The Annals of Probability
1996, Vol. 24, No. 3, 1531–1562

OCCUPATION MEASURES FOR CONTROLLED MARKOV PROCESSES: CHARACTERIZATION AND OPTIMALITY

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For controlled Markov processes taking values in a Polish space, control problems with ergodic cost, infinite-horizon discounted cost and finite-horizon cost are studied. Each is posed as a convex optimization problem wherein one tries to minimize a linear functional on a closed convex set of appropriately defined *occupation measures* for the problem. These are characterized as solutions of a linear equation asssociated with the problem. This characterization is used to establish the existence of optimal Markov controls. The *dual* convex optimization problem is also studied.

1. Introduction.

1.1. *An overview.* For many classical stochastic optimal control problems, an alternative to the more traditional *dynamic* approaches based on dynamic programming, maximum principle and so on is provided by posing these problems as a *static* optimization problem on a set of associated *occupation measures.* Under mild conditions, the latter set turns out to be closed convex (often compact) and the objective functional bounded linear. By investigating the extreme points of this set, one can deduce the existence of optimal controls in some desirable classes of controls. Further, one may dualize this convex programming (in fact, an infinite-dimensional linear programming) problem to exhibit the associated value function as the maximal subsolution of the associated Hamilton–Jacobi–Bellman inequality.

This approach has its roots in the linear programming approach to Markov decision theory. Developed originally in the finite setup by Manne (1960), it has undergone many refinements and extensions, a recent example being the work of Hernandez-Lerma, Hennet and Lasserre (1991). The idea was also exploited in deterministic and stochastic optimal control, prime examples of the two being the works of Vinter and Lewis (1978) and Fleming and Vermes (1989), respectively. A more recent contribution in the latter domain is the work on ergodic control problem by Stockbridge (1990a, 1990b) which serves as the immediate inspiration for the present work. Our aim here is threefold: to pose some classical stochastic control problems as optimization problems over sets of suitably defined occupation measures which are then characterized as being the solutions of certain equations, to establish the existence of optimal

Received June 1993; revised November 1995.
[1]Research supported by the National Board for Higher Mathematics, India.
[2]Research supported by Grant 26/01/92-G from the Department of Atomic Energy, India.
AMS 1991 *subject classifications*. Primary 93E20; secondary 60J25.
Key words and phrases. Controlled Markov processes, occupation measures, optimal control, infinite-dimensional linear programming.

controls in suitable classes of controls using the foregoing and to study the dual problems. A detailed outline follows. Recently the authors learned that a program similar to the one above for characterizing occupation measures when the state space is a locally compact separable metric space has been carried out in Kurtz and Stockbridge (1994).

We shall consider three different cost structures: the *ergodic* or "long-run average" cost, the infinite-horizon discounted cost and the finite-horizon cost. We shall call these respectively the ergodic problem, the discounted problem and the finite-horizon problem. Following some notational preliminaries in Section 1.2, we describe these problems in Section 1.3. Our controlled process will be specified as being a solution to the *controlled martingale problem* associated with a candidate "controlled generator." For each of the three problems mentioned above, a separate occupation measure will be defined in terms of the pair of state and control processes and the control problem recast as an optimization problem over these measures. Section 1.4 gives examples of such controlled processes—finite/infinite dimensional controlled diffusions, controlled nonlinear filters, and so forth. In Section 2, the aforementioned occupation measures are characterized as being solutions of certain linear equations. The ergodic problem serves as a model here in the sense that the other problems will be effectively reduced to this case. Our results on the ergodic problem extend those of Stockbridge (1990a) for the locally compact case, which in turn extend to the controlled setup the work of Echeverria (1982) [or rather its variant decribed in Chapter 4, Section 9, of Ethier and Kurtz (1986)] on stationary solutions to the (uncontrolled) martingale problems. The latter was extended to Polish space–valued processes in Bhatt and Karandikar (1993a). We combine the ideas of this work with those of Stockbridge to extend the latter's results to Polish space–valued processes.

Section 3 establishes the existence of optimal controls in certain desirable classes of controls (to be precise, *Markov* or *inhomogeneous Markov* controls) for each of these problems. We take specific instances of each from the "examples" described in Section 1.4. This is in order to facilitate ready reference to some previous work for details, which considerably simplifies and shortens this exposition. The possibility of extensions to other cases will be remarked upon. Specifically, we consider controlled nonlinear filters for the ergodic and the discounted problems and controlled, possibly degenerate, finite-dimensional diffusions for the finite-horizon problem. Section 4 studies the dual problems in the spirit of Fleming and Vermes (1989), again for these specific cases. Section 5 concludes by highlighting certain open issues.

1.2. *Notation.* The state space of our controlled process will be a Polish (i.e., separable and metrizable with a complete metric) space E. The control space U will be a compact metric space. For any Polish space S, $B(S)$ will denote the space of bounded measurable functions from S to $\mathbb{R}$, $C_b(S)$ that of bounded continuous functions from S to $\mathbb{R}$, $\mathscr{B}(S)$ the Borel σ-field of S and $\mathscr{P}(S)$ the Polish space of probability measures on $(S, \mathscr{B}(S))$ with the topology of weak convergence. Let $\mathscr{M}(S)$ denote the space of positive finite measures on

$(S, \mathscr{B}(S))$ with the topology of weak convergence. Let $D([0, \infty), S)$ denote the space of all r.c.l.l. functions (i.e., right-continuous functions having left limits) from $[0, \infty)$ to S, equipped with the Skorokhod topology. Let I_B denote the indicator function of the set B, ω a typical point in the underlying probability space and δ_x the Dirac measure at x. Here $\mathscr{R}(\cdots)$ will denote the "range of operator $\cdots$" and $\mathscr{L}(\cdots)$ will denote "the law of $\cdots$." A stochastic process $\xi(\cdot)$ will be time-indexed as $\xi(t)$ or ξ_t depending on convenience.

For f_k, f in $B(E)$, we say that $f_k \to_{\mathrm{bp}} f$ (where bp stands for boundedly and pointwise) if $\|f_k\| \leq M$ for all k, for some $M > 0$ and $f_k(x) \to f(x)$ for all $x \in E$. A set B is said to be bp-closed if $f_k \in B$, $f_k \to_{\mathrm{bp}} f$ implies $f \in B$. Define bp-closure (B) to be the smallest bp-closed set that contains B.

For the control space U, C_U will denote the space of measurable maps $[0, \infty) \to \mathscr{P}(U)$ with the compact metrizable topology, described on page 318 of Borkar (1991), defined on it.

1.3. *Problem framework.* Let A be an operator with $\mathscr{D}(A) \subset C_b(E)$ and $\mathscr{R}(A) \subset C_b(E \times U)$. Let $\nu \in \mathscr{P}(E)$.

DEFINITION 1.1. An $E \times U$-valued process $(X(\cdot), u(\cdot))$ defined on a probability space $(\Omega, \mathscr{F}, P)$ is said to be a solution to the controlled martingale problem for (A, ν) with respect to a filtration $\{\mathscr{F}_t,\ t \geq 0\}$ if:

(i) $(X(\cdot), u(\cdot))$ is $\{\mathscr{F}_t\}$-progressive;
(ii) $\mathscr{L}(X(0)) = \nu$;
(iii) for $f \in \mathscr{D}(A)$,

$$f(X(t)) - \int_0^t Af(X(s), u(s))\, ds$$

is an $\{\mathscr{F}_t\}$-martingale.

We shall generally work in the relaxed control framework which we describe next. Let $V = \mathscr{P}(U)$ (also a compact metric space).

DEFINITION 1.2. An $E \times V$-valued process $(X(\cdot), \pi(\cdot))$ defined on a probability space $(\Omega, \mathscr{F}, P)$ is said to be a solution to the relaxed controlled martingale problem for (A, ν) with respect to a filtration $\{\mathscr{F}_t,\ t \geq 0\}$ if:

(i) $(X(\cdot), \pi(\cdot))$ is $\{\mathscr{F}_t\}$-progressive;
(ii) $\mathscr{L}(X(0)) = \nu$;
(iii) for $f \in \mathscr{D}(A)$, for a.a. $t \geq 0$,

$$f(X(t)) - \int_0^t \int_U Af(X(s), u)\pi_s(du)\, ds \tag{1.1}$$

is an $\{\mathscr{F}_t\}$-martingale.

In both cases, we may omit the mention of the filtration $\{\mathscr{F}_t\}$ or the initial law ν when these are understood from the context. We may define $\overline{A}$: $\mathscr{D}(A) \to C_b(E \times V)$ by

$$\overline{A}f(x,\mu) = \int_U Af(x,u)\mu(du), \qquad f \in \mathscr{D}(A),\ x \in E,\ \mu \in V,$$

and rewrite (1.1) as

$$f(X(t)) - \int_0^t \overline{A}f(X(s),\pi(s))\,ds, \qquad t \ge 0. \tag{1.2}$$

The operator A will be required to satisfy the following conditions:

CONDITION 1. There exists a countable subset $\{g_k\} \subset \mathscr{D}(A)$ such that

$$\text{bp-closure}(\{(g_k, Ag_k)\colon\ k \ge 1\}) \supset \{(g, Ag)\colon\ g \in \mathscr{D}(A)\}.$$

CONDITION 2. $\mathscr{D}(A)$ is an algebra that separates points in E and contains constant functions. Furthermore, $A\mathbf{1} = 0$, where $\mathbf{1}$ is the constant function identically equal to 1.

CONDITION 3. For each $u \in U$, let $A^u f \equiv Af(\cdot, u)$. Then there exists an r.c.l.l. solution to the martingale problem for (A^u, δ_x) for all $u \in U$, $x \in E$.

The three control problems that we consider are associated with the following costs:

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}[k(X(s),u(s))]\,ds \qquad \text{(ergodic)}, \tag{1.3}$$

$$\mathbb{E}\left[\int_0^\infty e^{-\alpha s} k(X(s),u(s))\,ds\right], \qquad \alpha > 0 \qquad \text{(discounted)} \tag{1.4}$$

$$\mathbb{E}\left[\int_0^T k(X(s),u(s))\,ds\right], \qquad T > 0 \qquad \text{(finite horizon)} \tag{1.5}$$

where k: $E \times U \to [0,\infty]$ is a continuous *running cost* function. Of course, we assume that these quantities are finite for some $u(\cdot)$. It should be kept in mind that in case of the relaxed controlled martingale problem, one has to replace $k(X(t), u(t))$ by $\int_U k(X(t),\cdot)\,d\pi_t$ in (1.3)–(1.5).

Note that if $(X(\cdot), u(\cdot))$ is stationary with $\mathscr{L}(X(t), u(t)) = \mu$ for all $t \ge 0$, then the lim sup in (1.3) is a limit and equals $\int k\,d\mu$. In this case, we call μ the associated ergodic occupation measure. The ergodic occupation measure for nonstationary $(X(\cdot), u(\cdot))$ is left undefined. For the remaining problems, the corresponding occupation measures (i.e., discounted and finite-time occupation measures) are defined by

$$\int f\,d\mu = \alpha\mathbb{E}\left[\int_0^\infty e^{-\alpha t} f(X(t),u(t))\,dt\right], \tag{1.6}$$

$$\int f\,d\mu = T^{-1}\mathbb{E}\left[\int_0^T f(X(t),u(t))\,dt\right], \tag{1.7}$$

respectively for $f \in C_b(E \times U)$. Thus the above control problems amount to minimizing the functional $\mu \in \mathscr{P}(E \times U) \to \int k\,d\mu \in \mathbb{R}^+$ on the respective sets of occupation measures [with the proviso that, in the ergodic case, we consider only the stationary $(X(\cdot), u(\cdot))$; this will be justified later].

1.4. *Examples.* The following examples are seen to fit the above framework.

EXAMPLE 1. Consider the d-dimensional controlled diffusion process $X(\cdot) = [X_1(\cdot), \dots, X_d(\cdot)]^T$ described by

$$X(t) = X_0 + \int_0^t m(X(s), u(s))\,ds + \int_0^t \sigma(X(s))\,dW(s), \qquad t \geq 0. \tag{1.8}$$

Here

(i) $m(\cdot,\cdot) = [m_1(\cdot,\cdot), \dots, m_d(\cdot,\cdot)]^T$: $\mathbb{R}^d \times U \to \mathbb{R}^d$ is bounded continuous and Lipschitz in its first argument uniformly with respect to the second (U being a compact metric space);
(ii) $\sigma(\cdot) = [[\sigma_{ij}(\cdot)]]_{1\leq i,\, j\leq d}$: $\mathbb{R}^d \to \mathbb{R}^{d\times d}$ is bounded Lipschitz;
(iii) X_0 has a prescribed law;
(iv) $W(\cdot) = [W_1(\cdot), \dots, W_d(\cdot)]^T$ is a d-dimensional standard Wiener process independent of X_0;
(v) $u(\cdot)$ is a U-valued control process with measurable paths satisfying: for $t \geq s$, $W(t) - W(s)$ is independent of $\{u(r), W(r)\colon r \leq s\}$.

It should be remarked that the above conditions on m, σ can be relaxed. Let $\mathscr{D}(A) = C_0^2(\mathbb{R}^d)$, the space of twice continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$ which vanish at ∞ along with its first- and second-order partial derivatives. For $f \in \mathscr{D}(A)$, let $Af \in C_b(\mathbb{R}^d \times U)$ be defined by

$$Af(x, u) = \sum_{i=1}^d m_i(x, u)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,\,j,\,k=1}^d \sigma_{ik}(x)\sigma_{jk}(x)\frac{\partial^2 f}{\partial x_i\,\partial x_j}(x) \tag{1.9}$$

for $f \in \mathscr{D}(A)$, $x = [x_1, \dots, x_d]^T \in \mathbb{R}^d$, $u \in U$.

EXAMPLE 2. Let H be a real, separable Hilbert space and $X(\cdot)$ an H-valued controlled diffusion described by

$$X(t) = X_0 + \int_0^t m(X(s), u(s))\,ds + \int_0^t \sigma(X(s))\,dW(s), \qquad t \geq 0,$$

where $m\colon H \times U \to H$ is Lipschitz in its first argument uniformly with respect to the second, $\sigma\colon H \to L_2(H, H)$ is Lipschitz continuous and $W(\cdot)$ is an H-valued cylindrical Wiener process independent of X_0. Here $L_2(H, H)$ denotes the space of Hilbert–Schmidt operators on H with the HS-norm $\|\cdot\|_{\mathrm{HS}}$. Here $U, u(\cdot)$ are as in the previous example. Fix a CONS $\{e_i\colon i \geq 1\}$ in H and let

P_n: $H \to \mathbb{R}^n$ be the map defined by $P_n(x) = [\langle x, e_1\rangle, \ldots, \langle x, e_n\rangle]$. Let $\mathscr{D}(A) = \{f \circ P_n\colon f \in C_0^2(\mathbb{R}^n),\ n \geq 1\} \subset C_b(H)$ and define A: $\mathscr{D}(A) \to C_b(H \times U)$ by

$$[A(f \circ P_n)](h, u) = \sum_{i=1}^{n} \langle m(h, u), e_i\rangle \frac{\partial f}{\partial x_i} \circ P_n(h) + \frac{1}{2} \sum_{i,\, j=1}^{n} \langle \sigma^*(h)e_i, \sigma^*(h)e_j\rangle \frac{\partial^2 f}{\partial x_i\, \partial x_j} \circ P_n(h).$$

EXAMPLE 3. Let H, H' be real separable Hilbert spaces and let U be the closed unit ball of H' with the weak topology. Consider the H-valued controlled stochastic evolution equation (interpreted in the mild sense)

$$dX(t) = -LX(t)\, dt + (F(X(t)) + Bu(t))\, dt + dW(t),$$

where $-L$ is the infinitesimal generator of a differentiable semigroup of contractions on H such that L^{-1} is a bounded self-adjoint operator with discrete spectrum, F: $H \to H$ is bounded Lipschitz, B: $H' \to H$ is bounded linear, $W(\cdot)$ is an H-valued Wiener process independent of $X(0)$ with incremental covariance given by a trace class operator Q and $u(\cdot)$ is as before. Pick the CONS $\{e_i\}$ of H to be the eigenfunctions of L^{-1} with $\{\lambda_i^{-1}\}$ being the corresponding eigenvalues. Let $\mathscr{D}(A)$ be as in the preceding example and define A: $\mathscr{D}(A) \to C_b(H \times U)$ by

$$[A(f \circ P_n)](h, u) = \sum_{i=1}^{n} \langle e_i, (F(h) + Bu - \lambda_i h)\rangle \frac{\partial f}{\partial x_i} \circ P_n(h) + \frac{1}{2} \sum_{i,\, j=1}^{n} \langle e_i, Qe_j\rangle \frac{\partial^2 f}{\partial x_i\, \partial x_j} \circ P_n(h).$$

EXAMPLE 4. Consider $X(\cdot)$ as in (1.8) in conjunction with the m-dimensional *observation* process $Y(\cdot)$ given by

$$Y(t) = \int_0^t h(X(s))\, ds + W'(t),$$

where h: $\mathbb{R}^d \to \mathbb{R}^m$ is bounded, twice continuously differentiable with bounded first and second partial derivatives and $W'(\cdot)$ is an m-dimensional standard Wiener process independent of $W(\cdot)$ and X_0. Assume that there exists $\lambda_0 > 0$ such that $\|\sigma^*(x)y\|^2 \geq \lambda_0 \|y\|^2$ for all $x, y \in \mathbb{R}^d$. Let $(\Omega, \mathscr{F}, P)$ denote the underlying probability space and $\{\mathscr{F}_t\}$ the natural filtration of $(X(\cdot), u(\cdot), W(\cdot), W'(\cdot))$. Define a new probability measure P_0 on $(\Omega, \mathscr{F})$ as follows. If P_t, P_{0t} denote the restrictions of P, P_0 to $(\Omega, \mathscr{F}_t)$ for $t \geq 0$, then

$$\frac{dP_t}{dP_{0t}} = \exp\biggl(\int_0^t \langle h(X(s)), dY(s)\rangle - \frac{1}{2}\int_0^t \|h(X(s))\|^2\, ds\biggr).$$

We shall assume that $u(\cdot)$ satisfies the additional condition: under P_0, for each $t \geq 0$, $\{u(s), Y(s)\colon s \leq t\}$ are independent of $\{X_0, W(\cdot), Y(t + \cdot) -$

$Y(t)\}$. These are the so-called wide-sense admissible controls. Let $\{\mathscr{G}_t\} = \sigma\{Y(s), \int_a^b u_r\,dr\colon 0 \le s \le t,\ 0 \le a \le b \le t\}$ and let μ_t denote the regular conditional law of $X(t)$ given $\mathscr{G}_t$ for $t \ge 0$. Then $\{\mu_t\}$ is a $\mathscr{P}(\mathbb{R}^d)$-valued process whose evolution is given by

$$\begin{aligned} \mu_t(f) = \mu_0(f) &+ \int_0^t \mu_s(Lf(\cdot, u(s)))\,ds \\ &+ \int_0^t \langle \mu_s(fh) - \mu_s(f)\mu_s(h),\ d\hat{Y}(s)\rangle, \end{aligned} \tag{1.10}$$

where $f \in \mathscr{D}(A)$, L is the operator A of Example 1 above, $\mu(f) := \int f\,d\mu$ for $\mu \in \mathscr{P}(\mathbb{R}^d)$, $f \in C_b(\mathbb{R}^d)$ and $\hat{Y}(t) = Y(t) - \int_0^t \mu_s(h)\,ds$, $t \ge 0$, is the so-called *innovation process*. The original control problem with one of the above costs is equivalent to the *separated control problem* of controlling $\{\mu_t\}$ governed by the nonlinear filter (1.10) with the corresponding cost functional obtained by replacing $k(X(t), u(t))$ in the original by $\mu_t(k(\cdot, u(t)))$, $t \ge 0$. [See Borkar (1989), Chapter 5, for details.] Now let $\mathscr{D}(A) = \{f \in C_b(\mathscr{P}(\mathbb{R}^d))\colon f(\mu) = g(\int f_1\,d\mu, \dots, \int f_n d\mu), \mu \in \mathscr{P}(\mathbb{R}^d),$ for some $n \ge 1,\ g \in C_0^2(\mathbb{R}^n),\ f_1, \dots, f_n \in \mathscr{D}(L)\}$ and define $A\colon \mathscr{D}(A) \to C_b(\mathscr{P}(\mathbb{R}^d) \times U)$ by

$$\begin{aligned} Af(\mu, u) = &\sum_{i=1}^n \frac{\partial g}{\partial x_i}\left(\int f_1\,d\mu, \dots, \int f_n\,d\mu\right)\mu(Lf_i(\cdot, u)) \\ &+ \frac{1}{2}\sum_{i,\,j=1}^n \frac{\partial^2 g}{\partial x_i \partial x_j}\left(\int f_1\,d\mu, \dots, \int f_n\,d\mu\right) \\ &\times \langle \mu(f_i h) - \mu(f_i)\mu(h), \mu(f_j h) - \mu(f_j)\mu(h)\rangle. \end{aligned}$$

In each of the above cases, one easily verifies the hypotheses on A, the existence of r.c.l.l. solutions for $u(\cdot) \equiv \overline{u} \in U$ being guaranteed by known results on the uncontrolled versions of these processes. [See Stroock and Varadhan (1979), Yor (1974), Borkar and Govindan (1994) and Chapter 5 of Borkar (1989).]

2. Characterization of occupation measures.

2.1. *The ergodic problem.* Note that if $(X(\cdot), u(\cdot))$ is a stationary solution to the controlled martingale problem for A with $\mathscr{L}(X(t), u(t)) = \mu$, then, for all $f \in \mathscr{D}(A)$, $t > 0$,

$$\begin{aligned} 0 &= \mathbb{E}[f(X(t))] - \mathbb{E}[f(X(0))] \\ &= \int_0^t \mathbb{E}[Af(X(s), u(s))]\,ds \\ &= t\int Af\,d\mu, \end{aligned}$$

implying

$$\int Af\,d\mu = 0 \quad \text{for all } f \in \mathscr{D}(A). \tag{2.1}$$

We shall show that (2.1) characterizes all ergodic occupation measures (with the qualification that we move over to the relaxed controlled martingale problem) in Theorem 2.1 below. We need the following preliminary lemma which is a straightforward generalization of Lemma 4.9.16 of Ethier and Kurtz (1986) to the controlled setup. The proof follows by applying this result to the operator $A^u,\ u \in U$.

LEMMA 2.1. *Let* ϕ: $G \subset \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$, *be convex and continuously differentiable and* $f_1, \dots, f_m \in \mathscr{D}(A)$ *satisfy* $\mathscr{R}((f_1, \dots, f_m)) \subset G$, $\phi(f_1, \dots, f_m) \in \mathscr{D}(A)$. *Then*

$$A\phi(f_1, \dots, f_m) \geq \nabla\phi(f_1, \dots, f_m) \cdot (Af_1, \dots, Af_m).$$

THEOREM 2.1. *For each* $\mu \in \mathscr{P}(E \times U)$ *satisfying* (2.1), *there exists a stationary solution* $(X(\cdot), \pi(\cdot))$ *of the relaxed controlled martingale problem for* A *such that*

$$\mathbb{E}\left[g(X(t)) \int_U h\, d\pi(t)\right] = \int_{E \times U} gh\, d\mu \qquad \forall g \in C_b(E),\ h \in C_b(U),\ t \geq 0. \tag{2.2}$$

PROOF. The proof closely mimics the arguments in Bhatt and Karandikar (1993a) and Stockbridge (1990a). We give here only the main steps in detail.

For $n \geq 1$, define operators A_n as follows. Let $\mathscr{D}(A_n) = \mathscr{R}(I - n^{-1}A)$ and set $A_n g = n[(I - n^{-1}A)^{-1} - I]g$ for $g \in \mathscr{D}(A_n)$. It follows from Proposition 4.3.5 of Ethier and Kurtz (1986) that A_n is a well-defined bounded operator. Note that the A_n's approximate A in the following sense. Let $f \in \mathscr{D}(A)$, $f_n := (I - n^{-1}A)f$, $n \geq 1$. Then

$$\|f_n - f\| \to 0, \qquad \|A_n f_n - Af\| \to 0 \quad \text{as } n \to \infty. \tag{2.3}$$

(In fact, $A_n f_n = Af$, $n \geq 1$.) A straightforward verification shows that

$$\int A_n g\, d\mu = 0 \qquad \forall\ g \in \mathscr{D}(A_n),\ n \geq 1.$$

Step 1. *Construction of stationary solutions corresponding to* A_n: Fix n. Let $M \subset C_b(E \times E \times U)$ be the linear space of functions of the form

$$F(x, y, u) = \sum_{i=1}^m f_i(x) g_i(y, u) + f(y, u), \tag{2.4}$$

$f_1, \dots, f_m \in C_b(E)$, $f \in C_b(E \times U)$, $g_1, \dots, g_m \in \mathscr{R}(I - n^{-1}A)$, $m \geq 1$. Define a linear functional Λ on M as follows. For F as in (2.4),

$$\Lambda F = \int_{E \times U} \left[\sum_{i=1}^m f_i(x)[(I - n^{-1}A)^{-1} g_i](x) + f(x, u)\right] \mu(dx, du). \tag{2.5}$$

Using Lemma 2.1 and proceeding exactly as in Stockbridge (1990a), we get $|\Lambda F| \leq \|F\|$ for $F \in M$. From the definition of Λ it is clear that $\Lambda 1 = 1$.

Together, these imply that $\Lambda F \geq 0$ whenever $F \geq 0$. By the Hahn–Banach theorem, Λ extends to a bounded, positive linear functional on $C_b(E \times E \times U)$ which we again denote by Λ. Since $E \times E \times U$ need not be compact, the Riesz representation theorem cannot be invoked here. But note that

$$\Lambda F_f = \int f\, d\mu_1, \qquad \Lambda F^g = \int g\, d\mu, \tag{2.6}$$

where $F_f(x, y, u) = f(x)$, $F^g(x, y, u) = g(y, u)$ and μ_1 is the marginal of μ on E. Now we can apply Theorem 2.3 of Bhatt and Karandikar (1993a) to get a $\nu \in \mathscr{P}(E \times E \times U)$ such that

$$\Lambda F = \int_{E\times E\times U} F\, d\nu \qquad \forall\ F \in C_b(E \times E \times U). \tag{2.7}$$

It is known [see, e.g., Ethier and Kurtz (1986), Appendix] that there exists a transition probability function η: $E \times \mathscr{B}(E \times U) \to [0, 1]$ such that, for $B_1 \in \mathscr{B}(E)$, $B_2 \in \mathscr{B}(E \times U)$,

$$\nu(B_1 \times B_2) = \int_{B_1} \eta(x, B_2)\mu_1(dx). \tag{2.8}$$

From (2.4)–(2.8) it follows that, for $g \in \mathscr{R}(I - n^{-1}A)$,

$$\int_{E\times U} g(y, u)\eta(x, dy, du) = [(I - n^{-1}A)^{-1}g](x) \qquad \mu_1\text{-a.s.} \tag{2.9}$$

Let $\{(Y(m), u(m))\colon m \geq 0\}$ be a Markov chain on $E \times U$ with initial distribution μ and transition function η. Putting $B_1 = E$ in (2.8), we get

$$\int_E \eta(x, B_2)\mu_1(dx) = \mu(B_2).$$

Thus $\{(Y(m), u(m))\}$ is a stationary chain. We can verify that, for any $g \in \mathscr{R}(I - n^{-1}A)$,

$$g(Y(m), u(m)) - \sum_{j=0}^{m-1} n^{-1}A_n g(Y(j), u(j))$$

is a $\sigma(Y(i), u(i)\colon i \leq m)$-martingale. Let $V^n(\cdot)$ be a Poisson process with parameter n, independent of $\{(Y(m), u(m))\}$. Let

$$X^n(t) := Y(V^n(t)), \qquad u^n(t) := u(V^n(t)) \qquad \forall\ t \geq 0,\ n \geq 1, \tag{2.10}$$

and $\{\mathscr{F}_t^n\}$ be the filtration defined by $\mathscr{F}_t^n = \sigma(X_s^n, \int_a^b u_r^n dr\colon s \leq t,\ 0 \leq a \leq b \leq t)$. Then one can show that $(X^n(\cdot), u^n(\cdot))$ is a stationary solution to the (uncontrolled) martingale problem for (A_n, μ) with respect to the filtration $\{\mathscr{F}_t^n\}$ for every $n \geq 1$.

Step 2. *Convergence of marginals of* $(X^n(\cdot), u^n(\cdot))$: Define the (random) occupation measures π_n^* on $\mathscr{B}(U \times [0, \infty))$ by

$$\pi_n^*(C) = \int_{U\times[0,\infty)} I_C(u^n(s), s)\, ds \qquad \forall\ C \in \mathscr{B}(U \times [0, \infty)).$$

Argue as in Lemma 4.3 of Stockbridge (1990a) to conclude that there exists a subsequence of $\{\pi_n^*\}$ (which we relabel as $\{\pi_n^*\}$), a random measure π^* and a $\mathscr{P}(U)$-valued process $\pi(\cdot)$ such that $\int f\,d\pi_n^* \to \int f\,d\pi^*$ for compactly supported $f \in C_b(U \times [0,\infty))$ and

$$\pi^*(C) = \int_{U\times[0,\infty)} I_C(u,s)\pi_s(du)\,ds, \qquad C \in \mathscr{B}(U\times[0,\infty)). \tag{2.11}$$

Furthermore, since $u^n(\cdot)$ are stationary, $\pi(\cdot)$ may be taken to be a stationary process [cf. Lemma 4.4 of Stockbridge (1990a)].

For extracting a convergent subsequence of $\{X^n(\cdot)\}$, we proceed as follows: let $\{g_m\colon m \geq 1\}$ be the countable collection that features in Condition 1 stipulated for the operator A. Let $\|g_m\| = a_m,\ m \geq 1$, and $\hat{E} = \prod_{m=1}^{\infty}[-a_m, a_m]$. Define $\mathbf{g}\colon E \to \hat{E}$ by $\mathbf{g}(x) = (g_1(x), \ldots, g_m(x), \ldots)$. Since $\mathscr{D}(A)$ separates points in $\hat{E}$ and vanishes nowhere, so does $\{g_m\colon m \geq 1\}$. Hence it follows that $\mathbf{g}$ is a one–one continuous function and $\mathbf{g}(E)$ is a Borel subset of $\hat{E}$. Also $\mathbf{g}^{-1}$ defined on $\mathbf{g}(E)$ is measurable. Define $\mathbf{g}^{-1}(z) = e$ for $z \notin \mathbf{g}(E)$, where e is a prescribed point in E. By (2.3), $\xi^n(t) = f_n(X^n(t)),\ \phi_n(t) = A_n f_n(X^n(t), u^n(t)),\ n \geq 1$, satisfy the conditions of Theorem 3.9.4 of Ethier and Kurtz (1986). Along with Condition 2 on operator A, this leads to the relative compactness of $\mathscr{L}(f_1 \circ X^n(\cdot), \ldots, f_l \circ X^n(\cdot))$ in $\mathscr{P}(D([0,\infty), \mathbb{R}^l))$ for $f_1, \ldots, f_l \in \mathscr{D}(A),\ 1 \leq l \leq \infty$. In particular, $\{\mathscr{L}(\mathbf{g}(X^n(\cdot)))\}$ is relatively compact in $\mathscr{P}(D([0,\infty), \hat{E}))$. Thus each subsequence thereof has a further subsequence converging in law to a $D([0,\infty), \hat{E})$-valued random variable, say $Z(\cdot)$. Without loss of generality, we may take a common convergent subsequence for $\mathscr{L}(\pi_n^*)$, $\mathscr{L}(\mathbf{g}(X^n(\cdot))$ and further suppose that $\mathscr{L}((\mathbf{g}(X^n(\cdot)), \pi_n^*))$ converges along this subsequence to $\mathscr{L}((Z(\cdot), \pi^*))$. Using a Skorokhod representation [see e.g., page 102 of Ethier and Kurtz (1986)], we may suppose that this convergence is a.s. on a common probability space. From the stationarity of $(\mathbf{g}(X^n(\cdot)), u^n(\cdot))$ for $n \geq 1$, that of $(Z(\cdot), u(\cdot))$ follows. Also $\mathscr{L}(\mathbf{g}(X^n(t))) = \mu_1 \circ \mathbf{g}^{-1}$ for all n, t. Thus $\mathscr{L}(Z(t)) = \mu_1 \circ \mathbf{g}^{-1}$ for all $t \geq 0$, implying $P(Z(t) \in \mathbf{g}(E)) = 1$ for all t. Define $X(t) = \mathbf{g}^{-1}(Z(t)), t \geq 0$. Applying Lemma 2.2 of Bhatt and Karandikar (1993a) , we conclude that $X^n(t) \to X(t)$ in E in probability for each $t \geq 0$. This completes step 2.

Let $\mathscr{F}_t = \sigma(Z_s, \int_a^b \pi_r(f)\,dr\colon\ s \leq t,\ 0 \leq a \leq b \leq t,\ f \in C(U))$, where $\pi(f) = \int_U f\,d\pi$. Then $(X(\cdot), \pi(\cdot))$ is $\{\mathscr{F}_t\}$-progressive. We will now show that $(X(\cdot), \pi(\cdot))$ is a solution to the relaxed controlled martingale problem for (A, μ_1) with respect to $\{\mathscr{F}_t\}$. Clearly, $(X(\cdot), \pi(\cdot))$ is $\{\mathscr{F}_t\}$-progressive. Let $h \in C_b(E \times U)$. Then it is easy to see that

$$|h(X^n(s), u^n(s)) - h(X(s), u^n(s))| \to 0$$

as $n \to \infty$ in probability for each $s \geq 0$. Since $\pi_n^* \to \pi^*$ a.s.,

$$\int_0^t h(X(s), u^n(s))\,ds \to \int_0^t \int_U h(X(s), u)\pi_s(du)\,ds \quad \text{a.s.}$$

Thus, for all $t \geq 0$,

$$\int_0^t h(X^n(s), u^n(s))\, ds \to \int_0^t \int_U h(X(s), u)\pi_s(du)\, ds \tag{2.12}$$

in probability. Let $f \in \mathscr{D}(A)$, $\{f_n\}$ as in (2.3), $0 \leq t_1 < t_2 < \cdots < t_m < t_{m+1}$, $0 \leq \delta_i \leq t_i$, $1 \leq i \leq m$, $h_1, \ldots, h_m \in C_b(E \times U)$ for some $m \geq 1$. Then, using (2.3), (2.12) and the fact that $(X^n(\cdot), u^n(\cdot))$ is a solution to the martingale problem for A_n, we have

$$\begin{aligned}
&\mathbb{E}\biggl[\biggl(f(X(t_{m+1})) - f(X(t_m)) - \int_{t_m}^{t_{m+1}} \int_U Af(X(s), u)\pi_s(du)\, ds\biggr) \\
&\qquad\qquad \times \prod_{i=1}^m \int_{t_i-\delta_i}^{t_i} \int_U h_i(X(t_i), u)\pi_s(du)\, ds\biggr] \\
&\quad = \lim_{n\to\infty} \mathbb{E}\biggl[\biggl(f_n(X^n(t_{m+1})) - f_n(X^n(t_m)) \\
&\qquad\qquad - \int_{U\times[t_m, t_{m+1}]} A_n f_n(X^n(s), u)\pi_n^*(du, ds)\biggr) \\
&\qquad\qquad \times \prod_{i=1}^m \int_{U\times[t_i-\delta_i, t_i]} h_i(X^n(t_i), u)\pi_n^*(du, ds)\biggr] \\
&\quad = 0.
\end{aligned}$$

By standard monotone class arguments, it follows that $(X(\cdot), \pi(\cdot))$ is a solution to the relaxed controlled martingale problem for (A, μ_1). This completes the proof. □

COROLLARY 2.1. *The relaxed control process $\{\pi_t\}$ above may be taken to be of the form $\pi_t = v(X(t))$, $t \geq 0$, for a measurable v: $E \to \mathscr{P}(U)$.*

PROOF. Disintegrate μ as $\mu(dx, du) = \mu_1(dx)\overline{v}(x, du)$, where $\overline{v}(x, du)$ is defined μ_1-a.s. uniquely. Let v: $E \to \mathscr{P}(U)$ map $x \in E$ to $\overline{v}(x, du) \in \mathscr{P}(U)$. Now repeat the above argument with $\overline{A}$ replacing A, $\mathscr{P}(U)$ [respectively $\mathscr{P}(\mathscr{P}(U))$] replacing U [respectively $\mathscr{P}(U)$] and $\overline{\mu} \in \mathscr{P}(E \times \mathscr{P}(U))$ given by $\overline{\mu}(dx, dy) = \mu_1(dx)\delta_{v(x)}(dy)$ replacing μ.

Thus there exists an $E \times \mathscr{P}(\mathscr{P}(U))$-valued stationary solution $(X(\cdot), \pi(\cdot))$ to the relaxed controlled martingale problem for $\overline{A}$ with respect to some filtration $\{\mathscr{F}_t\}$, satisfying

$$\mathbb{E}\biggl[g(X(t)) \int_{\mathscr{P}(U)} h\, d\pi(t)\biggr] = \int_{E\times\mathscr{P}(U)} gh\, d\overline{\mu}$$

$$\forall g \in C_b(E),\ h \in C(\mathscr{P}(U)),\ t \in \mathbb{R}.$$

Define a $\mathscr{P}(\mathscr{P}(U))$-valued stationary process $\tilde{\pi}(\cdot)$ by

$$\int h\, d\tilde{\pi}(t) = \mathbb{E}\biggl[\int h\, d\pi(t) | \mathscr{F}_t^X\biggr], \qquad t \in \mathbb{R},$$

for h in a countable dense set in $C(\mathscr{P}(U))$, $\{\mathscr{F}_t^X\}$ being the natural filtration of $X(\cdot)$. Then it is easy to see that $(X(\cdot), \tilde{\pi}(\cdot))$ is a stationary solution to the relaxed controlled martingale problem for $\overline{A}$ with respect to $\{\mathscr{F}_t^X\}$, satisfying

$$\begin{aligned}\mathbb{E}\left[g(X(t))\int_{\mathscr{P}(U)} h\,d\tilde{\pi}(t)\right] &= \int_{E\times\mathscr{P}(U)} gh\,d\overline{\mu}\\ &= \int_{E\times\mathscr{P}(U)} g(x)h(y)\mu_1(dx)\delta_{v(x)}(dy)\\ &= \int_E g(x)h(v(x))\mu_1(dx)\\ &= \mathbb{E}[g(X(t))h(v(X(t)))]\end{aligned}$$

$\forall\ g \in C_b(E),\ \ h \in C(\mathscr{P}(U)),\ \ t \in \mathbb{R}$. This implies

$$\mathbb{E}\left[\int_{\mathscr{P}(U)} h\,d\tilde{\pi}(t)|X(t)\right] = h(v(X(t))) \quad \text{a.s. } \forall\ t$$

for all $h \in C(\mathscr{P}(U))$. In particular,

$$\mathbb{E}\left[\int_{\mathscr{P}(U)} h^2\,d\tilde{\pi}(t)|X(t)\right] = h^2(v(X(t))) \quad \text{a.s. } \forall\ t$$

and hence

$$\begin{aligned}&\mathbb{E}\left[\int_{\mathscr{P}(U)}[h(y) - h(v(X(t)))]^2\tilde{\pi}(t,dy)|X(t)\right]\\ &\quad= \mathbb{E}\left[\int_{\mathscr{P}(U)} h^2(y)\tilde{\pi}(t,dy)|X(t)\right] - 2\mathbb{E}\left[\int_{\mathscr{P}(U)} h(y)\tilde{\pi}(t,dy)|X(t)\right]h(v(X(t)))\\ &\qquad+ h^2(v(X(t)))\\ &\quad= 0.\end{aligned}$$

This implies $\tilde{\pi}(t) = \delta_{v(X(t))}$ a.s. This completes the proof. □

A relaxed control of this type will be called a relaxed Markov control.

2.2. *The discounted problem.* Let $(X(\cdot), u(\cdot))$ satisfy the controlled martingale problem for (A, ν_0) and let $\alpha > 0$. Then, for $f \in \mathscr{D}(A)$,

$$e^{-\alpha t}\mathbb{E}[f(X(t))] - \mathbb{E}[f(X(0))] = \mathbb{E}\left[\int_0^t e^{-\alpha s}(Af(X(s),u(s)) - \alpha f(X(s)))\,ds\right].$$

Multiplying throughout by α and letting $t \to \infty$, we obtain

$$\int Af\,d\mu = \alpha\left(\int f\,d\mu - \int f\,d\nu_0\right) \qquad \forall\ f \in \mathscr{D}(A) \tag{2.13}$$

for the discounted occupation measure μ [see (1.6)]. [We have identified $f \in C_b(E)$ with $\overline{f} \in C_b(E \times U)$ given by $\overline{f}(x,u) = f(x)$ by a slight abuse of notation.] Our aim here is to prove that (2.13) characterizes the discounted occupation measures.

We will require the following *imbedding* of the martingale problem for A into a compact space. [See Bhatt and Karandikar (1993a, 1993b).] Recall the definition of $\hat{E}$ constructed in the proof of Theorem 2.1. Let $\{g_m\colon\ m \geq 1\}$ be the countable set of functions featured in Condition 1 on the operator A, with $\|g_m\| = a_m, m \geq 1$. Without loss of generality, we may take $g_1 \equiv 1$. Let $\mathbf{g}\colon\ E \to \hat{E}$ be defined by $\mathbf{g}(x) = (g_1(x), \dots, g_m(x), \dots)$, where $\hat{E} = \prod_{m=1}^{\infty}[-a_m, a_m]$. Let $\mathscr{D}(\mathscr{A})$ be the algebra of functions on $C(\hat{E})$ generated by $\{\hat{f}_k \in C(\hat{E})\colon\ \hat{f}_k((z_1, \dots, z_k, \dots)) = z_k\}$. The operator $\mathscr{A}\colon\ \mathscr{D}(\mathscr{A}) \to B(\hat{E} \times U)$ is defined by

$$\mathscr{A}(c\hat{f}_{i_1}\hat{f}_{i_2}\cdots\hat{f}_{i_k})(z, u) = \begin{cases} c[A(g_{i_1}g_{i_2}\cdots g_{i_k})](x, u), & \text{if } z = \mathbf{g}(x), \\ 0, & \text{otherwise.} \end{cases} \tag{2.14}$$

Note that $\hat{f}_k(\mathbf{g}(x)) = g_k(x)$ and $\mathscr{A}\hat{f}_k(\mathbf{g}(x), u) = Ag_k(x, u),\ \ k \geq 1$. Let $(X_1(\cdot), \pi_1(\cdot))$ be a solution to the relaxed controlled martingale problem for A. Set $Z_1(t) = \mathbf{g}(X_1(t)),\ \forall\ t \geq 0$. Then

$$P(Z_1(t) \in \mathbf{g}(E)) = 1 \qquad \forall\ t \geq 0. \tag{2.15}$$

Let $\hat{f} \in \mathscr{D}(\mathscr{A})$ be such that $\hat{f} \circ \mathbf{g} = f$ for a prescribed $f \in \mathscr{D}(A)$. Let $0 \leq t_1 < \cdots < t_m < t_{m+1},\ 0 \leq \delta_i \leq t_i,\ \hat{h}_1, \dots, \hat{h}_m \in B(\hat{E} \times U)$, with $\hat{h}_i(\mathbf{g}(x), u) = h_i(x, u)$ for $h_1, \dots, h_m \in B(E \times U)$ and $m \geq 1$. Then

$$\begin{aligned} &\mathbb{E}\biggl[\biggl(\hat{f}(Z_1(t_{m+1})) - \hat{f}(Z_1(t_m)) - \int_{t_m}^{t_{m+1}} \int_U \mathscr{A}\hat{f}(Z_1(s), u)\pi_{1s}(du)\,ds\biggr) \\ &\qquad\qquad \times \prod_{i=1}^{m} \int_{t_i-\delta_i}^{t_i} \int_U \hat{h}_i(Z_1(t_i), u)\pi_{1s}(du)\,ds\biggr] \\ &\quad = \mathbb{E}\biggl[\biggl(f(X_1(t_{m+1})) - f(X_1(t_m)) - \int_{t_m}^{t_{m+1}} \int_U Af(X_1(s), u)\pi_{1s}(du)\,ds\biggr) \\ &\qquad\qquad \times \prod_{i=1}^{m} \int_{t_i-\delta_i}^{t_i} \int_U h_i(X_1(t_i), u)\pi_{1s}(du)\,ds\biggr] \\ &\quad = 0. \end{aligned}$$

It follows that $(Z_1(\cdot), \pi_1(\cdot))$ is a solution to the relaxed controlled martingale problem for $\mathscr{A}$. Conversely, for any such solution $(Z_1(\cdot), \pi_1(\cdot))$ to the relaxed controlled martingale problem for $\mathscr{A}$ satisfying (2.15), we define $X_1(t) = \mathbf{g}^{-1}(Z_1(t))$. Then $(X_1(\cdot), \pi_1(\cdot))$ is a solution to the relaxed controlled martingale problem for A.

Now we are ready to prove the result characterizing the discounted occupation measures.

THEOREM 2.2. *If* $\mu \in \mathscr{P}(E \times U)$ *satisfies* (2.13), *then there exists a solution* $(X(\cdot), \pi(\cdot))$ *to the relaxed controlled martingale problem for* (A, ν_0) *such that* μ *is the discounted occupation measure for this process.*

PROOF. We will prove the theorem in various steps.

Step 1. *Reduction of the problem to the ergodic problem for a related operator* B *and imbedding the new martingale problem into a compact space*: Let $\mathscr{D}(B) \subset C_b(E \times \{-1, 1\})$ be defined by

$$\mathscr{D}(B) = \{f_1 f_2 \colon f_1 \in \mathscr{D}(A),\ f_2 \in C(\{-1, 1\})\}. \tag{2.16}$$

Let $E_0 = E \times U \times \{-1, 1\}$. For $f = f_1 f_2 \in \mathscr{D}(B)$, $Bf \in C_b(E_0)$ is defined by

$$(Bf)(x, u, j) = f_2(j) A f_1(x, u) + \alpha\left(f_2(-j) \int_E f_1\, d\nu_0 - f_1(x) f_2(j)\right). \tag{2.17}$$

Define $\overline{\mu} \in \mathscr{P}(E_0)$ by

$$\overline{\mu} = \mu \otimes \left(\tfrac{1}{2}\delta_1 + \tfrac{1}{2}\delta_{-1}\right).$$

Then it follows from (2.13) that

$$\int_{E_0} Bf\, d\overline{\mu} = 0 \qquad \forall\, f \in \mathscr{D}(B).$$

Clearly, B satisfies Conditions 1 and 2 of Section 1.2. Condition 3 there follows from Theorem 4.10.2 of Ethier and Kurtz (1986). Theorem 2.1 now ensures the existence of a stationary solution $(X(\cdot), \pi(\cdot), Y(\cdot))$ to the relaxed controlled martingale problem for $(B, \tilde{\mu})$ with respect to some filtration $\{\mathscr{F}_t\}$, where $\tilde{\mu}$ is the marginal of $\overline{\mu}$ on $E \times \{-1, 1\}$. Note that if μ_1 denotes the marginal of μ on E, then

$$\tilde{\mu} = \mu_1 \otimes \left(\tfrac{1}{2}\delta_1 + \tfrac{1}{2}\delta_{-1}\right).$$

In general, we do not know about the regularity of paths of this solution. To overcome this, we shall imbed the controlled martingale problem for B into a compact space as was done for the operator A at the beginning of this subsection.

Recall the definition of $\hat{E}$ and $\mathscr{A}$. Let $\mathscr{D}(\mathscr{B}) = \{F = \hat{f} h \colon \hat{f} \in \mathscr{D}(\mathscr{A}), h \in C(\{-1, 1\})\}$ and define the operator $\mathscr{B} \colon \mathscr{D}(\mathscr{B}) \to B(\hat{E} \times U \times \{-1, 1\})$ by

$$(\mathscr{B}(\hat{f} h))(z, u, j) = h(j) \mathscr{A} \hat{f}(z, u) + \alpha\left(h(-j) \int_{\hat{E}} \hat{f}\, d\hat{\nu}_0 - \hat{f}(z) h(j)\right) \tag{2.18}$$

where $\hat{\nu}_0 \in \mathscr{P}(\hat{E})$ is defined by

$$\hat{\nu}_0(\Gamma) = \nu_0(\mathbf{g}^{-1}(\Gamma \cap \mathbf{g}(E)))$$

and $\hat{f}$, h are as in the definition of $\mathscr{D}(\mathscr{B})$. The controlled martingale problems for B and $\mathscr{B}$ are equivalent in the same sense as those for A and $\mathscr{A}$. Let $Z(t) = \mathbf{g}(X(t))$. Then $(Z(\cdot), \pi(\cdot), Y(\cdot))$ is a stationary solution to the relaxed controlled martingale problem for $(\mathscr{B}, \hat{\mu})$, where $\hat{\mu} = \mu_1 \circ \mathbf{g}^{-1} \otimes \left(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}\right)$. Without loss of generality, we assume that the process $(Z(\cdot), \pi(\cdot), Y(\cdot))$ is defined on the entire time axis $(-\infty, \infty)$. Since $\mathscr{D}(\mathscr{B})$ is an algebra that separates points in $\overline{\mathbf{g}(E)} \times \{-1, 1\}$, it is a measure determining set [see, e.g., Theorem 3.4.5 of Ethier and Kurtz (1986)]. From Theorem 4.3.6 of Ethier and Kurtz (1986), it follows that $(Z(\cdot), Y(\cdot))$ admit an r.c.l.l. modification $(\hat{Z}(\cdot), \hat{Y}(\cdot))$ in

$\overline{\mathbf{g}(E)} \times \{-1, 1\}$. Then $(\hat{Z}(\cdot), \pi(\cdot), \hat{Y}(\cdot))$ is a stationary solution to the relaxed controlled martingale problem for $(\mathscr{B}, \hat{\mu})$ with respect to the filtration $\{\mathscr{F}_t\}$, where $\mathscr{F}_t = \sigma(\hat{Z}_s, \hat{Y}_s, \int_a^b \pi_r(f)\,dr\colon\ s \le t, 0 \le a \le b \le t, f \in C(U))$.

Step 2. *Recovering a solution to the relaxed controlled martingale problem for A.* For $k \ge 0$, define inductively stopping times $\{\tau_k\}$ by

$$\tau_0 = 0, \qquad \tau_k = \inf\{t > \tau_{k-1}\colon\ \hat{Y}(t) = -\hat{Y}(\tau_{k-1})\}$$

and set $N(t) = k$ for $\tau_k \le t < \tau_{k+1},\ k \ge 0$. We want to show that, for $\hat{f} \in \mathscr{D}(\mathscr{A})$,

$$\begin{aligned} &\hat{f}(\hat{Z}(t))I_{\{N(t)=k\}} \\ &\quad - \int_0^t \int_U \biggl[\mathscr{A}\hat{f}(\hat{Z}(s), u)\pi_s(du)I_{\{N(s)=k\}} \\ &\qquad + \alpha\biggl(I_{\{N(s)+1=k\}} \int_{\hat{E}} \hat{f}\,d\hat{\nu}_0 - I_{\{N(s)=k\}}\hat{f}(\hat{Z}(s))\biggr)\biggr]\,ds, \qquad t \ge 0, \end{aligned} \tag{2.19}$$

is a martingale with respect to $\{\mathscr{F}_t\}$. It suffices to prove this for a solution $(\hat{Z}^y(\cdot), \pi^y(\cdot), \hat{Y}^y(\cdot))$ to the relaxed controlled martingale problem for $(\mathscr{B}, \mu_1 \circ \mathbf{g}^{-1} \otimes \delta_y)$, $y \in \{-1, 1\}$. (Note that $\mathbb{E}[(\hat{Z}(\cdot), \pi(\cdot), \hat{Y}(\cdot))|\hat{Y}(0) = y]$ is one such solution.) By the optional sampling theorem,

$$\begin{aligned} &\hat{f}(\hat{Z}^y(\tau_{k+1} \wedge t))h(\hat{Y}^y(\tau_{k+1} \wedge t)) - \hat{f}(\hat{Z}^y(\tau_{k-1} \wedge t))h(\hat{Y}^y(\tau_{k-1} \wedge t)) \\ &\qquad - \int_{\tau_{k-1}\wedge t}^{\tau_{k+1}\wedge t} \int_U (\mathscr{B}\hat{f}h)(\hat{Z}^y(s), u, \hat{Y}^y(s))\pi_s(du)\,ds, \qquad t \ge 0, \end{aligned} \tag{2.20}$$

is a martingale for $h \in C(\{-1, 1\})$. Equation (2.20) implies (2.19) on taking $h(\cdot) = I_{\{\cdot=(-1)^k y\}}$.

Define $(Z^2(\cdot), \pi^2(\cdot), N^2(\cdot)) := (\hat{Z}(\tau_2 + \cdot), \pi(\tau_2 + \cdot), N(\tau_2 + \cdot))$. It follows that

$$\begin{aligned} \hat{f}(Z^2(t))I_{\{N^2(t)=k\}} - \int_0^t \int_U \biggl[&\mathscr{A}\hat{f}(Z^2(s), u)\pi_s^2(du)I_{\{N^2(s)=k\}} \\ &+ \alpha\biggl(I_{\{N^2(s)+1=k\}} \int_{\hat{E}} \hat{f}\,d\hat{\nu}_0 - I_{\{N^2(s)=k\}}\hat{f}(Z^2(s))\biggr)\biggr]\,ds \end{aligned}$$

is an $\{\mathscr{F}_t^2\}$-martingale, where $\mathscr{F}_t^2 = \mathscr{F}_{\tau_2+t}$. This in turn implies that

$$I_{\{N^2(t)=N^2(0)\}} \exp(\alpha t) \tag{2.21}$$

is a mean-one martingale with respect to $\{\mathscr{F}_t^2\}$ and, more generally,

$$\begin{aligned} &\hat{f}(Z^2(t))I_{\{N^2(t)=N^2(0)\}}e^{\alpha t} \\ &\qquad - \int_0^t \int_U e^{\alpha s} I_{\{N^2(s)=N^2(0)\}}\mathscr{A}\hat{f}(Z^2(s), u)\pi_s^2(du)\,ds \end{aligned} \tag{2.22}$$

is an $\{\mathscr{F}_t^2\}$-martingale. This suggests that $(Z^2(s), \pi^2(s))$ is a solution under a transformed measure to the relaxed controlled martingale problem for $\mathscr{A}$, where the transformed measure has Radon–Nikodym derivative $I_{\{N^2(t)=N^2(0)\}}\exp(\alpha t)$ on $\mathscr{F}_t^2$ for $t > 0$ with respect to the original measure P. Indeed, this is true but this solution may not have the required occupation measure since without well-posedness of the relaxed controlled martingale problem we cannot show that $(\hat{Z}(\cdot), \pi(\cdot))$ have i.i.d. cycles between τ_k and τ_{k+1}.

To get the required solution, we follow an idea of Kurtz and Stockbridge (1994). Let

$$\tau_{-1} = \sup\{t < 0\colon\ \hat{Y}(t) \neq \hat{Y}(0)\}.$$

Then $W = [\alpha(\tau_1 - \tau_{-1})]^{-1}$ is an $\mathscr{F}_0^2$-measurable, positive random variable with mean 1. Thus, using (2.21) and (2.22), we get that

$$WI_{\{N^2(t)=N^2(0)\}}\exp(\alpha t) \tag{2.23}$$

is a mean-one martingale with respect to $\{\mathscr{F}_t^2\}$ and so is

$$\begin{aligned} &W\hat{f}(Z^2(t))I_{\{N^2(t)=N^2(0)\}}e^{\alpha t} \\ &\qquad - \int_0^t e^{\alpha s}WI_{\{N^2(s)=N^2(0)\}}\mathscr{A}\hat{f}(Z^2(s),u)\pi_s^2(du)\,ds. \end{aligned} \tag{2.24}$$

Let C_U be as in Section 1.2. Let (θ, π^*) denote the coordinate process on $D([0,\infty), \hat{E}) \times C_U$. Define a probability measure Q on $D([0,\infty), \hat{E}) \times C_U$ as follows. For every fixed $T > 0$ and $0 \le t_1 < \cdots < t_m \le T$, $0 \le \delta_i \le t_i$, $f_1, \dots, f_m \in C(\hat{E})$, $h_1, \dots, h_m \in C(\hat{E} \times U)$, $U_1, \dots, U_m \subset U$, $m \ge 1$,

$$\begin{aligned} &\mathbb{E}^Q\Biggl[\prod_{i=1}^m f_i(\theta(t_i))\int_{t_i-\delta_i}^{t_i}\int_{U_i} h_i(\theta(s),u)\pi_s^*(du)\,ds\Biggr] \\ &\qquad = \mathbb{E}\Biggl[\prod_{i=1}^m f_i(Z^2(t_i))\int_{t_i-\delta_i}^{t_i}\int_{U_i} h_i(Z^2(s),u)\pi_s^2(du)\,ds \\ &\qquad\qquad \times WI_{\{N^2(t_m)=N^2(0)\}}\exp(\alpha t_m)\Biggr]. \end{aligned} \tag{2.25}$$

Since (2.23) is a mean-one nonnegative martingale, it is easy to see that (2.25) defines a probability measure.

We claim that, under Q, $(\theta(\cdot), \pi^*(\cdot))$ is a solution to the relaxed controlled martingale problem for $(\mathscr{A}, \hat{\nu}_0)$ with respect to $\{\mathscr{G}_t\}$, where $\mathscr{G}_t = \sigma(\theta_s, \int_a^b \pi_r^*(h)\,dr\colon\ s \le t,\ 0 \le a \le b \le t,\ h \in C(U))$. To see this, let $0 \le t_1 < \cdots < t_m < t_{m+1}$, $0 \le \delta_i \le t_i$, $h_1, \dots, h_m \in C(\hat{E} \times U)$, $m \ge 1$. Since the process in (2.24) is an $\{\mathscr{F}_t^2\}$-martingale, we have

$$\begin{aligned}
&\mathbb{E}^Q\biggl[\biggl(\hat{f}(\theta(t_{m+1})) - \hat{f}(\theta(t_m)) - \int_{t_m}^{t_{m+1}}\int_U \mathscr{A}\hat{f}(\theta(s),u)\pi_s^*(du)\,ds\biggr)\\
&\qquad\qquad \times \prod_{i=1}^m \int_{t_i-\delta_i}^{t_i}\int_U h_i(\theta(t_i),u)\pi_s^*(du)\,ds\biggr]\\
&\quad = \mathbb{E}\biggl[\biggl(\hat{f}(Z^2(t_{m+1}))WI_{\{N^2(t_{m+1})=N^2(0)\}}e^{\alpha t_{m+1}}\\
&\qquad - \hat{f}(Z^2(t_m))WI_{\{N^2(t_m)=N^2(0)\}}e^{\alpha t_m}\\
&\qquad - \int_{t_m}^{t_{m+1}}\int_U e^{\alpha s}WI_{\{N^2(s)=N^2(0)\}}\mathscr{A}\hat{f}(Z^2(s),u)\pi_s^2(du)\,ds\biggr)\\
&\qquad\qquad \times \prod_{i=1}^m \int_{t_i-\delta_i}^{t_i}\int_U h_i(Z^2(s),u)\pi_s^2(du)\,ds\biggr]\\
&\quad = 0.
\end{aligned}$$

By standard monotone class arguments, it follows that $(\theta(\cdot), \pi^*(\cdot))$ is a solution to the relaxed controlled martingale problem for $(\mathscr{A}, \mathscr{L}(\theta(0)))$ with respect to $\{\mathscr{G}_t\}$. Now, for $\Gamma \in \mathscr{B}(\hat{E})$, $Q(\theta(0) \in \Gamma) = \mathbb{E}[WI_{\{\hat{Z}_{\tau_2}\in\Gamma\}}]$. Note that W is $\mathscr{F}_{\tau_1}$-measurable. Hence it follows from (2.19) and the optional sampling theorem that, for $\hat{f} \in \mathscr{D}(\mathscr{A})$,

$$\mathbb{E}[W\hat{f}(\hat{Z}(\tau_2))] = \mathbb{E}[\alpha(\tau_2-\tau_1)W]\int \hat{f}\,d\hat{\nu}_0.$$

For $\hat{f} = \mathbf{1}$, this implies $\mathbb{E}[\alpha(\tau_2-\tau_1)W] = \mathbb{E}[W] = 1$. Thus we get

$$\mathbb{E}[\hat{f}(\hat{Z}(\tau_2))W] = \int \hat{f}\,d\hat{\nu}_0, \qquad \hat{f} \in \mathscr{D}(\mathscr{A}).$$

Since $\mathscr{D}(\mathscr{A})$ is an algebra that separates points in $\overline{\mathbf{g}(E)}$, it is a measure-determining set [cf. Theorem 3.4.5 of Ethier and Kurtz (1986)]. Thus $Q(\theta(0) \in \Gamma) = \hat{\nu}_0(\Gamma)$, which completes the proof of the claim.

Also, since $\hat{Z}(t) \in \mathbf{g}(E)$ a.s. for every t, using Fubini's theorem, we get

$$\mathbb{E}\int_0^\infty I_{\mathbf{g}(E)^c}(\hat{Z}(t))\,dt = 0.$$

In particular,

$$\mathbb{E}\int_0^\infty I_{\mathbf{g}(E)^c}(\hat{Z}(t+\tau_2))\,dt = 0.$$

This implies that $Z^2(t) \in \mathbf{g}(E)$ for almost all $t > 0$, a.s. $[P]$. Thus

$$Q(\theta(t) \in \mathbf{g}(E)) = \mathbb{E}[WI_{\mathbf{g}(E)}(Z^2(t))I_{\{N^2(t)=N^2(0)\}}e^{\alpha t}] = 1$$

for a.a. $t \geq 0$. Let $X^2(t) = \mathbf{g}^{-1}(\theta(t))$. Then $(X^2(\cdot), \pi^*(\cdot))$ is a solution on $(\Omega, \mathscr{F}, Q)$ to the relaxed controlled martingale problem for (A, ν_0) with respect to $\{\mathscr{G}_t\}$. This completes step 2.

Step 3. *Finding the discounted occupation measure for* $(X^2(\cdot), \pi^*(\cdot))$: Note that, for $f \in C(\hat{E} \times U)$,

$$\begin{aligned}
&\alpha \int_0^\infty e^{-\alpha s} \mathbb{E}^Q \left[\int_U f(\theta(s), \cdot)\, d\pi^*(s) \right] ds \\
&\quad = \alpha \int_0^\infty e^{-\alpha s} \mathbb{E} \left[\int_U e^{\alpha s} f(Z^2(s), \cdot)\, d\pi^2(s) W I_{\{\tau_3 - \tau_2 > s\}} \right] ds \\
&\quad = \mathbb{E} \int_0^{\tau_3 - \tau_2} \frac{1}{\tau_1 - \tau_{-1}} \int_U f(Z^2(s), \cdot)\, d\pi^2(s)\, ds \\
&\quad = \mathbb{E} \left[\frac{1}{\tau_1 - \tau_{-1}} \int_{\tau_2}^{\tau_3} \int_U f(\hat{Z}(s), \cdot)\, d\pi(s)\, ds \right].
\end{aligned} \tag{2.26}$$

For simplicity of writing, let us denote $V(s) = \int_U f(\hat{Z}(s), \cdot)\, d\pi(s)$ for every s. Let, for $t \geq 0$,

$$\tau_{-1}^t = \sup\{s < t\colon\ \hat{Y}(s) \neq \hat{Y}(t)\},$$

$$\tau_1^t = \inf\{s > t\colon\ \hat{Y}(s) \neq \hat{Y}(t)\},$$

$$\tau_2^t = \inf\{s > \tau_1^t\colon\ \hat{Y}(s) \neq \hat{Y}(\tau_1^t)\}$$

and

$$\tau_3^t = \inf\{s > \tau_2^t\colon\ \hat{Y}(s) \neq \hat{Y}(\tau_2^t)\}.$$

Then

$$\frac{1}{\tau_1^t - \tau_{-1}^t} \int_{\tau_2^t}^{\tau_3^t} V(s)\, ds \tag{2.27}$$

is stationary in t. Also, for $t \in [\tau_k, \tau_{k+1})$, (2.27) is the same as

$$\frac{1}{\tau_{k+1} - \tau_k} \int_{\tau_{k+2}}^{\tau_{k+3}} V(s)\, ds.$$

Then, writing $\sigma_i = \tau_i$ for $i \geq 1$ and $\sigma_0 = \tau_{-1}$ and recalling that $N(T)$ denotes the number of jumps of $\hat{Y}$ in the interval $[0, T]$, we get

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{\tau_1-\tau_{-1}}\int_{\tau_2}^{\tau_3} V(s)\,ds\right]\\
&\quad=\mathbb{E}\left[\int_0^T \frac{1}{T(\tau_1^t-\tau_{-1}^t)}\int_{\tau_2^t}^{\tau_3^t} V(s)\,ds\,dt\right]\\
&\quad=\frac{1}{T}\mathbb{E}\left[\sum_{k=1}^{N(T)+1}\frac{1}{(\sigma_k-\sigma_{k-1})}\left\{\int_{\sigma_{k+1}}^{\sigma_{k+2}} V(s)\,ds\right\}(T\wedge\sigma_k-\sigma_{k-1}\vee 0)\right]\\
&\quad=\frac{1}{T}\mathbb{E}\int_0^T V(s)\,ds+\frac{1}{T}\mathbb{E}\frac{T\wedge\sigma_1}{\sigma_1-\sigma_0}\int_{\sigma_2}^{\sigma_3} V(s)\,ds-\frac{1}{T}\mathbb{E}\int_0^{T\wedge\sigma_3} V(s)\,ds\\
&\qquad+\frac{1}{T}\mathbb{E}\left[I_{\{N(T)>0\}}\frac{T-\sigma_{N(T)}}{\sigma_{N(T)+1}-\sigma_{N(T)}}\int_{\sigma_{N(T)+2}}^{\sigma_{N(T)+3}} V(s)\,ds\right]\\
&\qquad+\frac{1}{T}\mathbb{E}\left[I_{\{N(T)>1\}}\int_{\sigma_{N(T)+1}}^{\sigma_{N(T)+2}} V(s)\,ds\right]+\frac{1}{T}\mathbb{E}\left[I_{\{N(T)>2\}}\int_{\sigma_{N(T)}}^{\sigma_{N(T)+1}} V(s)\,ds\right]\\
&\qquad-\frac{1}{T}\mathbb{E}\left[I_{\{N(T)>3\}}\int_{\sigma_{N(T)}}^{T} V(s)\,ds\right].
\end{aligned}$$

Since this holds for every T, letting $T\to\infty$, we get

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{\tau_1-\tau_{-1}}\int_{\tau_2}^{\tau_3} V(s)\,ds\right]\\
&\quad=\lim_{T\to\infty}\mathbb{E}\left[\int_0^T \frac{1}{T(\tau_1^t-\tau_{-1}^t)}\int_{\tau_2^t}^{\tau_3^t} V(s)\,ds\,dt\right]\\
&\quad=\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\int_0^T V(s)\,ds+0\\
&\quad=\mathbb{E}\int_U f(\hat{Z}(s),\cdot)\pi_s(du)\\
&\quad=\mathbb{E}\int_U f(Z(s),\cdot)\pi_s(du).
\end{aligned}$$

Recalling that $Z(s)=\mathbf{g}(X(s))$ and $\theta(s)=\mathbf{g}(X^2(s))$, using (2.26) and a change of variables, we get, for any $f\in C_b(E\times U)$,

$$\alpha\int_0^\infty e^{-\alpha s}\mathbb{E}^Q\left[\int_U f(X^2(s),\cdot)\,d\pi^*(s)\right]ds=\int_{E\times U} f\,d\mu.$$

This completes the proof. □

The following corollary follows from Corollary 2.1.

COROLLARY 2.2. *The relaxed control process $\pi^*(\cdot)$ above may be taken to be a relaxed Markov control.*

2.3. *The finite-horizon problem.* Fix $T > 0$. Let $(X(\cdot), u(\cdot))$ be a solution to the controlled martingale problem for (A, ν_0). Let $f \in \mathscr{D}(A)$ and $g \in C_0^1([0, T])$, where

$$C_0^1([0,T]) := \left\{h \in C([0,T]) \colon \frac{\partial h}{\partial t}(\cdot) \text{ exists and is in } C([0,T]),\ h(T) = \frac{\partial h}{\partial t}(T) = 0\right\}.$$

Then

$$\mathbb{E}[f(X(T))g(T)] - \mathbb{E}[f(X(0))g(0)] = \mathbb{E}\left[\int_0^T f(X(s))\frac{\partial g}{\partial s}(s) + g(s)Af(X(s), u(s))\,ds\right],$$

implying

$$T\int_{[0,T]\times E\times U}\left[\left(\frac{\partial}{\partial t} + A\right)fg\right]d\mu = -g(0)\int_E f\,d\nu_0, \tag{2.28}$$

where μ is the corresponding finite-time occupation measure. Our aim is to show that (2.28) characterizes such measures.

Before stating the result, we define a metric on $[0, T] \times E$. Let d denote the original metric on this space. Then, for $(t_1, x_1), (t_2, x_2) \in [0, T] \times E$, define

$$d'((t_1, x_1), (t_2, x_2)) = \min\big(d((t_1, x_1), (t_2, x_2)), d\big((t_1, x_1), T \times E\big) + d(T \times E, (t_2, x_2))\big),$$

where $d((t, x), T\times E) = \inf_{y\in E} d((t, x), (T, y))$. Here d' is a pseudo-metric and any two points $(T, x), (T, y)$ for $x \neq y$ in E are zero d'-distance away from each other. The latter property defines an equivalence relation [viz. $(t, x) \sim (s, y)$ if and only if $d'((t, x), (s, y)) = 0$]. Passing to the quotient space under $\sim$, it is easy to see that d' becomes a complete metric on the same. By abuse of notation we continue to denote this space as $[0, T] \times E$, keeping in mind that $T \times E$ has now collapsed to a point. Note that a sequence (t_n, x_n) converges in the d'-metric if either (t_n, x_n) converges in the d-metric or $t_n \to T$. Now we are ready for the characterization of the finite-time occupation measures.

THEOREM 2.3. *Suppose* $\mu \in \mathscr{P}([0, T] \times E \times U)$ *satisfies* (2.28) *for* $f \in \mathscr{D}(A)$, $g \in C_0^1([0, T])$. *Then*:

(i) *the marginal of* μ *on* $[0, T]$ *is the normalized Lebesgue measure*;

(ii) *there exists a solution* $(X(\cdot), \pi(\cdot))$ *to the relaxed controlled martingale problem for* (A, ν_0) *such that*

$$\mathbb{E}\left[\int_0^T h(s, X(s), u)\pi_s(du)\,ds\right] = T\int_{[0,T]\times E\times U} h(s, x, u)\mu(ds, dx, du)$$

for all $h \in C_b([0, T] \times E \times U)$.

PROOF. The first claim follows easily on setting $f \equiv 1$ in (2.28). We now consider the complete, separable metric space $([0, T] \times E, d')$ defined above. By Δ we will denote the point $T \times E \in ([0, T] \times E, d')$.

Let $\mathscr{D}(B) \subset C_b([0, T] \times E)$ be the algebra of functions generated by constants and the set $\{gf\colon\ g \in C_0^1([0, T]),\ f \in \mathscr{D}(A)\}$. Define the operator $B\colon \mathscr{D}(B) \to C_b([0, T] \times E \times U)$ as follows: for $g \in C_0^1([0, T]),\ f \in \mathscr{D}(A)$,

$$
\begin{aligned}
[B(gf)](t, x, u) &= f(x)\frac{\partial g}{\partial t}(t) + g(t)Af(x, u) \qquad &&\text{if } (t, x) \in [0, T) \times E,\\
&= 0 &&\text{if } (t, x) = \Delta,\\
B\mathbf{1} &= 0.
\end{aligned}
$$

Note that B satisfies all the conditions of Theorem 2.2. Define $\tilde{\mu} \in \mathscr{P}([0, T] \times E \times U)$ by

$$
\begin{aligned}
&\int_{[0, T]\times E\times U} f(t, x, u)\tilde{\mu}(dt, dx, du)\\
&\qquad = \int_{[0, T)\times E\times U} e^{-t/T} f(t, x, u)\mu(dt, dx, du) + e^{-1}\int_U f(\Delta, u)\eta(du)\\
&\qquad\qquad\qquad \forall\, f \in C_b([0, T] \times E \times U),
\end{aligned}
$$

where η is the marginal of μ on U. Then, for $f \in \mathscr{D}(A),\ g \in C_0^1([0, T])$,

$$
\begin{aligned}
&\int_{[0, T]\times E\times U} B(fg)\, d\tilde{\mu}\\
&\qquad = \int_{[0,T)\times E\times U} e^{-t/T}\left(\frac{\partial}{\partial t} + A\right) fg\, d\mu\\
&\qquad = \int_{[0,T]\times E\times U}\left(\frac{\partial}{\partial t} + A\right) fe^{-t/T} g\, d\mu + \frac{1}{T}\int_{[0,T]\times E\times U} e^{-t/T} fg\, d\mu\\
&\qquad = -\frac{1}{T}g(0)\int_E f\, d\nu_0 + \frac{1}{T}\int_{[0, T]\times E} fg\, d\mu_1,
\end{aligned}
$$

where μ_1 is the marginal of $\tilde{\mu}$ on $[0, T] \times E$. This is of the form (2.13) and thus we can apply Theorem 2.2 to get a solution $(Y(\cdot), X(\cdot), \pi(\cdot))$ to the relaxed controlled martingale problem for $(B, \delta_0 \otimes \nu_0)$ such that, for $f \in C_b([0, T] \times E \times U)$,

$$
\begin{aligned}
&\int_0^\infty e^{-s/T}\mathbb{E}\left[\int_U f(Y(s), X(s), u)\pi_s(du)\right] ds\\
&\qquad = \int_{[0, T]\times E\times U} f(t, x, u)\tilde{\mu}(dt, dx, du).
\end{aligned} \tag{2.29}
$$

From the definition of B, it is clear that $Y(t) = t \wedge T$. Picking f above so that $f(\Delta, u) = 0$ for all u, we have

$$\int_0^T e^{-s/T}\mathbb{E}\left[\int_U f(s, X(s), u)\pi_s(du)\right] ds$$
$$= \int_{[0,T]\times E\times U} e^{-t/T} f(t, x, u)\mu(dt, dx, du).$$

The second claim follows. □

We shall call $\pi(\cdot)$ a relaxed time inhomogeneous Markov control if $\pi_t = v(X(t), t)$, $t \geq 0$, for some measurable v: $E \times [0, \infty) \to \mathscr{P}(U)$. The following corollary follows from Corollary 2.2.

COROLLARY 2.3. *In the above theorem, $\pi(\cdot)$ may be taken to be a relaxed time-inhomogeneous Markov control.*

The foregoing developments have an interesting offshoot. Suppose $\{\eta(\cdot)\}$ is a $\mathscr{P}(E \times U)$-valued process satisfying

$$\int_E f\, d\nu(t) = \int_E f\, d\nu(0) + \int_0^t \int_{E\times U} Af\, d\eta(s)\, ds, \quad \forall\, t \geq 0,\; f \in \mathscr{D}(A), \tag{2.30}$$

where $\nu(t)$ is the marginal of $\eta(t)$ on E for all $t \geq 0$.

THEOREM 2.4. *Suppose that $\{\mathscr{L}(X(\cdot), \pi(\cdot))|(X(\cdot), \pi(\cdot))$ solves the relaxed controlled martingale problem for $(A, \xi)\} \subset \mathscr{P}(C([0,\infty), E) \times C_U)$ is compact for each choice of ξ. Let $\eta(t)$ satisfy* (2.30). *Then there exists a solution $(\overline{X}(\cdot), \overline{\pi}(\cdot))$ of the relaxed controlled martingale problem for $(A, \nu(0))$ such that*

$$\mathbb{E}\left[\int_U f(\overline{X}(t), u)\overline{\pi}_t(du)\right] = \int_{E\times U} f\, d\eta(t) \qquad \forall\, t \geq 0,\; f \in C_b(E \times U). \tag{2.31}$$

Furthermore, $\overline{\pi}(\cdot)$ may be taken to be a relaxed time-inhomogeneous Markov control.

PROOF. For $t > 0$, define $\mu(t) \in \mathscr{M}(E \times U)$ by

$$\int_{E\times U} f\, d\mu(t) = \int_0^t \int_{E\times U} f\, d\eta(s)\, ds, \qquad \forall\, f \in C_b(E \times U)$$

and let $L(t) = \{\mathscr{L}(X(\cdot), \pi(\cdot))|(X(\cdot), \pi(\cdot))$ solves the relaxed controlled martingale problem for $(A, \nu(0))$ and its finite-time occupation measure on $[0, t]$ is $t^{-1}\mu(t)\}$. From our hypotheses and the foregoing, $L(t)$ is a compact, nonempty set for each $t > 0$. So is $L = \cap_{t>0} L(t)$. If $\mathscr{L}(X(\cdot), \pi(\cdot)) \in L$, then $(X(\cdot), \pi(\cdot))$ is a solution to the relaxed controlled martingale problem for $(A, \nu(0))$ and, for each $t > 0$, its finite-time occupation measure on $[0, t]$ is $t^{-1}\mu(t)$. Thus

$$\int_0^t \mathbb{E}\left[\int_U f(X(s), u)\pi_s(du)\right] ds = \int_0^t \int_{E\times U} f\, d\eta(s), \qquad \forall\, f \in C_b(E \times U).$$

Clearly, (2.31) holds for a.a. t. The qualification "a.a." can be dropped by suitably modifying $s \mapsto \pi_s$ on a set of zero Lebesgue measure. The last claim follows from Corollary 2.3. □

In particular, it follows that if $\eta(\cdot)$ is such that (2.31) holds for some solution $(X(\cdot), \pi(\cdot))$ to the relaxed controlled martingale problem for $(A, \nu(0))$, then there is another solution $(X'(\cdot), \pi'(\cdot))$ such that $\pi'(\cdot)$ is the relaxed time inhomogeneous Markov control and (2.31) holds with $(X'(\cdot), \pi'(\cdot))$ replacing $(X(\cdot), \pi(\cdot))$. This settles a conjecture of Borkar (1993) in the affirmative.

3. Existence of optimal controls.

3.1. *The ergodic problem.* We shall work with a specific *test case* viz. the controlled nonlinear filter described in Example 4. Recall the processes $(X(\cdot), Y(\cdot), \mu(\cdot))$ and the operators L and A defined there. We consider the following two conditions.

CONDITION 4. $\lim_{\|x\|\to\infty} \inf_u k(x, u) = \infty$.

CONDITION 5. There exists a twice continuously differentiable function w: $\mathbb{R}^d \to \mathbb{R}^+$ such that:

(i) $\lim_{\|x\|\to\infty} w(x) = \infty$ uniformly in $\|x\|$;
(ii) $\lim_{\|x\|\to\infty} \sup_u Lw(x, u) = -\infty$ uniformly in $\|x\|$;
(iii) $\mathbb{E}[\int_0^T \|\sigma^*(X(t))\nabla w(X(t))\|^2\, dt] < \infty$ for all $T > 0$ and all $X(\cdot)$ as above.

Let $\Gamma \subset \mathscr{P}(\mathscr{P}(\mathbb{R}^d) \times U)$ denote the set of all possible ergodic occupation measures. Then, by Theorem 2.1,

$$\Gamma = \{\eta \in \mathscr{P}(\mathscr{P}(\mathbb{R}^d) \times U)\colon \int Af\, d\eta = 0\ \forall\ f \in \mathscr{D}(A)\}.$$

Clearly, Γ is closed. Let $\beta = \inf_\Gamma \int \bar{k}\, d\eta$, where $\bar{k}$: $\mathscr{P}(\mathscr{P}(\mathbb{R}^d) \times U) \to \mathbb{R}$ is defined by $\bar{k}(\mu, u) = \int k(\cdot, u)\, d\mu$ for $\mu \in \mathscr{P}(\mathbb{R}^d)$, $u \in U$.

LEMMA 3.1. *Under either Condition* 4 *or* 5, *there exists a stationary solution* $(\mu(\cdot), \pi(\cdot))$ *of the relaxed controlled martingale problem for A such that if* η^* *is the corresponding ergodic occupation measure*, *then* $\int \bar{k} d\eta^* = \beta$.

PROOF. Condition 4 implies that the sets $\{\eta\colon \int \bar{k}\, d\eta < c\}$ are compact for all $c < \infty$. Since $\eta \mapsto \int \bar{k}\, d\eta$ is lower semicontinuous, the claim follows.

Now assume Condition 5 and let $(\mu(\cdot), \pi(\cdot))$be a stationary solution to the relaxed controlled martingale problem for A. Let $X(\cdot)$ be the actual controlled process in the background. If $\xi_t = \mathscr{L}(X(t))$, then $\int f\, d\xi_t = \mathbb{E}[\int f\, d\mu(t)]$ for $f \in C_b(\mathbb{R}^d)$, $t \geq 0$, implying in particular that $\xi_t \equiv \xi_0$ for all $t \geq 0$. It follows as in Corollary 5.1, page 174, of Borkar (1989) that ξ_0 belongs to a tight set of probability measures on $\mathbb{R}^d$. From Lemma 3.6, page 126, of the above reference, it follows that $\mathscr{L}(\mu(t))$ belongs to a tight set of $\mathscr{P}(\mathscr{P}(\mathbb{R}^d))$ regardless of the choice of the stationary solution. Since U is compact, it follows that Γ is tight and hence compact. The claim follows. □

Let $(\mu(\cdot), \pi(\cdot))$ be any solution of the relaxed controlled martingale problem for A. Define a $\mathscr{P}(\mathbb{R}^d)$-valued process $\{\phi_t\}$ by

$$\int f\, d\phi_t = \frac{1}{t}\int_0^t \mathbb{E}\left[\int_U f(X(s), u)\pi_s(du)\right] ds \qquad \forall\, t \geq 0,\ f \in C_b(\mathbb{R}^d \times U),$$

and a $\mathscr{P}(\mathscr{P}(\mathbb{R}^d))$-valued process $\{\Phi_t\}$ by

$$\int f\, d\Phi_t = \frac{1}{t}\int_0^t \mathbb{E}\left[\int_U f(\mu(s), u)\pi_s(du)\right] ds \qquad \forall\, t \geq 0,\ f \in C_b(\mathscr{P}(\mathbb{R}^d) \times U).$$

LEMMA 3.2. $\liminf_{t\to\infty} \int \bar{k}\, d\Phi_t \geq \beta$.

PROOF. Consider Condition 5 first. Then $\{\phi_t\}$ is tight as argued in Section 6.5 of Borkar (1989). By Lemma 3.6, page 126, of this reference, $\{\Phi_t\}$ is also tight. Now, for $f \in \mathscr{D}(A)$,

$$\mathbb{E}[f(\mu(t))] - \mathbb{E}[f(\mu(0))] = t\int Af\, d\Phi_t, \qquad t \geq 0.$$

Dividing by t and letting $t \to \infty$, we conclude that any limit point Φ of $\{\Phi_t\}$ must satisfy $\int Af\, d\Phi = 0$, implying that $\Phi \in \Gamma$. The claim follows.

Now, consider Condition 4. Suppose that along a subsequence $t_n \uparrow \infty$ of $[0, \infty)$, $\Phi(t_n)$ is tight. Then arguing as above, we conclude that $\liminf_{n\to\infty} \int \bar{k}\, d\Phi(t_n) \geq \beta$. If not, Condition 4 implies that $\lim \int \bar{k}\, d\Phi(t_n) = \infty$ and thus $\liminf \int \bar{k}\, d\Phi(t_n) \geq \beta$ trivially. The claim follows. □

This result justifies confining our attention to stationary solutions of the relaxed controlled martingale problem. Using Lemma 3.1 and Corollary 2.1, we conclude that there exists an optimal relaxed Markov control such that the corresponding pair $(X(\cdot), \pi(\cdot))$ is stationary. In fact, we may replace "stationary" by "ergodic" by invoking the ergodic decomposition thereof. Now, we shall show the existence of an optimal Markov solution along the lines of Borkar (1995). For this purpose, we need a result from Borkar and Sunilkumar (1995) which we outline below. Recall the space C_U. Let $\mathbf{a}' = \{\mathscr{L}(\mu(\cdot), \pi(\cdot)) | (\mu(\cdot), \pi(\cdot))$ is a solution of the relaxed controlled martingale problem for $(A, \nu_0)\}$, where ν_0 is prescribed. Then $\mathbf{a}' \in \mathscr{P}(\mathscr{P}(\mathbb{R}^d) \times C_U)$ is compact and convex. Define an equivalence relation $\sim$ on $\mathbf{a}'$ as follows: $\mathscr{L}(\mu(\cdot), \pi(\cdot)) \sim \mathscr{L}(\mu'(\cdot), \pi'(\cdot))$ if $\mathscr{L}(\mu(t), \pi(t)) = \mathscr{L}(\mu'(t), \pi'(t))$ for a.a. t and let $\mathbf{a}$ denote the equivalence classes under $\sim$. Then $\mathbf{a}$ is compact and convex in the quotient topology. We denote by $\langle\mu(\cdot), \pi(\cdot)\rangle^{\sim}$ the $\sim$-equivalence class that contains $\mathscr{L}(\mu(\cdot), \pi(\cdot))$. It is proved in Borkar and Sunilkumar (1995) that each representative of an extremal element of $\mathbf{a}$ is a Markov process.

Let $\mathscr{L}(\mu(\cdot), \pi(\cdot)), \mathscr{L}(\mu_i(\cdot), \pi_i(\cdot)) \in \mathbf{a}'$, $\alpha_i \in (0,1)$, $1 \le i \le m$, $m \ge 2$, be such that $\sum_i \alpha_i = 1$, $\pi(\cdot) = v(X(\cdot))$ for a measurable $v\colon \mathscr{P}(\mathbb{R}^d) \to \mathscr{P}(U)$ and

$$\langle \mu(\cdot), \pi(\cdot)\rangle^{\sim} = \sum_{i=1}^{m} \alpha_i \langle \mu_i(\cdot), \pi_i(\cdot)\rangle^{\sim}.$$

LEMMA 3.3. *For* $1 \le i \le m$, $\pi_i(t) = v(\mu_i(t))$ *a.s for a.a.* t.

PROOF. For all t outside a set of zero Lebesgue measure, the following holds. Let ξ_i, ξ, ψ_i, ψ denote the laws of $(\mu_i(t), \pi_i(t))$, $(\mu(t), \pi(t))$, $\mu_i(t)$, $\mu(t)$, respectively, for $1 \le i \le m$. Disintegrate ξ_i, ξ as

$$\xi_i(dx, dy) = \psi_i(dx) q_i(x, dy), \qquad 1 \le i \le m,$$
$$\xi(dx, dy) = \psi(dx)\delta_{v(x)}(dy),$$

where $q_i\colon \mathscr{P}(\mathbb{R}^d) \to \mathscr{P}(\mathscr{P}(U))$ are measurable. Clearly, $\psi = \sum_i \alpha_i \psi_i$. Let

$$\Lambda_i(x) = \alpha_i \frac{d\psi_i}{d\psi}(x), \qquad x \in \mathscr{P}(\mathbb{R}^d).$$

Then $\xi = \sum_i \alpha_i \xi_i$ must disintegrate as

$$\xi(dx, dy) = \psi(dx)\biggl(\sum_i \Lambda_i(x) q_i(x, dy)\biggr),$$

implying that, for all x outside a ψ-null set,

$$\sum_i \Lambda_i(x) q_i(x, dy) = \delta_{v(x)}(dy).$$

Since a Dirac measure cannot be a convex combination of two or more distinct probability measures, the claim follows. □

Fix $\mathscr{L}(\mu(\cdot), \pi(\cdot)) \in \mathbf{a}'$. By the result of Borkar and Sunilkumar (1995) mentioned above and Choquet's theorem [see pages 140–141 of Choquet (1969)], it follows that $\langle \mu(\cdot), \pi(\cdot)\rangle^{\sim}$ is the barycenter of a probability measure on $\{\langle \mu'(\cdot), \pi'(\cdot)\rangle^{\sim} \in \mathbf{a} | \mu'(\cdot)$ is a Markov process$\}$. Thus there exists a process $(\bar{\mu}(\cdot), \bar{\pi}(\cdot))$ such that $\mathscr{L}(\bar{\mu}(\cdot), \bar{\pi}(\cdot)) \in \langle \mu(\cdot), \pi(\cdot)\rangle^{\sim}$ and such that $\mathscr{L}(\bar{\mu}(\cdot), \bar{\pi}(\cdot))$ is the barycenter of a probability measure Φ on the set $M = \{\mathscr{L}(\mu'(\cdot), \pi'(\cdot)) \in \mathbf{a}' | \mu'(\cdot)$ is a Markov process$\}$. From Theorem 1.1.5, page 12, and the remarks at the beginning of page 132 in Borkar (1989), it follows that when $\mathscr{L}(\mu'(\cdot), \pi'(\cdot)) \in M$, then $\pi'(t) = \bar{v}(\mu'(t), t)$ a.s. for a measurable $\bar{v}\colon \mathscr{P}(\mathbb{R}^d) \times \mathbb{R}^+ \to \mathscr{P}(U)$. Suppose $\pi(\cdot) = v(\mu(\cdot))$ as before.

LEMMA 3.4. *For* Φ-*a.s.* γ, *where* $\gamma = \mathscr{L}(\hat{\mu}(\cdot), \hat{\pi}(\cdot))$ (*say*), $\hat{\pi}(t) = v(\hat{\mu}(t))$ *a.s. for a.a.* t.

PROOF. Construct on $\mathscr{P}(\mathbf{a}') \times C([0,\infty), \mathscr{P}(\mathbb{R}^d)) \times C_U$ the probability measure $\psi(d\rho, dx, dy) = \Phi(d\rho)\rho(dx, dy)$ and let $(\xi, \tilde{\mu}(\cdot), \tilde{\pi}(\cdot))$ be the canonical random variables on this space. That is, if $\omega = (\omega_1, \omega_2, \omega_3)$, with $\omega_1 \in \mathscr{P}(\mathbf{a}')$, $\omega_2 = \omega_2(\cdot) \in C([0,\infty), \mathscr{P}(\mathbb{R}^d))$ and $\omega_3 = \omega_3(\cdot) \in C_U$, denotes a typical element of $\mathscr{P}(\mathbf{a}') \times C([0,\infty), \mathscr{P}(\mathbb{R}^d)) \times C_U$, then $\xi(\omega) = \omega_1$, $\tilde{\mu}_t(\omega) = \omega_2(t)$ and $\tilde{\pi}_t(\omega) = \omega_3(t)$, $\forall\ t \geq 0$. For each such t, we define a measurable evaluation map $y(\cdot) \in C_U \mapsto y(t) \in \mathscr{P}(U)$ such that $t \mapsto y(t)$ a.e. agrees with $y(\cdot)$. [See the first paragraph of Section 3 of Borkar (1995).] Let $\xi_t(d\rho, dx, du) = \Phi(d\rho)\rho_t(dx, du)$, where ρ_t is the image of ρ under the map $(x(\cdot), y(\cdot)) \in C([0,\infty), \mathscr{P}(\mathbb{R}^d)) \times C_U \mapsto (x(\cdot), y(t)) \in C([0,\infty), \mathscr{P}(\mathbb{R}^d)) \times \mathscr{P}(U)$. Thus ξ_t is the law of $(\xi, \tilde{\mu}(\cdot), \tilde{\pi}(t))$, where the evaluation map $\tilde{\pi}(\cdot) \mapsto \tilde{\pi}(t)$ is in the above sense. For all t outside a set of Lebesgue measure 0, the following argument applies.

Since Φ is supported on M, by the remarks preceding the statement of this lemma, it follows that ξ_t must disintegrate as

$$\xi_t(d\rho, dx, du) = \Phi(d\rho)\nu_\rho(dx)\delta_{f(\rho, x(t), t)}(du),$$

where $\rho \mapsto \nu_\rho$ is the regular conditional law of $\tilde{\mu}(\cdot)$ given ξ and $f\colon \mathscr{P}(\mathbf{a}') \times \mathscr{P}(\mathbb{R}^d) \times \mathbb{R}^+ \to \mathscr{P}(U)$ is a measurable map. Of course, $x(t)$ is the evaluation of $x = x(\cdot) \in C([0,\infty), \mathscr{P}(\mathbb{R}^d))$ at t. Thus, for any $h \in C(\mathscr{P}(U))$,

$$\mathbb{E}[h(\tilde{\pi}(t)) | \xi, \tilde{\mu}(t)] = h(f(\xi, \tilde{\mu}(t), t)) \quad \text{a.s.}$$

for a.a. t. Let $\overline{A}_n = [A_{n1}, \ldots, A_{nm_n}]$, $n \geq 1$, be a sequence of finite partitions of $\mathbf{a}'$ such that:

(i) $\overline{A}_{n+1}$ refines $\overline{A}_n$;
(ii) A_{ni} are Borel with $\Phi(A_{ni}) > 0$;
and if $\sigma(\overline{A}_n)$ is the σ-field generated by $\overline{A}_n$ for $n \geq 1$, then
(iii) $\vee_n \sigma(\overline{A}_n)$ is the Borel σ-field of $\mathbf{a}'$.

Such a sequence of partitions exists as $\mathbf{a}'$ is Polish. Now

$$\begin{aligned} &\mathbb{E}[h(\tilde{\pi}(t)) | \tilde{\mu}(\cdot), I_{\{\xi \in A_{ni}\}},\ 1 \leq i \leq m_n] \\ &\qquad = \sum_{i=1}^{m_n} \Phi(A_{ni}) \mathbb{E}[h(\pi(t)) | \mu_i(\cdot) = \alpha(\cdot)]\Big|_{\alpha(\cdot) = \tilde{\mu}(\cdot)}, \end{aligned}$$

where $\mathscr{L}(\mu_i(\cdot), \pi_i(\cdot))$ is the barycenter of the probability measure

$$\frac{\Phi(d\rho) I_{\{\rho \in A_{ni}\}}}{\Phi(A_{ni})}, \qquad 1 \leq i \leq n.$$

(Recall that $\mathbf{a}'$ is convex.) Clearly, $\langle \tilde{\mu}(\cdot), \tilde{\pi}(\cdot) \rangle\tilde{} = \langle \mu(\cdot), \pi(\cdot) \rangle\tilde{}$ is a convex combination of $\{\langle \mu_i(\cdot), \pi_i(\cdot) \rangle\tilde{},\ 1 \leq i \leq n\}$. By the preceding lemma, $\pi_i(t) = v(\mu_i(t))$ a.s. Thus

$$\mathbb{E}[h(\tilde{\pi}(t)) | \tilde{\mu}(\cdot), I_{\{\xi \in A_{ni}\}},\ 1 \leq i \leq m_n] = h(v(\tilde{\mu}(t))) \quad \text{a.s.}$$

Let $n \to \infty$. The martingale convergence theorem yields

$$h(v(\tilde{\mu}(t))) \to h(f(\xi, \tilde{\mu}(t), t)) \quad \text{a.s.}$$

Since $h \in C(\mathscr{P}(U))$ was arbitrary, $v(\tilde{\mu}(t)) = f(\xi, \tilde{\mu}(t), t)$ a.s. Thus

$$\mathbb{E}[v(\tilde{\mu}(t))|\xi = \rho] = \mathbb{E}[f(\xi, \tilde{\mu}(t), t)|\xi = \rho] \quad \text{for } \Phi\text{-a.s. } \rho.$$

In other words, $v(\hat{\mu}(t)) = f(\rho, \hat{\mu}(t), t)$ a.s. for a.a. t and Φ a.s. $\rho = \mathscr{L}(\hat{\mu}(\cdot), \hat{\pi}(\cdot))$. The claim follows. □

If the $\mathscr{L}(\mu(\cdot), \pi(\cdot))$ that we started with was an optimal stationary solution, a little thought (in view of Lemma 3.2) shows that for Φ-a.s. γ, if $\gamma = \mathscr{L}(\hat{\mu}(\cdot), \hat{\pi}(\cdot))$, then $\mathscr{L}(\hat{\mu}(\cdot), \hat{\pi}(\cdot))$ is also optimal. Thus we have the existence of optimal solutions such that the control is relaxed Markov and the corresponding $\mu(\cdot)$ is either stationary (even ergodic) and/or Markov. Ideally, one would like to ensure both stationarity and the Markov property at the same time, but such a result eludes us at present.

For the finite-dimensional controlled diffusions described in Section 1.4, the foregoing goes over in toto and this program has, in fact, been carried out in Borkar (1995).

For the Hilbert space–valued controlled diffusions and stochastic evolutions described in Section 1.4, the above program can be carried out if one finds appropriate analogs of Conditions 4 and 5 above that enable us to generalize Lemmas 3.1 and 3.2 to this setup. Condition 5 clearly should be replaced by some uniform stability condition that ensures compactness of Γ and tightness for the *empirical measures* $\{\Phi_t\}$ defined by

$$\int f\, d\Phi_t = \frac{1}{t}\int_0^t \mathbb{E}\left[\int_U f(X(s), \cdot)\pi_s(du)\right] ds \qquad \forall\, t \geq 0,\ f \in C_b(E \times U),$$

for any control policy. As for Condition 4, one would like to mimic it in its original form, but this does not seem to work out. Instead, a more convenient condition viz. $\{(x, u)\colon\ k(x, u) \leq r\}$ is compact for all $r \in \mathbb{R}$, has to be imposed. Note, incidentally, that if this condition is true, k cannot be finite valued everywhere.

3.2. *Other problems.* As for the existence results for discounted and finite-time problems, we shall confine ourselves only to brief remarks for two reasons. The first is that the details mimic (and are generally simpler than) those for the ergodic problem. The second reason is that the end product, that is, the existence result, does not improve upon results already deduced by other means such as Krylov's Markov selection method. [See El Karoui, Nguyen and Jeanblanc-Pique (1987, 1988)].

Consider first the discounted problem for the nonlinear filter $\mu(\cdot)$. The existence of an optimal solution for a prescribed initial condition can be easily established by standard compactness methods. One proves that the solution measures $\mathscr{L}(\mu(\cdot), \pi(\cdot))$ for a prescribed law $\mathscr{L}(\mu(0))$ form a tight set [Lemma 3.7, page 128, of Borkar (1989)] and therefore so do the discounted occupation measures. But the latter set, characterized by (2.13), is clearly closed and hence compact. Since the cost functional is lower semicontinuous, it attains a minimum. By Corollary 2.2, the optimal control may be taken to be relaxed Markov. Now the arguments of Lemmas 3.3 and 3.4 above may be used to deduce that there exists an optimal solution $\mathscr{L}(\mu(\cdot), \pi(\cdot))$ (for a prescribed initial condition) such that $\mu(\cdot)$ is a Markov process and $\pi(\cdot)$ a relaxed Markov control. Similar arguments apply to the other examples. [For stochastic semilinear evolutions, see Borkar and Govindan (1994) for compactness results.]

Analogous comments apply to the finite-horizon problem.

4. Dual optimization problems.

4.1. *Infinite-dimensional linear programming.* We recall from Anderson and Nash (1987) some facts concerning infinite-dimensional linear programming. Two topological vector spaces X, Y will be said to form a dual pair if there exists a bilinear form $\langle\cdot,\cdot\rangle\colon X \times Y \to \mathbb{R}$ such that the functions $x \mapsto \langle x, y\rangle$ for $y \in Y$ separate points of X and the functions $y \mapsto \langle x, y\rangle$ for $x \in X$ separate points of Y. We shall endow X with the $\sigma(X, Y)$ topology which is the coarsest topology required to render continuous the maps $x \mapsto \langle x, y\rangle$, $y \in Y$, and endow Y with the dual topology. Let P be the positive cone in X and define the dual cone $P^* \subset Y$ by

$$P^* = \{y \in Y\colon \langle x, y\rangle \geq 0 \text{ for all } x \in P\}.$$

Let Z, W be a dual pair. Let $F\colon X \to Z$ be a $\sigma(X, Y) - \sigma(Z, W)$-continuous linear map. Define $F^*\colon W \to X^*$, the algebraic dual of X, by $\langle Fx, w\rangle = \langle x, F^*w\rangle$, $x \in X$, $w \in W$. The *primal* linear programming problem is

$$\begin{aligned}&\text{minimize } \langle x, c\rangle\\&\text{subject to } Fx = b,\ x \in P,\end{aligned}$$

where $b \in Z$, $c \in Y$ are given. Let β denote the infimum of $\langle x, c\rangle$ subject to these constraints. The dual problem is

$$\begin{aligned}&\text{maximize } \langle b, w\rangle\\&\text{subject to } -F^*w + c \in P^*,\ w \in W.\end{aligned}$$

Let β' denote the supremum of $\langle b, w\rangle$ subject to these constraints. It is known that $\beta \geq \beta'$. Let

$$\begin{aligned}C &= \{x \in P\colon Fx = b\},\\D &= \{(Fx, \langle x, c\rangle)\colon x \in P\}.\end{aligned}$$

We shall use the following result from Anderson and Nash (1987) (see page 53).

THEOREM 4.1. *If C is nonempty, D is closed and $x \mapsto \langle x, c\rangle$ attains its minimum on C, then $\beta = \beta'$.*

4.2. *The ergodic problem.* We assume that there exists a continuous function h: $E \to \mathbb{R}^+$ with $\inf_{x\in E} h(x) > 0$ and $\sup_{x,u} |k(x,u)/h(x)| < \infty$. Let X denote the space of finite signed measures μ on $E \times U$ satisfying $\int_{E\times U} h(x)|\mu(dx,du)| < \infty$ and let Y denote the space of continuous functions f: $E \times U \to \mathbb{R}$ satisfying $\sup_{x,u} |f(x,u)/h(x)| < \infty$. These form a dual pair under the bilinear form $\langle \mu, f\rangle_{X,Y} = \int_{E\times U} f\, d\mu$. Let A be the operator as in Section 2.1. Let $\overline{W} = \mathscr{D}(A)$, rendered a normed linear space with norm $\|f\| = \sup_x |f(x)| + \sup_{x,u} |Af(x,u)|$, and let $\overline{Z} = \overline{W}^*$ be the space of bounded linear functionals on $\overline{W}$. Let $Z = \overline{Z}\times\mathbb{R}$, $W = \overline{W}\times\mathbb{R}$, rendered a dual pair with the bilinear form $\langle z, w\rangle_{Z,W} = \langle \overline{z}, \overline{w}\rangle_{\overline{W}^*,W} + ab$, where $z = (\overline{z}, a)$, $w = (\overline{w}, b)$. Letting $\mathbf{1}$ denote the constant function identically equal to 1, define F: $X \to Z$ by $F(\mu) = (\nu, \int \mathbf{1}\, d\mu)$, where $\nu \in \overline{W}^*$ is defined by

$$\langle f, \nu\rangle_{\overline{W},\overline{W}^*} = -\int Af\, d\mu, \qquad f \in \overline{W} = \mathscr{D}(A).$$

Our primal problem is

$$\text{minimize } \langle \mu, k\rangle_{X,Y}$$
$$\text{subject to } F\mu = (\theta, 1),\ \mu \in P,$$

where θ is the zero element of $\overline{Z}$. In other words,

minimize $\int k\, d\mu$

subject to $\int Af\, d\mu = 0$ for $f \in \mathscr{D}(A)$, μ is a probability measure.

The dual problem becomes

$$\text{maximize } \langle(\theta, 1), (f, a)\rangle_{Z,W} = a$$
$$\text{subject to } Af - a + k \geq 0,\ f \in \mathscr{D}(A).$$

Note that β is now the optimal ergodic cost. We shall assume that C, the set of ergodic occupation measures, is compact in X [in the $\sigma(X,Y)$ topology]. We then have the following analog of Theorem 4.1.

THEOREM 4.2. $\beta = \sup\{a \in \mathbb{R}\colon \inf_u(Af(x,u) + k(x,u)) \geq a,\ f \in \mathscr{D}(A)\}$.

PROOF. We only need to verify that the set D is closed. Let $\{\mu_n\} \in C$ be such that $\nu_n = F\mu_n \to \nu \in Z$ and $\int k\, d\mu_n \to d \in \mathbb{R}$. Let $\mu_n \to \mu$ along a subsequence which we relabel by $\{\mu_n\}$. Then $\int Af\, d\mu_n \to \int Af\, d\mu$ for $f \in \mathscr{D}(A)$, implying $\nu = F\mu$. Also, $\int k\, d\mu_n \to \int k\, d\mu$. Thus D is closed. The above result now follows from Theorem 4.1. □

REMARK 1. The compactness condition on C could be relaxed to: $\{\mu \in C\colon \int k\, d\mu \leq r\}$ is compact in X for all $r \in \mathbb{R}$. Either condition will have to be verified for the problem at hand on a case-by-case basis. Compare these with Conditions 4 and 5 of Section 3.1.

REMARK 2. In particular, this theorem implies the existence of sequences $\{a_n\} \in \mathbb{R}$, $\{f_n\} \in \mathscr{D}(A)$ such that $a_n \to \beta$ and

$$\inf_u (Af_n(x, u) - k(x, u)) \geq a_n, \qquad n \geq 1. \tag{4.1}$$

Compare this with the results of Vinter and Lewis (1978).

4.3. *The discounted problem.* Consider the discounted problem with initial law $\nu_0 = \delta_{x_0}$ for some $x_0 \in E$. Let $V(x_0)$ denote the minimum discounted cost for the initial law ν_0. The function $V(\cdot)$ is recognized as the *value function* of the dynamic approach to this problem. [See Chapter 3 of Borkar (1989).] Let X, Y be as before and set $W = \mathscr{D}(A)$, $Z = \mathscr{D}(A)^*$ (our $\overline{W}$ in the preceding subsection). Our primal problem becomes

$$\text{minimize } \langle \mu, k \rangle_{X,Y}$$
$$\text{subject to } F\mu = \delta_{x_0}, \mu \in P,$$

where F: $X \to Z$ is defined by $F(\mu) = \nu$,

$$\langle f, \nu \rangle_{W, W^*} = -\int (Af - \alpha f)\, d\mu \qquad \forall\, f \in \mathscr{D}(A).$$

In other words, the problem is to minimize $\int k\, d\mu$ over all $\mu \in X$ satisfying

$$\int (Af - \alpha f)\, d\mu = -f(x_0) \qquad \forall\, f \in \mathscr{D}(A).$$

The dual problem becomes

$$\text{maximize } f(x_0)$$
$$\text{subject to } Af - \alpha f + k \geq 0,\ f \in \mathscr{D}(A).$$

Suppose C denotes the set of discounted occupation measures and is compact in X. Theorem 4.1 now becomes the following.

THEOREM 4.3. $V(x_0) = \sup\{f(x_0)\text{: } \inf_u (Af(x, u) - \alpha f(x) + k(x, u)) \geq 0,\ f \in \mathscr{D}(A)\}$.

The proof follows along the lines of that of Theorem 4.2.

4.4. *The finite-horizon problem.* Let $B = \partial/\partial t + A$ with $\mathscr{D}(B)$ being the algebra generated by constants and the set $\{gf\text{: } g \in C_0^1([0.T]),\ f \in \mathscr{D}(A)\}$. Let $V(x, t) = \min \mathbb{E}[\int_t^T k(X(s), u(s))\, ds | X(t) = x]$, where the minimum is over all solutions to the relaxed controlled martingale problem for (A, δ_x) on the time interval $[t, T]$. Assume that, for fixed initial conditions, the set of finite-time occupation measures is compact. By considering the control problem on the interval $[t, T]$, we can deduce the following along the lines of Theorem 4.2.

THEOREM 4.4. $V(x_0, t) = \sup\{f(x_0, t)\text{: } \inf_u (Bf(s, x, u) + k(x, u)) \geq 0,\ f \in \mathscr{D}(B)$ *with support* $(f) \subset [t, T] \times E\}$.

5. Open problems. We conclude by listing some open problems suggested by the foregoing.

1. Establish the existence of an optimal solution to the ergodic problem which is both Markov and ergodic.
2. It is conjectured that the extreme points of the closed convex sets of ergodic/discounted occupation measures correspond to time-homogeneous Markov processes [i.e., they are occupation measures for a solution $(X(\cdot), \pi(\cdot))$, possibly among others, for which $X(\cdot)$ is a time-homogeneous Markov process]. Prove or disprove the conjecture.
3. Find a *good* set of conditions under which the existence theory of Section 3.1 can be extended to Hilbert space-valued controlled diffusions and stochastic evolutions.
4. In (4.1), show that $V(x) = \lim_{n\to\infty} f_n(x)$ exists for $x \in E$, thus providing a candidate *value function* for the ergodic problem.

Acknowledgments. The authors would like to express their thanks to Professor Rajeeva L. Karandikar and Professor T. G. Kurtz for their useful comments. The first author is also thankful for the hospitality provided by the Department of Electrical Engineering, Indian Institute of Science, during his stay in Bangalore.

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