A STUDY OF THE ROBUSTNESS OF INFERENCE PROCEDURES IN LINEAR MODELS WITH SPECIFICATION ERRORS

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CONTENTS

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NOTATIONS

R^m : the m-dimensional Euclidean space.

For an $m \times n$ matrix A over the field of real numbers,

A^- : a generalised inverse (or g-inverse) of A, i.e. any matrix satisfying $AA^- A = A$.

$R(A)$: the rank of A.

$\underline{M}(A)$: the vector space spanned by the columns of A.

$\underline{N}(A)$: the null space of A, i.e., the set of vectors x satisfying $Ax = 0$.

$A^{\perp}$: a matrix of maximum rank satisfying $A' A^{\perp} = 0$.

$P_{A,M}$: $A(A'MA)^- A'M$, the orthogonal projection operator onto $\underline{M}(A)$, when the inner product in R^m is induced by M.

P_A : $P_{A,I}$, I being the identity matrix.

'non-negative definite' and 'positive definite' will be abbreviated as 'n.n.d.' and 'p.d.' respectively.

For an n.n.d. matrix N,

$N^{1/2}$: the unique n.n.d. square root of N.

$\oplus$ will denote the direct sum of vector spaces. For generalised inverses, the notations of Rao and Mitra (1971) will be followed.

INTRODUCTION

We consider the general linear model $Y = X\beta + e$, where Y is an $n \times 1$ random vector taking values in R^n, X is an $n \times m$ matrix (the design matrix), β is an $m \times 1$ vector of unknown parameters varying in R^m and e is an $n \times 1$ vector of errors with $E(e) = 0$ and $E(ee') = \sigma^2 V$, σ^2 being a positive scalar (known or unknown) and V is an $n \times n$ non-negative definite matrix. It is assumed that $m \leq n$. Such a model (also known as the Gauss-Markov model) is usually denoted by $(Y, X\beta, \sigma^2 V)$. The definitions of an estimable linear parametric function, simple least squares estimator (SLSE), best linear unbiased estimator (BLUE), linear minimum bias estimator (LIMBE) and best linear minimum bias estimator (BLIMBE) under the model $(Y, X\beta, \sigma^2 V)$ are well known and we refer to Rao and Mitra (1971, Chapters 7 and 8) for the details.

Early contributions towards estimating linear functionals of β are due to Legendre (1806), Gauss (1809) and Markov (1912), where attention was concentrated on the case where $R(X) = m$ and $V = I$, the identity matrix. Aitken (1934) considered the problem of best linear unbiased estimation under the setup where $R(X) = m$ and V is any positive definite matrix. Bose (1944) considered the case where $R(X) < m$ and $V = I$, while Rao (1945) generalised this to any positive definite V. Seal (1967) gives a good historical account of the linear model upto 1935 and Plackett (1949) gives a short historical note on the method of least squares.

When the covariance matrix $\sigma^2 V$ is nonsingular with V known and the $n \times m$ matrix X is of full rank, i.e. of rank m and when further the columns of the matrix X are all orthonormal eigenvectors of V, then it is an easily verifiable fact that the BLUE of β is identical with its SLSE. This fact was first pointed out by Anderson (1948) and notice of it was taken soon after by Durbin and Watson (1950). From this time onwards, the problem of deriving necessary and sufficient conditions under which the SLSE's are also corresponding BLUE's has received considerable attention, mainly due to the computational advantage of the SLSE over the BLUE.

The present work is devoted to the study of the robustness of estimation and testing procedures in linear models with incorrect design and dispersion matrices. Before giving a summary of the problems considered we shall present a brief review of the literature in this area.

A statement on various necessary and sufficient conditions for the equality of the SLSE's and corresponding BLUE's was made by Zyskind (1962) One of the conditions stated here is that there exists a subset of r eigenvectors of V that forms a basis of the vector space spanned by the columns of the design matrix X. A proof that the eigenvector condition is both necessary and sufficient for the corresponding BLUE and SLSE to have the same covariance matrix is presented with X of full rank and V nonsingular by Magness and McGuire (1962). Though not pointed out by Magness and McGuire, this in fact establishes that the eigenvector condition is necessary and sufficient for the SLSE to be corresponding BLUE under $(Y, X\beta, \sigma^2 V)$ with V nonsingular and X of full rank, since, if $L_1 Y$ is an unbiased estimator of $X\beta$ and $L_2 Y$ is BLUE of $X\beta$ under $(Y, X\beta, \sigma^2 V)$, then $L_1 Y$ and $L_2 Y$ coincide with probability one if and only if their covariance matrices are the same. In Zyskind (1967) the validity of the eigenvector condition for the equality of the SLSE and the corresponding BLUE is established for an arbitrary design matrix X and covariance matrix V. In the same paper, various equivalent conditions are proved and an explicit representation of dispersion matrices V for which the corresponding BLUE and SLSE coincide is also given. An equivalent characterisation of V is given by Rao (1967) which is proved in the context of a known and nonsingular V and a known full rank matrix X. Rao adds that the condition holds even if the matrix X is deficient in rank and V is singular. This statement is justified in Rao (1968). Watson (1967) gives another necessary and sufficient condition for the SLSE to be corresponding BLUE. Kruskal (1968) adopts a coordinate-free approach and shows that the SLSE is also BLUE if and only if $VX = XB$ for some matrix B. Thomas (1968) states that the BLUE of $X\beta$ under $(Y, X\beta, \sigma^2 V_1)$ is its BLUE under $(Y, X\beta, \sigma^2 V)$ if and only if $VV_1^{-1}X = XB$ for some matrix B. Here V_1 and V are positive definite matrices and this condition is easily seen to follow from Kruskal's result. Zyskind (1969) considers a class of design matrices of a particular rank less than n, further satisfying the condition that the column space of every such design matrix contains a particular vector subspace and gives a necessary and sufficient condition on the form of V so that the SLSE's are also corresponding BLUE's under $(Y, X\beta, \sigma^2V)$, for every such design matrix X. Zyskind's result is a generalisatio[n] of an earlier result of McElroy (1967). Mitra and Rao (1969) consider alternative linear models which differ in the expectatio[n] of the observations or both the expectation and dispersion of observations and examine to what extent the estimators based on one model are valid with respect to the other. Rao (1971) gives an explicit representation of a dispersion matrix σ^2V such that BLUE of $X\beta$ under $(Y, X\beta, \sigma^2V_1)$ is its BLUE under $(Y, X\beta, \sigma^2V)$ also. Here V_1 is any non-negative definite matrix and X may be deficient in rank. Anderson (1972) gives a rank criterion for the SLSE to be simultaneously BLUE for the case of V nonsingula[r] and X of full rank which has been extended by Styan (1973) to the case of arbitrary X and by Baksalary and Kala (1977) to the case of arbitrary X and V.

Mitra and Moore (1973) revive the problem of optimality of BLUE's computed under a singular dispersion matrix considered previously by Rao (1968, 1971). They observe that if V_1 is singular the BLUE of $X\beta$ may not have a unique linear representation under $(Y, X\beta, \sigma^2V_1)$ and proceed to characterise V such that a given linear representation/some linear representation/every linear representation of BLUE of $X\beta$ under $(Y, X\beta, \sigma^2V_1)$ is its BLUE under $(Y, X\beta, \sigma^2V)$ also.

Haberman (1975) gives a bound for the norm of the difference between the SLSE and the corresponding BLUE in the case of a nonsingular covariance structure. This gives an idea as to how large is the difference between the SLSE and the corresponding BLUE, when they do not coincide. Haberman's approach is coordinate free and his result has been extended by Baksalary and Kala (1978, 1980).

Now consider the linear model $(Y, X\beta, \sigma^2V)$ where V is positive definite and Y has multivariate normal distribution. Let $A\beta$ be estimable under this model. For testing the hypothesis $H_0 : A\beta = 0$, the likelihood ratio test (LRT) is known to have some optimal properties (see e.g. Scheffe (1959)). However, the LRT statistic under $(Y, X\beta, \sigma^2V)$ for testing H_0 involves the projection operator $P_{X,V^{-1}}$ to be computed under the hypothesis and otherwise. Hence it is natural to ask for conditions on V under which the LRT statistic has the same form as computed under $(Y, X\beta, \sigma^2I)$. Ghosh and Sinha (1980) took V to be the intraclass covariance matrix and derived necessary and sufficient conditions for the LRT statistics under $(Y, X\beta, \sigma^2V)$ and $(Y, X\beta, \sigma^2I)$

to be the same for testing H_0. An alternative proof of their result is given by Sinha and Mukhopadhyay (1980(a)). Sinha and Mukhopadhyay (1980(b)) also considered the covariance structure $V = aI + c1_n' + 1_nc'$, where a is a positive real number, c is a vector and 1_n is a vector with each component unity and obtained conditions under which the LRT statistic for testing H_0 computed under $(Y,X\beta,\sigma^2V)$ coincides with that computed under $(Y,X\beta,\sigma^2I)$. Later, Khatri (1980) developed a general solution to this problem, applicable to any form of V, positive definite.

The problem of robustness of BLUE's when there are specification errors in the dispersion matrix has received considerable attention in the literature. However, the problem of robustness of BLUE's when we have an incorrectly specified design matrix has not received much attention except for Mitra and Rao (1969) where the dispersion matrix considered is σ^2I. If we have a linear model $(Y,X\beta,\sigma^2V)$ with restrictions of the type $R\beta = 0$, then in the restricted model, the expectation of the observations is given by

$$E\begin{bmatrix} Y \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ R \end{bmatrix}\beta ,$$

whereas in the unrestricted model, the expectation can be written as

$$E\begin{bmatrix} Y \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \end{bmatrix}\beta .$$

Thus, we have two linear models which differ in their design matrices and it is of importance to characterise R such that the BLUE of $X\beta$ in the unrestricted model continues to be its BLUE in the restricted model also. Another situation where we come across linear models which differ in their design matrices is when we consider parametric augmentations to a given linear model. In the original model, we have $E(Y) = X_1\beta_1$, whereas in the augmented model we have $E(Y) = X_1\beta_1 + X_2\beta_2$. Here, it is of interest to characterise X_2 so that the BLUE of $X_1\beta_1$ in the original model continues to be its BLUE in the augmented model also.

Motivated by the above considerations, we devote the first chapter to the problem of optimality of BLUE's in a linear model with incorrect design matrix. We consider alternative linear models $(Y, X_1\beta, V)$ and $(Y, X\beta, V)$ which differ in the expectation of the random vector Y. Here V (we omit the scalar multiplier σ^2 since the estimator doesn't involve the scalar multiplier) is a known non-negative definite matrix and since we allow the dispersion matrix to be singular, the BLUE of any estimable parametric functional under $(Y, X_1\beta, V)$ may not have a unique linear representation. Also, since the models differ in their design matrices, parametric functionals estimable under one model need not be so under the other. We characterise design matrices X such that a given linear representation/some linear representation/every linear representation of BLUE of every estimable parametric functional under $(Y, X_1\beta, V)$ is its unbiased estimator, BLUE, LIMBE or BLIMBE under $(Y, X\beta, V)$. When we want the BLUE's under $(Y,X_1\beta,V)$ to be corresponding BLIMBE's under $(Y,X\beta,V)$, what is essentially demanded is the following : Let $p'\beta$ be estimable under $(Y,X_1\beta,V)$. If it is also estimable under $(Y,X\beta,V)$, its BLUE under the first model should also be its BLUE under the second model. If $p'\beta$ is not estimable under the second model, we want its BLUE under the first model to be an optimal estimator under the second model, namely its BLIMBE. The problem of robustness of BLUE's of a subclass of estimable parametric functionals is also analysed when we have an incorrectly specified design matrix. The results are illustrated using examples.

In chapter 2, we consider a more general setup. We consider alternative linear models $(Y,X_1\beta,V_1)$ and $(Y,X\beta,V)$ which differ in the expectation as well as the dispersion of the observations. Such a setup was considered by Mitra and Rao (1969) with $V_1 = I$. We proceed to characterise matrices X and V (non-negative definite) such that a given linear representation/some linear representation/every linear representation of BLUE of every estimable parametric functional under $(Y,X_1\beta,V_1)$ is its BLUE or BLIMBE under $(Y,X\beta,V)$. Here we don't look for conditions under which the BLUE's under $(Y,X_1\beta,V_1)$ is an unbiased estimator or a LIMBE for the corresponding parametric functional under $(Y,X\beta,V)$, since these involve conditions on X only and have been investigated in chapter 1. In the last section of chapter 2, we derive conditions for the optimality of BLIMBE's when we have a linear model with specification errors in the design matrix or dispersion matrix or both.

In chapter 3, we study the robustness of the LRT statistic. Here we assume that the random vector Y has multivariate normal distribution. Let V be a positive definite matrix and $A\beta$ be a vector of estimable parametric functionals under $(Y, X\beta, \sigma^2 V)$. Suppose L and L_V respectively denote the LRT statistics under $(Y, X\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$ for testing $H_0 : A\beta = 0$. In section 3.2 we obtain several equivalent necessary and sufficient conditions for the equality of the BLUE's of $A\beta$ under $(Y, X\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$ and also prove the interesting fact that the equality of L_V and L implies the equality of the BLUE's of $A\beta$ under both the models. Ghosh and Sinha (1980), Sinha and Mukhopadhyay (1980(a) and (b)) and Khatri (1980) have furnished necessary and sufficient conditions under which the LRT statistic retains the same form under various structural forms of V, positive definite. However, it is easy to observe that even if L and L_V are different, but is known that $L_V - L \geq 0$ (or ≤ 0), then the rejection (or acceptance) of H_0 under $(Y, X\beta, \sigma^2 I)$ will imply its rejection (respectively acceptance) under $(Y, X\beta, \sigma^2 V)$ also. In section 3.2 we derive necessary and sufficient conditions under which $L_V - L \geq 0$ or $L_V - L \leq 0$. The result derived by Khatri (1980) follows as a corollary. We also provide a simple proof to the result obtained by Ghosh and Sinha (1980). Illustrative examples are also given. In section 3.3 we give an error bound for the LRT statistic L_V, i.e. we give an upper bound for $|L_V - L|$. This gives an idea as to how much the LRT statistics can differ when they are not equal. In section 3.4 we consider the models $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 I)$ and in section 3.5 we consider $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$. In both cases, we obtain necessary and sufficient conditions under which the LRT statistic for testing $H_0 : A\beta = 0$ under the alternative model is the same as the LRT statistic for testing H_0 under $(Y, X_1\beta, \sigma^2 I)$. Here $A\beta$ is estimable under both the models and V is a positive definite matrix.

The last chapter is devoted to the study of the robustness of estimators and tests when we have a class of linear models. We consider linear models whose design matrices are such that the vector space spanned by the columns of the design matrix contains a particular vector subspace $\underline{M}(U)$ whose dimension is less than $R(X)$. Such models were considered by Zyskind (1969). We proceed to characterise n.n.d. matrices V such that every linear representation/some linear representation of BLUE of $X\beta$ under $(Y, X\beta, V_1)$ continues to be its BLUE under $(Y, X\beta, V)$ for every X of a specified rank satisfying $\underline{M}(U) \subset \underline{M}(X)$. In section 4.3, we consider design matrices X of a specified rank whose column space contains $\underline{M}(U)$ and whose row space contains $\underline{M}(A')$, where A is a given non-null matrix. We characterise n.n.d. matrices V such that for every such X, every linear representation or some linear representation of BLUE of $A\beta$ under $(Y, X\beta, V_1)$ continues to be its BLUE under $(Y, X\beta, V)$ also. Section 4.4 deals with the variance components model and the covariance components model. We obtain conditions under which $X\beta$ admits a BLUE under such a model for every X of a specified rank with $\underline{M}(U) \subset \underline{M}(X)$ or $A\beta$ admits a BLUE under such a model for every X of a specified rank with $\underline{M}(U) \subset \underline{M}(X)$ and $\underline{M}(A') \subset \underline{M}(X')$. In the last section we consider the setup described in section 4.3 and study the robustness of the LRT statistic for testing a hypothesis $H_0 : A\beta = 0$. Here we assume normality of the random vector Y and the problems considered are similar to those taken up in section 3.2 of chapter 3.

CHAPTER 1

SPECIFICATION ERRORS IN THE DESIGN MATRIX

1.1 Statement of the problems.

Mitra and Rao (1969) have investigated the problem of robustness of BLUE's computed under a linear model with an incorrect design matrix assuming that the dispersion matrix is $\sigma^2 I$. Here we consider alternative linear models $(Y, X_1\beta, V)$ and $(Y, X\beta, V)$ which differ in the expectation of the random vector Y, where V is a known n.n.d. matrix. Since we don't impose the condition of nonsingularity on V, the BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ will not have a unique linear representation, except when $R(VX_1^{\perp}) = R(X_1^{\perp})$, as pointed out by Mitra and Moore (1973). Rao (1971) and Mitra and Moore (1973) have established that every linear representation of BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ can be expressed as $X_1(X_1'GX_1)^- X_1'GY$, where G is a g-inverse of $V + X_1X_1'$. Mitra and Moore (1973) have also observed that G could be chosen to be n.n.d. without loss of generality. Hence, a specific linear representation of BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ is $P_{X_1, G}Y$ where G is a specific n.n.d. g-inverse of $V + X_1X_1'$. In the same spirit as in Mitra and Moore (1973), but by different methods, we obtain complete solutions to the following problems :

Problem (1). What is the class of all models $(Y, X\beta, V)$ such that a specific linear representation of BLUE of every estimable parametric functional $p'\beta$ under $(Y, X_1\beta, V)$ is (a) an unbiased estimator (UE) (b) a BLUE (c) a LIMBE and (d) a BLIMBE of $p'\beta$ under $(Y, X\beta, V)$?

Problem (2). What is the class of all models $(Y, X\beta, V)$ such that at least one linear representation of BLUE of every estimable parametric functional $p'\beta$ under $(Y, X_1\beta, V)$ is (a) an UE (b) a BLUE (c) a LIMBE and (d) a BLIMBE of $p'\beta$ under $(Y, X\beta, V)$?

Problem (3). What is the class of all models $(Y, X\beta, V)$ such that every linear representation of BLUE of every estimable parametric functional $p'\beta$ under $(Y, X_1\beta, V)$ is (a) an UE (b) a BLUE (c) a LIMBE and (d) a BLIMBE of $p'\beta$ under $(Y, X\beta, V)$?

Before proceeding further, we prove

Lemma 1.1.1. The BLUE of $X\beta$ has a unique linear representation under $(Y, X\beta, V)$ if and only if $V + XX'$ is positive definite.

Proof : Suppose $L_1'Y$ and $L_2'Y$ are two linear representations for the BLUE of $X\beta$ under $(Y, X\beta, V)$. We shall show that $L_1 = L_2$ if and only if $V + XX'$ is p.d. Since $L_1'Y$ and $L_2'Y$ are BLUE's of $X\beta$ under $(Y, X\beta, V)$, we have

$$L_1'X = X, \; L_2'X = X,$$

$$X^{\perp'} V L_1 = 0 \quad \text{and} \quad X^{\perp'} V L_2 = 0 .$$

From the above relations we get

$$L_1'(X : VX^{\perp}) = L_2'(X : VX^{\perp}) \qquad \ldots(1.1.1) .$$

Now if $V + XX'$ is p.d., then clearly,

$$R^n = \underline{M}(V + XX') = \underline{M}(X : V) = \underline{M}(X : VX^{\perp})$$

and from (1.1.1) it now follows that $L_1 = L_2$, which proves the 'if' part.

Conversely, if $V + XX'$ is not p.d., then $\underline{M}(X : VX^{\perp})$ is not the whole space and hence, we can find a non-null matrix K such that $K'(X : VX^{\perp}) = 0$. Choose $L_2 = L_1 + K$. Then $L_1 \neq L_2$ and $L_2'Y$ is BLUE of $X\beta$ under $(Y, X\beta, V)$, whenever $L_1'Y$ is so. Thus, the BLUE of $X\beta$ doesn't have a unique linear representation if $V + XX'$ is not p.d. and this proves the 'only if' part.

<u>Remark 1.1.1</u>. Mitra and Moore (1973) have observed that if $R(VX^{\perp}) = R(X^{\perp})$, then the BLUE of $X\beta$ has a unique linear representation under $(Y, X\beta, V)$. From the proof of lemma 1.1.1, it is clear that $V + XX'$ is p.d. if and only if $R(VX^{\perp}) = R(X^{\perp})$.

From lemma 1.1.1 it is clear that if $V + X_1X_1'$ is p.d., then the three problems stated above merge into one. We also observe that the three problems merge into one provided it is known that $\underline{M}(X) \subseteq \underline{M}(V : X_1)$.

The need for obtaining conditions under which the BLUE of every estimable parametric functional $p'\beta$ under $(Y, X_1\beta, V)$ is its BLIMBE under $(Y, X\beta, V)$ arises because of the fact that parametric functionals estimable under $(Y, X_1\beta, V)$ need not be so under $(Y, X\beta, V)$. But still, we want the BLUE of $p'\beta$ computed under $(Y, X_1\beta, V)$ to be an optimal estimator of $p'\beta$ under $(Y, X\beta, V)$, namely, its BLUE if $p'\beta$ is estimable under $(Y, X\beta, V)$ and its BLIMBE otherwise.

Sometimes one might be interested only in inferences regarding a few estimable parametric functionals and not all. The problem of robustness of BLUE's of a subset of estimable parametric functionals is studied in the last section.

For matrices X_1 and X, we will denote

$$D = X_1 - X,\ Z_1 = X_1^{\perp} \ \text{and} \ Z = X^{\perp}.$$

1.2 Solution to problem (1).

We first prove an algebraic lemma which we need in the sequel.

Lemma 1.2.1. Let V be an n.n.d. matrix of order $n \times n$, X_1 be a matrix of order $n \times m$ and G, a specific n.n.d. g-inverse of $V + X_1X_1'$. Then there exists a nonsingular matrix P of order $n \times n$ and an orthogonal matrix Q of order $m \times m$ such that

$$X_1 = P \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q'$$

$$V + X_1X_1' = P \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} P' \quad \text{and}$$

$$G = P'^{-1} \begin{bmatrix} \Lambda_1^{-1} & 0 & 0 \\ 0 & \Lambda_2^{-1} & 0 \\ 0 & 0 & S \end{bmatrix} P^{-1}$$

where Λ_1 and Λ_2 are diagonal p.d. matrices, S is n.n.d. and Λ_1 and I are of the same order.

<u>Proof</u> : Since X_1X_1' and $V+X_1X_1'$ are both n.n.d. and since $\underline{M}(X_1X_1') \subset \underline{M}(V+X_1X_1')$, there exists a nonsingular matrix T such that

$$X_1X_1' = T \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} T' \quad \text{and}$$

$$V+X_1X_1' = T \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} T'$$

where Λ_1 and Λ_2 are diagonal p.d. matrices (see Rao and Mitra, 1971, page 121). Consider the above representation for $V+X_1X_1'$. Then G is an n.n.d. g-inverse of $V+X_1X_1'$ if and only if

$$G = T'^{-1} \begin{bmatrix} \Lambda_1^{-1} & 0 & R_1 \\ 0 & \Lambda_2^{-1} & R_2 \\ R_1' & R_2' & R_3 \end{bmatrix} T^{-1}$$

where R_1, R_2 and R_3 are such that $R_3 - R_1' \Lambda_1 R_1 - R_2' \Lambda_2 R_2$ is n.n.d.

Clearly,

$$G = T'^{-1}\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ R_1'\Lambda_1 & R_2'\Lambda_2 & I \end{bmatrix}\begin{bmatrix} \Lambda_1^{-1} & 0 & 0 \\ 0 & \Lambda_2^{-1} & 0 \\ 0 & 0 & S \end{bmatrix}\begin{bmatrix} I & 0 & \Lambda_1 R_1 \\ 0 & I & \Lambda_2 R_2 \\ 0 & 0 & I \end{bmatrix} T^{-1}$$

where $S = R_3 - R_1'\Lambda_1 R_1 - R_2'\Lambda_2 R_2$.

Write $P = T\begin{bmatrix} I & 0 & -\Lambda_1 R_1 \\ 0 & I & -\Lambda_2 R_2 \\ 0 & 0 & I \end{bmatrix}$ and check that

$$X_1 X_1' = P\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P'$$

$$V + X_1 X_1' = P\begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} P' \quad \text{and}$$

$$G = P'^{-1}\begin{bmatrix} \Lambda_1^{-1} & 0 & 0 \\ 0 & \Lambda_2^{-1} & 0 \\ 0 & 0 & S \end{bmatrix} P^{-1}$$

where S is n.n.d.

Now

$$X_1 X_1' = P\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P \implies$$

there exists an orthogonal matrix Q of order $m \times m$ such that

$$X_1 = P \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q'$$

(See Rao and Mitra, 1971, page 17).

This completes the proof of lemma 1.2.1.

Remark 1.2.1. P in lemma 1.2.1 can be chosen such that $\Lambda_1 - I$ is of the form $\begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}$, where Λ is a diagonal p.d. matrix.

The following theorem gives a solution to the problem (1) (a) and (b).

Theorem 1.2.1. Consider the linear models $(Y, X_1\beta, V)$ and $(Y, X\beta, V)$ and let G be a given n.n.d. g-inverse of $V + X_1X_1'$. Then

(a) the linear representation $P_{X_1,G}Y$ of BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ is unbiased for $X_1\beta$ under $(Y, X\beta, V)$ if and only if

$$X = X_1 + (I - P_{X_1,G})A \qquad \ldots(1.2.1)$$

where A is arbitrary and

(b) $P_{X_1,G}Y$ is BLUE of $X_1\beta$ under $(Y, X\beta, V)$ if and only if X is given by (1.2.1), where A satisfies the condition

$$\underline{M}\begin{bmatrix} VGX_1 \\ 0 \end{bmatrix} \subset \underline{M}\begin{bmatrix} X_1 \\ (I - P_{X_1,G})A \end{bmatrix},$$

or equivalently, A satisfies

$(I - P_{X_1,G})A(X_1^- VGX_1 + (I - X_1^- X_1)B) = 0$ for some B, X_1^- being an arbitrary but fixed g-inverse of X_1.

Proof : $E\left[P_{X_1,G} Y \mid (Y, X\beta, V) \right] = X_1\beta \quad \forall \beta$

$\iff P_{X_1,G} D = 0$, where $D = X_1 - X$.

$\iff -D = (I - P_{X_1,G})A$, where is arbitrary.

This completes the proof of (a).

Now,

$P_{X_1,G} Y$ is BLUE of $X_1\beta$ under $(Y, X\beta, V)$ iff X is given by (1.2.1) and further

$$P_{X_1,G} V Z = 0, \quad \text{where } Z = X^{\perp}$$

We observe that $\underline{M}(VGX_1) \subset \underline{M}(X_1)$, since,

$$VGX_1 = (V + X_1X_1')GX_1 - X_1X_1'GX_1 = X_1 - X_1X_1'GX_1 .$$

Now,

$$P_{X_1,G} V Z = 0$$

$$\iff X_1'GVZ = 0$$

$$\iff VGX_1 = XC = X_1C + (I - P_{X_1,G})AC \quad \text{for some } C .$$

Premultiplying both sides by $(I - P_{X_1,G})$ and using the fact $\underline{M}(VGX_1) \subset \underline{M}(X_1)$, we see that the above equation holds for some C if and only if

$$VGX_1 = X_1 C \quad \text{and}$$
$$0 = (I - P_{X_1,G})AC \quad \text{for some } C$$

$$\iff \quad \underline{M}\begin{bmatrix} VGX_1 \\ 0 \end{bmatrix} \subset \underline{M}\begin{bmatrix} X_1 \\ (I - P_{X_1,G})A \end{bmatrix}$$

Further, $VGX_1 = X_1 C$

$\iff C = X_1^- VGX_1 + (I - X_1^- X_1)B$, where B is arbitrary.

Substituting this expression for C in

$0 = (I - P_{X_1,G})AC$, we get

$(I - P_{X_1,G})A(X_1^- VGX_1 + (I - X_1^- X_1)B) = 0$.

This completes the proof of (b).

Remark 1.2.2. The equation

$$(I - P_{X_1,G})A(X_1^- VGX_1 + (I - X_1^- X_1)B) = 0$$

is consistent in A for every B. Consider the set of solutions for A in the above equation for every fixed B. With A obtained like this, the class of design matrices X for which $P_{X_1,G}Y$ is BLUE of $X_1\beta$ under $(Y, X\beta, V)$ is given by

$$X = X_1 + (I - P_{X_1,G})A.$$

Corollary 1.2.1 (Mitra and Rao, 1969).

If in theorem 1.2.1 $V = I$, then the BLUE of $X_1\beta$ under $(Y, X_1\beta, I)$ is its BLUE under $(Y, X\beta, .)$ if and only if

$$X = X_1 + (I - P_{X_1})A, \quad \text{where } A \text{ satisfies}$$

$$\underline{M}(X_1') \cap \underline{M}(A'(I - P_{X_1})) = \{0\}.$$

Proof : If $V = I$, then $G = (I + X_1X_1')^{-1}$ and $P_{X_1, G} = P_{X_1}$.

In this case, the conditions in theorem 1.2.1 reduce to

$$X = X_1 + (I - P_{X_1})A, \text{ where } A \text{ satisfies}$$

$$\underline{M}\begin{bmatrix} GX_1 \\ 0 \end{bmatrix} \subset \underline{M}\begin{bmatrix} X_1 \\ (I - P_{X_1})A \end{bmatrix}$$

$$\iff \quad \underline{M}(X_1') \cap \underline{M}(A'(I - P_{X_1})) = \{0\}$$

(use the fact that $a'X_1 = 0 \iff a'GX_1 = 0$).

The corollary is thus established.

We shall now obtain solutions to problem (1) (c) and (d).

We state

Lemma 1.2.2. For any vector $z \in R^n$, let $\|z\|^2 = z'Mz$, where M is an n.n.d. matrix. Then a linear estimator $\ell'Y$ is LIMBE of a parametric functional $p'\beta$ under $(Y, X\beta, V)$ if and only if

$$XMX'\ell = XMp$$

and $\ell'Y$ is BLIMBE of $p'\beta$ if and only if

$$XMX'\ell = XMp$$

$$\text{and} \quad V\ell \in \underline{M}(X)$$

The proof of the lemma, being straightforward, is omitted.

From lemma 1.2.2, it is clear that if M is p.d., then $\ell'Y$ is LIMBE (or BLIMBE) of $p'\beta$ under $(Y, X\beta, V)$ with respect to the norm defined as $||z||^2 = z'Mz$ if and only if $\lambda' Y$ is LIMBE (respectively BLIMBE) of $p'M^{1/2}\beta$ under $(Y, XM^{1/2}\beta, V)$ with respect to the norm defined as $||z||^2 = z'z$. Hence, from now onwards, whenever we speak about LIMBE or BLIMBE, we will be considering only the case where the norm defining the bias is the Euclidean norm, since if the norm is defined through a positive definite matrix M, we can bring it down to the case of the Euclidean norm by means of a non-singular transformation as described above. We now prove

Theorem 1.2.2. Consider the linear models $(Y, X_1\beta, V)$ and $(Y, X\beta, V)$ and let G be a given n.n.d. g-inverse of $V + X_1X_1'$. Consider the representations of X_1, $V + X_1X_1'$ and G as specified in lemma 1.2.1. Then (a) the representation $p'(X_1'GX_1)^- X_1'GY$ of BLUE of every estimable parametric functional $p'\beta$ under $(Y, X_1\beta, V)$ is its LIMBE under $(Y, X\beta, V)$ if and only if

$$D = P \begin{bmatrix} B & C_1 & C_2 \\ E_1 & E_2 & E_3 \\ F_1 & F_2 & F_3 \end{bmatrix} Q' \qquad \ldots(1.2.2)$$

where, B is an arbitrary n.n.d. matrix with eigenvalues in $[0,1]$, C_1 and C_2 are arbitrary matrices satisfying $C_1C_1' + C_2C_2' = B - B^2$ and E_1, E_2, E_3, F_1, F_2 and F_3 are arbitrary matrices satisfying $\underline{M}(E_1 \vdots E_2 \vdots E_3)' \subset \underline{N}(B : C_1 : C_2)$ and $\underline{M}(F_1 \vdots F_2 \vdots F_3)' \subset \underline{N}(B : C_1 : C_2)$ and (b) the representation

$p'(X_1'GX_1)^- X_1'GY$ of BLUE of every estimable parametric functional $p'\beta$ under $(Y, X_1\beta, V)$ is its BLIMBE under $(Y, X\beta, V)$ if and only if D is as given in part (a), where P is chosen as in remark 1.2.1, B is given by

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{bmatrix}$$

with $(I - B_{11}) - B_{12} \; (I - B_{22})^- \; B_{12}'$ p.d., B_{11} being the top left hand corner submatrix of B having the same order as that of Λ in remark 1.2.1 and $\underline{M}(E_1 \vdots E_2 \vdots E_3)'$ and $\underline{M}(F_1 \vdots F_2 \vdots F_3)'$ are also subspaces of $\underline{N}(U_1' \vdots U_2' \vdots U_3')'$, U_1, U_2 and U_3 being arbitrary matrices satisfying $(I - B)U_1 - C_1U_2 - C_2U_3 = I - \Lambda_1^{-1}$.

<u>Proof</u> : (a) From lemma 1.2.2 it follows that for every estimable $p'\beta$ under $(Y, X_1\beta, V)$, its BLUE $p'(X_1'GX_1)^- X_1'GY$ is its LIMBE under $(Y, X\beta, V)$ if and only if

$$XX'GX_1(X_1'GX_1)^- X_1' = XX_1'$$

$$\Longleftrightarrow \quad XD'GX_1 = 0$$

$$\Longleftrightarrow \quad X_1'GDX_1' = X_1'GDD' \qquad \ldots(1.2.3)$$

Write $$P^{-1}DQ = T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$ where the partitioning is clear from the context. Using the above partitioning of $P^{-1}DQ$ and lemma 1.2.1, we see that (1.2.3) holds if and only if

$$T_{11} = T_{11}T_{11}' + T_{12}T_{12}' + T_{13}T_{13}' \qquad \ldots(1.2.4)$$

$$0 = T_{11}T_{21}' + T_{12}T_{22}' + T_{13}T_{23}' \qquad \ldots(1.2.5)$$

$$\text{and} \quad 0 = T_{11}T_{31}' + T_{12}T_{32}' + T_{13}T_{33}' \qquad \ldots(1.2.6)$$

(1.2.4) holds if and only if $T_{11} = B$, $T_{12} = C_1$ and $T_{13} = C_2$, where B is n.n.d. with eigenvalues in $[0,1]$ and $C = (C_1 \vdots C_2)$ is such that $CC' = B - B^2$. (1.2.5) and (1.2.6) hold if and only if $T_{2i} = E_i$ and $T_{3i} = F_i$, $i = 1,2,3$ where E_i and F_i are as specified in the statement of part (a) of the theorem. The proof of (a) is now complete.

(b) For every estimable $p'\beta$ under $(Y, X_1\beta, V)$, its BLUE $p'(X_1'GX_1)^- X_1'GY$ is its BLIMBE under $(Y, X\beta, V)$ if and only if

$$X_1'GDX_1' = X_1'GDD'$$

$$\text{and} \quad Z'VGX_1 = 0$$

$\Longleftrightarrow$ $\underline{M}(VGX_1) \subset \underline{M}(X_1 - D)$, where D is given by part (a)

$$\Longleftrightarrow \begin{bmatrix} I-\Lambda_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I-B & -C_1 & -C_2 \\ -E_1 & -E_2 & -E_3 \\ -F_1 & -F_2 & -F_3 \end{bmatrix} U \qquad \ldots(1.2.7)$$

for some matrix U. Let $(U_1' \vdots U_2' \vdots U_3')'$ denote the first r columns of U, where $r = R(X_1)$ is the order of B. Then (1.2.7) holds if and only if

$$(I-B)U_1 - C_1U_2 - C_2U_3 = I - \Lambda_1^{-1} \qquad \ldots(1.2.8)$$

$$E_1U_1 + E_2U_2 + E_3U_3 = 0 \qquad \ldots(1.2.9)$$

and $$F_1U_1 + F_2U_2 + F_3U_3 = 0 \qquad \ldots(1.2.10)$$

The equation (1.2.8) is consistent if and only if

$$\underline{M}(I - \wedge_1^{-1}) \subset \underline{M}(I - B) \qquad \ldots(1.2.11)$$

Let $$\wedge_1^{-1} = \begin{bmatrix} \wedge_{11}^{-1} & 0 \\ 0 & \wedge_{22}^{-1} \end{bmatrix}$$ and

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{bmatrix}$$

where $\wedge_{11}$ and B_{11} are of the same order equal to that of $\wedge$ in remark 1.2.1. Then we have

$$I - \wedge_1^{-1} = (\wedge_1 - I)\wedge_1^{-1} = \begin{bmatrix} \wedge & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \wedge_{11}^{-1} & 0 \\ 0 & \wedge_{22}^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \wedge\wedge_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

$\wedge\wedge_{11}^{-1}$ is a p.d. matrix and (1.2.11) holds if and only if

$$\underline{M} \begin{bmatrix} \wedge\wedge_{11}^{-1} \\ 0 \end{bmatrix} \subset \underline{M} \begin{bmatrix} I - B_{11} & -B_{12} \\ -B_{12}' & I-B_{22} \end{bmatrix} \qquad \ldots(1.2.12)$$

Now, $$\underline{M} \begin{bmatrix} I-B_{11} & -B_{12} \\ -B_{12}' & I-B_{22} \end{bmatrix} = \underline{M} \begin{bmatrix} I-B_{11} & -B_{12} \\ -B_{12}' & I-B_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ (I-B_{22})^- B_{12}' & I \end{bmatrix}$$

$$= \underline{M} \begin{bmatrix} (I-B_{11}) - B_{12}(I-B_{22})^- B_{12}' & -B_{12} \\ 0 & I-B_{22} \end{bmatrix}$$

Hence it follows that (1.2.12) holds if and only if $(I - B_{11}) - B_{12}(I - B_{22})^{-} B_{12}'$ is p.d. From (1.2.9) and (1.2.10), it follows that E_1, E_2, E_3, F_1, F_2 and F_3 satisfy the conditions stated in the theorem and this completes the proof of theorem 1.2.2.

Remark 1.2.3. In theorem 1.2.2, we have encountered a matrix

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{bmatrix}$$

with the properties B and $I - B$ are n.n.d. and $(I - B_{11}) - B_{12}(I - B_{22})^{-} B_{12}'$ is p.d. We shall give below a class of matrices which satisfy the above properties.

Since we want $(I-B_{11}) - B_{12}(I-B_{22})^{-} B_{12}'$ to be p.d., $I - B_{11}$ itself should be p.d. and since $I - B$ is n.n.d., $I - B_{22}$ is also n.n.d. Hence we can write

$$B_{11} = R \begin{bmatrix} \Delta_1 & 0 \\ 0 & 0 \end{bmatrix} R'$$

$$\text{and} \quad B_{22} = T \begin{bmatrix} \Delta_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} T'$$

where R and T are orthogonal matrices and Δ_1 and Δ_2 are diagonal matrices whose diagonal elements lie in $(0,1)$. If $R = (R_1 : R_2)$ and $T = (T_1 : T_2 : T_3)$, then we can write $B_{12} = R_1 S T_1'$, since $\underline{M}(B_{12})$ is a subspace of $\underline{M}(B_{11})$ and $\underline{M}(I - B_{11})$ and $\underline{M}(B_{12}')$ is a subspace of $\underline{M}(B_{22})$ and $\underline{M}(I - B_{22})$. Since we have chosen B_{22} to be n.n.d., B is n.n.d. if and only if

$$B_{11} - B_{12}B_{22}^{-}B_{12}' \text{ is n.n.d.}$$

$$\Longleftrightarrow \quad \Delta_1 - S\Delta_2^{-1}S' \text{ is n.n.d.}$$

$$\Longleftrightarrow \quad I - \Delta_1^{-\frac{1}{2}} S\Delta_2^{-1}S'\Delta_1^{-\frac{1}{2}} \text{ is n.n.d.}$$

$(I - B_{11}) - B_{12}(I - B_{22})^{-}B_{12}'$ is p.d. if and only if

$$R \begin{bmatrix} I - \Delta_1 & 0 \\ 0 & I \end{bmatrix} R' - R \begin{bmatrix} S(I - \Delta_2)^{-1}S' & 0 \\ 0 & 0 \end{bmatrix} R' \text{ is p.d.}$$

$$\Longleftrightarrow \quad I - (I - \Delta_1)^{-1/2} S(I - \Delta_2)^{-1}S'(I - \Delta_1)^{-1/2} \text{ is p.d.}$$

Thus we want to characterise S so that

$$I - \Delta_1^{-1/2} S\Delta_2^{-1}S'\Delta_1^{-1/2} \text{ is n.n.d.}$$

$$\text{and} \quad I - (I - \Delta_1)^{-1/2} S(I - \Delta_2)^{-1}S'(I - \Delta_1)^{-1/2} \text{ is p.d.}$$

Let δ_1 and λ_1 denote the minimum and maximum of the diagonal elements of Δ_1 and δ_2 and λ_2 denote those of Δ_2. From the choice of Δ_1 and Δ_2, it is clear that $0 < \delta_i < 1$ and $0 < \lambda_i < 1$ for $i = 1, 2$. Then,

$$\| \Delta_1^{-1/2} S\Delta_2^{-1}S'\Delta_1^{-1/2} \| \leq \delta_1^{-1}\delta_2^{-1} \|S\|^2 ,$$

where for any matrix M, $\|M\|$ is defined as $\|M\| = \sup_{\|z\|_p \leq 1} \|Mz\|_p$, $1 \leq p \leq \infty$. Hence if we choose $\|S\|^2 \leq \delta_1\delta_2$, then $I - \Delta_1^{-1/2} S\Delta_2^{-1}S'\Delta_1^{-1/2}$ is n.n.d.

$$\|(I - \Delta_1)^{-1/2} S(I - \Delta_2)^{-1}S'(I - \Delta_1)^{-1/2}\| \leq (1-\lambda_1)^{-1}(1-\lambda_2)^{-1} \|S\|^2 .$$

Hence if we choose $\|S\|^2 < (1-\lambda_1)(1-\lambda_2)$, then

$$I - (I-\Delta_1)^{-1/2} S (I-\Delta_2)^{-1} S' (I-\Delta_1)^{-1/2} \quad \text{is p.d.}$$

Thus, a class of matrices S with the desired properties is obtained by choosing S such that

$$\|S\|^2 < \min(\delta_1\delta_2, (1-\lambda_1)(1-\lambda_2)).$$

<u>Corollary 1.2.2</u>. Let $X_1 = R\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}Q'$ be the singular value decomposition of X_1. Then,

(a) the BLUE of every estimable parametric functional under $(Y, X_1\beta, \sigma^2 I)$ is its LIMBE under $(Y, X\beta, \sigma^2 I)$ if and only if

$$D = R\begin{bmatrix}\Delta B & \Delta C\\ E_1 & E_2\end{bmatrix}Q' \qquad \ldots(1.2.13)$$

where B is an arbitrary n.n.d. matrix with eigenvalues in $[0,1]$, C is any matrix satisfying $CC' = B - B^2$, and E_1 and E_2 are arbitrary matrices satisfying $\underline{M}(E_1 : E_2)' \subset \underline{N}(B : C)$.

(b) the BLUE of every estimable parametric functional under $(Y, X_1\beta, \sigma^2 I)$ is its BLIMBE under $(Y, X\beta, \sigma^2 I)$ if and only if D is as given in part (a), where the eigenvalues of B lie in $[0,1)$ and $\underline{M}(E_1 : E_2)'$ is also a subspace of $\underline{N}(U_1' : U_2')$, U_1 and U_2 being arbitrary matrices satisfying

$$(I-B)U_1(I+\Delta^2) - CU_2(I+\Delta^2) = I .$$

<u>Proof</u> : Write $P = R\begin{bmatrix}\Delta & 0\\ 0 & I\end{bmatrix}$. Then, $X_1 = P\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}Q'$

and part (a) of the corollary follows from part (a) of theorem 1.2.2. We can write

$$I + X_1X_1' = P\begin{bmatrix}\Lambda_1 & 0\\ 0 & \Lambda_2\end{bmatrix}P' \text{ which is a p.d. matrix, where}$$

$\Lambda_1 = \Delta^{-2}(I+\Delta^2)$ and $\Lambda_2 = I$. From part (b) of theorem 1.2.2, it follows that $I - B$ should be p.d. or in other words the eigen-values of B should lie in $[0,1)$ and U_1 and U_2 should satisfy

$$(I-B)U_1 - CU_2 = I - \Lambda_1^{-1} = I - \Delta^2(I+\Delta^2)^{-1}$$

$$\Longleftrightarrow \quad (I-B)U_1(I+\Delta^2) - CU_2(I+\Delta^2) = I \, .$$

This proves part (b) of the corollary.

<u>Remark 1.2.4</u>. Theorem 1.2.2 gives an explicit characterisation of design matrices X for which a given linear representation of BLUE of every estimable parametric functional under $(Y, X_1\beta, V)$ is corresponding LIMBE (or BLIMBE) under $(Y, X\beta, V)$. However, given two linear models $(Y, X_1\beta, V)$ and $(Y, X\beta, V)$, in order to verify whether the linear representation $p'(X_1'GX_1)^- X_1'GY$ of BLUE of every estimable $p'\beta$ under $(Y, X_1\beta, V)$ is BLIMBE under $(Y, X\beta, V)$ one need only verify the conditions

$$X_1'GDX_1' = X_1'GDD'$$

$$\text{and} \quad Z'VGX_1 = 0 \, .$$

1.3. Solution to problem (2).

The class of all linear models $(Y, X\beta, V)$ which satisfy the conditions given in problem (2) is precisely the union of linear models $(Y, X\beta, V)$ which satisfy the conditions in problem (1) the union being taken over all n.n.d. g-inverses of $V + X_1X_1'$. However, given X, one does not know from the above whether there is atleast one linear representation of BLUE of every estimable $p'\beta$ under $(Y, X_1\beta, V)$ with the desired optimality condition. We give below several methods for finding this and also obtaining one such linear representation whenever it exists.

We state a result of Mitra (1973) which we need in the sequel.

Lemma 1.3.1 (Mitra). Let A_1, A_2, B_1, B_2 be n.n.d. matrices. A necessary and sufficient condition for the consistent equations $A_1WB_1 = C_1$ and $A_2WB_2 = C_2$ to have a common solution is $A_1(A_1+A_2)^- C_2(B_1+B_2)^- B_1 = A_2(A_1+A_2)^- C_1(B_1+B_2)^- B_2$ in which case, a general solution is

$$W = (A_1+A_2)^- (C_1+R+S+C_2) (B_1+B_2)^-$$
$$+ U - (A_1+A_2)^- (A_1+A_2)U(B_1+B_2)(B_1+B_2)^-$$

where U is arbitrary, R and S are arbitrary matrices satisfying the equations

$$A_2(A_1+A_2)^- R = A_1(A_1+A_2)^- C_2,$$
$$R(B_1+B_2)^- B_1 = C_1(B_1+B_2)^- B_2$$

and $A_1(A_1+A_2)^- S = A_2(A_1+A_2)^- C_1$,

$$S(B_1+B_2)^- B_2 = C_2(B_1+B_2)^- B_1 .$$

At least one linear representation of the BLUE of $X_1\beta$ under $(Y,X_1\beta,V)$ is unbiased for $X_1\beta$ under $(Y,X\beta,V)$ if and only if there exists G, a g-inverse of $V+X_1X_1'$ satisfying $X_1'GD = 0$, which is equivalent to demanding that the equations

$$X_1X_1'GDD' = 0$$

$$\text{and} \quad (V+X_1X_1')G(V+X_1X_1') = V+X_1X_1' \qquad \ldots(1.3.1)$$

should have a common solution in G, which can be verified using lemma 1.3.1.

Similarly, in order to verify if every estimable parametric functional under $(Y,X_1\beta,V)$ has at least one linear representation for its BLUE under $(Y,X_1\beta,V)$, a LIMBE under $(Y,X\beta,V)$, one need only examine the consistency of the equations

$$X_1X_1'GDX'XD = 0$$

$$\text{and} \quad (V+X_1X_1')G(V+X_1X_1') = V+X_1X_1' . \qquad \ldots(1.3.2)$$

To examine if at least one linear representation of the BLUE of every estimable parametric functional under $(Y,X_1\beta,V)$ is also its BLUE (or BLIMBE) under $(Y,X\beta,V)$, in addition to examining the consistency of (1.3.1) (respectively (1.3.2)), it is enough to check the condition $X_1'GVZ = 0$. Here G is any g-inverse of $V+X_1X_1'$, since $X_1'GVZ$ is invariant under the choice of g-inverse of $V+X_1X_1'$.

The proceeding discussion suggests a method of verifying whether the requirements stated in problem (2) are satisfied for two given linear models. In what follows, we shall present several equivalent solutions to problem (2).

Let P be an $n \times n$ nonsingular matrix and Q be an $m \times m$ orthogonal matrix such that

$$X_1 = P \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q' \qquad \ldots(1.3.3)$$

and

$$V + X_1 X_1' = P \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} P' \qquad \ldots(1.3.4)$$

where Λ_1 and Λ_2 are diagonal p.d. matrices. We now prove

Theorem 1.3.1. Consider the partitioned forms of X_1 and $V+X_1X_1'$ given by (1.3.3) and (1.3.4). Then,

(a) at least one linear representation of BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ is unbiased for $X_1\beta$ under $(Y, X\beta, V)$ if and only if

$$D = P \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} Q' \qquad \ldots(1.3.5)$$

where $\underline{M}(T_{11} : T_{12} : T_{13})' \subset \underline{M}(T_{31} : T_{32} : T_{33})' \qquad \ldots(1.3.6)$

(b) at least one linear representation of BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ is also BLUE of $X_1\beta$ under $(Y, X\beta, V)$ if and only if

D is given by (1.3.5) and (1.3.6) with the further condition

$$\underline{M}\begin{bmatrix} T_{21}(\Lambda_1 - I) \\ T_{31}(\Lambda_1 - I) \end{bmatrix} \subset \underline{M}\begin{bmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{bmatrix} \qquad \ldots(1.3.7)$$

Proof :

$$\text{Let} \quad P^{-1}DQ = T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

$$\text{and} \qquad G = P'^{-1}\begin{bmatrix} \Lambda_1^{-1} & 0 & R_1 \\ 0 & \Lambda_2^{-1} & R_2 \\ R_1' & R_2' & S \end{bmatrix} P^{-1}$$

where $S - R_1'\Lambda_1 R_1 - R_2'\Lambda_2 R_2$ is n.n.d., be any n.n.d. g-inverse of $V + X_1X_1'$. We want conditions on D such that $X_1'GD = 0$ for some n.n.d. g-inverse G of $V + X_1X_1'$

$$\Longleftrightarrow \quad \Lambda_1^{-1} T_{11} + R_1 T_{31} = 0,$$

$$\Lambda_1^{-1} T_{12} + R_1 T_{32} = 0$$

$$\text{and} \quad \Lambda_1^{-1} T_{13} + R_1 T_{33} = 0, \text{ for some } R_1$$

$$\Longleftrightarrow \quad \underline{M}(T_{11} : T_{12} : T_{13})' \subset \underline{M}(T_{31} : T_{32} : T_{33})',$$

which proves part (a).

Now we want to obtain conditions on D such that $X_1'GVZ = 0$ or equivalently

$\underline{M}(VGX_1) \subset \underline{M}(X_1 - D)$, where D is given by (1.3.5) and (1.3.6)

$$\iff \quad (\Lambda_1 - I)\Lambda_1^{-1} = (I - T_{11})K_1 - T_{12}K_2 - T_{13}K_3 \qquad \ldots(1.3.8)$$

$$0 = T_{21}K_1 + T_{22}K_2 + T_{23}K_3 \qquad \ldots(1.3.9)$$

$$\text{and} \quad 0 = T_{31}K_1 + T_{32}K_2 + T_{33}K_3 \qquad \ldots(1.3.10)$$

for some K_1, K_2 and K_3. Using (1.3.6) and (1.3.10) in (1.3.8) we get $(\Lambda_1 - I)\Lambda_1^{-1} = K_1$. Hence (1.3.8), (1.3.9) and (1.3.10) hold if and only if

$$\underline{M}\begin{bmatrix} T_{21} & (\Lambda_1 - I) \\ T_{31} & (\Lambda_1 - I) \end{bmatrix} \subset \underline{M}\begin{bmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{bmatrix}$$

which proves part (b).

<u>Theorem 1.3.2</u>(a) The conditions (1.3.5) and (1.3.6) are equivalent to anyone of the following :

(i) $\underline{M}(X_1 : 0 : 0)' \subset \underline{M}(X_1 : VZ_1 : D)'$

(ii) $\underline{M}(X_1) \cap \underline{M}(VZ_1 : D) = \{0\}$

(iii) $\underline{M}\begin{bmatrix} Z_1'V \\ D' \end{bmatrix} = \underline{M}\begin{bmatrix} Z_1'VZ_1 \\ D'Z_1 \end{bmatrix}$.

(b) The conditions (1.3.5), (1.3.6) and (1.3.7) are equivalent to anyone of the following :

(i) $\underline{M}(X_1 : 0 : 0 : 0)' \subset \underline{M}(X_1 : VZ_1 : VZ : D)'$

(ii) $\underline{M}(X_1) \cap \underline{M}(VZ_1 : VZ : D) = \{0\}$

(iii) $\underline{M}\begin{bmatrix} Z_1'V \\ Z'V \\ D' \end{bmatrix} = \underline{M}\begin{bmatrix} Z_1'VZ_1 \\ Z'VZ_1 \\ D'Z_1 \end{bmatrix}$

Proof : It is fairly easy to establish the equivalence of (i), (ii) and (iii) in theorem 1.3.2 (a) or (b). We shall prove (a) by establishing the equivalence of (1.3.5) and (1.3.6) with condition (ii) in (a).

Let X_1 and $V + X_1X_1'$ be given by (1.3.3) and (1.3.4). Then one choice of Z_1 is

$$Z_1 = P'^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and then we have

$$VZ_1 = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let D be given by (1.3.5) and let $a = (a_1' \; a_2' \; a_3')'$, $b = (b_1' \; b_2' \; b_3')'$ and $c = (c_1' \; c_2' \; c_3')'$ be any three vectors. Then $\underline{M}(X_1) \cap \underline{M}(VZ_1 : D) = \{0\}$ if and only if " $X_1a + VZ_1b + Dc = 0 \Longrightarrow X_1a = 0$ " which is equivalent to requiring that

$$\text{"} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

$\Longrightarrow$ $a_1 = 0$," which happens if and only if

$$\underline{M}(T_{11} : T_{12} : T_{13})' \subset \underline{M}(T_{31} : T_{32} : T_{33})'$$

This completes the proof of (a).

Next we shall show the equivalence of (1.3.5), (1.3.6) and (1.3.7) with condition (i) in theorem 3.2 (b). Condition (i) in theorem 3.2 (b) holds if and only if there exists a matrix L such that

$$X_1'L = X_1' \qquad \ldots(1.3.11)$$

$$Z_1'VL = 0 \qquad \ldots(1.3.12)$$

$$D'L = 0 \qquad \ldots(1.3.13)$$

$$\text{and} \quad Z'VL = 0 \qquad \ldots(1.3.14)$$

(which is equivalent to saying that $L'Y$ is BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ and $(Y, X\beta, V)$). (1.3.11), (1.3.12) and (1.3.13) hold if and only if $L'Y$ is a BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ and is unbiased for $X_1\beta$ under $(Y, X\beta, V)$, which, in view of theorem 3.2 (a) is equivalent to the conditions (1.3.5) and (1.3.6).

$$(1.3.11) \text{ holds } \Longleftrightarrow \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P'L = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P' \qquad \ldots(1.3.15)$$

$$(1.3.12) \text{ holds } \Longleftrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} P'L = 0 \qquad \ldots(1.3.16$$

$$(1.3.14) \text{ holds } \Longleftrightarrow \underline{M}(VL) \subset \underline{M}(X_1 - D)$$

$$\Longleftrightarrow \begin{bmatrix} \Lambda_1 - I & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} P'L = \begin{bmatrix} T_{11}-I & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} U$$

for some matrix U.

$$\Longleftrightarrow \begin{bmatrix} \Lambda_1 - I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P' L = \begin{bmatrix} T_{11}-I & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} U,$$

using (1.3.16)

$$\Longleftrightarrow \begin{bmatrix} \Lambda_1 - I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P' L = \begin{bmatrix} T_{11}-I & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} U$$

$$\Longleftrightarrow \begin{bmatrix} \Lambda_1 - I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} T_{11}-I & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} U_1 , \quad \text{using (1.3.15)}$$

where $U_1 = UP'^{-1}$.

$$\Longleftrightarrow \underline{M} \begin{bmatrix} T_{21} & (\Lambda_1 - I) \\ T_{31} & (\Lambda_1 - I) \end{bmatrix} \subset \underline{M} \begin{bmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{bmatrix}, \text{ in view of (1.3.6).}$$

The proof of (b) is now complete.

Let $V + X_1X_1' = \sum_{i=1}^{k} E_i$, $k \leq n$

and $DD' = \sum_{i=1}^{n} \lambda_i E_i$, be a spectral representation of DD' relative to $V + X_1X_1'$ as defined by Mitra and Moore (1973). Let G_o be a p.d. matrix such that G_o is a p.d. g-inverse of $V + X_1X_1'$ and $E_i G_o E_j = \delta_{ij} E_i$. We now prove

Theorem 1.3.3. (a) At least one linear representation of BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ is unbiased for $X_1\beta$ under $(Y, X\beta, V)$ if and only if

$$X = X_1 + (I - P_{X_1, G_o})A, \qquad \ldots(1.3.17)$$

where A is arbitrary and (b) at least one linear representation of BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ is also BLUE of $X_1\beta$ under $(Y, X\beta, V)$ if and only if X is given by (1.3.17), where A satisfies

$$\underline{M}\begin{bmatrix} VG_oX_1 \\ 0 \end{bmatrix} \subset \underline{M}\begin{bmatrix} X_1 \\ (I-P_{X_1, G_o})A \end{bmatrix}$$

or equivalently A satisfies

$$(I-P_{X_1, G_o})A(X_1^- VG_oX_1 + (I-X_1^-X_1)B) = 0$$

where B is arbitrary and X_1^- is an arbitrary but fixed g-inverse of X_1.

Proof : In view of theorem 1.2.1, sufficiency is obvious. To prove the necessity of (1.3.17), assume that there exists a g-inverse G of $V + X_1X_1'$ such that $X_1(X_1'GX_1)^- X_1'GY$ is unbiased for $X_1\beta$ under $(Y, X\beta, V)$. Then it is necessary and sufficient that

$$X_1'GD = 0$$

$$\iff \sum_{i=1}^{n} \lambda_i X_1'GE_i = 0 \qquad \ldots(1.3.18)$$

$$\iff \lambda_i X_1'GE_i = 0 \quad (i = 1, 2, \ldots, n), \qquad \ldots(1.3.19)$$

using $E_iG_oE_j = \delta_{ij}E_i$.

In view of (1.3.18), (1.3.19) and the fact that $\underline{M}(E_i) \subset \underline{M}(V \; X_1X_1')$ for $i = 1,2,\ldots,k$ and $X_1'G_oE_i = 0$ for $i > k$, we get

$$\sum_{i=1}^{n} \lambda_i X_1' G E_i = 0 \implies \sum_{i=1}^{n} \lambda_i X_1' G_o E_i = 0$$

$$\iff X_1' G_o D = 0 \iff (1.3.17) \text{ holds.}$$

This proves the necessity of the condition given in (a). The necessity of the condition given in part (b) of the theorem is similarly proved.

We state the following lemma due to Bhimasankaram (1976) which we need in the sequel.

<u>Lemma 1.3.2</u> (Bhimasankaram).

Let A and B be matrices such that $\underline{M}(A') \subset \underline{M}(B')$. Then the/matrix equations individually consistent $AW = C$ and $BWE = F$ have a common solution if and only if $AB^-F = CE$ for some g-inverse B^- of B, in which case the general common solution is

$$W = A^-C + (I-A^-A)B^-FE^- + (I-B^-B)R + (I-A^-A)S(I-EE^-)$$

where R and S are arbitrary.

We now prove

<u>Theorem 1.3.4</u> (a) At least one linear representation of BLUE of $X_1\beta$ under $(Y,X_1\beta,V)$ is unbiased for $X_1\beta$ under $(Y,X\beta,V)$ if and only if

$$\underline{M}(X_1) \subset \underline{M}((V+X_1X_1')(I-DD^+))$$

(b) At least one linear representation of BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ is also its BLUE under $(Y, X\beta, V)$ if and only if

$$\underline{M}(X_1) \subset \underline{M}((V + X_1X_1')(I - MM^+ - DD^+)),$$

where $M = (I - DD^+)VZ$.

<u>Proof</u> : (a) We want to obtain conditions under which the equations

$$X_1'GD = 0$$

$$\text{and} \quad (V + X_1X_1')G(V + X_1X_1') = V + X_1X_1'$$

have a common solution in G. Now

$X_1'GD = 0 \iff X_1'G = K(I - DD^+)$ for some K. Hence we want conditions under which

$$X_1'G = K(I - DD^+)$$

$$\text{and} \quad (V + X_1X_1')G(V + X_1X_1') = V + X_1X_1'$$

have a common solution. Using lemma 1.3.2, we see that the required condition is

$$X_1' = K(I - DD^+)(V + X_1X_1'), \quad \text{for some } K$$

$$\iff \underline{M}(X_1) \subset \underline{M}((V + X_1X_1')(I - DD^+)).$$

This proves (a).

(b) Here we want conditions under which the equations

$$X_1'G = K(I - DD^+) \qquad \ldots(1.3.20)$$

$$X_1'GVZ = 0 \qquad \ldots(1.3.21)$$

and $(V+X_1X_1')G(V+X_1X_1') = V+X_1X_1'$ ...(1.3.22)

have a common solution. Using (1.3.20) and writing $(I-DD^+)VZ = M$, we get from (1.3.21),

$$K = T(I-MM^+).$$

Hence
$$X_1'G = T(I-MM^+)(I-DD^+)$$
$$= T(I-MM^+-DD^+) \quad ...(1.3.23)$$

Applying lemma 1.3.2, we see that (1.3.22) and (1.3.23) have a common solution if and only if

$$\underline{M}(X_1) \subset \underline{M}((V+X_1X_1')(I-MM^+-DD^+)).$$

This completes the proof of theorem 1.3.4.

Remark 1.3.1. The g-inverses G of $V+X_1X_1'$, for which the BLUE $X_1(X_1'GX_1)^- X_1'GY$ of $X_1\beta$ under $(Y,X_1\beta,V)$ is an unbiased estimator (or a BLUE) of $X_1\beta$ under $(Y,X\beta,V)$ can also be obtained using lemma 1.3.2. In fact theorems 1.3.1, 1.3.2, 1.3.3 and 1.3.4 not only provde a method of verifying whether there exists a linear representation for BLUE of $X_1\beta$ under $(Y,X_1\beta,V)$ which is an unbiased estimator (or a BLUE) of $X_1\beta$ under $(Y,X\beta,V)$, but also suggests a procedure for constructing such a linear representation whenever one exists.

Remark 1.3.2. The conditions given in theorem 1.3.2 and 1.3.3 are analogous to those given in theorem 3.2 and 3.1 respectively in Mitra and Moore (1973).

Remark 1.3.3. The conditions stated in theorem 1.3.1 or the equivalent conditions stated in theorems 1.3.2, 1.3.3 and 1.3.4 give necessary and sufficient conditions for the existence of a g-inverse G of $V + X_1X_1'$ such that for every estimable parametric functional $p'\beta$, its BLUE $p'(X_1'GX_1)^- X_1'GY$ under $(Y, X_1\beta, V)$ is an unbiased estimator or a BLUE of $p'\beta$ under $(Y, X\beta, V)$. It is interesting to note that the same conditions are also necessary and sufficient for every estimable parametric functional $p'\beta$ under $(Y, X_1\beta, V)$ to have a linear representation for its BLUE under $(Y, X_1\beta, V)$ (not necessarily defined through the same g-inverse G of $V + X_1X_1'$) an unbiased estimator or a BLUE of $p'\beta$ under $(Y, X\beta, V)$. To prove this, observe that every $p'\beta$ estimable under $(Y, X_1\beta, V)$ has a linear representation for its BLUE under $(Y, X_1\beta, V)$ which is an unbiased estimator for $p'\beta$ under $(Y, X\beta, V)$ if and only if for every $p \in \underline{M}(X_1')$, there exists a vector ℓ such that $X_1'\ell = p$, $Z_1'V\ell = 0$ and $D'\ell = 0$, which is equivalent to the conditions given in theorem 1.3.2 (a). A similar argument applies to the conditions given in theorem 1.3.2(b also. The same observation holds good for the solutions obtained by Mitra and Moore (1973) in their theorem 3.1 and theorem 3.2, though this fact is not stated in their paper.

Next, we shall obtain solutions to problem (2) (c) and (d).

Theorem 1.3.5. (a) At least one linear representation for BLUE of every estimable parametric functional under $(Y, X_1\beta, V)$ is its LIMBE under $(Y, X\beta, V)$ if and only if any one of the following equivalent conditions holds :

(i) $\underline{M}(X_1 : 0 : 0)' \subset \underline{M}(X_1 : VZ_1 : DX')'$

(ii) $\underline{M}(X_1) \cap \underline{M}(VZ_1 : DX') = \{0\}$

(iii) $\underline{M}\begin{bmatrix} Z_1'V \\ XD' \end{bmatrix} = \underline{M}\begin{bmatrix} Z_1'VZ_1 \\ XD'Z_1 \end{bmatrix}$

and (b) at least one linear representation for BLUE of every estimable parametric functional under $(Y, X_1\beta, V)$ is its BLIMBE under $(Y, X\beta, V)$ if and only if any one of the following equivalent conditions holds :

(i) $\underline{M}(X_1 : 0 : 0 : 0)' \subset \underline{M}(X_1 : VZ_1 : VZ : DX')'$

(ii) $\underline{M}(X_1) \cap \underline{M}(VZ_1 : VZ : DX') = \{0\}$

(ii) $\underline{M}\begin{bmatrix} Z_1'V \\ Z'V \\ XD' \end{bmatrix} = \underline{M}\begin{bmatrix} Z_1'VZ_1 \\ Z'VZ_1 \\ XD'Z_1 \end{bmatrix}$

Theorem 1.3.5 is proved in a straight forward manner applying lemma 1.2.2.

Now let $V + X_1X_1' = \sum_{i=1}^{k} E_i$, $k \leq n$

and $DX'XD' = \sum_{i=1}^{n} \lambda_i E_i$ be a spectral representation of $DX'XD'$ relative to $V + X_1X_1'$. Let G_0 be positive definite g-inverse of $V + X_1X_1'$ satisfying $E_iG_0E_j = \delta_{ij}E_i$. The following theorem gives an alternate solution to problem (2) (c) and (d).

Theorem 1.3.6. At least one linear representation for BLUE of every estimable parametric functional $p'\beta$ under $(Y, X_1\beta, V)$ is a LIMBE (or BLIMBE) for $p'\beta$ under $(Y, X\beta, V)$ if and only if its BLUE $p'(X_1'G_oX_1)^- X_1'G_oY$ under $(Y, X_1\beta, V)$ is a LIMBE (respectively BLIMBE) for $p'\beta$ under $(Y, X\beta, V)$.

The proof of theorem 1.3.6 is similar to that of theorem 1.3.3.

We give yet another solution to problem (2) (c) and (d), similar to theorem 1.3.4. The proof proceeds along the same lines as that of theorem 1.3.4 and hence, is omitted.

Theorem 1.3.7. (a) At least one linear representation of BLUE of every estimable parametric functional under $(Y, X_1\beta, V)$ is its LIMBE under $(Y, X\beta, V)$ if and only if

$$\underline{M}(X_1) \subset \underline{M}((V + X_1X_1')(I - (DX')(DX')^+))$$

and (b) at least one linear representation of BLUE of every estimable parametric functional under $(Y, X_1\beta, V)$ is its BLIMBE under $(Y, X\beta, V)$ if and only if

$$\underline{M}(X_1) \subset \underline{M}((V + X_1X_1')(I - MM^+ - (DX')(DX')^+))$$

where $\underline{M} = (I - (DX')(DX')^+)VZ$.

Remark 1.3.4. Theorems 1.3.5, 1.3.6 and 1.3.7 also provide a method of constructing a linear representation for BLUE of every estimable parametric functional under $(Y, X_1\beta, V)$ which satisfy the conditions in problem (2), (c) and (d) whenever one such linear representation exists.

Remark 1.3.5. The arguments given in remark 1.3.3 apply to the results given in theorems 1.3.5, 1.3.6 and 1.3.7 also.

1.4. Solution to problem (3).

We prove

Theorem 1.4.1 (a) The BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ irrespective of its linear representation is unbiased for $X_1\beta$ under $(Y, X\beta, V)$ if and only if

$$X = X_1 + VZ_1A \qquad \ldots(1.4.1)$$

where A is arbitrary and (b) the BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ irrespective of its linear representation is its BLUE under $(Y, X\beta, V)$ if and only if X is as in (1.4.1), where A satisfies

$$\underline{M}\begin{bmatrix} VGX_1 \\ 0 \end{bmatrix} \subset \underline{M}\begin{bmatrix} X_1 \\ VZ_1A \end{bmatrix},$$ or equivalently A satisfies

$VZ_1A(X_1^- VGX_1 + (I - X_1^- X_1)B) = 0$, where G is any g-inverse of $V + X_1X_1'$, B is arbitrary and X_1^- is an arbitrary but fixed g-inverse of X_1.

Proof : (a) Every linear representation of BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ is unbiased for $X_1\beta$ under $(Y, X\beta, V)$ if and only if

$$X_1'GD = 0 \qquad \ldots(1.4.2)$$

for every g-inverse G of $V + X_1X_1'$, which happens if and only if

$$-D = (V + X_1X_1')K \text{ for some } K.$$

Hence, from (1.4.2), we get $K = Z_1A$ for some A and thus

$X = X_1 + VZ_1A$, which completes the proof of (a). Proof of (b) is similar to that of theorem 2.1 (b).

Now consider the representations (1.3.3) and (1.3.4) for X_1 and $V + X_1X_1'$ respectively and let P be chosen in such a way that

$$\Lambda_1 - I = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \qquad \ldots(1.4.1)$$

where Λ is a diagonal positive definite matrix. We now prove

<u>Theorem 1.4.2.</u>(a) The BLUE of every estimable parametric functional under $(Y, X_1\beta, V)$, irrespective of its linear representation is its LIMBE under $(Y, X\beta, V)$ if and only if

$$D = P \begin{bmatrix} B & C_1 & C_2 \\ E_1 & E_2 & E_3 \\ 0 & 0 & 0 \end{bmatrix} Q' \qquad \ldots(1.4.4)$$

where B is an arbitrary n.n.d. matrix with eigenvalues in $[0,1]$, C_1 and C_2 are arbitrary matrices satisfying $C_1C_1' + C_2C_2' = B - B^2$ and E_1, E_2, E_3 are arbitrary matrices satisfying

$$\underline{M}(E_1 : E_2 : E_3)' \subset \underline{N}(B : C_1 : C_2) \quad \text{and}$$

(b) The BLUE of every estimable parametric functional under $(Y, X_1\beta, V)$ irrespective of its linear representation is its BLIMBE under $(Y, X\beta, V)$ if and only if D is as given in part (a), where B is given by $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}' & B_{22} \end{bmatrix}$ with $(I-B_{11}) - B_{12}(I-B_{22})^{-} B_{12}'$ positive definite, B_{11} being of the same order as that of Λ in (1.4.3) and $\underline{M}(E_1 : E_2 : E_3)'$ is also a subspace of $\underline{M}(U_1' : U_2' : U_3')$, U_1, U_2 and U_3 being arbitrary matrices satisfying

$$(I - B)U_1 - C_1U_2 - C_2U_3 = I - \Lambda_1^{-1} .$$

Proof : The BLUE of every estimable parametric functional $p'\beta$ under $(Y, X_1\beta, V)$, irrespective of its linear representation is a LIMBE of $p'\beta$ under $(Y, X\beta, V)$ if and only if

$$X_1'GDX_1' = X_1'GDD' \qquad \ldots(1.4.5)$$

for every n.n.d. g-inverse G of $V + X_1X_1'$. Using (1.3.3), (1.3.4) and the partitioned forms of $T = P^{-1}DQ$ and G as given in the proof of theorem 1.3.1, we see that (1.4.5) holds for every n.n.d. g-inverse G of $V + X_1X_1'$ if and only if the following equations hold for every R_1.

$$\Lambda_1^{-1}T_{11} + R_1T_{31} = \Lambda_1^{-1}(T_{11}T_{11}' + T_{12}T_{12}' + T_{13}T_{13}') + R_1(T_{31}T_{11}' + T_{32}T_{12}' + T_{33}T_{13}') \qquad \ldots(1.4.6)$$

$$0 = \Lambda_1^{-1}(T_{11}T_{21}' + T_{12}T_{22}' + T_{13}T_{23}') + R_1(T_{31}T_{21}' + T_{32}T_{22}' + T_{33}T_{23}') \qquad \ldots(1.4.7)$$

and

$$0 = \Lambda_1^{-1}(T_{11}T_{31}' + T_{12}T_{32}' + T_{13}T_{33}') \quad R_1(T_{31}T_{31}' + T_{32}T_{32}' + T_{33}T_{33}') \qquad \ldots(1.4.8)$$

Equation (1.4.8) holds for every R_1 if and only if

$$T_{31} = 0,\ T_{32} = 0 \text{ and } T_{33} = 0.$$

Equations (1.4.6) and (1.4.7) now reduce to

$$T_{11} = T_{11}T_{11}' + T_{12}T_{12}' + T_{13}T_{13}'$$

and $$0 = T_{11}T_{21}' + T_{12}T_{22}' + T_{13}T_{23}'$$

The proof of theorem 1.4.2 can now be completed along the same lines as that of theorem 1.2.2.

Remark 1.4.1. Theorem 1.4.2 gives an explicit characterisation of design matrices X for which BLUE of $X_1\beta$ under $(Y,X_1\beta,V)$ irrespective of its linear representation is its LIMBE (or BLIMBE) under $(Y,X\beta,V)$. The condition (1.4.5) holds for every G, a g-inverse of $V+X_1X_1'$ if and only if $DX' = VZ_1A$ for some A. Thus, given two linear models $(Y,X_1\beta,V)$ and $(Y,X\beta,V)$ in order to verify if BLUE of $X_1\beta$ under $(Y,X_1\beta,V)$ irrespective of its linear representation is its LIMBE (or BLIMBE) under $(Y,X\beta,V)$ one need only verify if $\underline{M}(DX') \subset \underline{M}(VZ_1)$ (respectively $\underline{M}(DX') \subset \underline{M}(VZ_1)$ and $X_1'GVZ = 0$, G being any g-inverse of $V+X_1X_1'$).

1.5 Examples.

Example 1. The restricted general linear model.

Consider the linear model $(Y,X_1\beta,V)$, where β satisfies the consistent equation $R\beta = s$, R being a known matrix and s is a known vector. We will denote such a linear model by

$(Y, X_1\beta, V | R\beta = s)$. This model can be equivalently written as $(Y - X_1R^-s\ ,\ X_1U\delta, V)$, where $U = I - R^-R$ (see Rao and Mitra, 1971, p. 144). Then it is easy to see that any linear representation $X_1(X_1'GX_1)^- X_1'GY$ for BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ is unbiased for $X_1\beta$ under $(Y, X_1\beta, V | R\beta = s)$ and hence is its BLUE under $(Y, X_1\beta, V | R\beta = s)$ if and only if

$$X_1(X_1'GX_1)^- X_1'GV(X_1U)^{\perp} = 0$$

$$\Longleftrightarrow \quad \underline{M}(VGX_1) \subset \underline{M}(X_1U)$$

$$\Longleftrightarrow \quad \underline{M}(V) \cap \underline{M}(X_1) \subset \underline{M}(X_1(I - R^-R))\ , \qquad \ldots(1.5.1)$$

since $\underline{M}(VGX_1) = \underline{M}(VGX_1X_1') = \underline{M}(V) \cap \underline{M}(X_1)$, using a result of Anderson and Duffin (1969) (see Rao and Mitra, 1971, section 10.1.6). If $\underline{M}(X_1) \subset \underline{M}(V)$, then (1.5.1) reduces to

$$\underline{M}(X_1) = \underline{M}(X_1(I - R^-R))$$

$$\Longleftrightarrow \quad \underline{M}(X_1') \cap \underline{M}(R') = \{0\} \qquad \ldots(1.5.2)$$

Thus we have proved

<u>Theorem 1.5.1</u>. The BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ continues to be its BLUE under $(Y, X_1\beta, V | R\beta = s)$ if and only if $\underline{M}(V) \cap \underline{M}(X_1) \subset \underline{M}(X_1(I - R^-R))$. This condition reduces to $\underline{M}(X_1') \cap \underline{M}(R') = \{0\}$ if $\underline{M}(X_1) \subset \underline{M}(V)$.

Recently, Baksalary and Kala (1979) have stated that the BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ is also its BLUE under $(Y, X_1\beta, V | R\beta = s)$ if and only if $\underline{M}(X_1') \cap \underline{M}(R') = \{0\}$. But this result is not true unless $\underline{M}(X_1) \subset \underline{M}(V)$. The error in their proof is that they equate the BLUE's of $X_1\beta$ under $(Y, X_1\beta, V)$ and $(Y, X_1\beta, V | R\beta = s)$ and claim that the equality should hold for every $\underline{Y} \in \underline{M}(X_1 : V)$. However, under $(Y, X_1\beta, V | R\beta = s)$, $E(Y) = X_1\beta$ where β is subject to the restriction $R\beta = s$ and hence it is *not* true that $\underline{Y} \in \underline{M}(X_1 : V)$ with probability one under $(Y, X_1\beta, V | R\beta = s)$.

If we have restrictions of the type $R\beta = 0$ then theorem 1.2.1 can be used to obtain the above results. The restricted model can be written as

$$\left(\begin{bmatrix} Y \\ 0 \end{bmatrix}, \begin{bmatrix} X_1 \\ R \end{bmatrix} \beta, \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} \right),$$

whereas the unrestricted model is

$$\left(\begin{bmatrix} Y \\ 0 \end{bmatrix}, \begin{bmatrix} X_1 \\ 0 \end{bmatrix} \beta, \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} \right).$$

Thus essentially we have two linear models which differ in their design matrices. Applying theorem 1.2.1, we see that any linear representation $X_1(X_1'GX_1)^- X_1'GY$ for BLUE of $X_1\beta$ under the unrestricted model continues to be its BLUE under the restricted model if and only if

$$\underline{M} \begin{bmatrix} VGX_1 \\ 0 \end{bmatrix} \subset \underline{M} \begin{bmatrix} X_1 \\ R \end{bmatrix}$$

$$\Longleftrightarrow \quad VGX_1 = X_1C \text{ and } RC = 0 \text{ for some } C$$

$\Longleftrightarrow$ $VGX_1 = X_1(I - R^-R)C_1$ for some C_1 which is equivalent to the condition (1.5.1).

Example 2. Parametric augmentations to a given linear model.

We consider the linear model $(Y, X_1\beta_1, V)$. Suppose we include a few more parameters to get the linear model $(Y, X_1\beta_1 + X_2\beta_2, V)$. If $\beta = \begin{bmatrix}\beta_1 \\ \beta_2\end{bmatrix}$, then the original model is $(Y, (X_1 \vdots 0)\beta, V)$ and the new model is $(Y, (X_1 \vdots X_2)\beta, V)$. Our purpose is to obtain conditions on X_2 so that BLUE of $X_1\beta_1$ under the first model is an optimal estimator of $X_1\beta_1$ under the second model. First we shall prove that if any linear representation of BLUE of every estimable $p'\beta_1$ under $(Y, X_1\beta_1, V)$ is its LIMBE under $(Y, (X_1 \vdots X_2)\beta, V)$, then it is infact an unbiased estimator of $p'\beta_1$ under $(Y, (X_1 \vdots X_2)\beta, V)$. If $D = (0 \vdots - X_2)$, then $D(X_1 \vdots 0)' = 0$ and hence from (1.2.3) it follows that any linear representation $p'(X_1'GX_1)^-X_1'GY$ of BLUE of $p'\beta_1$ under $(Y, X_1\beta_1, V)$ is its LIMBE under $(Y, (X_1 \vdots X_2)\beta, V)$ if and only if $X_1'GD = 0$, i.e. $X_1'GX_2 = 0$, which is equivalent to saying that $X_1(X_1'GX_1)^- X_1'GY$ is unbiased for $X_1\beta_1$ under $(Y, (X_1 \vdots X_2)\beta, V)$. If $Z_1 = X_1^{\perp}$ and $Z = (X_1 \vdots X_2)^{\perp}$, then, since $\underline{M}(Z) \subset \underline{M}(Z_1)$ and $X_1'GVZ_1 = 0$, we see that $X_1'GVZ = 0$. Thus if $X_1(X_1'GX_1)^- X_1'GY$ is unbiased for $X_1\beta_1$ under $(Y, (X_1 \vdots X_2)\beta, V)$, then it is infact a BLUE of $X_1\beta_1$ under $(Y, (X_1 \vdots X_2)\beta, V)$. Applying theorem 1.2.1, theorem 1.3.2 (ii) and theorem 1.4.1, we see that

(a) if G is an n.n.d. g-inverse of $V + X_1X_1'$, then

$X_1(X_1'GX_1)^- X_1'GY$ is BLUE of $X_1\beta_1$ under $(Y, X_1\beta_1 + X_2\beta_2, V)$ if and only if $X_2 = (I - P_{X_1,G})A$, where A is arbitrary ,

(b) at least one linear representation of BLUE of $X_1\beta_1$ under $(Y, X_1\beta_1, V)$ is its BLUE under $(Y, X_1\beta_1 + X_2\beta_2, V)$ if and only if $\underline{M}(X_1) \cap \underline{M}(VZ_1 : X_2) = \{0\}$ and

(c) the BLUE of $X_1\beta_1$ under $(Y, X_1\beta_1, V)$ irrespective of its linear representation is its BLUE under $(Y, X_1\beta_1 + X_2\beta_2, V)$ if and only if $X_2 = VZ_1A$, where A is arbitrary.

Example 3. Linear models with $\underline{M}(V) \subset \underline{M}(X_1)$.

Here we consider a linear model $(Y, X_1\beta, V)$ where $\underline{M}(V) \subset \underline{M}(X_1)$. If $\underline{M}(X_1)$ contains the vector $1_n = (1,1,\ldots,1)'$ and if $\mathrm{Cov}\,(y_i, y_j) = \sigma^2$ for all i and j, y_i being the i^{th} coordinate of Y, then $V = \sigma^2 1_n 1_n'$ and we have a situation where $\underline{M}(V) \subset \underline{M}(X_1)$. If $\underline{M}(V) \subset \underline{M}(X_1)$, then $V + X_1X_1'$ is not p.d. and the BLUE of $X_1\beta$ does not have a unique linear representation under $(Y, X_1\beta, V)$. We shall characterise design matrices X such that BLUE of $\ell' X_1\beta$ under $(Y, X_1\beta, V)$ is an optimal estimator of $\ell' X_1\beta$ under $(Y, X\beta, V)$ in the sense of being its BLUE or BLIMBE under $(Y, X\beta, V)$. Since $\underline{M}(V) \subset \underline{M}(X_1)$, $\underline{M}(V + X_1X_1') = \underline{M}(X_1)$. If $G = (V + X_1X_1')^+$, then applying theorem 1.2.1 (a) we see that $X_1(X_1'GX_1)^- X_1'GY$ is unbiased for $X_1\beta$ under $(Y, X\beta, V)$ if and only if

$$X_1'GD = 0$$

$$\iff \quad X_1'D = 0$$

$\iff$ $D = (I - P_{X_1})A$, where A is arbitrary. Since $\underline{M}(V) \subset \underline{M}(X_1)$, $\underline{M}(VGX_1) = \underline{M}(V)$ and applying theorem 1.2.1 (b), we get $X_1(X_1'GX_1)^- X_1'GY$ is BLUE of $X_1\beta$ under $(Y, X\beta, V)$ if and only if

$$X = X_1 + (I - P_{X_1})A, \text{ where } A \text{ satisfies}$$

$$\underline{M}\begin{bmatrix} V \\ 0 \end{bmatrix} \subset \underline{M}\begin{bmatrix} X_1 \\ (I - P_{X_1})A \end{bmatrix}.$$

Using part (iii) of theorem 1.3.2 (a), we get, at least one linear representation for BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ is unbiased for $X_1\beta$ under $(Y, X\beta, V)$ if and only if $\underline{M}(X_1) \cap \underline{M}(D) = \{0\}$ (since $VZ_1 = 0$ owing to the fact that $\underline{M}(V) \subset \underline{M}(X_1)$). Condition (ii) of theorem 1.3.2 (b) simplifies to

$$\underline{M}(X_1) \cap \underline{M}(VZ : D) = \{0\}$$

$$\iff \quad \underline{M}(X_1) \cap \underline{M}(D) = \{0\}$$

$$\text{and} \qquad VZ = 0$$

which are the necessary and sufficient conditions for at least one linear representation of BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ to be its BLUE under $(Y, X\beta, V)$. From theorem 1.4.1, it follows that every linear representation of BLUE of $X_1\beta$ under $(Y, X_1\beta, V)$ is unbiased for $X_1\beta$ under $(Y, X\beta, V)$ if and only if $X = X_1$.

If $G = (V + X_1X_1')^+$, then for every estimable $p'\beta$ under $(Y, X_1\beta, V)$, $p'(X_1'GX_1)^- X_1'GY$ is LIMBE of $p'\beta$ under $(Y, X\beta, V)$ if and only if

$$X_1'GDX_1' = X_1'GDD'$$

$$\Longleftrightarrow \quad X_1' D X_1' = X_1' D D' \qquad \ldots(1.5.3)$$

and is BLIMBE of $p'\beta$ under $(Y, X\beta, V)$ if and only if in addition to (1.5.3) the condition $X_1' G V Z = 0$ or equivalently $VZ = 0$ holds At least one linear representation of BLUE of every estimable parametric functional under $(Y, X_1\beta, V)$ is its LIMBE under $(Y, X\beta, V)$ if and only if $\underline{M}(X_1) \cap \underline{M}(DX') = \{0\}$ and is its BLIMBE under $(Y, X\beta, V)$ if and only if $\underline{M}(X_1) \cap \underline{M}(DX') = \{0\}$ and $VZ = 0$. Since $\underline{M}(V) \subset \underline{M}(X_1)$, from remark 1.4.1 we see that the BLUE of every estimable parametric functional under $(Y, X_1\beta, V)$, irrespective of its linear representation is its LIMBE under $(Y, X\beta, V)$ if and only if $DX_1' = DD'$ and is its BLIMBE under $(Y, X\beta, V)$ if and only if $DX_1' = DD'$ and $VZ = 0$.

1.6 Optimality of BLUE's of a subclass of parametric functionals.

In practice it happens that one may not be interested in inferences on all estimable parametric functionals, but may be interested in only a subset of them, say, for example, the treatment contrasts in design of experiments. In this section we shall study the robustness regarding the BLUE's of a subclass of parametric functionals.

Let A be a specified $k \times m$ $(k \leq n)$ matrix such that $\underline{M}(A') \subset \underline{M}(X_1')$. We shall obtain conditions on X such that a specific linear representation/some linear representation/every linear representation of BLUE of $A\beta$ under $(Y, X_1\beta, V)$ is its BLUE under $(Y, X\beta, V)$ also.

Consider the BLUE $A(X_1'GX_1)^- X_1'GY$ of $A\beta$ under $(Y, X_1\beta, V)$, where G is a specified g-inverse of $V + X_1X_1'$. Let $B = A(X_1'GX_1)^- X_1'G$. BY is also BLUE of $A\beta$ under $(Y, X\beta, V)$ if and only if $BD = 0$ $(D = X_1 - X)$ and $BVZ = 0$ which are equivalent to the conditions

$$D = (I - B^-B)T \qquad \ldots(1.6.1)$$

and $$BV = W(X_1' - T'(I - B'B^{-'})) \qquad \ldots(1.6.2)$$

for some T and W. It is easily verified that, for any fixed W, (1.6.2) is consistent in T if and only if

$$\underline{M}(BV) \subset \underline{M}(W) \qquad \ldots(1.6.3)$$

and $$BVB' = WA' \, . \qquad \ldots(1.6.4)$$

Once we obtain matrices W satisfying (1.6.3) and (1.6.4), T can be solved for from (1.6.2). The following lemma gives a characterisation of matrices W satisfying (1.6.3) and (1.6.4).

<u>Lemma 1.6.1</u>. W satisfies (1.6.3) and (1.6.4) if and only if

$$W = (L_1 : L_2) \begin{bmatrix} S_1 \\ S_2 \end{bmatrix},$$

where $(L_1 : L_2)$ is an orthogonal matrix such that the columns of L_1 form a basis for $\underline{M}(BV)$, S_1 is any solution (of full row rank) of the equation $L_1'BVB' = S_1A'$ and S_2 is any solution of the equations $S_2A' = 0$, $S_2R = 0$, R being an arbitrary right inverse of S_1.

<u>Proof</u> : If $(L_1 : L_2)$ is as specified in the lemma, then the matrix W can be written as

$$W = (L_1 : L_2) \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \text{ for some } S_1 \text{ and } S_2 .$$

(1.6.3) holds if and only if there exists a matrix R satisfying

$$(L_1 : L_2) \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} R = L_1$$

$$\Longleftrightarrow \quad S_1 R = I$$

$$\text{and} \quad S_2 R = 0 .$$

(1.6.4) holds if and only if

$$BVB' = (L_1 : L_2) \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} A'$$

$$\Longleftrightarrow \quad L_1' BVB' = S_1 A' \qquad \ldots(1.6.5)$$

$$\text{and} \quad 0 = S_2 A' .$$

Since $R(L_1' BVB') = R(BV) = R(L_1) =$ number of columns of L_1, S_1 satisfying (1.6.5) is of full row rank. This completes the proof of lemma 1.6.1.

If $V = I$, then $B = A(X_1' X_1)^- X_1'$ and the equations (1.6.1) and (1.6.2) can be written as

$$D = (I - P_{B'})T \qquad \ldots(1.6.6)$$

$$\text{and} \quad B = W[X_1' P_{B'} + (X_1' - T')(I - P_{B'})] \qquad \ldots(1.6.7)$$

for some T and W. (1.6.7) holds if and only if

$$B = W X_1' P_{B'}$$

$$\text{and} \quad 0 = W(X_1' - T')(I - P_{B'}) ,$$

for some W, which is equivalent to the condition

$$\underline{M}\begin{bmatrix} B' \\ 0 \end{bmatrix} \subset \underline{M}\begin{bmatrix} P_{B'}X_1 \\ (I - P_{B'})(X_1 - T) \end{bmatrix} \qquad \ldots(1.6.8)$$

$$\iff \quad \underline{M}(X_1' P_{B'}) \cap \underline{M}((X_1' - T')(I - P_{B'})) = \{0\}. \qquad \ldots(1.6.9)$$

Hence D is given by (1.6.6), where T satisfies (1.6.8) or equivalently (1.6.9). Matrices T satisfying (1.6.8) or (1.6.9) are also solutions of the equation

$$(I - P_{B'})(X_1 - T)\left[(P_{B'}X_1)^- B' + (I - (P_{B'}X_1)^- (P_{B'}X_1))Q\right] = 0,$$

where Q is arbitrary. The solution obtained here is analogous to that given in corollary 1.2.1. An alternate necessary and sufficient condition for the BLUE of $A\beta$ under $(Y, X_1\beta, \sigma^2 I)$ to be its BLUE under $(Y, X\beta, \sigma^2 I)$ assuming that A is of full rank has been given by Mitra and Rao (1969).

Now consider the positive definite g-inverse G_0 of $V + X_1 X_1'$ introduced in theorem 1.3.3. Arguments similar to those given in the proof of theorem 1.3.3 lead us to the conclusion that at least one linear representation of BLUE of $A\beta$ under $(Y, X_1\beta, V)$ is an unbiased estimator (or a BLUE) of $A\beta$ under $(Y, X\beta, V)$ if and only if $A(X_1' G_0 X_1)^- X_1' G_0 Y$ is an unbiased estimator (respectively BLUE) of $A\beta$ under $(Y, X\beta, V)$. Analogous to theorem 1.3.2 we can show that at least one linear representation of BLUE of $A\beta$ under $(Y, X_1\beta, V)$ is unbiased for $A\beta$ under $(Y, X\beta, V)$ if and only if

$$\underline{M}(A : 0 : 0)' \subset \underline{M}(X_1 : VZ_1 : D)'$$

and is BLUE of $A\beta$ under $(Y, X\beta, V)$ if and only if

$$\underline{M}(A : 0 : 0 : 0)' \subset \underline{M}(X_1 : VZ_1 : VZ : D)' .$$

Corresponding to theorem 1.3.4, we can prove that at least one linear representation of BLUE of $A\beta$ under $(Y, X_1\beta, V)$ is unbiased for $A\beta$ under $(Y, X\beta, V)$ if and only if

$$\underline{M}(X_1(X_1'GX_1)^- A') \subset \underline{M}((V + X_1X_1')(I - DD^+))$$

and is BLUE of $A\beta$ under $(Y, X\beta, V)$ if and only if

$$\underline{M}(X_1(X_1'GX_1)^- A') \subset \underline{M}((V + X_1X_1')(I - MM^+ - DD^+))$$

where $M = (I - DD^+)VZ$ and G is any g-inverse of $V + X_1X_1'$.

Now, let G be any g-inverse of $V + X_1X_1'$ and let $C = A(X_1'GX_1)^- X_1'$. Every linear representation of BLUE of $A\beta$ under $(Y, X_1\beta, V)$ is its BLUE under $(Y, X\beta, V)$ if and only if

$$D = (V + X_1X_1')(I - C^- C)T$$

$$\text{and} \quad CGV = W(X_1' - T'(I - C'C^{-\prime})(V + X_1X_1')) \qquad \ldots(1.6.10)$$

for some W and T. Let K be a matrix of maximum rank such that $(I - C'C^{-\prime})(V + X_1X_1')K = 0$. Observe that K satisfies $\underline{M}((V + X_1X_1')K) = \underline{M}(C')$. Then for every fixed W, (1.6.10) is consistent in T if and only if

$$\underline{M}(CGV) \subset \underline{M}(W) \qquad \ldots(1.6.11)$$

$$\text{and} \quad CGVK = WX_1'K . \qquad \ldots(1.6.12)$$

It is fairly easy to observe that

$$\underline{M}(K'VGC') \subset \underline{M}(K'X_1) \qquad \ldots(1.6.13)$$

and $\quad R(CGVK) = R(CGV)$. $\qquad \ldots(1.6.14)$

Using (1.6.13), (1.6.14) and arguments similar to those given in the proof of lemma 1.6.1, one can easily arrive at the fact that W satisfies (1.6.11) and (1.6.12) if and only if

$$W = (L_1 : L_2)\begin{bmatrix} S_1 \\ S_2 \end{bmatrix},$$

where $(L_1 : L_2)$ is an orthogonal matrix such that the columns of L_1 form a basis for $\underline{M}(CGV)$, S_1 is any solution (of full row rank) of $L_1'CGVK = S_1X_1'K$ and S_2 is any solution of the equations $S_2X_1'K = 0$, $S_2R = 0$, R being an arbitrary right inverse of S_1. After solving for W, T could be obtained from (1.6.10). We thus have a characterisation of the class of design matrices X such that every linear representation of BLUE of $A\beta$ under $(Y,X_1\beta,V)$ remains its BLUE under $(Y,X\beta,V)$ also.

As an example, consider the model

$$y_{ij} = \mu + t_j + e_{ij}\ , \quad i = 1,2,\ldots,b\ ,\ j = 1,2,\ldots,k$$

where the e_{ij}'s are uncorrelated random variables with zero mean and common variance σ^2. The above model can also be written as

$$E(Y) = X_1\beta_1$$

and $\quad D(Y) = \sigma^2 I \qquad$ where

$$Y = (y_{11}\ y_{12} \cdots y_{1k}\ y_{21}\ y_{22} \cdots y_{2k} \cdots y_{b_1}\ y_{b_2} \cdots y_{bk})$$

$$X_1 = (1_b \otimes 1_k : 1_b \otimes I_k)$$

and $\beta_1 = (\mu, t_1, \ldots, t_k)'$, 1_b and 1_k being $b \times 1$ and $k \times 1$ vector with each component unity and $\otimes$ denotes the Kronecker product. Consider the estimable parametric function

$$A\beta_1 = (t_1 - t_2,\ t_1 - t_3, \ldots, t_1 - t_k)'.$$

(If t_j, for $j = 1, 2, \ldots, k$ denote treatment effects then $A\beta_1$ is the vector of treatment contrasts). Then $A = (0 : 1_{k-1} : -I_{k-1})$ and the BLUE of $A\beta_1$ is given by BY, where

$$B = \frac{1}{b} 1_b' \otimes (1_{k-1} : -I_{k-1}). \qquad \ldots(1.6.15)$$

We shall characterise design matrices X of the form $X = (X_1 : X_2)$ such that BY continues to be the BLUE of $A\beta_1$ under $(Y, X_1\beta_1 + X_2\beta_2, \sigma^2 I)$. Since $\underline{M}(X_1) \subset \underline{M}(X)$, it is clear that if BY is unbiased for $A\beta_1$ under $(Y, X_1\beta_1 + X_2\beta_2, \sigma^2 I)$, then it will be the BLUE of $A\beta_1$ under $(Y, X_1\beta_1 + X_2\beta_2, \sigma^2 I)$. Hence we need only characterise X_2 so that BY is unbiased for $A\beta_1$ under $(Y, X_1\beta_1 + X_2\beta_2, \sigma^2 I)$. From equation (1.6.1), we see that BY is unbiased for $A\beta_1$ under $(Y, X_1\beta_1 + X_2\beta_2, \sigma^2 I)$ if and only if $X_2 = B'^{\perp} T$ where T is arbitrary and B is given by (1.6.15).

Now $$B'^{\perp} = \left[\frac{1}{b} 1_b \otimes \begin{bmatrix} 1_{k-1}' \\ -I_{k-1} \end{bmatrix}\right]^{\perp}$$

$$= \left[1_b^{\perp} \otimes \begin{bmatrix} 1_{k-1}' \\ -I_{k-1} \end{bmatrix} : 1_b \otimes \begin{bmatrix} 1_{k-1}' \\ -I_{k-1} \end{bmatrix}^{\perp} : 1_b^{\perp} \otimes \begin{bmatrix} 1_{k-1}' \\ -I_{k-1} \end{bmatrix}^{\perp}\right] \quad ..(1.6$$

Thus matrices X_2 for which the BLUE of $A\beta_1$ under $(Y, X_1\beta_1, \sigma^2 I)$ continues to be its BLUE under $(Y, X_1\beta_1 + X_2\beta_2, \sigma^2 I)$ are given by $X_2 = B'^{\perp} T$, where T is arbitrary and $B'^{\perp}$ is given by (1.6.16). In particular, $X_2 = I_b \otimes 1_k$ satisfies $BX_2 = 0$ and hence $\underline{M}(X_2) \subset \underline{M}(B'^{\perp})$. The design matrix corresponding to $X_2 = I_b \otimes 1_k$ is $X = (X_1 : X_2) = (1_b \otimes 1_k : 1_b \otimes I_k : I_b \otimes 1_k)$ which is the design matrix corresponding to the model

$$y_{ij} = \mu + \alpha_i + t_j + e_{ij}, \quad i = 1, 2, \ldots, b, \quad j = 1, 2, \ldots, k$$

and β_2 in this case is the vector $(\alpha_1, \alpha_2, \ldots, \alpha_b)'$. Thus we have proved

<u>Theorem 1.6.1</u>. Let e_{ij}'s $(i = 1, 2, \ldots, b;\ j = 1, 2, \ldots, k)$ be uncorrelated random variables with zero mean and common variance σ^2. Then the BLUE of the estimable parametric function $(t_1 - t_2,\ t_1 - t_3, \ldots, t_1 - t_k)'$ under the model $y_{ij} = \mu + \alpha_i + t_j + e_{ij}$ $(i = 1, 2, \ldots, b;\ j = 1, 2, \ldots, k)$ is same as its BLUE under the model $y_{ij} = \mu + t_j + e_{ij}$ $(i = 1, 2, \ldots, b;\ j = 1, 2, \ldots, k)$.

<u>Remark 1.6.1</u>. If $e = (e_{11}, e_{12}, \ldots, e_{1k}, e_{21}, e_{22}, \ldots, e_{2k}, \ldots, e_{b1}, e_{b2}, \ldots, e_{bk})'$, then we have assumed that $D(e) = \sigma^2 I$ in theorem 1.6.1. We would like to mention that theorem 1.6.1 remains true if $D(e) = \sigma^2 [(1-\rho) I + \rho 1_n 1_n']$, where $-\frac{1}{n-1} \leq \rho \leq 1$, $1_n = (1, 1, \ldots, 1)'$ and $n = bk$. This happens because every BLUE under any of the models considered in theorem 1.6.1 with $D(e) = \sigma^2 I$ is also BLUE under the corresponding model with $D(e) = \sigma^2 \left[(1-\rho) I + \rho 1_n 1_n' \right]$.

CHAPTER 2

SPECIFICATION ERRORS IN THE DESIGN AND DISPERSION MATRICES

2.1 Statement of the problems

We now consider the most general setup where we have two linear models which differ in the expectations as well as the dispersions of the random vector Y. Furthermore, we allow the design and dispersion matrices to be deficient in rank. Let the two linear models be $(Y,X_1\beta,V_1)$ and $(Y,X\beta,V)$. Our first objective is to obtain solutions to the following problems :

Problem (1). What is the class of all models $(Y,X\beta,V)$ such that a specific linear representation of BLUE of every estimable parametric functional $p'\beta$ under $(Y,X_1\beta,V_1)$ is (a) a BLUE and (b) a BLIMBE of $p'\beta$ under $(Y,X\beta,V)$?

Problem (2). What is the class of all models $(Y,X\beta,V)$ such that at least one linear representation of BLUE of every estimable parametric functional $p'\beta$ under $(Y,X_1\beta,V_1)$ is (a) a BLUE and (b) a BLIMBE of $p'\beta$ under $(Y,X\beta,V)$?

Problem (3). What is the class of all models $(Y,X\beta,V)$ such that every linear representation of BLUE of every estimable parametric functional $p'\beta$ under $(Y,X_1\beta,V_1)$ is (a) a BLUE and (b) a BLIMBE of $p'\beta$ under $(Y,X\beta,V)$?

Here, we are not looking for conditions under which BLUE of every estimable parametric functional under $(Y,X_1\beta,V_1)$ is its unbiased estimator or LIMBE under $(Y,X\beta,V)$ since, these involve conditions on X only and have been investigated in chapter 1. As was the case in chapter 1, whenever we speak about BLIMBE in this chapter we will be considering only the situation where the bias of the estimator is defined through the Euclidean norm since, if the norm is defined through a positive definite matrix, we can bring it down to the case of the Euclidean norm by means of a non-singular transformation, as pointed out in chapter 1.

In the next three sections, we provide solutions to the three problems stated above. The last section is devoted to the study of robustness of BLIMBE's when we have linear models with specification errors. In this section, we also characterise parametric functionals which admit a common BLIMBE under two given linear models. We denote

$$D = X_1 - X, \; Z_1 = X_1^{\perp} \text{ and } Z = X^{\perp}.$$

2.2 Solution to problem (1).

Let G_1 be a specific g-inverse of $V_1 + X_1X_1'$. We now prove

Theorem 2.2.1. Let W_1 be such that $\underline{M}(W_1) = \underline{N}(X_1'G_1)$. Then, the BLUE $X_1(X_1'G_1X_1)^- X_1'G_1Y$ of $X_1\beta$ under $(Y, X_1\beta, V_1)$ is also its BLUE under $(Y, X\beta, V)$ if and only if

$$D = W_1C$$

$$\text{and} \quad V = X_1AX_1' + W_1BW_1' - X_1AD' - DAX_1',$$

where C is arbitrary and A and B are arbitrary subject to the condition that V is n.n.d.

<u>Proof</u> : $X_1(X_1'G_1X_1)^- X_1'G_1Y$ is BLUE of $X_1\beta$ under $(Y, X\beta, V)$ if and only if

$$X_1'G_1D = 0 \qquad \ldots(2.2.1)$$

$$\text{and} \quad X_1'G_1VZ = 0 \qquad \ldots(2.2.2)$$

(2.2.1) holds if and only if $D = W_1C$ for some matrix C. Since $\underline{M}(W_1) \oplus \underline{M}(X_1)$ is the whole space (see Mitra and Moore, 1973) we can write

$$V = X_1\Lambda_1X_1' + W_1\Lambda_2W_1' + X_1\Lambda_3W_1' + W_1\Lambda_3'X_1'$$

for some Λ_1, Λ_2 and Λ_3. Then, (2.2.2) holds if and only if

$$X_1'G_1(X_1\Lambda_1X_1' + X_1\Lambda_3W_1')Z = 0$$

$$\Longleftrightarrow \quad X_1\Lambda_1X_1'G_1X_1 + W_1\Lambda_3'X_1'G_1X_1 = (X_1 - D)K, \quad \text{for some } K$$

$$\Longleftrightarrow \quad X_1\Lambda_1X_1'G_1X_1 = X_1K$$

$$\text{and} \quad W_1\Lambda_3'X_1'G_1X_1 = -DK,$$

since, by (2.2.1), $\underline{M}(D) \subset \underline{M}(W_1)$ and $\underline{M}(W_1) \cap \underline{M}(X_1) = \{0\}$

$$\Longleftrightarrow \quad X_1\Lambda_1X_1' = X_1K(X_1'G_1X_1)^- X_1'$$

$$\text{and} \quad W_1\Lambda_3'X_1' = -DK(X_1'G_1X_1)^- X_1'.$$

Putting $A = K(X_1'G_1X_1)^-$ and $B = \Lambda_2$, we get the result.

<u>Remark 2.2.1</u>. Matrices A, B and $D = W_1C$ which make $V = X_1AX_1' + W_1BW_1' - X_1AD' - DAX_1'$ an n.n.d. matrix could be characterised as follows :

Let P be a nonsingular matrix and Q be an orthogonal matrix such that X_1 has the partitioned form given in Lemma 1.2.1 and $V_1 + X_1X_1'$ and G_1 have partitioned forms corresponding to those of $V + X_1X_1'$ and G respectively given in Lemma 1.2.1. Then, one choice of W_1 is

$$W_1 = P\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} Q'$$

Let $$Q'AQ = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \quad Q'BQ = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}$$

and $$Q'CQ = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}.$$

Then $$V = X_1AX_1' + W_1BW_1' - X_1AD' - DAX_1'$$

$$= P\begin{bmatrix} A_{11} & -\sum_{i=1}^{3} A_{1i}C_{2i}' & -\sum_{i=1}^{3} A_{1i}C_{3i}' \\ -\sum_{i=1}^{3} C_{2i}A_{1i}' & B_{22} & B_{23} \\ -\sum_{i=1}^{3} C_{3i}A_{1i}' & B_{32} & B_{33} \end{bmatrix} P'$$

Write $A_1 = (A_{11} \vdots A_{12} \vdots A_{13})$

$$C_1 = \begin{bmatrix} C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

and $$B_1 = \begin{bmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{bmatrix}$$

Then $$V = P \begin{bmatrix} A_{11} & -A_1 C_1' \\ -C_1 A_1' & B_1 \end{bmatrix} P' .$$

Matrices A_1, C_1 and B_1 which make V n.n.d. could be chosen as follows :

Let A_{11} be any n.n.d. matrix. For V to be n.n.d., we should have $\underline{M}(A_1C_1') \subset \underline{M}(A_{11})$ or equivalently, $A_1C_1' = A_{11}T$ for some T. Since $\underline{M}(A_{11}) \subset \underline{M}(A_1)$, the equation $A_1C_1' = A_{11}T$ is consistent in C_1 for any T. Hence for every fixed T, we can solve for C_1 from this equation. To have an n.n.d. V, we now have to choose B_1 such that $B_1 - C_1A_1'A_{11}^{-}A_1C_1'$ is n.n.d. Hence $B_1 = C_1A_1'A_{11}^{-}A_1C_1' + E$, where E is an arbitrary n.n.d. matrix.

<u>Corollary 2.2.1</u> (Mitra and Rao, 1969).

The BLUE of $X_1\beta$ under $(Y, X_1\beta, \sigma^2 I)$ is its BLUE under $(Y, X\beta, V)$ if and only if

$$D = Z_1 C$$

and $$V = X_1AX_1' + Z_1BZ_1' - X_1AD' - DAX_1' ,$$

where C is arbitrary and A and B are arbitrary subject to the condition that V is n.n.d.

<u>Proof</u> $G_1 = (I + X_1X_1')^{-1}$ and if W_1 satisfies $\underline{M}(W_1) = \underline{M}(X_1G_1)$,

then
$$X_1'(I + X_1X_1')^{-1}W_1 = 0$$
$$\iff \quad X_1'W_1 = 0$$

and hence $W_1 = X_1^{\perp} = Z_1$. Corollary 2.2.1 now follows from theorem 2.2.1.

Let P be a nonsingular matrix and Q be an orthogonal matrix such that X_1 has the partitioned form given in lemma 1.2.1 and $V_1 + X_1X_1'$ and G_1 have partitioned forms corresponding to those of $V + X_1X_1'$ and G respectively given in lemma 1.2.1. We now prove

<u>Theorem 2.2.2</u>. The BLUE $p'(X_1'G_1X_1)^- X_1'G_1Y$ of every estimable parametric functional $p'\beta$ under $(Y, X_1\beta, V_1)$ is its BLIMBE under $(Y, X\beta, V)$ if and only if

$$D = P\begin{bmatrix} B & C_1 & C_2 \\ E_1 & E_2 & E_3 \\ F_1 & F_2 & F_3 \end{bmatrix} Q'$$

and
$$V = P\begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{12}' & W_{22} & W_{23} \\ W_{13}' & W_{23}' & W_{33} \end{bmatrix} P'$$

where B is an arbitrary n.n.d. matrix with eigenvalues in $[0,1]$, C_1 and C_2 are arbitrary matrices satisfying

$C_1C_1' + C_2C_2' = B - B^2$ and $E = (E_1 : E_2 : E_3)$ and $F = (F_1 : F_2 : F_3)$ are arbitrary matrices satisfying

$$\underline{M}(E_1 : E_2 : E_3)' \subseteq \underline{M}(B : C_1 : C_2)$$

$$\text{and} \quad \underline{M}(F_1 : F_2 : F_3)' \subseteq \underline{M}(B : C_1 : C_2)$$

and if $A = (I - B : -C_1 : -C_2)$, then

$$W_{11} = ATA',$$

$$W_{12}' = (E - E_1(I - B)^- A)HTA' - ETA'$$

$$\text{and} \quad W_{13}' = (F - F_1(I - B)^- A)HTA' - FTA'$$

where H is an arbitrary matrix and T is an arbitrary n.n.d. matrix. For $i,j = 2,3$, W_{ij} are arbitrary subject to the condition that V is n.n.d.

Proof : The expression for D given in the theorem is derived in the same way as in the proof of theorem 1.2.2. All that remains now is to characterise n.n.d. matrices V satisfying

$$X_1'G_1VZ = 0.$$

Let V have the partitioned form given in the statement of the theorem. Then, $X_1'G_1VZ = 0$ if and only if

$$\underline{M}(VG_1X_1) \subseteq \underline{M}(X_1 - D)$$

$$\Longleftrightarrow \quad W_{11} = (I - B)K_1 - C_1K_2 - C_2K_3, \qquad \ldots(2.2.3)$$

$$W_{12}' = - E_1K_1 - E_2K_2 - E_3K_3 \qquad \ldots(2.2.4)$$

$$\text{and} \quad W_{13}' = - F_1K_1 - F_2K_2 - F_3K_3 \qquad \ldots(2.2.5)$$

for some K_1, K_2 and K_3. Since W_{11} is n.n.d., it follows from (2.2.3) that $(K_1' : K_2' : K_3')' = TA' + (I - A^+A)R$, where $A = (I - B : -C_1 : -C_2)$ and T is an arbitrary n.n.d. matrix and R is arbitrary so that

$$W_{11} = ATA'.$$

Since $A^+A = A'(AA')^- A = A'(I - B)^- A$, we get

$$W_{12}' = -ETA' - E(I - A'(I - B)^- A)R$$
$$= -ETA' - (E - E_1(I - B)^- A)R,$$

using the fact that $\underline{M}(E') \subset \underline{M}(B : C_1 : C_2)$. Also, since V is n.n.d., $\underline{M}(W_{12}) \subset \underline{M}(W_{11}) = \underline{M}(AT)$, and we get from (2.2.4)

$$W_{12}' = - ETA' + (E - E_1(I - B)^- A)HTA'$$

Similarly $W_{13}' = - FTA' + (F - F_1(I - B)^- A)HTA'$.

The proof of theorem 2.2.2 is now complete.

<u>Remark 2.2.1</u>. In theorem 2.2.1 and theorem 2.2.2, the equation which characterises V is $X_1'C_1VZ = 0$. Hence, once D is known, is known and n.n.d. solutions to $X_1'C_1VZ = 0$ can also be obtained using the results of Khatri and Mitra (1976).

<u>Corollary 2.2.2</u>. Let $X_1 = R \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} Q'$ be the singular value decomposition of X_1. Then, the BLUE of every estimable parametric functional under $(Y, X_1\beta, \sigma^2 I)$ is its BLIMBE under $(Y, X\beta, V)$ if and only if

$$D = P \begin{bmatrix} \Delta B & \Delta C \\ E_1 & E_2 \end{bmatrix} Q'$$

and $$V = R \begin{bmatrix} W_{11} & W_{12} \\ W'_{12} & W_{22} \end{bmatrix} R'$$

where B is an arbitrary n.n.d. matrix with eigenvalues in $[0,1]$, C is any matrix satisfying $CC' = B - B^2$, $E = (E_1 : E_2)$ is an arbitrary matrix satisfying $\underline{M}(E_1 : E_2)' \subset \underline{M}(B : C)$, and if $A = (I - B : - C)$, then

$$W_{11} = ATA', \quad W'_{12} = (E - E_1(I - B)^- A)HTA' - ETA',$$

H being an arbitrary matrix and T is an arbitrary n.n.d. matrix and W_{22} is arbitrary subject to the condition that V is n.n.d.

<u>Proof</u> : The expression for D is derived in a manner similar to the proof of corollary 1.2.2 (a). We have to characterise n.n.d. matrices V satisfying

$$X_1' G_1 VZ = 0, \quad \text{where} \quad G_1 = (1 + X_1 X_1')^{-1}$$

$$\Longleftrightarrow \quad X_1' VZ = 0$$

$\Longleftrightarrow$ $\underline{M}(VX_1) \subset \underline{M}(X_1 - D)$, where D is as given in the statement of the corollary

$$\Longleftrightarrow \quad W_{11} = (I - B)X_1 - CK_2$$

and $$W'_{12} = - E_1 K_1 - E_2 K_2$$

for some K_1 and K_2. The rest of the proof is similar to the proof of theorem 2.2.?.

2.3 Solution to problem (2).

The class of all linear models $(Y, X\beta, V)$ which satisfy the conditions stated in problem (2) is obtained by taking the union of linear models $(Y, X\beta, V)$ which satisfy the conditions in problem (1), the union being taken over all n.n.d. g-inverses of $V_1 + X_1X_1'$. Hence, our objective in this section is to provide methods of verifying whether two given linear models satisfy the requirements in problem (2).

At least one linear representation of BLUE of every estimable parametric functional under $(Y, X_1\beta, V_1)$ is its BLUE under $(Y, X\beta, V)$ if and only if the equations

$$X_1'G_1(D : VZ) = 0$$

and $$(V_1+X_1X_1')G_1(V_1+X_1X_1') = V_1+X_1X_1'$$

are jointly consistent in G_1 and is its BLIMBE under $(Y, X\beta, V)$ if and only if the equations

$$X_1'G_1(DX' : VZ) = 0$$

and $$(V_1+X_1X_1')G_1(V_1+X_1X_1') = V_1+X_1X_1'$$

are jointly consistent in G_1. The joint consistency of any of the above systems of equations can be checked using lemma 1.3.1.

Theorem 2.3.1. At least one linear representation of BLUE of $X_1\beta$ under $(Y, X_1\beta, V_1)$ is its BLUE under $(Y, X\beta, V)$ if and only if anyone of the following equivalent conditions holds :

(i) $\underline{M}(X_1 : 0 : 0 : 0)' \subset \underline{M}(X_1 : V_1Z_1 : VZ : D)'$

(ii) $\underline{M}(X_1) \cap \underline{M}(V_1Z_1 : VZ : D) = \{0\}$

(iii) $$\underline{M}\begin{bmatrix} Z_1'V_1 \\ Z'V \\ D' \end{bmatrix} = \underline{M}\begin{bmatrix} Z_1'V_1Z_1 \\ Z'VZ_1 \\ D'Z_1 \end{bmatrix}$$

Proof : We want conditions under which there exists an L satisfying

$$X_1'L = X_1', \quad Z_1'V_1L = 0, \quad D'L = 0 \text{ and } Z'VL = 0$$

which is equivalent to (i). The equivalence of (i), (ii) and (iii) is easily established.

We shall now state a few more theorems which give solutions to problem (2). The proofs, being similar to those of the corresponding results in section 1.3, are omitted.

Theorem 2.3.2. At least one linear representation of BLUE of $X_1\beta$ under $(Y, X_1\beta, V_1)$ is its BLUE under $(Y, X\beta, V)$ if and only if

$$\underline{M}(X_1) \subset \underline{M}((V_1 + X_1X_1')(I - RR^+ - DD^+))$$

where $R = (I - DD^+)VZ$.

Theorem 2.3.3. At least one linear representation of BLUE of every estimable parametric functional under $(Y, X_1\beta, V_1)$ is its BLIMBE under $(Y, X\beta, V)$ if and only if anyone of the following equivalent conditions holds :

(i) $\underline{M}(X_1 : 0 : 0 : 0)' \subset \underline{M}(X_1 : V_1Z_1 : VZ : DX')'$

(ii) $\underline{M}(X_1) \cap \underline{M}(V_1Z_1 : VZ : DX') = \{0\}$

(iii) $$\underline{M}\begin{bmatrix} Z_1'V \\ Z'V \\ XD' \end{bmatrix} = \underline{M}\begin{bmatrix} Z_1'VZ_1 \\ Z'VZ_1 \\ XD'Z_1 \end{bmatrix}$$

(iv) $\underline{M}(X_1) \subset \underline{M}((V + X_1X_1')(I - SS^+ - (DX')(DX')^+))$

where $S = (I - (DX')(DX')^+)VZ$.

<u>Remark 2.3.1</u>. The theorems stated in this section not only provide a method of verifying whether there exists a linear representation of BLUE of every estimable parametric functional under $(Y, X_1\beta, V_1)$ which is its BLUE (or BLIMBE) under $(Y, X\beta, V)$, but also suggests a procedure for constructing such a linear representation whenever one exists.

<u>Remark 2.3.2</u>. The observation made in remark 1.3.3 holds good for the results in this section also.

2.4 <u>Solution to problem (3)</u>.

Let P be a nonsingular matrix and Q an orthogonal matrix such that

$$X_1 = P\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}Q' \qquad \ldots(2.4.1)$$

and $$V_1 + X_1X_1' = P\begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}P' \qquad \ldots(2.4.2)$$

We now prove

Theorem 2.4.1. Consider the partitioned forms of X_1 and $V_1 + X_1X_1'$ given above. Then, the BLUE of $X_1\beta$ under $(Y, X_1\beta, V_1)$, irrespective of its linear representation is its BLUE under $(Y, X\beta, V)$ if and only if

$$D = P\begin{bmatrix} 0 & 0 & 0 \\ E_1 & E_2 & E_3 \\ 0 & 0 & 0 \end{bmatrix} Q' \qquad \ldots(2.4.3)$$

and $$V = P\begin{bmatrix} W_{11} & W_{12} & 0 \\ W_{12}' & W_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} P',$$

where E_1, E_2, E_3 are arbitrary, W_{11} is arbitrary n.n.d. $W_{12}' = (-E_1 + E_2S_2 + E_3S_3)W_{11}$, S_2, S_3 being arbitrary and W_{22} is arbitrary subject to the condition that V is n.n.d.

Proof : Write

$$D = P\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} Q'$$

and $$V = P\begin{bmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{bmatrix} P'$$

We want $X_1'G_1D = 0$ and $X_1'G_1VZ = 0$ or equivalently $\underline{M}(VG_1X_1) \subset \underline{M}(X_1 - D)$ for every n.n.d. g-inverse G_1 of $V_1 + X_1X_1'$. G_1 can be written as

$$G_1 = P'^{-1}\begin{bmatrix} \Lambda_1^{-1} & 0 & R_1 \\ 0 & \Lambda_2^{-1} & R_2 \\ R_1' & R_2' & S \end{bmatrix} P^{-1},$$

for some R_1, R_2 and S satisfying $S - R_1'\Lambda_1 R_1 - R_2'\Lambda_2 R_2$ is n.n.d. Using the partitioned forms of $X_1, V_1 + X_1X_1'$ and G_1, we get the conditions given in the theorem.

Remark 2.4.1. V obtained in theorem 2.4.1 satisfies $\underline{M}(V) \subset \underline{M}(V_1 + X_1X_1')$. Hence, if $V_1 + X_1X_1'$ is not p.d., then V cannot be p.d.

Corallary 2.4.1. D and V obtained in theorem 2.4.1 can be represented equivalently as

$$D = V_1 Z_1 C$$

and $$V = X_1 A X_1' + V_1 Z_1 B Z_1' V_1 - X_1 A D' - D A' X_1',$$

where C is arbitrary and A and B are arbitrary subject to the condition that V is n.n.d.

Proof : Since X_1 is given by (2.3.1), we can write

$$Z_1 = P'^{-1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Now it is easy to see that $D = V_1 Z_1 C$ for some C if and only if D is given by (2.4.3). From theorem 2.4.1, we get $\underline{M}(V) \subset \underline{M}(V_1 + X_1X_1')$. Since $\underline{M}(V_1 + X_1X_1') = \underline{M}(V_1 Z_1 : X_1)$, we can write $V = X_1\Delta_1 X_1' + V_1 Z_1 \Delta_2 Z_1' V_1 + X_1 \Delta_3 Z_1' V_1 + V_1 Z_1 \Delta_3' X_1'$ for some

Δ_1, Δ_2 and Δ_3. $X_1'G_1VZ = 0$ <==>

$$X_1\Delta_1X_1'G_1X_1 + V_1Z_1\Delta_3'X_1'G_1X_1 = (X_1 - D)K, \text{ for some } K$$

$$\iff X_1\Delta_1X_1'G_1X_1 = X_1K$$

$$\text{and } V_1Z_1\Delta_3'X_1'G_1X_1 = -DK,$$

(since $\underline{M}(D) \subset \underline{M}(V_1Z_1)$ and $\underline{M}(X_1) \cap \underline{M}(V_1Z_1) = \{0\}$)

$$\iff X_1\Delta_1X_1' = X_1K(X_1'G_1X_1)^-X_1'$$

$$\text{and } V_1Z_1\Delta_3'X_1' = -DK(X_1'G_1X_1)^-X_1'.$$

Putting $A = K(X_1'G_1X_1)^-$ and $B = \Delta_2$ the result follows.

Remark 2.4.2. Let X_1 and $V_1+X_1X_1'$ be as given in (2.4.1) and (2.4.2) respectively. Then matrices C, A and B which make the matrix V in corollary 2.4.1 an n.n.d. matrix could be characterised in a manner similar to the characterisation procedure considered in remark 2.2.1.

Theorem 2.4.2. Consider the partitioned forms of X_1 and $V_1+X_1X_1'$ given by (2.4.1) and (2.4.2). Then the BLUE of every estimable parametric functional under $(Y, X_1\beta, V_1)$ irrespective of its linear representation is its BLIMBE under $(Y, X\beta, V)$ if and only if

$$D = P\begin{bmatrix} B & C_1 & C_2 \\ E_1 & E_2 & E_3 \\ 0 & 0 & 0 \end{bmatrix} Q'$$

$$\text{and} \qquad V = P \begin{bmatrix} W_{11} & W_{12} & 0 \\ W_{12}' & W_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} P'$$

where B is an arbitrary n.n.d. matrix with eigenvalues in $[0,1]$, C_1 and C_2 are arbitrary matrices satisfying $C_1C_1' + C_2C_2' = B - B^2$ and $E = (E_1 : E_2 : E_3)$ is an arbitrary matrix satisfying

$$\underline{M}(E') \subset \underline{M}(B : C_1 : C_2)$$

and if $A = (I - B : - C_1 : - C_2)$, then

$$W_{11} = ATA', \quad W_{12} = (E - E_1(I - B)^{-} A)HTA' - ETA'$$

where T is an arbitrary n.n.d. matrix and H is an arbitrary matrix. W_{22} is arbitrary subject to the condition that V is n.n.d.

Proof : The theorem is proved in a manner similar to theorem 1.2.2 and theorem 2.2.2 and hence we omit the proof.

2.5 Best linear minimum bias estimation.

First, we shall prove an algebraic lemma which we need in the sequel. Even though the lemma is stated and proved for matrices over the field of real numbers, it holds good for matrices over the field of complex numbers as well.

Lemma 2.5.1. Let A and B be two $m \times n$ matrices and N an $n \times n$ positive definite matrix. Write $B = A + C$ and let $U' = (A')^{\perp}$. Then $A^{+} = B^{+}_{IN}$ if and only if

(i) $C = AEU$

and (ii) $N = A'\wedge_1 A + U'\wedge_2 U + A'\wedge_1 C + C'\wedge_1' A$

where E is arbitrary and $\wedge_1$ and $\wedge_2$ are arbitrary subject to the condition that N is p.d.

<u>Proof</u> : 'Only if' part.

Let $A^+ = B^+_{IN}$. Then,

$BA^+B = B$, $A^+BA^+ = A^+$, $(NA^+B)' = NA^+B$ and $(BA^+)' = BA^+$.

The last equation implies

$$(CA^+)' = CA^+ . \qquad \ldots(2.5.1)$$

$BA^+B = B \iff$

$$AA^+C + CA^+A + CA^+C = C \qquad \ldots(2.5.2)$$

$A^+BA^+ = A^+ \iff A^+CA^+ = 0 \iff A^+A^{+'}C' = 0$, using (2.5.1)

$$\iff AC' = 0 \qquad \ldots(2.5.3)$$

(2.5.2) and (2.5.3) together imply

$$AA^+C = C . \qquad \ldots(2.5.4)$$

From (2.5.3) and (2.5.4) it now follows that

$C = AEU$, E being arbitrary.

Since $\underline{M}(A') \oplus \underline{M}(U')$ is the whole space, we can write

$$N = A'\wedge_1 A + U'\wedge_2 U + A'\wedge_3 U + U'\wedge_3' A .$$

Then $(NA^+B)' = NA^+B \iff A^+AN + C'A^{+'}N = NA^+A + NA^+C$

$$\Longleftrightarrow \quad A'\Lambda_1 A + A'\Lambda_3 U + C'\Lambda_1 A + C'\Lambda_3 U = A'\Lambda_1 A + U'\Lambda_3' A + A'\Lambda_1 C + U'\Lambda_3' C$$

$\Longrightarrow \quad A'\Lambda_3 U = A'\Lambda_1 C$, using the fact that $\underline{M}(C') \subset \underline{M}(U')$ and $\underline{M}(A') \cap \underline{M}(U') = \{0\}$. This completes the proof of the 'only if' part. The 'if' part is easily verified.

Remark 2.5.1. If N is not p.d., but only n.n.d, then A^+ is B^+_{IN} if and only if $BA^+B = B$, $NA^+BA^+ = NA^+$, $(NA^+B)' = NA^+B$ and $(BA^+)' = BA^+$. (see Mitra and Rao, 1974, theorem 3.4). It is easy to verify that if an n.n.d. matrix N and the matrix B have the form specified in lemma 2.5.1, then A^+ is B^+_{IN}.

It is easy to see that the BLIMBE of every parametric functional under $(Y, X_1\beta, \sigma^2 I)$ is its BLIMBE under $(Y, X\beta, \sigma^2 I)$ if and only if $X = X_1$. We shall now consider linear models which differ in the dispersion matrices or both in the design and dispersion matrices.

Theorem 2.5.1. The BLIMBE of every parametric functional under $(Y, X\beta, V_1)$ is its BLIMBE under $(Y, X\beta, V)$ if and only if V satisfies $\underline{M}(VZ) \subset \underline{M}(V_1 Z)$.

In view of lemma 1.2.2, the proof of theorem 2.5.1 is similar to that of theorem 5.2 in Rao (1971) and hence is omitted.

Remark 2.5.2. Rao (1971) has proved that BLUE of every estimable parametric functional under $(Y, X\beta, V_1)$ (irrespective of its linear representation) is its BLUE under $(Y, X\beta, V)$ if and only if

$\underline{M}(VZ) \subset \underline{M}(V_1 Z)$. Hence from theorem 2.5.1 we see that, BLIMBE of every parametric functional under $(Y, X\beta, V_1)$ is its BLIMBE under $(Y, X\beta, V)$ if and only if BLUE of every estimable parametric functional under $(Y, X\beta, V_1)$ is its BLUE under $(Y, X\beta, V)$. This result is expected, since any BLIMBE is a BLUE of its expectation. In theorem 2.5.2 we consider every possible linear representation of BLIMBE of a parametric functional. The general linear representation of BLIMBE of a parametric functional is given in Rao (1978).

Theorem 2.5.3. If V is positive definite, then the BLIMBE of every parametric functional under $(Y, X_1\beta, \sigma^2 I)$ is its BLIMBE under $(Y, X\beta, V)$ if and only if

$$X = X_1 + Z_1 E X_1$$

and $$V = X_1 \Lambda_1 X_1' + Z_1 \Lambda_2 Z_1' + X_1 \Lambda_1 X_1' E' Z_1' + Z_1 E X_1 \Lambda_1 X_1'$$

where Λ_1, Λ_2 and E are arbitrary subject to the condition that V is positive definite. If V is non-negative definite then the above conditions are sufficient.

Proof : The BLIMBE of every parametric functional under $(Y, X_1\beta, \sigma^2 I)$ is corresponding BLIMBE under $(Y, X\beta, V)$ if and only if $(X_1')^+$ is $(X')^+_{IV}$. Theorem 2.5.3 follows from lemma 2.5.1 and remark 2.5.1.

Remark 2.5.3. Matrices Λ_1, Λ_2 and E which make $V = X_1 \Lambda_1 X_1' + Z_1 \Lambda_2 Z_1' + X_1 \Lambda_1 X_1' E' Z_1' + Z_1 E X_1 \Lambda_1 X_1'$ a p.d. matrix could be characterised as follows :

Let $X_1 = P \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} Q'$ be the singular value decomposition of X_1. Then

$$Z_1 = P \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

Write $Q' \Lambda_1 Q = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $\Lambda_2 = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$

and $EP = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$. Then

$$V = X_1 \Lambda_1 X_1' + Z_1 \Lambda_2 Z_1' + X_1 \Lambda_1 X_1' E' Z_1' + Z_1 E X_1 \Lambda_1' X_1'$$

$$= P \begin{bmatrix} \Delta A_{11} \Delta & \Delta A_{11} \Delta C_{21}' \\ C_{21} \Delta A_{11}' \Delta & B_{22} \end{bmatrix} P' = P \begin{bmatrix} A_1 & A_1 C_{21}' \\ C_{21} A_1' & B_{22} \end{bmatrix} P'$$

where $A_1 = \Delta A_{11} \Delta$. Choose any p.d. matrix A_1 and an arbitrary matrix C_{21}. Then V is p.d. if and only if $B_{22} - C_{21} A_1^{-} C_{21}'$ is p.d. Hence choose B_{22} so that $B_{22} - C_{21} A_1^{-} C_{21}'$ is p.d. The other matrices in the partitioned forms of $Q' \Lambda_1 Q$, Λ_2 and EP could be chosen arbitrarily.

If we are given two linear models there may exist parametric functionals which admit common BLIMBE under both the models though every parametric functional may not be so. Our next attempt is to characterise such parametric functionals.

Theorem 2.5.4. Consider the linear models $(Y, X_1\beta, V)$ and $(Y, X\beta, V)$. Let the matrices W_1, Z_1, W and Z be such that $\underline{N}(X_1) = \underline{M}(W_1')$, $\underline{N}(X_1') = \underline{M}(Z_1)$, $\underline{N}(X) = \underline{M}(W')$ and $\underline{N}(X') = \underline{M}(Z)$. Then the parametric functional $p'\beta$ has a common BLIMBE under both the models if and only if p admits the representation

$$p = X_1'X_1\theta_1 + W_1'\mu_1 = X'X\theta + W'\mu, \text{ where } \mu_1 \text{ and } \mu$$

are arbitrary and θ_1 and θ satisfy

$$\begin{bmatrix} X_1 & -X \\ Z_1'VX_1 & 0 \\ 0 & Z'VX \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta \end{bmatrix} \in \underline{M} \begin{bmatrix} Z_1 & -Z \\ Z_1'VZ_1 & 0 \\ 0 & Z'VZ \end{bmatrix}.$$

Proof : Let $\ell'Y$ be a common BLIMBE of $p'\beta$ under both the models and let

$$\ell = X_1\theta_1 - Z_1\alpha_1 = X\theta - Z\alpha .$$

Then $X_1X_1'\ell = X_1p$ and $XX'\ell = Xp$ <==>

$$p = X_1'X_1\theta_1 + W_1'\mu_1 = X'X\theta + W'\mu , \text{ for some } \mu_1 \text{ and } \mu .$$

Since $\ell'Y$ is BLIMBE under both the models

$$Z_1'V(X_1\theta_1 - Z_1\alpha_1) = 0 \qquad \ldots(2.5.5)$$

$$Z'V(X\theta - Z\alpha) = 0 . \qquad \ldots(2.5.6)$$

Also $$X_1\theta_1 - Z_1\alpha_1 = X\theta - Z\alpha \qquad \ldots(2.5.7)$$

(2.5.5), (2.5.6) and (2.5.7) hold if and only if the condition given in the theorem holds .

<u>Corollary 2.5.1</u>. Using the notations of theorem 2.5.4, $p'\beta$ has a common BLIMBE under the models $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 I)$ if and only if p admits the representation

$$p = X_1'X_1\theta_1 + W_1'\mu_1 = X'X\theta + W'\mu ,$$

where μ_1 and μ are arbitrary and $X_1\theta_1 = X\theta$.

<u>Proof</u> : From the proof of theorem 2.5.4, we see that when $V = \sigma^2 I$, (2.5.5) and (2.5.6) hold if and only if $Z_1\alpha_1 = 0$ and $Z\alpha = 0$ and hence (2.5.7) holds if and only if $X_1\theta_1 = X\theta$ which proves the corollary.

<u>Corollary 2.5.2</u>. (a) $p'\beta$ has a common BLUE under $(Y, X_1\beta, V)$ and $(Y, X\beta, V)$ if and only if p is of the form $p = X_1'X_1\theta_1 = X'X\theta$, where θ_1 and θ satisfy the condition stated in theorem 2.5.4.

(b) $p'\beta$ has a common BLUE under $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 I)$ if and only if p is of the form

$$p = X_1'X_1\theta_1 = X'X\theta \quad \text{where} \quad X_1\theta_1 = X\theta .$$

<u>Theorem 2.5.5</u>. Let matrices W and Z be such that $\underline{N}(X) = \underline{M}(W')$ and $\underline{N}(X') = \underline{M}(Z)$. Then the parametric functional $p'\beta$ has a common BLIMBE under $(Y, X\beta, V_1)$ and $(Y, X\beta, V)$ if and only if $p = X'X\theta + W\mu$, where μ is arbitrary and $X\theta$ satisfies

$$\begin{bmatrix} Z'V_1X\theta \\ Z'VX\theta \end{bmatrix} \subset \underline{M} \begin{bmatrix} Z'V_1Z \\ Z'VZ \end{bmatrix}$$

<u>Proof</u> : If $\lambda'Y$ is a BLIMBE of $p'\beta$ under $(Y, X\beta, V_1)$, then by lemma 1.2.2, $XX'\lambda = Xp$ and $Z'V_1\lambda = 0$. Write $\lambda = X\theta + Z\alpha$.

Then $XX'\ell = Xp \iff XX'X\theta = Xp \iff p = X'X\theta + W'\mu$, μ being arbitrary. If $\ell'Y$ is a common BLIMBE of $p'\beta$ under both the models, then $Z'V_1\ell = 0$ and $Z'V\ell = 0 \iff$

$$Z'V_1(X\theta + Z\alpha) = 0$$
$$\text{and} \quad Z'V(X\theta + Z\alpha) = 0$$

$\iff \begin{bmatrix} Z'V_1X\theta \\ Z'VX\theta \end{bmatrix} \in \underline{M} \begin{bmatrix} Z'V_1Z \\ Z'VZ \end{bmatrix}$, which completes the proof of theorem 2.5.5.

Corollary 2.5.3. If $V_1 = I$, then $p'\beta$ has common BLIMBE under $(Y, X\beta, \sigma^2 I)$ and $(Y, X\beta, V)$ if and only if $p = X'X\theta + W'\mu$, where μ is arbitrary and $X\theta$ satisfies the condition $VX\theta \in \underline{M}(X)$.

Corollary 2.5.4. If $p \in \underline{M}(X')$, then $p'\beta$ has a common BLUE under $(Y, X\beta, V_1)$ and $(Y, X\beta, V)$ if and only if $p = X\, X\theta$, where $X\theta$ satisfies

$$\begin{bmatrix} Z'V_1X\theta \\ Z'VX\theta \end{bmatrix} \in \underline{M} \begin{bmatrix} Z'V_1Z \\ Z'VZ \end{bmatrix}.$$

Theorem 2.5.6. Consider the linear models $(Y, X_1\beta, V_1)$ and $(Y, X\beta, V)$. Let the matrices W_1, Z_1, W and Z be such that $\underline{N}(X_1) = \underline{M}(W_1')$, $\underline{N}(X_1') = \underline{M}(Z_1)$, $\underline{N}(X) = \underline{M}(W')$ and $\underline{N}(X') = \underline{M}(Z)$. Then the parametric functional $p'\beta$ has a common BLIMBE under both the models if and only if p admits the representation $p = X_1'X_1\theta_1 + W_1'\mu_1 = X'X\theta + W'\mu$, where μ_1 and μ are arbitrary and θ_1 and θ satisfy

$$\begin{bmatrix} X_1 & -X \\ Z_1'V_1X_1 & 0 \\ 0 & Z'VX \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta \end{bmatrix} \in \underline{M} \begin{bmatrix} Z_1 & -Z \\ Z_1'V_1Z_1 & 0 \\ 0 & Z'VZ \end{bmatrix} .$$

Proof : The proof of theorem 2.5.6 is similar to that of theorem 2.5.4 and is omitted.

Remark 2.5.4. Theorems 2.5.4, 2.5.5 and 2.5.6 not only characterise parametric functionals which admit a common BLIMBE under both the models, but also provide a method of obtaining the common BLIMBE, whenever it exists.

CHAPTER 3

THE LIKELIHOOD RATIO TEST

3.1 Introduction.

Consider the linear model $(Y, X\beta, \sigma^2 I)$ and a hypothesis $H_o : A\beta = 0$, where $A\beta$ is a set of estimable parametric functionals. In this chapter, we assume that Y has a multivariate normal distribution and our purpose is to study the robustness of the likelihood ratio test (LRT) procedure for testing H_o. Recently, Ghosh and Sinha (1980) and Sinha and Mukhopadhyay (1980(a) and (b)) undertook this problem and obtained conditions under which the LRT statistic retains the same form under various structural forms of the dispersion matrix of Y. Khatri (1980) undertook this problem and gave a characterisation of positive definite matrices V such that the LRT statistic for testing H_o under $(Y, X\beta, \sigma^2 V)$ is sam as the LRT statistic for testing H_o under $(Y, X\beta, \sigma^2 I)$. Here we consider alternative linear models which differ from $(Y, X\beta, \sigma^2 I)$ in the dispersion of the observations or expectation or both and derive conditions for the LRT statistic under $(Y, X\beta, \sigma^2 I)$ for testing H_o to be valid under the alternative model also. Our attention will be concentrated only on the situation where Y has a positive definite covariance structure.

If we have a hypothesis $H_o : A\beta = b$ to be tested under $(Y, X\beta, \sigma^2 V)$, where $A\beta$ is a vector of estimable parametric functionals and b is a vector which makes $A\beta = b$ consistent, the under H_o , we can write

$$\beta = A^- b + \eta$$

and $$X\beta = XA^- b + X\eta ,$$

where η is an arbitrary vector which satisfies $A\eta = 0$. Thus, testing the hypothesis $H_o : A\beta = \eta$ under $(Y, X\beta, \sigma^2 V)$ is equivalent to testing the hypothesis $H_o : A\eta = 0$ under $(Y - XA^- b, X\eta, \sigma^2 V)$. Hence we will be considering only hypotheses of the form $H_o : A\beta = 0$.

3.2 Specification errors in the dispersion matrix.

Let L_V and L denote the LRT statistics for testing $H_o : A\beta = 0$ under $(Y, X\beta, \sigma^2 V)$ and $(Y, X\beta, \sigma^2 I)$ respectively, where V is a positive definite matrix. Under H_o, the model $(Y, X\beta, \sigma^2 V)$ can be rewritten as $(Y, X_o\beta_o, \sigma^2 V)$, where $X_o = X(I - A^- A)$ is an $n \times m$ matrix, and β_o is an $m \times 1$ vector. If $R(X) = r \leq m$ and $R(A) = k$, then it can be shown that $R(X_o) = r - k$. If $X_o\hat{\beta}_o$ and $X\hat{\beta}$ denote the BLUE's of $X_o\beta_o$ and $X\beta$ respectively under $(Y, X_o\beta_o, \sigma^2 V)$ and $(Y, X\beta, \sigma^2 V)$, then L_V is given by

$$L_V = \frac{(Y - X_o\hat{\beta}_o)' V^{-1} (Y - X_o\hat{\beta}_o)}{(Y - X\hat{\beta})' V^{-1} (Y - X\hat{\beta})} ,$$

which, after simplifications, can be written as

$$L_V = \frac{Y' V^{-1} (I - P_{X_o, V^{-1}}) Y}{Y' V^{-1} (I - P_{X, V^{-1}}) Y} .$$

We notice that L_V can be defined since $P(Y \in \underline{M}(X)) = 0$. Under H_o, $\frac{n-r}{k}(L_V - 1)$ has $F_{k,n-r}$ distribution and the LRT is given by the critical region $L_V > c$ where c is chosen so as to satisfy the level condition.

Our main objective in this section is to derive conditions on V under which $L_V - L \geq 0$ (or ≤ 0) with probability 1. If $L_V - L \geq 0$ (or ≤ 0), then the rejection (or acceptance) of H_o under $(Y, X\beta, \sigma^2 I)$ will imply its rejection (respectively acceptance) under $(Y, X\beta, \sigma^2 V)$ also. Thus we are considering a situation where L_V and L may be different, but still, L can be used to test H_o under $(Y, X\beta, \sigma^2 V)$. We achieve this by considering two separate cases. In case (1), we assume V to satisfy the condition that the BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 I)$ is its BLUE under $(Y, X\beta, \sigma^2 V)$ also. In case (2), we consider an arbitrary p.d. V.

Now let Z be a matrix of order $n \times (n-r)$ and $Z_o = (Z \vdots Z_1)$ be a matrix of order $n \times (n-r+k)$ satisfying $Z'X = 0$, $Z_o'X_o = 0$, $Z'Z = I_{n-r}$ and $Z_o'Z_o = I_{n-r+k}$. We now state

<u>Lemma 3.2.1.</u> $\underline{M}(A') = \underline{M}(X'Z_o) = \underline{M}(X'Z_1)$.

The lemma can be easily established by showing that A and $Z_o'X$ have the same null spaces.

It is known that the BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 I)$ is also its BLUE under $(Y, X\beta, \sigma^2 V)$ if and only if

$$V = I + X\Lambda_1 X' + Z\Lambda_2 Z' + X\Lambda_4 Z' + Z\Lambda_4' X' ,$$

where Λ_1, Λ_2 and Λ_4 are arbitrary except that $A\Lambda_4 Z' = 0$ and V is p.d. (see Rao and Mitra, 1971 p.159). V can be equivalently represented as

$$V = I + X\Lambda_1 X' + Z\Lambda_2 Z' + X_0\Lambda_3 Z' + Z\Lambda_3' X_0' \qquad \ldots(3.2.1)$$

where Λ_1, Λ_2 and Λ_3 are arbitrary subject to the condition that V is p.d. It can be verified that if a p.d. (or a n.n.d) matrix V admits the representation (3.2.1), then the matrices $X\Lambda_1 X'$ and Λ_2 are symmetric and unique.

The following lemma gives further necessary and sufficient conditions for the representation (3.2.1) to hold.

<u>Lemma 3.2.2</u>. The BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 I)$ is its BLUE under $(Y, X\beta, \sigma^2 V)$, or equivalently, the representation (3.2.1) holds if and only if anyone of the following equivalent conditions holds :

(i) $Z'VZ_1 = 0$

(ii) $P_X V^{-1}(I - P_{X_0,V^{-1}})$ is symmetric

(iii) $(I - P_{X_0,V^{-1}})(I - P_{X,V^{-1}})'$ is symmetric

(iv) There exists an orthogonal matrix T such that $T'(I - P_X)T$, $T'(I - P_{X_0})T$, $T'V^{-1}(I - P_{X,V^{-1}})T$ and $T'V^{-1}(I - P_{X_0,V^{-1}})T$ are diagonal matrices.

In order to prove lemma 3.2.2, we need

Lemma 3.2.3. Let V be a p.d. matrix and let $Z = X^{\perp}$ Then

$$V^{-1}(I - P_{X,V^{-1}}) = Z(Z'VZ)^{-} Z'.$$

Proof : It is easily verified that

$$V^{-1}(I - P_{X,V^{-1}})(X : VZ) = Z(Z'VZ)^{-} Z'(X : VZ),$$

which proves the lemma, since, $\underline{M}(X : VZ)$ is the whole space.

Proof of lemma 3.2.2. $A(X'X)^{-} X'Y$ is the BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 V)$ if and only if $A(X'X)^{-} X'VZ = 0 \iff Z_1'X(X'X)^{-} X'VZ = 0$, (using lemma 3.2.1.) which is equivalent to the condition $Z_1'VZ = 0$, since $\underline{M}(Z_1) \subset \underline{M}(X)$, using the fact that $Z'Z_1 = 0$. This shows that the condition (i) is necessary and sufficient for the BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 I)$ to be its BLUE under $(Y, X\beta, \sigma^2 V)$ also.

$$P_X V^{-1}(I - P_{X_o, V^{-1}}) \text{ is symmetric}$$

$$\iff (I - P_X)V^{-1}(I - P_{X_o, V^{-1}}) \text{ is symmetric}$$

$$\iff ZZ'Z_o(Z_o'VZ_o)^{-1}Z_o' = Z_o(Z_o'VZ_o)^{-1}Z_o'ZZ' \text{, using lemma 3.2.3}$$

$$\iff Z_o'VZZ'Z_o = Z_o'ZZ'VZ_o$$

$$\iff \begin{bmatrix} Z'VZ & 0 \\ Z_1'VZ & 0 \end{bmatrix} = \begin{bmatrix} Z'VZ & Z'VZ_1 \\ 0 & 0 \end{bmatrix}$$

$$\iff Z'VZ_1 = 0.$$

This proves the equivalence of (i) and (ii).

$$(I - P_{X_o,V^{-1}})(I - P_{X,V^{-1}})' \text{ is symmetric}$$

$$\Longleftrightarrow \quad Z_o(Z_o'VZ_o)^{-1}Z_o'Z(Z'VZ)^{-1} Z' = Z(Z'VZ)^{-1} Z'Z_o(Z_o'VZ_o)^{-1} Z_o'$$

$$\Longleftrightarrow \quad Z_o(Z_o'VZ_o)^{-1} Z_o'Z = Z(Z'VZ)^{-1}$$

$$\Longleftrightarrow \quad Z_o'Z = Z_o'VZ(Z'VZ)^{-1}$$

$$\Longleftrightarrow \quad \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} Z_1'VZ \\ Z_1'VZ \end{bmatrix} (Z'VZ)^{-1}$$

$$\Longleftrightarrow \quad Z'VZ_1 = 0 .$$

The equivalence of (i) and (iii) is thus established.

The equivalent conditions (ii) and (iii) hold if and only if the matrices $I - P_X$, $I - P_{X_o}$, $V^{-1}(I - P_{X,V^{-1}})$ and $V^{-1}(I - P_{X_o,V^{-1}})$ commute pairwise, which is necessary and sufficient for the existence of an orthogonal matrix which diagonalises them simultaneously (see Rao and Mitra, 1971, p. 124). The proof of lemma 3.2.2 is now complete.

Corollary 3.2.1. The BLUE of every estimable parametric functional under $(Y, X\beta, \sigma^2 I)$ is its BLUE under $(Y, X\beta, \sigma^2 V)$ if and only if anyone of the following equivalent conditions holds :

(i) $X'VZ = 0$ (Rao, 1967)

(ii) VP_X is symmetric (Zyskind, 1967)

(iii) $P_{X,V^{-1}}$ is symmetric.

Remark 3.2.1. Let $A_1 = I-P_{X_o}$, $A_2 = I-P_X$, $A_3 = V^{-1}(I-P_{X_o,V^{-1}})$ and $A_4 = V^{-1}(I-P_{X,V^{-1}})$. Then condition (iv) in lemma 3.2.2 is equivalent to demanding that there exists a nonsingular matrix T satisfying $T'A_iT$ is diagonal for $i = 1,2,3,4$. To justify this claim, we appeal to a theorem due to Bhimasankaram (1971), which states that if $\underline{M}(A_i) \subset \underline{M}(A_1)$, then there exists a nonsingular matrix T with the desired property if and only if for some g-inverse A_1^- of A_1, (a) $A_iA_1^-$ is semisimple with real eigen-values for all i and (b) $A_iA_1^-A_j$ is symmetric for all i a j. Choosing $A_1^- = (I-P_{X_o}) = A_1$, we see that (a) is satisfied and (b) is satisfied $\forall$ i and j if and only if the equivalent con tions (ii) and (iii) in lemma 3.2.2 hold.

Next we shall prove

Lemma 3.2.4. If the LRT statistics for testing $H_o : A\beta = 0$ ar the same with probability one under $(Y, X\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$ then the BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 I)$ is its BLUE under $(Y, X\beta, \sigma^2 V)$ also.

Proof :

$$\frac{Y'V^{-1}(I-P_{X_o,V^{-1}})Y}{Y'V^{-1}(I-P_{X,V^{-1}})Y} = \frac{Y'(I-P_{X_o})Y}{Y'(I-P_X)Y}, \quad \forall\ Y \notin \underline{M}(X)$$

$$\Longleftrightarrow \quad Y'Z_o(Z_o'VZ_o)^{-1}Z_o'Y.Y'ZZ'Y = Y'Z(Z'VZ)^{-1}Z'Y.Y'Z_oZ_o'Y, \ \forall\ Y \notin \underline{M}($$

Putting $Y = VZ\theta$, we get

$$\theta' Z' V Z Z' V Z \theta = \theta' Z' V Z_o Z_o' V Z \theta \quad \forall \theta$$

$$\iff \quad Z' V Z_1 = 0 .$$

Lemma 3.2.4 stands proved, in view of lemma 3.2.2 (i).

Remark 3.2.2. The interesting observation made in lemma 3.2.4 is implicit in the main result derived by Khatri (1980), but this fact is not stated in his paper.

We now proceed to derive the necessary and sufficient conditions for $L_V - L \geq 0$ with probability one.

Case 1 : Here we assume that V belongs to the class of p.d. matrices for which the simple least squares estimator of $A\beta$ continues to be its BLUE under $(Y, X\beta, \sigma^2 V)$. In other words, V has the representation (3.2.1).

Since V admits the representation (3.2.1), the condition (iv) in lemma 3.2.2 is satisfied. Let T be the orthogonal matrix which reduces $I-P_X$, $I-P_{X_o}$, $V^{-1}(I-P_{X,V^{-1}})$ and $V^{-1}(I-P_{X_o V^{-1}})$ simultaneously to diagonal forms. The columns of T are the common eigenvectors of these four matrices. It can be shown that each column of T belongs to $\underline{M}(Z)$, $\underline{M}(Z_1)$ or $\underline{M}(X_o)$. Rearrange the columns of T so that the first $n-r$ columns belong to $\underline{M}(Z)$, the next k columns belong to $\underline{M}(Z_1)$ and the last $r-k$ columns belong to $\underline{M}(X_o)$. (Recall that $r = R(X)$, $k = R(A)$ and $r-k = R(X_o)$). Let λ_i and λ_{oi} $(i = 1, 2, \ldots, n-r)$ denote the nonzero eigenvalues of $V^{-1}(I-P_{X,V^{-1}})$ and

$V^{-1}(I-P_{X_o,V^{-1}})$ respectively corresponding to the same eigenvectors belonging to $\underline{M}(Z)$ and let $\lambda_{oi}(i = n-r+1,\dots,n-r+k)$ denote the nonzero eigenvalues of $V^{-1}(I-P_{X_o,V^{-1}})$ corresponding to the eigenvectors belonging to $\underline{M}(Z_1)$. Note that the number of nonzero eigenvalues of $V^{-1}(I-P_{X,V^{-1}})$ and $V^{-1}(I-P_{X_o,V^{-1}})$ are respectively $n-r$ and $n-r+k$, their ranks. Let $T'Y = t = (t_1 t_2 \dots t_n)'$. Then, recalling the manner in which the columns of T have been arranged, we get

$$L = \frac{t'T'(I-P_{X_o})Tt}{t'T'(I-P_X)Tt} = \frac{\sum_{i=1}^{n-r+k} t_i^2}{\sum_{i=1}^{n-r} t_i^2}$$

Similarly

$$L_V = \frac{\sum_{i=1}^{n-r+k} \lambda_{oi} t_i^2}{\sum_{i=1}^{n-r} \lambda_i t_i^2} .$$

Thus we have proved

<u>Lemma 3.2.5</u>. Let L_V, L, λ_{oi}, λ_i and t_i be as defined above. Then

$$L_V - L = \frac{\sum_{i=1}^{n-r+k} \sum_{j=1}^{n-r} (\lambda_{oi} - \lambda_j) t_i^2 t_j^2}{\sum_{i=1}^{n-r} \sum_{j=1}^{n-r} \lambda_i t_i^2 t_j^2} .$$

From lemma 3.2.5, one can easily conclude that $L_V - L \geq 0$ with probability one if and only if

$$\lambda_{oi} \geq \lambda_i \quad \text{for} \quad i = 1,2,\ldots,n-r$$

$$\text{and} \quad \lambda_{oi} \geq \lambda_j \quad \text{for} \quad i = n-r+1,\ldots,n-r+k$$
$$j = 1,2,\ldots,n-r .$$

Since V is assumed to have the representation (3.2.1), the condition (i) in lemma 3.2.2 holds and hence we have

$$Z_o(Z_o'VZ_o)^{-1}Z_o'ZZ' = Z(Z'VZ)^{-1}Z'$$

or equivalently,

$$V^{-1}(I-P_{X_o,V^{-1}})(I-P_X) = V^{-1}(I-P_{X,V^{-1}})$$

$$\Longleftrightarrow \quad \lambda_{oi} = \lambda_i \quad \text{for} \quad i = 1,2,\ldots,n-r .$$

Using this observation and lemma 3.2.5, we have

Lemma 3.2.6. $L_V - L \geq 0$ with probability one if and only if

$$\lambda_{oi} \geq \lambda_{oj} , \; i = n-r+1,\ldots,n-r+k$$
$$j = 1,2,\ldots,n-r .$$

Next, we shall derive conditions on V such that the eigenvalues $\lambda_{oi}(i = 1,2,\ldots,n-r+k)$ of $Z_o(Z_o'VZ_o)^{-1}Z_o'$ satisfy the condition stated in lemma 3.2.6. Using the representation (3.2.1) for V and recalling that $Z_o = (Z \vdots Z_1)$ satisfies $Z_o'Z_o = I$, we get

$$Z_o'VZ_o = \begin{bmatrix} I+\Lambda_2 & 0 \\ 0 & I+Z_1'X\Lambda_1X'Z_1 \end{bmatrix} .$$

Hence

$$Z_o(Z_o'VZ_o)^{-1}Z_o' = Z(I+\Lambda_2)^{-1}Z' + Z_1(I+Z_1'X\Lambda_1X'Z_1)^{-1}Z_1' \qquad \ldots(3.2.2).$$

For $i = 1,2,\ldots,n-r$, λ_{oi} are the eigenvalues of $Z_o(Z_o'VZ_o)^{-1}Z_o'$ corresponding to eigenvectors belonging to $\underline{M}(Z)$ and for $i = n-r+1,\ldots,n-r+k$, λ_{oi} correspond to eigenvectors belonging to $\underline{M}(Z_1)$. Using this, and the fact that Z and Z_1 are chosen to satisfy $Z'Z = I_{n-r}$, and $Z_1'Z_1 = I_{n-r+k}$, it follows from (3.2.2) that λ_{oi} $(i = 1,2,\ldots,n-r)$ are the eigenvalues of $(I+\Lambda_2)^{-1}$ and λ_{oi} $(i = n-r+1,\ldots,n-r+k)$ are the eigenvalues of $(I+Z_1'X\Lambda_1X'Z_1)^{-1}$. Hence, it follows that $\lambda_{oi} \geq \lambda_{oj}$ for $i = n-r+1,\ldots,n-r+k$ and $j = 1,2,\ldots,n-r$ if and only if the minimum eigenvalue of $I+\Lambda_2$ is greater than or equal to the maximum eigenvalue of $I+Z_1'X\Lambda_1X'Z_1$. Since V is given by (3.2.1), we see that $I+\Lambda_2 = Z'VZ$ and $I+Z_1'X\Lambda_1X'Z_1 = Z_1'VZ_1$, and the eigenvalues of $I+\Lambda_2$ and $I+Z_1'X\Lambda_1X'Z_1$ are respectively the non-null eigenvalues of $ZZ'VZZ'$ and $Z_1Z_1'VZ_1Z_1'$. But $ZZ' = I-P_X$ and $Z_1Z_1' = P_X - P_{X_o}$. Thus we have proved

Theorem 3.2.1. Let L and L_V respectively denote the LRT statistics for testing $H_o : A\beta = 0$ under $(Y,X\beta,\sigma^2I)$ and $(Y,X\beta,\sigma^2V)$, where V is a positive definite matrix admitting the representation (3.2.1) and $\underline{M}(A') \subset \underline{M}(X')$. Then $L_V - L \geq 0$ (or ≤ 0) with probability one if and only if the minimum non-null eigenvalue (or the maximum eigenvalue) of $(I-P_X)V(I-P_X)$ is greater than or equal to (or less than or equal to) the maximum eigenvalue (respectively the minimum non-null eigenvalue) of $(P_X - P_{X_o})V(P_X - P_{X_o})$. Under this condition, the rejection (or acceptance) of H_o under $(Y, X\beta, \sigma^2 I)$ will imply its rejection (respectively acceptance) under $(Y, X\beta, \sigma^2 V)$ also.

In theorem 3.2.1, we have assumed that V admits the representation (3.2.1) or equivalently $Z'VZ_1 = 0$. However, if we are interested in conditions under which $\lambda_V = \lambda$ with probability one, then this assumption always holds, in view of lemma 3.2.4. Thus we have also proved

<u>Theorem 3.2.2</u>. Consider the linear models $(Y, X\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$, where V is positive definite. For testing $H_o : A\beta = 0$, the LRT statistic under $(Y, X\beta, \sigma^2 V)$ is the same as the LRT statistic under $(Y, X\beta, \sigma^2 I)$ if and only if V satisfies

(i) $(I - P_X)V(P_X - P_{X_o}) = 0$,

(ii) $(I - P_X)V(I - P_X) = s(I - P_X)$

and (iii) $(P_X - P_{X_o})V(P_X - P_{X_o}) = s(P_X - P_{X_o})$,

where s is a positive real number, or equivalently

$$V = I + X\Lambda_1 X' + (s-1)(I - P_X) + X_o\Lambda_3 Z' + Z\Lambda_3' X_o',$$

where Λ_1 and Λ_3 are arbitrary and s is an arbitrary positive real number subject to the conditions (i) V is positive definite and (ii) $(P_X - P_{X_o})X\Lambda_1 X'(P_X - P_{X_o}) = (s-1)(P_X - P_{X_o})$.

Corollary 3.2.2. The condition on V given in theorem 3.2.2 is equivalent to anyone of the following equivalent conditions :

(i) $(I - P_{X_o})V(I - P_{X_o}) = a(I - P_{X_o})$ for some $a > 0$

(ii) $V^{-1}(I - P_{X_o, V^{-1}}) = a(I - P_{X_o})$ for some $a > 0$

(iii) $\begin{bmatrix} I - P_X \\ LP_X \end{bmatrix} (V - aI)(I - P_X : P_X L') = 0$, for some $a > 0$, where L is such that $LX = A$.

Proof : (i) is equivalent to the condition $Z_o' V Z_o = aI$ for some $a > 0$ $\iff$ $Z'VZ = aI$, $Z_1' V Z_1 = aI$ and $Z'VZ_1 = 0$ which are equivalent to the conditions on V given in theorem 3.2.2. Condition (ii) in the corollary holds if and only if

$Z_o(Z_o' V Z_o)^{-1} Z_o' = a Z_o Z_o'$ $\iff$ $Z_o' V Z_o = \frac{1}{a} I$ $\iff$ (i) holds.

Since L is such that $LX = A$, from lemma 3.2.1 it follows that $LP_X = Z_1'$. Hence the condition (iii) in the corollary can be written as

$$\begin{bmatrix} ZZ' \\ Z_1' \end{bmatrix} V(ZZ' : Z_1) = a \begin{bmatrix} ZZ' & 0 \\ 0 & I \end{bmatrix}$$

which is equivalent to (i). The proof of corollary 3.2.2 is now complete.

Note 3.2.1. Corollary 3.2.2 (iii) is the result obtained by Khatri (1980).

From corollary 3.2.2 (ii), we have

Corollary 3.2.3. $L_V = L$ with probability one if and only if the numerator of L_V is proportional to the numerator of L.

We have derived conditions for $L_V - L \geq 0$ with probability one assuming that V has the representation (3.2.1). Now we shall consider

Case 2 : V is an arbitrary positive definite matrix.

We prove

Theorem 3.2.3. Consider the linear model $(Y, X\beta, \sigma^2 V)$ and a hypothesis $H_o : A\beta = 0$, where $\underline{M}(A') \subset \underline{M}(X')$. Let X_o, L_V and L be as defined before. Let $\mu_{oi}(i = 1,2,\ldots,n-r+k)$ and $\mu_i(i = 1,2,\ldots,n-r)$ respectively denote the non-null eigenvalues of $(I-P_{X_o})V(I-P_{X_o})$ and $(I-P_X)V(I-P_X)$, where $r = R(X)$ and $k = R(A)$. Suppose there exists exactly $p(0 \leq p \leq n-r)$ eigenvalues of $(I-P_{X_o})V(I-P_{X_o})$, say $\mu_{oi}(i = 1,2,\ldots,p)$ corresponding to eigenvectors belonging to $\underline{M}(I-P_X)$. (Then $\mu_{oi}(i = 1,2,\ldots,p)$ are also eigenvalues of $(I-P_X)V(I-P_X)$ and we can assume without loss of generality that $\mu_{oi} = \mu_i$ $(i = 1,2,\ldots,p)$). Then

(i) $L_V - L \geq 0$ with probability one if and only if

$$\min_{1 \leq i \leq n-r} \mu_i \geq \max_{p+1 \leq i \leq n-r+k} \mu_{oi}$$

and $$\min_{p+1 \leq i \leq n-r} \mu_i \geq \max_{1 \leq i \leq n-r+k} \mu_{oi} \text{ , and}$$

(ii) $L_V - L \leq 0$ with probability one if and only if

$$\max_{1 \leq i \leq n-r} \mu_i \leq \min_{p+1 \leq i \leq n-r+k} \mu_{oi}$$

and $\max_{p+1 \leq i \leq n-r} \mu_i \leq \min_{1 \leq i \leq n-r+k} \mu_{oi}$.

Proof : We have

$$L_V = \frac{Y'Z_o(Z_o'VZ_o)^{-1}Z_o'Y}{Y'Z(Z'VZ)^{-1}Z'Y} \text{ and } L = \frac{Y'Z_oZ_o'Y}{Y'ZZ'Y} ,$$

where Z_o and Z are as defined in case 1. Let $\lambda_{oi}(i=1,2,\ldots,n-r+k)$ and $\lambda_i(i=1,2,\ldots,n-r)$ respectively denote the non-null eigenvalues of $Z_o(Z_o'VZ_o)^{-1}Z_o'$ and $Z(Z'VZ)^{-1}$ It is easy to see that λ_{oi} is an eigenvalue of $Z_o(Z_o'VZ_o)^{-1}Z_o'$ corresponding to an eigenvector in $\underline{M}(Z_o)$ if and only if $\frac{1}{\lambda_{oi}}$ is an eigenvalue of $(I-P_{X_o})V(I-P_{X_o})$ corresponding to the same eigenvector. The same relationship exists between $Z(Z'VZ)^{-1}Z'$ and $(I-P_X)V(I-P_X)$ also. One can also verify that if λ_o is an eigenvalue of $Z_o(Z_o'VZ_o)^{-1}Z_o'$ corresponding to an eigenvector in $\underline{M}(Z)$, then λ_o is also an eigenvalue of $Z(Z'VZ)^{-1}Z'$ correspond ing to the same eigenvector. Let T and U be orthogonal matrices such that $T'Z_o(Z_o'VZ_o)^{-1}Z_o'T$, $T'Z_oZ_o'T$, $U'Z(Z'VZ)^{-1}Z'U$ and $U'ZZ'U$ are diagonal matrices. Let

$$t = (t_1,t_2,\ldots,t_n)' = T'Y \text{ and } u = (u_1,u_2,\ldots,u_n)' = U'Y .$$

In view of the above discussion, we can assume that $\lambda_{oi} = \lambda_i(i=1,2,\ldots,p)$ and the i^{th} column of T and U are identical for $i=1,2,\ldots,p$, if $Z_o(Z_o'VZ_o)^{-1}Z_o'$ has p eigenvalues corresponding to eigenvectors in $\underline{M}(Z)$. After simplifications, we can write

$$L_V - L = \frac{\sum_{i=1}^{n-r+k} \sum_{j=1}^{n-r} (\lambda_{oi} - \lambda_j) u_j^2 t_i^2}{Y' Z(Z'VZ)^{-1} Z' Y . Y' ZZ' Y} .$$

Now using the fact that $\lambda_{oi} = \lambda_i$ and $t_i = u_i$ for $i = 1, 2, \ldots, p$ we get

$L_V - L \geq 0$ with probability one if and only if $\lambda_{oi} \geq \lambda_j$ for $i = 1, 2, \ldots, n-r+k$, $j = p+1, \ldots, n-r$ and $\lambda_{oi} \geq \lambda_j$ for $i = p+1, \ldots, n-r+k$, $j = 1, 2, \ldots, n-r$. Since $\mu_{oi} = \frac{1}{\lambda_{oi}}$ and $\mu_i = \frac{1}{\lambda_i}$, the theorem follows.

Remark 3.2.3. Theorems 3.2.1, 3.2.2 and 3.2.3 have been proved without assuming that the matrix A is of full row rank.

Examples.

We now consider a few examples to illustrate the results so far obtained.

Example 1. Let $n > 1$, and let V_ρ be the intraclass covariance matrix given by

$$V_\rho = (1 - \rho) I_n + \rho \, 1_n 1_n', \; -\frac{1}{n-1} < \rho < 1 ,$$

where 1_n is the $n \times 1$ column vector with each element equal to 1. The condition $-\frac{1}{n-1} < \rho < 1$, guarantees the positive definiteness of V_ρ. With V_ρ defined like this we consider the linear model $(Y, X\beta, \sigma^2 V_\rho)$, where Y has multivariate normal distribution. Suppose we want to test the hypothesis $H_o : A\beta = 0$, where $\underline{M}(A') \subset \underline{M}(X')$. Let X_o be as defined before. For the intraclass covariance matrix V_ρ, Ghosh and Sinha (1980) proved

<u>Theorem 3.2.4.</u> The LRT statistic L_ρ for testing H_0 under $(Y, X\beta, \sigma^2 V_\rho)$ has the same value for every fixed $\rho \in (-\frac{1}{n-1}, 1)$ if and only if $1_n \in \underline{M}(X_0)$.

It is an easy matter to deduce theorem 3.2.4 from condition (i) in corollary 3.2.2. The proof given by Ghosh and Sinha is lengthy and involved. We present here a simple alternative proof of theorem 3.2.4, which, we feel, is of independent interest.

<u>Proof of theorem 3.2.4.</u> Let Z and Z_0 be as defined in the beginning of this section. Then we have

$$L_\rho = \frac{Y'Z_0(Z_0'V_\rho Z_0)^{-1}Z_0'Y}{Y'Z(Z'V_\rho Z)^{-1}Z'Y}$$

$$= \frac{Y'Z_0\left[I_{n-r+k} - \dfrac{\frac{\rho}{1-\rho}\, Z_0'1_n1_n'Z_0}{1+\frac{1}{1-\rho}\, 1_n'Z_0Z_0'1_n}\right]Z_0'Y}{Y'Z\left[I_{n-r} - \dfrac{\frac{\rho}{1-\rho}\, Z'1_n1_n'Z}{1+\frac{1}{1-\rho}\, 1_n'ZZ'1_n}\right]Z'Y},$$

upon simplification; where $r = R(X)$ and $k = R(A)$.

$L_\rho = L_0$ with probability one for every fixed $\rho \in (-\frac{1}{n-1}, 1)$ if and only if $\forall Y$ and $\forall \rho \in (-\frac{1}{n-1}, 1)$,

$$\frac{Y'Z_0Z_0'1_n1_n'Z_0Z_0'Y}{1+\frac{1}{1-\rho}\, 1_n'Z_0Z_0'1_n} \cdot Y'ZZ'Y = \frac{Y'ZZ'1_n1_n'ZZ'Y}{1+\frac{1}{1-\rho}\, 1_n'ZZ'1_n} \cdot Y'Z_0Z_0'Y \quad ..(3.2.3)$$

Sufficiency of the condition $1_n \in \underline{M}(X_o)$ is now obvious. To prove its necessity, notice that if $1_n'Z = 0$, then from (3.2.3) we get

$$(Y'Z_oZ_o'1_n1_n'Z_oZ_o'Y).Y'ZZ'Y = 0 \qquad ...(3.2.4)$$

for all Y. Putting $Y = Z_o\theta$ in (3.2.4), we get $1_n'Z_o = 0$ and this proves the theorem. Hence it is enough to show that $1_n'Z = 0$. Assume that $1_n'Z \neq 0$ and let ξ_i denote the columns of Z for $i = 1,2,\ldots,n-r$. Then for atleast one i, $1_n'ZZ'\xi_i \neq 0$, and putting $Y = \xi_i$ in (3.2.3), we get $1_n'Z_oZ_o'1_n = 1_n'ZZ'1_n \iff Z_1'1_n = 0$, since $Z_o = (Z \,\vdots\, Z_1)$. Using this condition, (3.2.3) simplifies to

$$(Y'ZZ'1_n)^2.Y'ZZ'Y = (Y'ZZ'1_n)^2.Y'Z_oZ_o'Y \qquad ...(3.2.5)$$

for all Y. Putting $Y = Zb + Z_1b_1$ in (3.2.5), we get $Z'1_n = 0$. This completes the proof of theorem 3.2.4.

Next, we shall examine the situation when $L_\rho \neq L_o$. In this case one can think of applying theorem 3.2.3. First, we shall show that theorem 3.2.3 is applicable only if $1_n'Z = 0$ (i.e. $P_X1_n = 1_n$) or $1_n'Z_1 = 0$ (i.e. $P_X1_n = P_{X_o}1_n$). If $1_n'Z = 0$ or $1_n'Z_1 = 0$ we observe that $Z'V_\rho Z_1 = 0$ or equivalently V_ρ admits the representation (3.2.1) so that theorem 3.2.1 applies.

Assume that $1_n'Z \neq 0$ and $1_n'Z_1 \neq 0$. Since $Z_o'Z_o = I_{n-r+k}$ and $Z_o'V_\rho Z_o = (1-\rho)\, I_{n-r+k} + \rho Z_o'1_n1_n'Z_o$, the non-null eigenvalues of $(I-P_{X_o})V_\rho(I-P_{X_o})$ are $1-\rho$ of multiplicity $n-r+k-1$ and

$1-\rho+\rho 1_n'Z_oZ_o'1_n$ of multiplicity one. If $Z\theta$ is an eigenvector of $(I-P_{X_o})V_\rho(I-P_{X_o})$ corresponding to the eigenvalue $1-\rho$, then we get $Z_oZ_o'1_n1_n'Z\theta = 0 \iff 1_n'Z\theta = 0$. Hence, since $1_n'Z \neq 0$, among the multiplicity $n-r+k-1$ of $1-\rho$ as an eigenvalue of $(I-P_{X_o})V_\rho(I-P_{X_o})$, $n-r-1$ correspond to eigenvectors in $\underline{M}(Z)$. One can also show that the eigenvalue $1-\rho+\rho 1_n'Z_oZ_o'1_n$ corresponds to an eigenvector of $(I-P_{X_o})V_\rho(I-P_{X_o})$ in $\underline{M}(Z)$ if and only if $Z_1'1_n = 0$. Thus, when $1_n'Z \neq 0$ and $1_n'Z_1 \neq 0$, $(I-P_{X_o})V_\rho(I-P_{X_o})$ has non-null eigenvalue $1-\rho$ of multiplicity $n-r-1$ corresponding to eigenvectors in $\underline{M}(Z)$ and $1-\rho$ of multiplicity k and $1-\rho+\rho 1_n'Z_oZ_o'1_n$ of multiplicity one corresponding to eigenvectors in $\underline{M}(Z_o)$ that are not in $\underline{M}(Z)$. One can see that in this situation, the inequalities given in theorem 3.2.3 (i) or (ii) lead to a contradiction. Thus, when $1_n'Z \neq 0$ and $1_n'Z_1 \neq 0$, we cannot have $L_\rho - L_o \geq 0$ (or ≤ 0) with probability one.

Now we proceed under the assumption that $1_n'Z = 0$ or $1_n'Z_1 = 0$. Notice that $1_n'Z = 0$ implies $X'V_\rho Z = 0$ and $1_n'Z_1 = 0$ implies $Z'V_\rho Z_1 = 0$. Using lemma 3.2.2 and corollary 3.2.1, we see that if either $Z'1_n = 0$ or $Z_1'1_n = 0$, then the BLUE of $A\beta$ under $(Y, X\beta, \sigma^2 I)$ is also its BLUE under $(Y, X\beta, \sigma^2 V_\rho)$. Hence V_ρ admits the representation (3.2.1). Assuming that $L_\rho \neq L_o$, or equivalently, $Z_o'1_n \neq 0$, we shall examine the applicability of theorem 3.2.1 in this set up. We consider the following situations :

(a) $Z'1_n = 0,\ Z_1'1_n \neq 0$. In this case $(P_X - P_{X_0})V_\rho(P_X - P_{X_0}) = Z_1Z_1'V_\rho Z_1Z_1'$ has non-null eigenvalues $1-\rho$ of multiplicity $k-1$ and $1-\rho+\rho 1_n'Z_1Z_1'1_n$ of multiplicity one. $(I-P_X)V_\rho(I-P_X)$ has non-null eigenvalues $1-\rho$ of multiplicity $n-r$. Applying theorem 3.2.1, we get

$L_\rho - L_0 > 0$ with probability one if and only if $\rho < 0$

and $L_\rho - L_0 < 0$ with probability one if and only if $\rho > 0$.

Thus when $1_n \in \underline{M}(X)$, but $1_n \notin \underline{M}(X_0)$, we see that the rejection (or acceptance) of H_0 under $(Y, X\beta, \sigma^2 I)$ implies its rejection (respectively acceptance) under $(Y, X\beta, \sigma^2 V_\rho)$ also if $\rho < 0$ (or $\rho > 0$).

(b) $Z'1_n \neq 0,\ Z_1'1_n = 0$. In this case $(P_X - P_{X_0})V_\rho(P_X - P_{X_0})$ has non-null eigenvalues $1-\rho$ of multiplicity k and $(I-P_X)V_\rho(I-P_X)$ has non-null eigenvalues $1-\rho$ of multiplicity $n-r-1$ and $1-\rho+\rho 1_n'ZZ'1_n$ of multiplicity one. Applying theorem 4.2.1, we get

$L_\rho - L_0 > 0$ with probability one if and only if $\rho > 0$

and $L_\rho - L_0 < 0$ with probability one if and only if $\rho < 0$.

Thus when $1_n \notin \underline{M}(X_0)$, but $P_{X_0}1_n = P_X1_n$, then the rejection or acceptance of H_0 under $(Y, X\beta, \sigma^2 I)$ implies its rejection or acceptance under $(Y, X\beta, \sigma^2 V_\rho)$ also according as $\rho > 0$ or $\rho < 0$. It is interesting to observe that the conclusions in (a) and (b) above do not involve the matrix A, apart from the requirements

$P_X 1_n = 1_n$ or $P_X 1_n = P_{X_0} 1_n$, and hence is valid for any testing problem $H_0 : A\beta = 0$ with $\underline{M}(A') \subset \underline{M}(X')$ and satisfying this requirement.

Example 2. We now consider the linear model $(Y, X\beta, \sigma^2 V_c)$, where V_c is given by

$$V_c = \alpha I_n + c 1_n' + 1_n c' \quad \text{with} \quad \alpha > 0 ,$$
$$c = (c_1, c_2, \ldots c_n)' \quad \text{and} \quad 1_n = (1, 1, \ldots 1)' .$$

One can show that the eigenvalues of V_c are $\alpha + 1_n' c + (nc'c)^{1/2}$, $\alpha + 1_n' c - (nc'c)^{1/2}$ and α (of multiplicity $n-2$). The vectors c which make V_c positive definite are precisely those which make $\alpha + 1_n' c + (nc'c)^{1/2}$ and $\alpha + 1_n' c - (nc'c)^{1/2}$ positive. Baldessari (1966) studied the validity of χ^2- and F-tests for dependent normal data and came up with a model having the dispersion matrix considered here. Suppose we have a testing problem $H_0 : A\beta = 0$ and let L_c and L_0 denote the LRT statistics for testing H_0 under $(Y, X\beta, \sigma^2 V_c)$ and $(Y, X\beta, \sigma^2 I)$ respectively. Let X_0, Z and Z_0 be as defined before. From corollary 3.2.2 (i), we see that for every fixed vector c (which makes V_c p.d.), $L_c = L_0$ with probability one if and only if

$$(I - P_{X_0}) V_c (I - P_{X_0}) = a(I - P_{X_0}) \quad \text{for some} \quad a > 0$$

$$\iff \quad Z_0' (\alpha I + 1_n c' + c 1_n') Z_0 = aI$$

$$\iff \quad Z_0' (1_n c' + c 1_n') Z_0 = (a - \alpha) I . \qquad \ldots (3.2.6)$$

If $n > 2$, then (3.2.6) leads to a contradiction if $(a-\alpha) \neq 0$, since, $(a-\alpha)I$ in such a case has rank greater than 2, whereas the matrix $Z_o'(1_n c' + c 1_n')Z_o$ has rank less than or equal to 2. If $n = 2$, then writing $a_1 = Z_o' 1_n$ and $a_2 = Z_o' c$, (3.2.6) becomes $a_1 a_2' + a_2 a_1' = (a-\alpha)I$. Here a_1 and a_2 are 2×1 vectors and it can be easily shown that the above relation leads to a contradiction if $a-\alpha \neq 0$. Thus $a-\alpha = 0$ in either case and we get

$L_c = L_o$ with probability one

$\Longleftrightarrow \quad Z_o'(1_n c' + c 1_n')Z_o = 0$

$\Longleftrightarrow \quad Z_o' 1_n c' Z_o = 0$

$\Longleftrightarrow \quad Z_o' 1_n = 0$ or $c' Z_o = 0$ or both are zeros

$\Longleftrightarrow \quad 1_n$ or c or both belong to $\underline{M}(X_o)$.

The above result was also obtained by Sinha and Mukhopadhyay (1980 (b)) by a different approach.

Our next attempt is to apply theorem 3.2.3 when L_c and L_o are different, i.e. when neither 1_n nor c belong to $\underline{M}(X_o)$. The non-null eigenvalues of $(I-P_X)V_c(I-P_X)$ are precisely the eigenvalues of $Z'V_c Z = \alpha I + Z'(1_n c' + c 1_n')Z$. Also, $Z'(1_n c' + c 1_n')Z$ and $(c : 1_n)' ZZ' (1_n : c)$ have the same non-null eigenvalues. Direct computation shows that the eigenvalues of $(c : 1_n)' ZZ' (1_n : c)$ are $1_n' ZZ' c \pm ((1_n' ZZ' 1_n)(c' ZZ' c))^{1/2}$. Hence, the non-null eigenvalues of $(I-P_X)V_c(I-P_X)$ are α (of multiplicity $n-r-2$), $\alpha + 1_n' ZZ' c + ((1_n' ZZ' 1_n)(c' ZZ' c))^{1/2}$ and $\alpha + 1_n' ZZ' c - ((1_n' ZZ' 1_n)(c' ZZ' c))^{1/2}$

of multiplicity one each. It can be verified that the eigenvectors of $(I-P_X)V_c(I-P_X)$ corresponding to the eigenvalues $\alpha+1_n'ZZ'c \pm ((1_n'ZZ'1_n)(c'ZZ'c))^{1/2}$ are respectively $(1_n'ZZ'1_n)^{1/2}ZZ'c \pm (c'ZZ'c)^{1/2}ZZ'1_n$. Similarly it can be shown that the non-null eigenvalues of $(I-P_{X_o})V_c(I-P_{X_o})$ are $\alpha+1_n'Z_oZ_o'c \pm ((1_n'Z_oZ_o'1_n)(c'Z_oZ_o'c))^{1/2}$ of multiplicity one each (with corresponding eigenvectors $(1_n'Z_oZ_o'1_n)^{1/2}Z_oZ_o'c \pm (c'Z_oZ_o'c)^{1/2}Z_oZ_o'1_n$ respectively) and α (of multiplicity $n-r+k-2$). Assuming that neither 1_n nor c belong to $\underline{M}(X_o)$, we now consider

Case 1. Either 1_n or c or both belong to $\underline{M}(X)$.

We consider only the case $1_n \varepsilon \underline{M}(X)$. The other cases can be dealt with similarly. In this situation, the non-null eigenvalues of $(I-P_X)V_c(I-P_X)$ are each equal to α. We shall show that, when $1_n \varepsilon \underline{M}(X)$, the eigenvectors $(1_n'Z_oZ_o'1_n)^{1/2}Z_oZ_o'c \pm (c'Z_oZ_o'c)^{1/2}Z_oZ_o'1_n$ of $(I-P_{X_o})V_c(I-P_{X_o})$ corresponding to the eigenvalues $\alpha+1_n'Z_oZ_o'c \pm (1_n'Z_oZ_o'1_n)^{1/2}(c'Z_oZ_o'c)^{1/2}$ cannot belong to $\underline{M}(Z)$. For, suppose $(1_n'Z_oZ_o'1_n)^{1/2}Z_oZ_o'c + (c'Z_oZ_o'c)^{1/2}Z_oZ_o'1_n$ belong to $\underline{M}(Z)$. Then the corresponding eigenvalue is α (see the discussion given at the beginning of the proof of theorem 3.2.3). Hence $1_n'Z_oZ_o'c + (1_n'Z_oZ_o'1_n)^{1/2}(c'Z_oZ_o'c)^{1/2} = 0$
$\Longrightarrow (1_n'Z_oZ_o'c)^2 = (1_n'Z_oZ_o'1_n)(c'Z_oZ_o'c) \Longrightarrow Z_o'c = aZ_o'1_n = a(0:1_n'Z_1)$
where a is a real number. The above equation implies $Z'c = 0$

and hence $(1_n'Z_oZ_o'1_n)^{1/2}Z_oZ_o'c + (c'Z_oZ_o'c)^{1/2}Z_oZ_o'1_n =$
$(1_n'Z_1Z_1'1_n)^{1/2}Z_1Z_1'c + (c'Z_1Z_1'c)^{1/2}Z_1Z_1'1_n \in \underline{M}(Z_1)$, a contradiction. Thus, if $(I-P_{X_o})V_c(I-P_{X_o})$ has an eigenvector in $\underline{M}(Z)$, then the corresponding eigenvalue is α. Applying theorem 3.2.3, we get $L_c - L_o \geq 0$ with probability one if and only if

$$1_n'Z_oZ_o'c \pm (1_n'Z_oZ_o'1_n)^{1/2}(c'Z_oZ_o'c)^{1/2} \leq 0$$

$$\iff \quad Z_o'c = aZ_o'1_n \text{ for some } a \leq 0$$

$$\iff \quad (I-P_{X_o})c = a(I-P_{X_o})1_n \text{ for some } a \leq 0 .$$

Similarly it can be shown that $L_c - L_o \leq 0$ with probability one if and only if

$$(I-P_{X_o})c = a(I-P_{X_o})1_n \text{ for some } a \geq 0 .$$

Thus, when 1_n(or c or both) belong to $\underline{M}(X)$, we get,

$L_c - L_o \geq 0$ with probability one if and only if

$$(I-P_{X_o})c = a(I-P_{X_o})1_n \text{ for some } a \leq 0$$

and $L_c - L_o \leq 0$ with probability one if and only if

$$(I-P_{X_o})c = a(I-P_{X_o})1_n \text{ for some } a \geq 0 .$$

Notice that the conditions $1_n \in \underline{M}(X)$ and $(I-P_{X_o})c = a(I-P_{X_o})1_n$ implies $c \in \underline{M}(X)$. Also, the inequalities $L_c - L_o \geq 0$ (or ≤ 0) and $a \leq 0$ (or ≥ 0) can be replaced by $L_c - L_o > 0$ (or < 0) and $a < 0$ (or > 0), since, when $L_c \neq L_o$, $(I-P_{X_o})c$ and $(I-P_{X_o})1_n$ are both non-zero. Next, we shall consider

Case 2. Neither 1_n nor c belong to $\underline{M}(X)$.

In general, when neither 1_n nor c is in $\underline{M}(X)$, it is difficult to give conditions under which $L_c - L_o \geq 0$ (or ≤ 0) with probability one. In this case, there are situations where we cannot have $L_c - L_o \geq 0$ (or ≤ 0) with probability one. Here, we consider the case when $P_X 1_n = P_{X_o} 1_n$ and $P_X c = P_{X_o} c$ or equivalently $Z_1' 1_n = 0$ and $Z_1' c = 0$. Another way of putting this condition is that $Z_1' 1_n = 0$ and the least squares estimator of $A\beta$ is its BLUE under $(Y, X\beta, \sigma^2 V_c)$, since the least squares estimator of $A\beta$ is its BLUE under $(Y, X\beta, \sigma^2 V_c)$ if and only if $Z' V_c Z_1 = Z'(1_n c' + c 1_n') Z_1 = 0$, by lemma 3.2.2. From the expression for the eigenvalues and eigenvectors of $(I-P_X)V_c(I-P_X)$ and $(I-P_{X_o})V_c(I-P_X)$ given in case 1, it can be seen that when $Z_1' 1_n = 0 = Z_1' c$, the non-null eigenvalues and corresponding eigen vectors of $(I-P_X)V_c(I-P_X)$ are also eigenvalues and corresponding eigenvectors of $(I-P_{X_o})V_c(I-P_{X_o})$. Applying theorem 3.2.1 or theorem 3.2.3, we get $L_c - L_o \geq 0$ (or ≤ 0) with probability one if and only if $1_n' Z Z' c \pm (1_n' Z Z' 1_n)^{1/2} (c' Z Z' c)^{1/2} \geq 0$ (or ≤ 0) $\Longleftrightarrow (I-P_X)c = a(I-P_X)1_n$ for some $a \geq 0$ (or ≤ 0). Thus when neither 1_n nor c in $\underline{M}(X)$, but we have $P_X 1_n = P_{X_o} 1_n$ and $P_X c = P_{X_o} c$, then

$L_c - L_o > 0$ with probability one if and only if

$(I-P_X)c = a(I-P_X)1_n$ for some $a > 0$

and $L_c - L_o < 0$ with probability one if and only if

$$(I-P_X)c = a(I-P_X)1_n \quad \text{for some} \quad a < 0 .$$

Example 3. Consider the linear model $(Y, X\beta, \sigma^2 V_\theta)$, where V_θ is given by

$$V_\theta = \begin{bmatrix} 1+\theta^2 & \theta & 0 & 0 \ldots 0 & 0 \\ \theta & 1+\theta^2 & \theta & 0 \ldots 0 & 0 \\ 0 & \theta & 1+\theta^2 & \theta \ldots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \ldots \theta & 1+\theta^2 \end{bmatrix}$$

θ being a real number. Such a variance covariance matrix arises in time series when the errors are generated by a moving average process of order 1. (See O.D. Anderson, 1976, p.32). It can be shown that

$$|V_\theta| = \frac{1-\theta^{2n+2}}{1-\theta^2} \quad \text{if } \theta \text{ is not equal to } \pm 1$$

$$= 1+n, \quad \text{otherwise.}$$

Thus V_θ is positive definite for all θ. Consider the hypothesis $H_o : A\beta = 0$, where $\underline{M}(A') \subset \underline{M}(X')$ and let L_θ and L_o denote the LRT statistics for testing H_o under $(Y, X\beta, \sigma^2 V_\theta)$ and $(Y, X\beta, \sigma^2 I)$ respectively. We shall obtain conditions under which $L_\theta = L_o$ with probability one, for every fixed θ.

Notice that V_θ can be written as $V_\theta = (1+\theta^2)I + \theta C$, where

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

From corollary 3.2.2(i), we see that $L_\theta = L_0$ with probability one for every fixed θ

$\iff (I-P_{X_0})((1+\theta^2)I+\theta C)(I-P_{X_0}) = a_1(I-P_{X_0})$ for some $a_1 > 0$

$\iff (I-P_{X_0}) + \frac{\theta}{1+\theta^2}(I-P_{X_0})C(I-P_{X_0}) = a(I-P_{X_0})$. for some $a > 0$

$\iff \frac{\theta}{1+\theta^2}(I-P_{X_0})C(I-P_{X_0}) = (a-1)(I-P_{X_0})$ for some $a > 0$

$\iff (I-P_{X_0})C(I-P_{X_0}) = b(I-P_{X_0})$ for some b satisfying $-2 < b <$

in view of the fact that $-\frac{1}{2} \leq \frac{\theta}{1+\theta^2} \leq \frac{1}{2}$. Thus $L_\theta = L_0$ with probability one for every fixed θ if and only if

$(I-P_{X_0})C(I-P_{X_0}) = b(I-P_{X_0})$ for some b satisfying $-2 < b < 2$.

Remark 3.2.4. In theorems 3.2.1, 3.2.2 and 3.2.3 and corollary 3.2.2 we have obtained conditions under which $L_V = L$, $L_V - L \geq$ or $L_V - L \leq 0$, with probability one, where L_V and L are the LRT statistics for testing a hypothesis $H_0 : A\beta = 0$ under $(Y, X\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$ respectively. These conditions involve the projection operators P_X and P_{X_0}. If V satisfi any of these conditions, then we can think of using L inste of L_V for testing H_0 under $(Y, X\beta, \sigma^2 V)$. However, if V doesn't satisfy the conditions under which $L_V = L$, $L_V - L \geq 0$

$L_V - L \leq 0$ with probability one, then one has to compute L_V to test H_o and L_V involves the projection operators $P_{X,V^{-1}}$ and $P_{X_o,V^{-1}}$. Here we would like to mention that if V fails to satisfy the conditions which enable us to use L instead of L_V to test H_o, then the projection operators P_X and P_{X_o} already computed to verify these conditions can be used to compute $P_{X,V^{-1}}$ and $P_{X_o,V^{-1}}$. This is achieved by using the relation

$$P_{X,V^{-1}} = P_X - P_X V(I-P_X)((I-P_X)V(I-P_X))^{-} (I-P_X) \qquad \ldots(3.2.7)$$

and a similar relation for $P_{X_o,V^{-1}}$. Notice that the expression given by (3.2.7) doesn't involve V^{-1}, whereas the expression $P_{X,V^{-1}} = X(X'V^{-1}X)^{-} X'V^{-1}$ does. (3.2.7) can be established by showing that by postmultiplying both sides of (3.2.7) by X or $V(I-P_X)$, equality holds and further, using the fact that $\underline{M}(X) \oplus \underline{M}(V(I-P_X))$ is the whole space.

3.3 An error bound for the LRT statistic.

So far we have explored the possibilities of using L instead of L_V to test a hypothesis $H_o : A\beta = 0$ under $(Y, X\beta, \sigma^2 V)$. Our motivation for considering this problem has been the computational advantage enjoyed by L over L_V. Haberman (1975) and Baksalary and Kala (1978, 1980) have given bounds for the norm of the difference between the SLSE and the BLUE of $X\beta$ under $(Y, X\beta, \sigma^2 V)$. If the difference is small, one can estimate

$X\beta$ by its SLSE since the computation of the SLSE is much simpler compared to the BLUE. In this section, we take up the task of obtaining an upper bound for $|L_V - L|$. We shall obtain a non-negative random variable t (depending on the random vector Y) that satisfies $|L_V - L| \leq t$ with probability one. Once we observe Y, we can compute t and this will give an idea as to how much L_V and L can differ if they are not equal. If t is small enough, one can still think of using L instead of L_V. The F-tests under $(Y, X\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$ for testing H_o are given by the critical regions $L > c$ and $L_V > c$ respectively. If $L > c$ (or $L < c$) and $|L_V - L| \leq t$ where t is small enough so that $L - t \geq c$ (or $L + t \leq c$), then $L_V \geq L - t \geq c$ (or $L_V \leq c$) and the rejection (or acceptance) of H_o under $(Y, X\beta, \sigma^2 I)$ will imply its rejection (respectively acceptance) under $(Y, X\beta, \sigma^2 V)$ also. Thus, there is still the possibility of using L to test H_o under $(Y, X\beta, \sigma^2 V)$. Here, since we have to compute t, we make use of the knowledge of an observed Y. These facts suggest that there are situations where the knowledge of an upper bound for $|L_V - L|$ can be of help to avoid the computation of L_V to test H_o under $(Y, X\beta, \sigma^2 V)$. We now prove

<u>Theorem 3.3.1</u>. Consider the linear models $(Y, X\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$, where V is positive definite and a hypothesis $H_o : A\beta = 0$, where $\underline{M}(A') \subset \underline{M}(X')$. Let L and L_V denote the LRT statistics for testing H_o under $(Y, X\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$ respectively. Suppose $k = R(A)$ and $r = R(X)$ and $X_o = X(I - A^-A$

If $\frac{1}{\lambda_{oi}}$ $(i = 1,2,\ldots,n-r+k)$ and $\frac{1}{\lambda_i}$ $(i = 1,2,\ldots,n-r)$ respectively denote the non-null eigenvalues of $(I-P_{X_o})V(I-P_{X_o})$ and $(I-P_X)V(I-P_X)$, then

(a) $$|L_V - L| \leq \frac{(Y'Y)^2}{(Y'(I-P_X)Y)^2} \cdot \frac{1}{\lambda} \sum_{i=1}^{n-r+k} \sum_{j=1}^{n-r} |\lambda_{oi} - \lambda_j|$$

with probability one, where $\lambda = \min_i \lambda_i$

and (b) $$|L_V - L| \leq \frac{(Y'Y)^2}{(Y'(I-P_X)Y)^2} \cdot \frac{(n-r+k)(n-r)}{\lambda} \cdot \max(a,b)$$

with probability one where $a = |\max_i \lambda_{oi} - \min_j \lambda_j|$ and $b = |\min_i \lambda_{oi} - \max_j \lambda_j|$.

<u>Proof</u> : Using the notations and expressions given in the proof of theorem 3.2.3, we get

$$L_V - L = \frac{\sum_{i=1}^{n-r+k} \sum_{j=1}^{n-r} (\lambda_{oi} - \lambda_j) u_j^2 t_i^2}{Y'(I-P_X)Y . Y'V^{-1}(I-P_{X,V^{-1}})Y} . \qquad \ldots(3.3.1)$$

Since λ_i's are eigenvalues of $V^{-1}(I-P_{X,V^{-1}}) = Z(Z'VZ)^{-1}Z'$ and $\lambda = \min_i \lambda_i$, we get

$$Y'V^{-1}(I-P_{X,V^{-1}})Y = Y'Z(Z'VZ)^{-1}Z'Y \geq \lambda Y'ZZ'Y = \lambda Y'(I-P_X)Y .$$

Since $t = (t_1, t_2, \ldots, t_n)'$ and $u = (u_1, u_2, \ldots, u_n)$ are orthogonal transformations of Y, we see that each u_j^2 and t_i^2 is less than or equal to $Y'Y$. Using these facts, the bound given in (a) follows from equation (3.3.1). The bound given in (b) follows from (a).

Remark 3.3.1. The bound given in (b) is less sharp compared to the bound in (a), but the bound given in (b) involves only the maximum and minimum eigenvalues of $(I-P_{X_0})V(I-P_{X_0})$ and $(I-P_X)V(I-P_X)$.

Remark 3.3.2. From the results given in theorem 3.2.2, we see that $L_V = L$ with probability one if and only if $\lambda_{oi} = \lambda_j$ for $i = 1,2,\ldots,n-r+k$ and $j = 1,2,\ldots,n-r$, i.e. if and only if the bound given in (a) is zero.

3.4 Specification errors in the design matrix.

In this section, we consider two alternative linear models $(Y,X_1\beta,\sigma^2I)$ and $(Y,X\beta,\sigma^2I)$ which differ in the design matrices and not in the dispersion of observations and obtain conditions on X such that the LRT statistic under $(Y,X\beta,\sigma^2I)$ for testing a hypothesis $H_0 : A\beta = 0$ is same as the LRT statistic under $(Y,X_1\beta,\sigma^2I)$ for testing H_0. Here, $A\beta$ is a parametric functional estimable under both the models. Let $R(A) = k$, $R(X_1) = r_1$ and $R(X) = r$. Then the LRT statistics L_1 and L for testing H_0 under $(Y,X_1\beta,\sigma^2I)$ and $(Y,X\beta,\sigma^2I)$ are respectively defined as

$$L_1 = \frac{n-r_1}{k}\,\frac{Y'(I-P_{X_1(I-A^-A)})Y}{Y'(I-P_{X_1})Y}$$

and $$L = \frac{n-r}{k}\,\frac{Y'(I-P_{X(I-A^-A)})Y}{Y'(I-P_X)Y}$$

L_1 is defined for all Y satisfying $(I-P_{X_1})Y \neq 0$ and L_2 is defined for all Y satisfying $(I-P_X)Y \neq 0$. Hence for the equality of L and L_1 to be meaningful, $(I-P_{X_1})$ and $(I-P_X)$ should be equal, or in other words, $\underline{M}(X) = \underline{M}(X_1)$. We now prove

<u>Theorem 3.4.1</u>. Consider the linear models $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 I)$, where $\underline{M}(X) = \underline{M}(X_1)$ and let L_1 and L be as defined above. Then $L = L_1$ with probability one if and only if $\underline{M}(X(I-A^- A)) = \underline{M}(X_1(I-A^- A))$.

<u>Proof</u>: Let $X_{10} = X_1(I-A^- A)$ and $X_0 = X(I-A^- A)$ and let $W_0 = (W \vdots W_1)$ and $Z_0 = (Z \vdots Z_1)$ be matrices satisfying $W_0' X_{10} = 0$, $W' X_1 = 0$, $Z_0' X_0 = 0$, $Z' X = 0$, $W_0' W_0 = I$ and $Z_0' Z_0 = I$. Since $\underline{M}(X_1) = \underline{M}(X)$, $r_1 = r$ and we have

$$L_1 = \frac{n-r}{k} \frac{Y' W_0 W_0' Y}{Y' WW' Y} = \frac{n-r}{k} \left(1 + \frac{Y' W_1 W_1' Y}{Y' WW' Y}\right)$$

and
$$L = \frac{n-r}{k} \left(1 + \frac{Y' Z_1 Z_1' Y}{Y' ZZ' Y}\right).$$

$L = L_1$ with probability one if and only if

$$\frac{Y' Z_1 Z_1' Y}{Y' ZZ' Y} = \frac{Y' W_1 W_1' Y}{Y' WW' Y} \quad \forall\ Y \notin \underline{M}(X_1)$$

$$\Longleftrightarrow \qquad Y' Z_1 Z_1' Y = Y' W_1 W_1' Y \quad \forall\ Y \notin \underline{M}(X_1), \qquad \ldots(3.4.1)$$

since $WW' = I-P_{X_1} = I-P_X = ZZ'$. Also since $\underline{M}(W) = \underline{M}(Z)$,

$$Z_1' W = 0. \qquad \ldots(3.4.2)$$

Let W_2 be such that $(W : W_1 : W_2)$ is an orthogonal matrix. Putting $Y = (W : W_2)\theta$ in (3.4.1), we get $Z_1' W_2 = 0$. This together with (3.4.2) implies $\underline{M}(Z_1) \subset \underline{M}(W_1)$. Similarly one can show that $\underline{M}(W_1) \subset \underline{M}(Z_1)$. Hence we get $\underline{M}(Z_1) = \underline{M}(W_1)$ $\Longleftrightarrow Z_1 Z_1' = W_1 W_1' \Longleftrightarrow Z_1 Z_1' + ZZ' = W_1 W_1' + WW' \Longleftrightarrow I - P_{X(I-A^-A)} = I - P_{X_1(I-A^-A)} \Longleftrightarrow \underline{M}(X(I-A^-A)) = \underline{M}(X_1(I-A^-A))$. This concludes the proof of theorem 3.4.1.

Suppose X_1 is a given matrix satisfying $\underline{M}(A') \subset \underline{M}(X_1')$. Our next attempt is to characterise matrices X satisfying $\underline{M}(X) = \underline{M}(X_1)$ and $\underline{M}(X(I-A^-A)) = \underline{M}(X_1(I-A^-A))$. We need

<u>Lemma 3.4.1</u>. Let C be a given matrix and let $(R_1 : R_2)$ be an orthogonal matrix, where the columns of R_1 form an orthonormal basis for $\underline{M}(C)$. Then matrices B satisfying $\underline{M}(C) \subset \underline{M}(B)$ are given by

$$B = (R_1 : R_2) \begin{bmatrix} K_1 \\ K_2 \end{bmatrix},$$

where K_1 is any matrix of full row rank and K_2 is such that $\underline{M}(K_1') \cap \underline{M}(K_2') = \{0\}$.

<u>Proof</u> : We can always write $B = (R_1 : R_2)(K_1' : K_2')'$ for some and K_2. $\underline{M}(C) \subset \underline{M}(B)$ if and only if there exists a matrix D satisfying $(R_1 : R_2)(K_1' : K_2')' D = R_1 \Longleftrightarrow (K_1' : K_2')' D = (I : 0)'$ $\Longleftrightarrow K_1$ is a matrix of full row rank and $\underline{M}(K_1') \cap \underline{M}(K_2') = \{0\}$ This completes the proof of lemma 3.4.1.

Now let E_1' and E_2' be such that the columns of E_1' form an orthonormal basis for $\underline{M}(A')$ and those of E_2' form an orthonormal basis for the orthogonal complement of $\underline{M}(A')$. In view of lemma 3.4.1 we can write $X_1 = (S_{11} : S_{21})(E_1' : E_2')'$ and $X = (S_1 : S_2)(E_1' : E_2')'$, where S_{11} is a matrix of full column rank and $\underline{M}(S_{11}) \cap \underline{M}(S_{21}) = \{0\}$ and S_1 and S_2 satisfy analogous conditions. It is easy to see that $\underline{M}(X_1) = \underline{M}(S_{11} : S_{21})$, $\underline{M}(X_1(I-A^-A)) = \underline{M}(S_{21})$, $\underline{M}(X) = \underline{M}(S_1 : S_2)$ and $\underline{M}(X(I-A^-A)) = \underline{M}(S_2)$. From the proof of lemma 3.4.1, we see that in order to characterise X with the desired properties, we have to characterise S_1 and S_2 satisfying $S_1'R = I$, $S_2'R = 0$ for some R, $\underline{M}(S_2) = \underline{M}(S_{21})$ and $\underline{M}(S_1 : S_2) = \underline{M}(S_{11} : S_{21})$. Let $(F_1 : F_2 : F_3)$ be an orthogonal matrix such that the columns of F_2 form an orthonormal basis for $\underline{M}(S_{21})$ and those of $(F_1 : F_2)$ form an orthonormal basis for $\underline{M}(S_{11} : S_{21})$. Choose S_2, any matrix satisfying $\underline{M}(S_2) = \underline{M}(S_{21})$. Then $S_2'R = 0 \implies S_{21}'R = 0$ $\implies F_2'R = 0 \implies R = F_1K_1 + F_3K_3$ for some K_1 and K_3. Since we want $\underline{M}(S_1) \subset \underline{M}(S_{11} : S_{21})$, let $S_1 = F_1M_1 + F_2M_2$ for some M_1 and M_2. Then $S_1'R = I \implies M_1'K_1 = I$ and hence M_1 is a nonsingular matrix. Thus S_2 is any matrix satisfying $\underline{M}(S_2) = \underline{M}(S_{21})$ (i.e., $\underline{M}(X(I-A^-A)) = \underline{M}(X_1(I-A^-A))$) and S_1 is any matrix of the form $F_1M_1 + F_2M_2$ where M_1 is an arbitrary nonsingular matrix and M_2 is arbitrary. It is easy to see that S_1 and S_2 so chosen satisfy $\underline{M}(S_1 : S_2) = \underline{M}(S_{11} : S_{21})$, i.e. $\underline{M}(X) = \underline{M}(X_1)$.

Remark 3.4.1. It is not true that under the set up considered in theorem 3.4.1, the equality of L and L_1 implies the equality of the BLUE's of $A\beta$ under both the models. However, if the BLUE of $A\beta$ under $(Y, X_1\beta, \sigma^2 I)$ is unbiased for $A\beta$ under $(Y, X\beta, \sigma^2 I)$, then the equality of L and L_1 implies the equality of the BLUE's of $A\beta$ under both the models.

3.5 Specification errors in the design and dispersion matrices.

Consider the linear models $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$, which differ both in the expectation and in the dispersion of observations. Here V is a p.d. matrix. We are interested in testing the hypothesis $H_0 : A\beta = 0$, where $A\beta$ is estimable under both the models. Let $X_{10} = X_1(I-A^-A)$ and $X_0 = X(I-A^-A)$. We assume that $\underline{M}(X) = \underline{M}(X_1)$. The LRT statistics for testing H_0 under $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$ are respectively given by

$$L_1 = \frac{Y'(I-P_{X_{10}})Y}{Y'(I-P_{X_1})Y}$$

and

$$L_V = \frac{Y'V^{-1}(I-P_{X_0, V^{-1}})Y}{Y'V^{-1}(I-P_{X, V^{-1}})Y} .$$

(The scalar multiplier $\frac{n-R(X_1)}{R(A)}$ is omitted since it is common to both L_1 and L_V). We now prove

Theorem 3.5.1. Consider the linear models $(Y, X_1\beta, \sigma^2 I)$ and $(Y, X\beta, \sigma^2 V)$, where $\underline{M}(X) = \underline{M}(X_1)$ and V is positive definite.

Let L_1 and L_V be as defined above. Then $L_V = L_1$ with probability one if and only if (i) $\underline{M}(X(I-A^-A)) = \underline{M}(X_1(I-A^-A))$, and (ii) $(I-P_{X_{10}})V(I-P_{X_{10}}) = a(I-P_{X_{10}})$ for some $a > 0$, where $X_{10} = X_1(I-A^-A)$.

Proof : Let Z_o and W_o be as defined in the proof of theorem 4.4.1. Then

$$L_1 = \frac{Y'W_oW_o'Y}{Y'WW'Y} \quad \text{and} \quad L_V = \frac{Y'Z_o(Z_o'VZ_o)^{-1}Z_o'Y}{Y'Z(Z'VZ)^{-1}Z'Y} .$$

$L_V = L_1$ with probability one if and only if

$$\frac{Y'Z_o(Z_o'VZ_o)^{-1}Z_o'Y}{Y'Z(Z'VZ)^{-1}Z'Y} = \frac{Y'W_oW_o'Y}{Y'WW'Y} \quad \forall\, Y \notin \underline{M}(X_1). \quad \ldots(3.5.1)$$

Now, $Z_o'VZ_o = \begin{bmatrix} Z'VZ & Z'VZ_1 \\ Z_1'VZ & Z_1'VZ_1 \end{bmatrix}$ and the top left hand corner submatrix in $(Z_o'VZ_o)^{-1}$ is

$$(Z'VZ)^{-1} + (Z'VZ)^{-1}Z'VZ_1(Z_1'VZ_1 - Z_1'VZ(Z'VZ)^{-1}Z'VZ_1)^{-1}Z_1'VZ(Z'VZ)^{-1}.$$

Putting $Y = W\theta$ in (3.5.1) and observing that $Z_1'W = 0$, since $\underline{M}(W) = \underline{M}(Z)$, we get

$$Z_1'VZ = 0 \qquad \ldots(3.5.2)$$

Using (3.5.2), (3.5.1) simplifies to

$$\frac{Y'Z_1(Z_1'VZ_1)^{-1}Z_1'Y}{Y'Z(Z'VZ)^{-1}Z'Y} = \frac{Y'W_1W_1'Y}{Y'WW'Y} . \qquad \ldots(3.5.3)$$

Now using arguments similar to those given in the proof of theorem 3.4.1, one can show from (3.5.3) that $\underline{M}(Z_1) = \underline{M}(W_1)$ and hence $Z_1 Z_1' = W_1 W_1' \iff Z_1 Z_1' + ZZ' = W_1 W_1' + WW' \iff \underline{M}(X(I-A^-A)) = \underline{M}(X_1(I-A^-A))$. Using this observation, (3.5.1) simplifies to

$$\frac{Y' W_0 (W_0' V W_0)^{-1} W_0' Y}{Y' W (W' V W)^{-1} W' Y} = \frac{Y' W_0 W_0' Y}{Y' W W' Y} .$$

Condition (ii) given in the theorem is similar to the condition given in corollary 3.2.2 (i) and the proof can be completed along the same lines as the proof of theorem 3.2.2 and corollary 3.2.2(i)

Remark 3.5.1. Equivalent conditions on V can be derived as in the case of theorem 3.2.2 and corollary 3.2.2.

Remark 3.5.2. The observation made in remark 4.4.2 is valid here also.

CHAPTER 4

LINEAR MODELS WITH A COMMON LINEAR PART

4.1 Introduction.

So far we have considered two linear models which differ in the expectations or dispersions or both of the random vector Y and obtained conditions under which the best linear unbiased estimators and the tests of hypotheses involving estimable parametric functionals under one model remain robust in some suitable sense under the other model. Our next attempt is to study the robustness of estimators and tests when we have a class of linear models, namely linear models with a common linear part. In other words, we consider linear models whose design matrices are such that the column space of each of them contains a specified vector subspace. In practical situations,(for example when we have linear models with a constant term) we frequently encounter design matrices whose column spaces contain the vector with every element unity. Here it is natural to ask for conditions under which the simple least squares estimators of the linear parametric functionals are also corresponding best linear unbiased estimators for every such model. This problem has been considered by McElroy (1967) for full rank linear models with nonsingular covariance structure. He proved a necessary and sufficient condition on the form of the covariance matrix so that in all models having design matrices of a particular rank > 1 with every element of the first column of the design matrix unity, the simple least squares estimators are also best linear unbiased. This problem has been considered in a more general and detailed fashion by Zyskind (1969). He considered linear models with design matrice X of a particular rank r such that $\underline{M}(X)$ contains a particul vector subspace $\underline{M}(U)$ of dimension $< r$ and has exhibited general conditions on the form of the covariance matrix so that for all such models every SLSE is also corresponding BLUE. Zyski has not made any assumption regarding the rank of the design mat or the dispersion matrix. One important consequence of Zyskind' result is that if the covariance matrix satisfies his conditions then all parametric augmentations of the linear model

$$Y = U\beta + e$$

will give rise to models for which all SLSE's are also correspond ing BLUE's.Zyskind has given a number of interesting examples of linear models which enjoy this property.

We start with the set up described by Zyskind (1969). We consider design matrices X of a specified rank $r < n$ such that $\underline{M}(U) \subset \underline{M}(X)$, U being a known matrix with $R(U) < R(X) = r$ The class of such matrices X will be denoted by $C^r(U)$. In section 4.2, we characterise dispersion matrices V such that every linear representation/some linear representation of BLUE of $X\beta$ under $(Y, X\beta, V_1)$ continues to be its BLUE under $(Y, X\beta, V)$ for every $X \in C^r(U)$. Here V_1 is a given n.n.d. matrix.

In section 4.3 we consider design matrices $X \in C^r(U)$ further satisfying $M(A') \subseteq M(X')$ where, A is a given matrix of rank $k(1 \leq k \leq r)$. The class of such matrices X will be denoted by $C_A^r(U)$ and we proceed to characterise dispersion matrices V such that every linear representation/some linear representation of BLUE of $A\beta$ under $(Y, X\beta, V_1)$ continues to be its BLUE under $(Y, X\beta, V)$ for every $X \in C_A^r(U)$. It turns out, surprisingly, that if for some non-null matrix A every linear representation of the BLUE of $A\beta$ under $(Y, X\beta, V_1)$ continues to be its BLUE under $(Y, X\beta, V)$ for every $X \in C_A^r(U)$, then every linear representation of the BLUE of $X\beta$ under $(Y, X\beta, V_1)$ continues to be its BLUE under $(Y, X\beta, V)$ for every $X \in C^r(U)$.

In section 4.4 we consider the variance components model and the covariance components model. We obtain conditions under which $X\beta$ admits a BLUE under such a model for every $X \in C^r(U)$ or $A\beta$ admits a BLUE under such a model for every $X \in C_A^r(U)$, when the variance components and covariance components are unknown. For a given linear model, i.e. for a fixed X, this problem has been considered for the variance components model by Seely and Zyskind (1971) and Mitra and Moore (1973, 1976) and for the covariance components model by Mitra and Moore (1973).

The last section deals with the robustness of the likelihood ratio test statistic for testing a hypothesis $H_0 : A\beta = 0$ when we have design matrices $X \in C_A^r(U)$. Here we assume normality of the random vector Y and take up problems similar to those considered in section 3.2 of chapter 3.

4.2 BLU estimation with an incorrect dispersion matrix.

We prove

Lemma 4.2.1. If V_1 and V are non-negative definite matrices, then $\underline{M}(VX^{\perp}) \subseteq \underline{M}(V_1X^{\perp})$ for all $X \in C^r(U)$ if and only if

$$VU^{\perp} = \lambda V_1 U^{\perp} \text{ for some } \lambda \geq 0 .$$

Proof : The 'if' part is obvious. To prove the 'only if' part observe that $\underline{M}(VX^{\perp}) \subseteq \underline{M}(V_1X^{\perp})$ if and only if " $w'V_1X^{\perp} = 0 \Longrightarrow w'VX^{\perp} = 0$" which is equivalent to demanding that whenever $V_1w \in \underline{M}(X)$, $Vw \in \underline{M}(X)$ for all $X \in C^r(U)$. It is easy to see that if $V_1w \in \underline{M}(U)$ and $Vw \notin \underline{M}(U)$, then, since $r < n$, we can choose $X \in C^r(U)$ such that $V_1w \in \underline{M}(X)$ and $Vw \notin \underline{M}(X)$. Thus we necessarily have

$$\underline{M}(VU^{\perp}) \subseteq \underline{M}(V_1U^{\perp}) . \qquad \ldots(4.2.1)$$

Now, let $w_1 = U^{\perp}z$. Let the columns of C span the orthogonal complement of $\underline{M}(U : V_1w_1 : Vw_1)$ and consider the matrix $X = (U : V_1w_1 : C_1)$ where C_1 is chosen in such a way that $\underline{M}(C_1) \subseteq \underline{M}(C)$ and $R(X) = r$. For such an X, $V_1w_1 \in \underline{M}(X)$, and hence we should have $Vw_1 \in \underline{M}(X)$. Thus $Vw_1 = U\alpha_1 + \lambda V_1w_1 + C_1\alpha_2$ for some vectors α_1 and α_2 and scalar λ. From the choice of C_1, it then follows that $Vw_1 = U\alpha_1 + \lambda V_1w_1 \Longrightarrow Vw_1 = \lambda V_1w_1$, since, from (4.2.1), $Vw_1 = VU^{\perp}z \in \underline{M}(V_1U^{\perp})$ and $\underline{M}(V_1U^{\perp}) \cap \underline{M}(U) = \{0\}$. Thus, $VU^{\perp}z = \lambda V_1U^{\perp}z$ for every z and hence $VU^{\perp} = \lambda V_1 U^{\perp}$ for some $\lambda \geq 0$.

The proof of lemma 4.2.1 is now complete.

Note 4.2.1. The proof of lemma 4.2.1 is similar to the proof of theorem 1 of Zyskind (1969).

Theorem 4.2.1. Every linear representation of the BLUE of every estimable parametric functional under $(Y, X\beta, V_1)$ continues to be its BLUE under $(Y, X\beta, V)$ for every $X \in C^r(U)$ if and only if any one of the following equivalent conditions holds :

(i) $VU^{\perp} = \lambda V_1 U^{\perp}$ for some $\lambda \geq 0$

(ii) Every vector in $\underline{M}(U^{\perp})$ is an eigenvector of V with respect to V_1.

(iii) $V = \lambda V_1 + UBU'$ for some $\lambda \geq 0$ and B such that V is n.n.d.

Proof : It is well known that $w'Y$ is BLUE of its expectation under $(Y, X\beta, V_1)$ if and only if $V_1 w \in \underline{M}(X)$. Hence we want to obtain conditions on V such that "$V_1 w \in \underline{M}(X) \Longrightarrow Vw \in \underline{M}(X)$" for every $X \in C^r(U)$. In other words, we want to characterise V so that "$X^{\perp'} V_1 w = 0 \Longrightarrow X^{\perp'} Vw = 0$" or equivalently $\underline{M}(VX^{\perp}) \subset \underline{M}(V_1 X^{\perp})$ for every $X \in C^r(U)$. Part (i) of the theorem now follows from lemma 4.2.1. (ii) is a restatement of (i) and the equivalence of (i) and (iii) is easily established.

Corollary 4.2.1. (Zyskind, 1969).

Consider the linear model $(Y, X\beta, V)$ where $X \in C^r(U)$.

Then, in each of these models all SLSE's are also corresponding BLUE's if and only if any one of the following equivalent conditions holds :

(i) $VU^{-} = \lambda U^{\perp}$ for some $\lambda \geq 0$.

(ii) Every vector in $\underline{M}(U^{\perp})$ is an eigenvector of V.

(iii) $V = \lambda I + UBU'$ for some $\lambda \geq 0$ and B such that V is n.n.d.

Putting $U = 1_n = (1,1,\ldots,1)'$ the representation given in corollary 4.2.1 (iii) becomes $V = \lambda I + b\, 1_n 1_n'$ where $\lambda \geq 0$ and b is a scalar such that V is n.n.d. This representation for V has been obtained by McElroy (1967) under full rank of the design matrix and nonsingularity of the dispersion matrix.

<u>Theorem 4.2.2</u>. (a) If $\underline{M}(V_1) \subset \underline{M}(U)$, then at least one linear representation of BLUE of every estimable parametric functional under $(Y,X\beta,V_1)$ continues to be its BLUE under $(Y,X\beta,V)$ for every $X \varepsilon C^r(U)$ and for arbitrary non-negative definite V.

(b) If $\underline{M}(V_1) \not\subset \underline{M}(U)$, then at least one linear representation of BLUE of every estimable parametric functional under $(Y,X\beta,V_1)$ continues to be its BLUE under $(Y,X\beta,V)$ for every $X \varepsilon C^r(U)$ if and only if $VU^{\perp} = \lambda V_1 U^{\perp}$ for some $\lambda \geq 0$.

<u>Proof</u> : We want conditions on V so that for every $X \varepsilon C^r(U)$ there exists a matrix L which satisfies $X'L = X'$, $X^{\perp'} V_1 L = 0$ and $X^{\perp'} VL = 0$. These conditions hold if and only if

$$\underline{M}(X : 0 : 0)' \subset \underline{M}(X : V_1X^{\perp} : VX^{\perp})' \qquad \ldots(4.2.2)$$

for every $X \in C^r(U)$. If $\underline{M}(V_1) \subset \underline{M}(U)$, then $U^{\perp'}V_1 = 0$ and hence $X^{\perp'}V_1 = 0$ for every $X \in C^r(U)$. In this case (4.2.2) simplifies to $\underline{M}(X : 0)' \subset \underline{M}(X : VX^{\perp})'$ and it is easy to see that this condition holds for any n.n.d. matrix V and any $X \in C^r(U)$ This proves part (a) of the theorem. Now we proceed to prove part (b).

The 'if' part is clear, since if $VU^{\perp} = \lambda V_1U^{\perp}$, then from theorem 4.2.1, we see that every linear representation of the BLUE of $X\beta$ under $(Y, X\beta, V_1)$ continues to be its BLUE under $(Y, X\beta, V)$. To prove the 'only if' part, we have to show that if (4.2.2) holds for every $X \in C^r(U)$, then $VU^{\perp} = \lambda V_1U^{\perp}$ for some $\lambda \geq 0$. We shall first show that for (4.2.2) to hold, the condition

$$\underline{M}(VX^{\perp}) \subset \underline{M}((V_1 + V)X^{\perp}) \qquad \ldots(4.2.3)$$

is necessary. Suppose (4.2.3) does not hold. Then we can find a vector a such that

$$a'(V_1 + V)X^{\perp} = 0 \qquad \ldots(4.2.4)$$

and $$a'VX^{\perp} \neq 0 . \qquad \ldots(4.2.5)$$

Consider the vector $(a'X : 0 : 0)'$. If (4.2.2) holds, then there exists a vector b such that

$$X'b = X'a \qquad \ldots(4.2.6)$$

$$X^{\perp'}V_1b = 0 \qquad \ldots(4.2.7)$$

and $X^{\perp'} Vb = 0$ . ...(4.2.8)

Solving for b from (4.2.6) and substituting in (4.2.7) and (4.2.8), we see that for some vector c

$$X^{\perp'} V_1 a + X^{\perp'} V_1 X^{\perp} c = 0 \quad \ldots(4.2.9)$$

$$\text{and } X^{\perp'} Va + X^{\perp'} VX^{\perp} c = 0 . \quad \ldots(4.2.10)$$

Adding the above two equations and using (4.2.4), we get $X^{\perp'} (V_1 + V)X^{\perp} c = 0 \iff V_1 X^{\perp} c = 0$ and $VX^{\perp} c = 0$. Hence, from (4.2.9) and (4.2.10), we get $X^{\perp'} V_1 a = 0$ and $X^{\perp'} Va = 0$, which contradicts (4.2.5). Thus the condition (4.2.3) is necessary for (4.2.2) to hold. From lemma 4.2.1, we see that (4.2.3) holds for all $X \in C^r(U)$ if and only if

$$VU^{\perp} = \alpha(V_1 + V)U^{\perp} \quad \ldots(4.2.11)$$

for some $\alpha \geq 0$. If $\underline{M}(V_1) \not\subset \underline{M}(U)$, then $V_1 U^{\perp} \neq 0$ and from (4.2.11) we get

$$(1 - \alpha)VU^{\perp} = \alpha V_1 U^{\perp} . \quad \ldots(4.2.12)$$

Hence $\alpha < 1$, since $V_1 U^{\perp} \neq 0$ and $\alpha \geq 0$. From (4.2.12), we get $VU^{\perp} = \lambda V_1 U^{\perp}$, where $\lambda = \frac{\alpha}{1-\alpha} \geq 0$, which completes the proof of part (b) of the theorem.

In view of theorem 4.2.1 and theorem 4.2.2 (b), we have

Corollary 4.2.2. If $\underline{M}(V_1) \not\subset \underline{M}(U)$, then at least one linear representation of BLUE of every estimable parametric functional under $(Y, X\beta, V_1)$ continues to be its BLUE under $(Y, X\beta, V)$ for every $X \in C^r(U)$ if and only if every linear representation is so.

Now let G_X be any least squares g-inverse of $V_1 + XX'$ and consider the linear representation $X(X'G_XX)^- X'G_XY$ of the BLUE of $X\beta$ under $(Y,X\beta,V_1)$. If $\underline{M}(V_1) \not\subset \underline{M}(U)$, then from corollary 4.2.2 it follows that if $X(X'G_XX)^- X'G_XY$ is BLUE of $X\beta$ under $(Y,X\beta,V)$ for every $X \in C^r(U)$, then every linear representation will be so and $VU^{\perp} = \lambda V_1 U^{\perp}$ for some $\lambda \geq 0$. Now we assume that $\underline{M}(V_1) \subset \underline{M}(U)$ and we shall obtain conditions on V so that $X(X'G_XX)^- X'G_XY$ is BLUE of $X\beta$ under $(Y,X\beta,V)$ for every $X \in C^r(U)$.

Theorem 4.2.3. Suppose $\underline{M}(V_1) \subset \underline{M}(U)$ and let G_X be a least squares g-inverse of $V_1 + XX'$. Then $X(X'G_XX)^- X'G_XY$ is BLUE of $X\beta$ under $(Y,X\beta,V)$ for every $\lambda \in C^r(U)$ if and only if $VU^{\perp} = \lambda U^{\perp}$ for some $\lambda \geq 0$.

Proof : Since we have assumed that $\underline{M}(V_1) \subset \underline{M}(U) \subset \underline{M}(X)$, $\underline{M}(V_1+XX') = \underline{M}(X)$ for every $X \in C^r(U)$. Also, since G_X is a least squares g-inverse of $V_1 + XX'$, $(V_1 + XX')G_X$ is symmetric. We want conditions on V so that

$$X'G_XVX^{\perp} = 0 \quad \text{for every } X \in C^r(U)$$

$\iff$ $(V_1+XX')G_XVX^{\perp} = 0$ for every $X \in C^r(U)$, using the fact that $\underline{M}(V_1+XX') = \underline{M}(X)$.

$\iff$ $G_X'(V_1+XX')VX^{\perp} = 0$ for every $X \in C^r(U)$, using the fact that $(V_1+XX')G_X$ is symmetric.

$\iff$ $X'VX^{\perp} = 0$ for every $X \in C^r(U)$ (premultiplying by X').

$\iff$ $\underline{M}(VX^{\perp}) \subset \underline{M}(X^{\perp})$ for every $X \in C^r(U)$.

$\Longleftrightarrow$ $VU^{\perp} = \lambda U^{\perp}$ for some $\lambda \geq 0$, using lemma 4.2.1.

Theorem 4.2.3 is thus established.

Corollary 4.2.3. Suppose $\underline{M}(V_1) \subset \underline{M}(U)$ and let $G_1 = (V_1+XX')^{-}$. Then $X(X'G_1X)^{-}X'G_1Y$ is BLUE of $X\beta$ under $(Y,X\beta,V)$ for every $X \varepsilon C^r(U)$ if and only if $VU^{\perp} = \lambda U^{\perp}$ for some $\lambda \geq 0$.

4.3 Optimality of BLUE's of a subclass of parametric functionals.

Recall that $C_A^r(U)$ denotes the class of matrices $X \varepsilon C^r(U)$, which further satisfy $\underline{M}(A') \subset \underline{M}(X')$ where A is a given matrix of rank $k(1 \leq k \leq r)$ and $R(U) < r = R(X)$. In this section our purpose is to characterise V such that every linear representation or some linear representation of BLUE of $A\beta$ under $(Y,X\beta,V_1)$ continues to be its BLUE under $(Y,X\beta,V)$ for every $X \varepsilon C_A^r(U)$. First we shall prove

Lemma 4.3.1. Let A,U and $C_A^r(U)$ be as defined before. Then for non-negative definite matrices V_1 and V,

$$\underline{M}\begin{bmatrix} AX^+ \\ I-XX^+ \end{bmatrix} VX^{\perp} \subset \underline{M}\begin{bmatrix} AX^+ \\ I-XX^+ \end{bmatrix} V_1X^{\perp}$$

for every $X \varepsilon C_A^r(U)$ if and only if $VU^{\perp} = \lambda V_1U^{\perp}$ for some $\lambda \geq$

Proof : The 'if' part is obvious, since, if $VU^{\perp} = \lambda V_1U^{\perp}$, then $VX^{\perp} = \lambda V_1X^{\perp}$ for every $X \varepsilon C_A^r(U)$ and consequently

$$\begin{bmatrix} AX^+ \\ I-XX^+ \end{bmatrix} VX^{\perp} = \lambda \begin{bmatrix} AX^+ \\ I-XX^+ \end{bmatrix} V_1X^{\perp} .$$

We shall now prove the 'only if' part.

Let $(E_1' : E_2')$ be an orthogonal matrix such that the columns of E_1' form an orthonormal basis for $\underline{M}(A')$. Then using lemma 3.4.1, every X with $\underline{M}(A') \subset \underline{M}(X')$ can be written as $X = (S_1 : S_2)(E_1' : E_2')'$ where S_1 is an $n \times k$ matrix of rank k and $\underline{M}(S_1) \cap \underline{M}(S_2) = \{0\}$. Then $\underline{M}(X) = \underline{M}(S_1 : S_2)$. Using the facts that $(S_1 : S_2)^{\perp}$ is $X^{\perp}$, $X^+ = (E_1' : E_2')(S_1 : S_2)^+$, $E_1E_1' = I$, $E_1E_2' = 0$ and $\underline{M}(I - XX^+) = M(X^{\perp}) = \underline{M}((S_1 : S_2)^{\perp})$, we see that

$$\underline{M}\begin{bmatrix} AX^+ \\ I-XX^+ \end{bmatrix} VX^{\perp} \subset \underline{M}\begin{bmatrix} AX^+ \\ I-XX^+ \end{bmatrix} V_1X^{\perp} \text{ if and only if}$$

$$\underline{M}\begin{bmatrix} (I:0)(S_1 : S_2)^+ \\ (S_1 : S_2)^{\perp'} \end{bmatrix} V(S_1 : S_2)^{\perp} \subset \underline{M}\begin{bmatrix} (I:0)(S_1 : S_2)^+ \\ (S_1 : S_2)^{\perp'} \end{bmatrix} V_1(S_1 : S_2)^{\perp}. \quad \ldots(4.3.1)$$

We want conditions under which (4.3.1) holds for all S_1 and S_2 with $\underline{M}(U) \subset \underline{M}(S_1 : S_2)$, S_1 is of full column rank and $\underline{M}(S_1) \cap \underline{M}(S_2) = \{0\}$. Choose S_2 so that $S_1'S_2 = 0$. Then $(S_1 : S_2)^+ = (S_1^{+'} : S_2^{+'})'$ and (4.3.1) implies

$$\underline{M}\begin{bmatrix} S_1' \\ (S_1 : S_2)^{\perp'} \end{bmatrix} V(S_1 : S_2)^{\perp} \subset \underline{M}\begin{bmatrix} S_1' \\ (S_1 : S_2)^{\perp'} \end{bmatrix} V_1(S_1 : S_2)^{\perp} \quad ..(4.3.2)$$

where S_1 is of full column rank, $S_1' S_2 = 0$ and $\underline{M}(U) \subset \underline{M}(S_1 : S_2)$. If $R(A) = r = R(X)$, then $S_2 = 0$ and (4.3.2) implies $\underline{M}((S_1 : S_1^{\perp})' VS_1^{\perp}) \subset \underline{M}((S_1 : S_1^{\perp})' V_1S_1^{\perp})\ \forall\ S_1 \in C^r(U)$ or equivalently, $\underline{M}(VS_1^{\perp}) \subset \underline{M}(V_1S_1^{\perp})\ \forall\ S_1 \in C^r(U)$ and the result follows from lemma 4.2.1. In the rest of the proof, we assume that $R(A) < r$.

Case 1. $R(U) > 1$, $R(A) > 1$.

Let $(U_1 : U_2)$ and $(U^{\perp}_{(1)} : U^{\perp}_{(2)})$ be semiorthogonal matrices such that $\underline{M}(U) = \underline{M}(U_1 : U_2)$ and $\underline{M}(U^{\perp}) = \underline{M}(U^{\perp}_{(1)} : U^{\perp}_{(2)})$. Choose $S_1 = (U_1 : U^{\perp}_{(1)})$ and $\underline{M}(S_2) = \underline{M}(U_2)$. (The partitioning of U and $U^{\perp}$ and the choice of S_1 and S_2 are done in such a way that S_1 is of full column rank, $S_1'S_2 = 0$ and $R(S_1 : S_2) = r$) With this choice of S_1 and S_2, (4.3.2) implies

$$\underline{M}((U_1 : U^{\perp}_{(1)} : U^{\perp}_{(2)})' V U^{\perp}_{(2)}) \subset \underline{M}((U_1 : U^{\perp}_{(1)} : U^{\perp}_{(2)})' V_1 U^{\perp}_{(2)})$$

or equivalently

$$\underline{M}((U_1 : U^{\perp})' V U^{\perp}_{(2)}) \subset \underline{M}((U_1 : U^{\perp})' V_1 U^{\perp}_{(2)}) \qquad \ldots(4.3.3)$$

for every semiorthogonal matrix U_1 of appropriate rank satisfying $\underline{M}(U_1) \subset \underline{M}(U)$. Suppose $R(U_1) = q$. We shall show that (4.3.3) holds for every U_1 satisfying $R(U_1) = q$, $U_1'U_1 = I_q$ and $\underline{M}(U_1) \subset \underline{M}(U)$ if and only if

$$\underline{M}((U : U^{\perp})' V U^{\perp}_{(2)}) \subset \underline{M}((U : U^{\perp})' V_1 U^{\perp}_{(2)}). \qquad \ldots(4.3.4)$$

If (4.3.4) does not hold, we can find vectors a and b satisfying

$$\begin{aligned} &(a'U' + b'U^{\perp\prime}) V_1 U^{\perp}_{(2)} = 0 \\ \text{and} \quad &(a'U' + b'U^{\perp\prime}) V U^{\perp}_{(2)} \neq 0. \end{aligned} \qquad \ldots(4.3.5)$$

Choose a matrix K such that $R(UK) = q$ and $a = Kc$ for some vector c. Let U_1 be a semiorthogonal matrix such that $\underline{M}(U_1) = \underline{M}(UK)$ and let d be such that $UKc = U_1 d$. Then, since $a = Kc$ and $UKc = U_1 d$, from (4.3.5) we get

$$(c'K'U'+b'U^{\perp'})V_1U^{\perp}_{(2)} = 0 \text{ and } (c'K'U'+b'U^{\perp'})VU^{\perp}_{(2)} \neq 0$$

$$\iff (d'U_1'+b'U^{\perp'})V_1U^{\perp}_{(2)} = 0 \text{ and } (d'U_1'+b'U^{\perp'})VU^{\perp}_{(2)} \neq 0$$

which means, there exists a U_1 for which (4.3.3) does not hold. Thus if (4.3.3) holds for every semiorthogonal matrix U_1 of rank q with $\underline{M}(U_1) \subset \underline{M}(U)$, then (4.3.4) holds and (4.3.4) is equivalent to the condition $\underline{M}(VU^{\perp}_{(2)}) \subset \underline{M}(V_1U^{\perp}_{(2)})$ for every $U^{\perp}_{(2)}$ of a specified rank (less then $R(U^{\perp})$) satisfying $\underline{M}(U^{\perp}_{(2)}) \subset \underline{M}(U^{\perp})$. Applying lemma 4.2.1, we get $VU^{\perp} = \lambda V_1U^{\perp}$ for some $\lambda \geq 0$.

Case 2. $R(U) = 1$, $R(A) > 1$.

We take U to be a column vector and let $(U^{\perp}_{(1)} : U^{\perp}_{(2)} : U^{\perp}_{(3)})$ be a semiorthogonal matrix with column space same as that of $U^{\perp}$. Choosing $S_1 = (U : U^{\perp}_{(1)})$ and $\underline{M}(S_2) = \underline{M}(U^{\perp}_{(3)})$ in (4.3.2), we get

$$\underline{M}((U : U^{\perp}_{(1)} : U^{\perp}_{(2)})'VU^{\perp}_{(2)}) \subset \underline{M}((U : U^{\perp}_{(1)} : U^{\perp}_{(2)})'V_1U^{\perp}_{(2)}) \,. \qquad \ldots(4.3.6)$$

ng arguments similar to those given in case 1, we see that (4.3.6) holds for every semiorthogonal matrix $(U^{\perp}_{(1)} : U^{\perp}_{(2)} : U^{\perp}_{(3)})$ ng the same column space as that of $U^{\perp}$ if and only if $((U : U^{\perp})'VU^{\perp}_{(2)}) \subset \underline{M}((U : U^{\perp})'V_1U^{\perp}_{(2)})$ or equivalently

$\underline{M}(VU^{\perp}_{(2)}) \subset \underline{M}(V_1U^{\perp}_{(2)})$ for every $U^{\perp}_{(2)}$ of a specified rank (less than $R(U^{\perp})$) satisfying $\underline{M}(U^{\perp}_{(2)}) \subset \underline{M}(U^{\perp})$. The result now follows by applying lemma 4.2.1.

Case 3. $R(A) = 1$.

Let U_i and $U^{\perp}_{(i)}$ $(i = 1,2)$ be as defined in case 1 where U_1 is a single column. Further, let $\underline{M}(U^{\perp}_{(1)}) = \underline{M}(U^{\perp}_{(1),1} : U^{\perp}_{(1),2})$ where $U^{\perp}_{(1),1}$ is a single column and $(U^{\perp}_{(1),1} : U^{\perp}_{(1),2})$ is a semiorthogonal matrix. Choosing $S_1 = U_1$ and $\underline{M}(S_2) = \underline{M}(U_2 : U^{\perp}_{(1)})$ in (4.3.2), we get $\underline{M}((U_1 : U^{\perp}_{(2)})' VU^{\perp}_{(2)}) \subset \underline{M}((U_1 : U^{\perp}_{(2)})' V_1U^{\perp}_{(2)})$ for every $U_1 \in \underline{M}(U)$, or equivalently

$$\underline{M}((U : U^{\perp}_{(2)})' VU^{\perp}_{(2)}) \subset \underline{M}((U : U^{\perp}_{(2)})' V_1U^{\perp}_{(2)}) \; . \qquad \ldots(4.3.7)$$

Now choosing $S_1 = U^{\perp}_{(1),1}$ and $\underline{M}(S_2) = \underline{M}(U : U^{\perp}_{(1),2})$ in (4.3.2), we get $\underline{M}((U^{\perp}_{(1),1} : U^{\perp}_{(2)})' VU^{\perp}_{(2)}) \subset \underline{M}((U^{\perp}_{(1),1} : U^{\perp}_{(2)})' V_1U^{\perp}_{(2)})$ for every vector $U^{\perp}_{(1),1} \in \underline{M}(U^{\perp}_{(1)})$, or equivalently

$$\underline{M}((U^{\perp}_{(1)} : U^{\perp}_{(2)})' VU^{\perp}_{(2)}) \subset \underline{M}((U^{\perp}_{(1)} : U^{\perp}_{(2)})' V_1U^{\perp}_{(2)}) \qquad \ldots(4.3.8)$$

(4.3.7) and (4.3.8) together imply $\underline{M}((U : U^{\perp})' VU^{\perp}_{(2)}) \subset \underline{M}((U : U^{\perp})' V_1U^{\perp}_{(2)})$ and the result follows by applying lemma 4.2.1. The proof of lemma 4.3.1 is now complete.

Remark 4.3.1. Lemma 4.3.1 remains true if we replace the matrix

$\begin{bmatrix} AX^+ \\ I-XX^+ \end{bmatrix}$ by $\begin{bmatrix} AX^-_{\ell} \\ I-XX^-_{\ell} \end{bmatrix}$, where X^-_{ℓ} is any least squares g-inverse of X, since, $XX^+ = P_X = XX^-_\ell$ and $AX^+ = AX^-_\ell$ using the fact that A satisfies $\underline{M}(A') \subseteq \underline{M}(X')$.

<u>Theorem 4.3.1</u>. Every linear representation of BLUE of $A\beta$ under $(Y, X\beta, V_1)$ is its BLUE under $(Y, X\beta, V)$ for every $X \in C_A^r(U)$ if and only if $VX^\perp = \lambda V_1 X^\perp$ for some $\lambda \geq 0$.

<u>Proof</u> : The 'if' part is clear in view of theorem 4.2.1. We shall now prove the 'only if' part. We want conditions on V so that the conditions

$$X'\ell = A'\mu \qquad \ldots(4.3.9)$$

$$\text{and} \quad X^{\perp'} V_1 \ell = 0 \qquad \ldots(4.3.10)$$

$$\text{imply} \quad X^{\perp'} V \ell = 0 . \qquad \ldots(4.3.11)$$

for every vector μ and for every $X \in C_A^r(U)$. From (4.3.9), we get $\ell = X'^+ A'\mu + (I - XX^+)z$, for some z. Hence from (4.3.10), and (4.3.11), we see that the problem reduces to characterising V so that

$$\text{"} (\mu' : z') \begin{bmatrix} AX^+ \\ I-XX^+ \end{bmatrix} V_1 X^\perp = 0 \implies (\mu' : z') \begin{bmatrix} AX^+ \\ I-XX^+ \end{bmatrix} VX^\perp = 0 \text{"}$$

which is equivalent to saying that

$$\underline{M} \begin{bmatrix} AX^+ \\ I-XX^+ \end{bmatrix} VX^\perp \subseteq \underline{M} \begin{bmatrix} AX^+ \\ I-XX^+ \end{bmatrix} V_1 X^\perp \quad \text{for every } X \in C_A^r(U) .$$

The theorem now follows from lemma 4.3.1. Putting $U = 0$, we get

<u>Corollary 4.3.1</u>. Every linear representation of BLUE of $A\beta$ under $(Y, X\beta, V_1)$ is its BLUE under $(Y, X\beta, V)$ for every matrix X of

rank r satisfying $\underline{M}(A') \subset \underline{M}(X')$ if and only if $V = \lambda V_1$.

It is interesting to note that the conditions on V stated in theorem 4.3.1 doesn't involve the matrix A. Thus we have

Corollary 4.3.2. If for some non-null matrix A every linear representation of BLUE of $A\beta$ under $(Y, X\beta, V_1)$ continues to be its BLUE under $(Y, X\beta, V)$ for every $X \varepsilon C_A^r(U)$, then every linear representation of BLUE of $X\beta$ under $(Y, X\beta, V_1)$ continues to be its BLUE under $(Y, X\beta, V)$ for every $X \varepsilon C^r(U)$.

Theorem 4.3.2. (a) If $\underline{M}(V_1) \subset \underline{M}(U)$ then atleast one linear representation of BLUE of $A\beta$ under $(Y, X\beta, V_1)$ is its BLUE under $(Y, X\beta, V)$ for every $X \varepsilon C_A^r(U)$ and for arbitrary V.

(b) If $\underline{M}(V_1) \not\subset \underline{M}(U)$, then at least one linear representation of BLUE of $A\beta$ under $(Y, X\beta, V_1)$ is its BLUE under $(Y, X\beta, V)$ for every $X \varepsilon C_A^r(U)$ if and only if $VU^{\perp} = \lambda V_1 U^{\perp}$ for some $\lambda \geq 0$.

Proof : We want conditions on V under which there exists a matrix L which satisfies $X'L = A'$, $X^{\perp'} V_1 L = 0$ and $X^{\perp'} VL = 0$ for every $X \varepsilon C_A^r(U)$, which are equivalent to the condition

$$\underline{M}(A : 0 : 0)' \subset \underline{M}(X : V_1 X^{\perp} : VX^{\perp})' \qquad \ldots(4.3.12)$$

for every $X \varepsilon C_A^r(U)$. The proof of part (a) of the theorem is completed along the same lines as that of theorem 4.2.2(a). Using arguments similar to those given in the proof of theorem 4.2.2(b) one can show that a necessary condition for (4.3.12) to hold is

$$\underline{M}\begin{bmatrix} AX^{+} \\ I-XX^{+} \end{bmatrix} VX^{\perp} \subset \underline{M}\begin{bmatrix} AX^{+} \\ I-XX^{+} \end{bmatrix} (V_1+V)X^{-} .$$

The rest of the proof is similar to that of theorem 4.2.2(b)' in view of lemma 4.3.1. From theorem 4.3.2(b), theorem 4.3.1 and corollary 4.3.2, we have

Corollary 4.3.3. Suppose $\underline{M}(V_1) \not\subset \underline{M}(U)$ and let A be a non-null matrix. Then, if some linear representation of BLUE of $A\beta$ under $(Y,X\beta,V_1)$ is its BLUE under $(Y,X\beta,V)$ for every $X \in C^r_A(U)$, then every linear representation is so and also every linear representation of BLUE of $X\beta$ under $(Y,X\beta,V_1)$ is its BLUE under $(Y,X\beta,V)$ for every $X \in C^r(U)$.

4.4 The variance components model and the covariance components model.

First we shall consider the variance components model, i.e. a linear model with $E(Y) = X\beta$ and $D(Y) = \sum_{i=1}^{p} \sigma_i^2 V_i$, where the σ_i^2 are unknown positive parameters and V_i are known n.n.d. matrices $(i = 1,2,\ldots,p)$. Our purpose is to obtain conditions on $V_i (i = 1,2,\ldots,p)$ so that $X\beta$ admits a BLUE under the model $(Y,X\beta, \sum_{i=1}^{p} \sigma_i^2 V_i)$, for every $X \in C^r(U)$. We prove

Theorem 4.4.1. $X\beta$ admits a BLUE under the model $(Y,X\beta, \sum_{i=1}^{p} \sigma_i^2 V_i)$ for every $X \in C^r(U)$ if and only if for $i = 1,2,\ldots,p$ $V_i U^{\perp} = \lambda_i V_0 U^{\perp}$ for some $\lambda_i \geq 0$ where $V_0 = \sum_{i=1}^{p} V_i$.

<u>Proof</u> : $X\beta$ admits a BLUE under $(Y, X\beta, \sum_{i=1}^{p} \sigma_i^2 V_i)$ if and only if there exists L satisfying $X'L = X'$ and $X^{\perp'} V_i L = 0$ for $i = 1, 2, \ldots, p$. Using arguments similar to those given in the proof of theorem 4.2.2 (b) it can be shownthat a necessary condition for the above to hold is $\underline{M}(V_i X^{\perp}) \subset \underline{M}(V_o X^{\perp})$, $i = 1, 2, \ldots, p$. The 'only if' part of the theorem now follows from lemma 4.2.1. To prove the 'if' part, observe that if $V_i U^{\perp} = \lambda_i V_o U^{\perp}$ for $i = 1, 2, \ldots, p$ then $X(X'G_oX)^- X'G_oY$ is BLUE of $X\beta$ under $(Y, X\beta, \sum_{i=1}^{p} \sigma_i^2 V_i)$, where G_o is a g-inverse of $V_o + XX'$.

<u>Remark 4.4.1</u>. If at least one of the V_i's satisfies $\underline{M}(V_i) \subset \underline{M}(U)$, then $\sum_{i=1}^{p} \lambda_i = 1$.

<u>Remark 4.4.2</u>. When $X\beta$ admits a BLUE under $(Y, X\beta, \sum_{i=1}^{p} \sigma_i^2 V_i)$ for every $X \in C^r(U)$ its BLUE could be computed as $X(X'G_oX)^- X'G_oY$, where G_o is any g-inverse of $V_o + XX'$.

<u>Corollary 4.4.1</u>. If $\underline{M}(V_i) \subset \underline{M}(U)$ for $i = 1, 2, \ldots, p$ or if $\underline{M}(V_i) \subset \underline{M}(U)$ for $i = 2, \ldots, p$ then $X\beta$ always admits a BLUE under $(Y, X\beta, \sum_{i=1}^{p} \sigma_i^2 V_i)$ for every $X \in C^r(U)$.

We now state

<u>Theorem 4.4.2</u>. If A is a non-null matrix, then $A\beta$ admits a BLUE under $(Y, X\beta, \sum_{i=1}^{p} \sigma_i^2 V_i)$ for every $X \in C_A^r(U)$ if and only if for $i = 1, 2, \ldots, p$ $V_i U^{\perp} = \lambda_i V_o U^{\perp}$ for some $\lambda_i \geq 0$, where

$V_o = \sum_{i=1}^{p} V_i$. Hence, if $\lambda\beta$ admits a BLUE under $(Y, X\beta, \sum_{i=1}^{p} \sigma_i^2 V_i)$ for every $X \in C_\lambda^r(U)$, then $\lambda\beta$ admits a BLUE under $(Y, X\beta, \sum_{i=1}^{p} \sigma_i^2 V_i)$ for every $X \in C^r(U)$. When $\lambda\beta$ admits a BLUE under $(Y, X\beta, \sum_{i=1}^{p} \sigma_i^2 V_i)$ for every $X \in C_\lambda^r(U)$, its BLUE could be computed as $\lambda(X' G_o X)^- X' G_o Y$, G_o being an arbitrary g-inverse of $V_o + XX'$.

The proof of the theorem is omitted.

We now turn our attention to the covariance components model, namely, a linear model with $E(Y) = X\beta$ and $D(Y) = \sum_{\eta=1}^{p} U_\eta' W U_\eta$, $U_1, U_2, \ldots, U_p$ being known $s \times n$ matrices and $W = ((w_{ij}))$ is an unknown n.n.d. matrix of order $s \times s$. We will consider the following canonical form of the covariance components model due to Mitra and Moore (1973) :

For $i \leq j = 1, 2, \ldots, s$ let the column vectors e_{ij} be defined as follows. e_{ii} is a vector with its i^{th} coordinate equal to 1 and other $s-1$ coordinates equal to zero. $e_{ij} = e_{ii} + e_{jj}$ if $i < j$. Write $B_{ij} = e_{ij} e_{ij}'$. Then the $\frac{s(s+1)}{2}$ matrices B_{ij} provide a basis for the linear space spanned by real symmetric matrices of order $s \times s$ and we can write $D(Y) = \sum_{i \leq j} \overline{w}_{ij} V_{ij}$, where $\overline{w}_{ii} = 2w_{ii} - \sum_j w_{ij}$, $\overline{w}_{ij} = w_{ij}$ for $i < j$ and $V_{ij} = \sum_{\eta=1}^{p} U_\eta' B_{ij} U_\eta$ is n.n.d. for each ij. Similar to theorem 4.4.1 and theorem 4.4.2, we now have

Theorem 4.4.3. Let V_{ij} be as defined above. Then $X\beta$ admits a BLUE under $(Y, X\beta, \sum_{\eta=1}^{p} U_\eta' W U_\eta)$ for every $X \in C^r(U)$ if and only if for each $i \leq j = 1,2,\ldots,s$, $V_{ij}U^\perp = \lambda_{ij} V_o U^\perp$ for some $\lambda_{ij} \geq 0$, where $V_o = \sum_{i \leq j} V_{ij}$. The same condition is also necessary and sufficient for $A\beta$ to admit a BLUE under $(Y, X\beta, \sum_{\eta=1}^{p} U_\eta' W U_\eta)$ for every $X \in C_A^r(U)$, A being a non-null matrix. When $X\beta$ admits a BLUE under $(Y, X\beta, \sum_{\eta=1}^{p} U_\eta' W U_\eta)$ for every $X \in C^r(U)$, (or $A\beta$ admits a BLUE under $(Y, X\beta, \sum_{\eta=1}^{p} U_\eta' W U_\eta)$ for every $X \in C_A^r(U)$) its BLUE could be computed as $X(X' G_o X)^- X' G_o Y$ (respectively $A(X' G_o X)^- X' G_o Y$) where G_o is any g-inverse of $V_o + XX'$.

We now consider the general linear model $(Y, X\beta, V)$ where V is any n.n.d. matrix in the p-dimensional linear space $L(V_1, V_2, \ldots, V_p)$ spanned by the n.n.d. matrices $V_1, V_2, \ldots, V_p$ (known). Thus V can be expressed as $V = \sum_{i=1}^{p} a_i V_i$, where all the a_i's may not be non-negative. Such a model has been considered by Mitra and Moore (1976). We state

Theorem 4.4.4. Let V be any n.n.d. matrix in $L(V_1, V_2, \ldots, V_p)$. Then $X\beta$ admits a BLUE under $(Y, X\beta, V)$ for every $X \in C^r(U)$ if and only if for $i = 1,2,\ldots,p$ $V_i U^\perp = \lambda_i V_o U^\perp$ for some $\lambda_i \geq 0$, where $V_o = \sum_{i=1}^{p} V_i$. The same condition is also necessary and sufficient for $A\beta$ to admit a BLUE under $(Y, X\beta, V)$ for every $X \in C_A^r(U)$, A being a non-null matrix.

Remark 4.4.2. Just as in the case of the variance components model or the covariance components model, when $X\beta$ admits a BLUE under $(Y, X\beta, V)$ for every $X \in C^r(U)$, V being any n.n.d. matrix in $L(V_1, V_2, \ldots, V_p)$, one choice of its BLUE is given by $X(X'G_oX)^- X'G_oY$, G_o being any g-inverse of $V_o + XX'$. Mitra and Moore (1976) have pointed out that even if $X\beta$ doesn't admit a BLUE under $(Y, X\beta, V)$, the estimator $\ell' X(X'G_oX)^- X'G_oY$ is an admissible estimator of $\ell' X\beta$, i.e. no linear unbiased estimator of $\ell' X\beta$ can be better than $\ell' X(X'G_oX)^- X'G_oY$.

Remark 4.4.3. Putting $U = 0$ in theorem 4.4.1 - theorem 4.4.4, one can see that $X\beta$ admits a BLUE under any of the models described in this section for every X of a particular rank $r < n$ (or $A\beta$ admits a BLUE under any of the models described in this section for every X of a particular rank $r < n$ satisfying $\underline{M}(A') \subset \underline{M}(X')$, A being a non-null matrix) if and only if the dispersion of Y is known either completely or upto a positive scalar multiplier.

As an example, consider the dispersion matrix $V_{a,b} = aI + b\, 1_n 1_n'$, where a and b are any two real numbers which make $V_{a,b}$ n.n.d. and 1_n is the column vector with each element unity. If $U = 1_n$, $V_1 = I$ and $V_2 = 1_n 1_n'$, then it is clear that $V_1 U^{\perp} = (V_1 + V_2) U^{\perp}$ and $V_2 U^{\perp} = 0$. Applying theorem 4.4.4, we see that $X\beta$ admits a BLUE under the model $(Y, X\beta, V_{a,b})$ for every $X \in C^r(1_n)$ and the BLUE can be computed as

$$X(X'(I+1_n 1_n')^{-1}X)^- X'(I+1_n 1_n')^{-1}\, Y.$$

4.5 Robustness of the LRT statistic.

Here we consider the linear model $(Y, X\beta, V)$, where Y has a multivariate normal distribution, V is a p.d. matrix and $X \in C_A^r(U)$, A being a non-null matrix. Our purpose is to obtain conditions on V so that the LRT statistic for testing $H_o : A\beta = 0$ under $(Y, X\beta, V)$ is same as the LRT statistic for testing H_o under $(Y, X\beta, I)$ for every $X \in C_A^r(U)$. We now show that $V = \sigma^2 I$ is the only dispersion matrix with this property.

Theorem 4.5.1. Consider the linear model $(Y, X\beta, V)$, where Y has a multivariate normal distribution and V is a positive definite matrix. Let A be a non-null matrix. Then, the LRT statistic for testing $H_o : A\beta = 0$ under $(Y, X\beta, V)$ is same as the LRT statistic for testing H_o under $(Y, X\beta, I)$ for every $X \in C_A^r(U)$ if and only if $V = \sigma^2 I$.

Proof : From corollary 3.2.2 (i) we see that we have to characterise p.d. matrices V which satisfy

$$(I-P_{X_o})V(I-P_{X_o}) = a(I-P_{X_o}) \qquad \ldots(4.5.1)$$

for some $a > 0$, where $X_o = X(I-A^- A)$. Let $X = (S_1 \vdots S_2)(E_1' \vdots E_2')'$ where S_1, S_2, E_1 and E_2 are as defined in the proof of lemma 4.3.1. Then $X_o = S_2 E_2$ and $P_{X_o} = P_{S_2}$. Thus (4.5.1) holds if and only if

$$(I-P_{S_2})V(I-P_{S_2}) = a(I-P_{S_2}) \text{ for some } a > 0. \qquad \ldots(4.5.2)$$

If $R(A) = R(X)$ then $S_2 = 0$ and the result is immediate from

(4.5.2). If $U = 0$, then (4.5.2) should hold for every S_2 with $R(S_2) = r - R(A)$ and choosing S_2 suitably, we can show that (4.5.2) implies $V = aI$. We now proceed under the assumption that $R(A) < R(X)$ and $R(U) \geq 1$.

For $i = 1,2$ let U_i and $U^{\perp}_{(i)}$ be semiorthogonal matrices so that $\underline{M}(U) = \underline{M}(U_1 : U_2)$ and $\underline{M}(U^{\perp}) = \underline{M}(U^{\perp}_{(1)} : U^{\perp}_{(2)})$. By lemma 3.2.4, equality of the LRT statistics for testing H_0 implies equality of the BLUE's of $A\beta$ and hence from theorem 4.3.1, we necessarily have $V = \lambda I + UBU'$. Choosing S_2 with $\underline{M}(S_2) = \underline{M}(U_2 : U^{\perp}_{(2)})$ in (4.5.2), we get

$$(U_1U_1' + U^{\perp}_{(1)}U^{\perp'}_{(1)})UBU'(U_1U_1' + U^{\perp}_{(1)}U^{\perp'}_{(1)}) = (a-\lambda)(U_1U_1' + U^{\perp}_{(1)}U^{\perp'}_{(1)})$$

$$\Longrightarrow \quad U_1U_1'UBU'U_1U_1' = (a-\lambda)U_1U_1' \quad \text{and} \quad 0 = (a-\lambda)U^{\perp}_{(1)}U^{\perp'}_{(1)}$$

$$\Longrightarrow \quad a-\lambda = 0 \quad \text{and} \quad U_1'UBU'U_1 = 0 \Longrightarrow UBU' = 0 \Longrightarrow V = \lambda I \quad \text{for}$$

some $\lambda > 0$.

The proof of the theorem is thus complete.

Now, let $L_V(X)$ and $L(X)$ respectively denote the LRT statistics for testing $H_0 : A\beta = 0$ under $(Y, X\beta, V)$ and $(Y, X\beta, I)$, where X is any matrix in $C^r_A(U)$. Our purpose is to obtain conditions under which $L_V(X) - L(X) \geq 0$ (or ≤ 0) with probability one for every $X \in C^r_A(U)$. We have been able to derive the result only when V is of the form $V = \lambda I + UBU'$, where $\lambda > 0$ and B is arbitrary subject to the condition that V is p.d., i.e. the BLUE of $A\beta$ under $(Y, X\beta, V)$ is same as its SLSE for every $X \in C^r_A(U)$. We prove

<u>Theorem 4.5.2</u>. Let $L_V(X)$ and $L(X)$ be as defined above where V is any positive definite matrix of the form $V = \lambda I + UBU'$. Then

(a) $L_V(X) - L(X) \geq 0$ with probability one for every $X \varepsilon C_A^r(U)$ if and only if $\lambda I - V$ is non-negative definite, or equivalently $V = \lambda I - UCC'U'$, where C is arbitrary subject to the condition that V is p.d.

(b) $L_V(X) - L(X) \leq 0$ with probability one for every $X \varepsilon C_A^r(U)$ if and only if $V - \lambda I$ is non-negative definite, or equivalently $V = \lambda I + UCC'U'$, where C is arbitrary.

<u>Proof</u> : Let $X = (S_1 : S_2)\,(E_1' : E_2')'$ where S_1, S_2, E_1 and E_2 are as defined in the proof of lemma 4.3.1. Then $P_X = P_{(S_1 : S_2)}$ and if $X_o = X(I - A^- A)$, then $P_{X_o} = P_{S_2}$. When $V = \lambda I + UBU'$, $(I-P_X)V(I-P_X) = \lambda(I - P_{(S_1:S_2)})$ and $(P_X - P_{X_o})V(P_X - P_{X_o}) = \lambda(P_{(S_1:S_2)} - P_{S_2}) + (P_{(S_1:S_2)} - P_{S_2})UBU'(P_{(S_1:S_2)} - P_{S_2})$.

Applying theorem 3.2.1 we see that $L_V(X) - L(X) \geq 0$ for every $X \varepsilon C_A^r(U)$ if and only if the eigenvalues of $(P_{(S_1:S_2)} - P_{S_2})UBU'(P_{(S_1:S_2)} - P_{S_2})$ are ≤ 0 for every $(S_1 : S_2)$ of rank r with $\underline{M}(U) \subset \underline{M}(S_1 : S_2)$, $\underline{M}(S_1) \cap \underline{M}(S_2) = \{0\}$ and $R(S_1) = R(A)$. But $(P_{(S_1:S_2)} - P_{S_2})UBU'(P_{(S_1:S_2)} - P_{S_2}) = (I - P_{S_2})UBU'(I - P_{S_2})$. Thus we want conditions on UBU' so that $(I - P_{S_2})UBU'(I - P_{S_2})$ is non-positive definite for every S_2 with $\underline{M}(U) \subset \underline{M}(S_1 : S_2)$ and $R(S_2) = r - R(A)$. If $R(A) = r$, then

$S_2 = 0$ and the required condition is that UBU' is non-positive definite. If $R(A) < r$, choose $U_i, U_{(i)}^{\perp}$ $(i = 1, 2)$ and S_2 as in the proof of theorem 4.5.1. Then $(I-P_{S_2})UBU'(I-P_{S_2}) = U_1U_1'UBU'U_1U_1'$ which is non-positive definite for every semiorthogonal matrix U_1 of appropriate rank satisfying $\underline{M}(U_1) \subset \underline{M}(U)$ if and only if UBU' itself is non-positive definite. Thus we can write $UBU' = -UCC'U'$ for some C. This proves part (a) of the theorem. Part (b) is proved similarly.

In chapter 3 we considered the covariance matrix

$$V_\rho = (1-\rho)I_n + \rho 1_n 1_n', \quad -\frac{1}{n-1} < \rho < 1 ,$$

and we proved that if $1_n \in \underline{M}(X)$ and $L_\rho \neq L_0$, then $L_\rho > L_0$ (or $L_\rho < L_0$) with probability one if and only if $\rho < 0$ (respectively $\rho > 0$) where we use the same notations as in chapter 3. In view of the discussion given in section 3.2 (see pp.103-106) it is clear that $L_\rho(X) > L_0(X)$ (or $L_\rho(X) < L_0(X)$) with probability one for every $X \in C_A^r(1_n)$ if and only if $\rho < 0$ (or $\rho > 0$), where $L_\rho(X)$ and $L_0(X)$ denote that L_ρ and L_0 are computed under $(Y, X\beta, V_\rho)$ and $(Y, X\beta, I)$ respectively. When $\rho < 0$ (or $\rho > 0$), V_ρ has the structure given in theorem 4.5.2 (a) (respectively theorem 4.5.2 (b)).

In section 3.2 of chapter 3 we also considered the covariance matrix $V_c = \alpha I + 1_n c' + c 1_n'$, where c is a vector and α is a positive real number (see pp. 106-111). We proved that if

$1_n \in \underline{M}(X)$ and $L_c \neq L_o$, then $L_c - L_o > 0$ (or $L_c - L_o < 0$) with probability one if and only if $(I-P_{X_o})c = a(I-P_{X_o})1_n$ for some $a < 0$ (respectively $a > 0$). Let S_1, S_2, E_1 and E_2 be as defined in the proof of lemma 4.3.1. Then in the proof of theorem 4.5.1 we observed that $P_{X_o} = P_{S_2}$ and hence $(I-P_{X_o})c = a(I-X_o)1_n$ for every $X \in C_A^r(1_n) \iff (I-P_{S_2})c = a(I-P_{S_2})1_n$ for every S_2 with $R(S_2) = r - R(A) \iff c = a1_n$. Thus if $L_c(X)$ and $L_o(X)$ denote L_c and L_o computed under $(Y, X\beta, V_c)$ and $(Y, X\beta, I)$ respectively, then $L_c(X) - L_o(X) > 0$ (or < 0) with probability one for every $X \in C_A^r(1_n)$ if and only if $c = a1_n$ where $a < 0$ (respectively $a > 0$).

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