (2). Suppose $\leq_*$- order topology on Y is weaker than T'. Then there exists a continuous inverse of f.

Proof. Define, for a fixed $x \in Y$ and a fixed $z \in f^{-1}(x)$, a map $g_{xz} : Y \to X$ as follows: $g_{xz}(y) = h_{xy}(z)$ for all $y \in Y$. We prove that g_{xz} is continuous with respect to $\leq_*$-order topology on Y and $<$- order_topology on X [cf. condition 2(d)] whence g_{xz} is a continuous inverse of f. To show that, for any $w \in X$, $O = g_{xz}^{-1}\{u: u > w, u \in X\}$ is a $\leq_*$- open set in Y.

Case 1. $z = w$.

$O = \{y: y \in Y$ such that $h_{xy}(z) > w = z\} = \{y: y \in Y$ such that $h_{xy}(z') > z'$ for all $z' \in f^{-1}(x)\}$ [by 2(b)], $= \{y : x \lneq_* y\}$, by definition of $\leq_*$.

Case 2. $z < w$.

If $w \in f^{-1}(x)$, then as $z \in f^{-1}(x)$, by 2(c), $z < w$ implies that $h_{xy}(z) < w$ for all $y \in Y$ and hence $O = \phi$.

Assume, then, $w \notin f^{-1}(x)$ and consider $h_{xw'}(z)$ where $w' = f(w)$.

Subcase 2(a). $h_{xw'}(z) < w$.

By 2(c), $h_{xy}(z) = h_{w'y} \circ h_{xw'}(z) < w$ for all $y \in Y$ since both $h_{xw'}(z)$ and $w \in f^{-1}(w')$. So $O = \phi$.

Subcase 2(b). $h_{xw'}(z) > w$.

By 2(c), $h_{xy}(z) = h_{w'y} \circ h_{xw'}(z) > w$ for all $y \in Y$.

Again, by 2(a), $h_{xx}(z) = z > w$ which is a contradiction, and so, $h_{xw'}(z) \not> w$.

Subcase 2(c). $h_{xw'}(z) = w$.

$$0 = \{y : y \in Y \text{ such that } h_{xy}(z) = h_{w'y} \circ h_{xw'}(z) > w\}$$
$$= \{y : y \in Y \text{ such that } h_{x'y}(w) > w\}$$
$$= \{y : y \in Y \text{ such that } y \underset{*}{>} w'\}\text{, by 2(b) and the}$$

definition of $\underset{*}{\leq}$.

Case 3. $z > w$. This can be verified in a way similar to that of Case 2.

By making use of Remarks 6.2.1 and the definitions of $\underset{*}{\leq}$ and using arguments similar to those above it can be shown that for any $w \in X$, $0 = g^{-1}_{xz}\{u : u < w, u \in X\}$ is $\underset{*}{\leq}$- open in Y. This proves the result.

It may be noted that there exist more than one continuous inverses of f under the hypotheses of Proposition 6.2.2, one continuous inverse of f for each fixed pair $x \in Y$ and $z \in f^{-1}(x)$. However, the cardinality of the set C of all continuous inverses of f under the same hypotheses is that of $f^{-1}(x)$ for any $x \in Y$. This is because the set

$C_x = \{g_{xz} : z \in f^{-1}(x)\}$ for $x \in Y$ is same for all choices of x and so equals C.

The following example illustrates the above discussions.

6.2.3. Example. Let T be a topology for the set X of reals whose base is $\{$usual base$\} \cup \{[n, x) : x > n$ and n integral$\}$. Let $Y = [0, 1)$ with usual topology. If $f : X \to Y$ is defined as : $f(x) =$ fractional part of x for all $x \in X$, then the hypotheses of Proposition 6.2.2 are satisfied and there exist continuous inverses of f. In this example, for $x, y \in X$, $h_{xy} : f^{-1}(x) \to f^{-1}(y)$ is defined by

$$h_{xy}(z) = z - x + y \quad \text{for all } z \in f^{-1}(x).$$

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