Construction of some asymmetrical orthogonal arrays

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Abstract

A method of construction of asymmetrical orthogonal arrays of strength two is described. The method exploits difference matrices and a special type of orthogonal arrays. Several families of asymmetrical orthogonal arrays are obtained through the method. An open problem relating to the construction of a class of asymmetrical orthogonal arrays is proposed and comments on partial solutions to the problem are made. A conjecture on an upper bound to the number of constraints in a class of arrays is also made.

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1. Introduction and preliminaries

The practical utility of orthogonal arrays in industrial research for quality improvement has given added impetus to the study of these arrays, as exemplified by the recent works of Cheng (1989), Wang and Wu (1991,1992), P.C. Wang (1990), Wu et al. (1992) and Hedayat et al. (1992). The papers of Wang and Wu (1991) and Hedayat et al. (1992) are the major ones, as these give some very general methods of construction of asymmetrical orthogonal arrays. In this communication, we first present a further method of construction of asymmetrical orthogonal arrays of strength two. The method is essentially a modification of that of Wang and Wu (1991). Application of the method to specific cases leads to several families of asymmetrical orthogonal arrays. Some new arrays are also obtained.

An open problem concerning the construction of a family of asymmetrical orthogonal arrays is posed and comments are made on partial solutions to this problem. We also make a conjecture on an upper bound to the number of constraints in a class of asymmetrical orthogonal arrays.

Asymmetrical orthogonal arrays were introduced by Rao (1973) as a generalization of the conventional (symmetrical) orthogonal arrays, also introduced by Rao (1947). Consider an $N \times r$ array A with entries in the ith column from a set Σ_i with cardinality $s_i (\geqslant 2), i = 1, 2, \ldots, r$, not all s_i's being equal. The array A is said to be an (asymmetrical) orthogonal array of strength two, if in any $N \times 2$ submatrix of A, say, with one column based on Σ_i and the other on $\Sigma_{i'}$, every element from $\Sigma_i \times \Sigma_{i'}$ appears equally often as a row. We shall denote an asymmetrical orthogonal array of strength two by $L_N(s_1 \times s_2 \times \cdots \times s_r)$. Here N stands for the number of runs, r the number of constraints (or factors) and s_i $(i = 1,2,\ldots,r)$ is the number of levels of the ith factor. If in an $N \times r$ orthogonal array, there are n_i factors $(i = 1,2,\ldots,k)$ at s_i levels, $n_1 + n_2 + \cdots + n_k = r$, we denote the array by $L_N(s_1^{n_1} \times s_2^{n_2} \times \cdots \times s_k^{n_k})$. If $s_1 = s_2 = \cdots = s_k = s$, the array reduces to the conventional (symmetrical) orthogonal array, denoted by $L_N(s^r)$.

Wang and Wu (1991) constructed several families of asymmetrical orthogonal arrays using a technique based on difference matrices. Since the method of construction suggested here also depends on the use of difference matrices, we recall the definition of a difference matrix.

Suppose G is an additive Abelian group on p elements, $\{0,1,2,\ldots,p-1\}$. A $\lambda p \times r$ matrix with entries from G is called a difference matrix if among the differences modulo p of the corresponding elements of any two columns, each element of G appears exactly λ times.

A difference matrix will be denoted by $D(\lambda p, r; p)$ with the implicit understanding that p is the order of the group G on which the difference matrix is based.

Note that a Hadamard matrix of order n is equivalent to a difference matrix $D(n,n;2)$. For a review on difference matrices, see de Launey (1986).

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices of orders $n \times r$ and $m \times s$ respectively and let A and B have entries from a finite additive group G of order p. The Kronecker sum of A and B, denoted by $A * B$, is defined to be

$$A * B = (B^{a_{ij}})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant r}$$

where

$$B^k = (B + kJ) \bmod p$$

is obtained by adding k mod p to the elements of B and J is an $m \times s$ matrix of all ones.

Finally, we need the following version of a result from Dey and Agrawal (1985).

Theorem A. *Let there exist an orthogonal array A,* $L_N(s^k)$ *and let it be possible to partition A as*

$$A = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_{u-1} \end{pmatrix}$$

such that each A_i *is an orthogonal array with* $r = N/u$ *runs, k constraints, s symbols and strength* unity. *Then it is possible to construct an orthogonal array,* $L_{N^*}(t \times s^k)$, *where* $N^* = Nm, t = mu$ *and* $m \geqslant 1$ *is an integer.*

2. A method of construction

To begin with, we describe a procedure of obtaining an asymmetrical orthogonal array from a symmetrical one. Subsequently, this procedure is generalized. Let A be an orthogonal array $L_N(s^{k_1})$, and suppose it is possible to partition A as in Theorem A.

Next, let $D(M,k;s)$ be a difference matrix and define

$$B = D(M,k;s) * A. \tag{2.1}$$

Then, following the lines of Wang and Wu (1991), one can show that

(a) B is an orthogonal array, $L_{MN}(s^{kk_1})$, and

(b) the rows of B can be partitioned into Mu sets of rows $\{c_{ij}, i = 0,1,2,\ldots,u-1, j = 0,1,2,\ldots,M-1\}$ such that each c_{ij} is an orthogonal array with r runs, kk_1 factors and strength unity.

The following result then follows from Theorem A.

Theorem 1. *Let $\mathbf{a}_i = (i,i,\ldots,i)'$ where the symbol i appears r times in $\mathbf{a}_i$. Then, the array C given by*

$$C' = \begin{pmatrix} c'_{00} & c'_{01} & \cdots & c'_{u-1,M-1} \\ \mathbf{a}'_0 & \mathbf{a}'_1 & \cdots & \mathbf{a}'_{Mu-1} \end{pmatrix}$$

is an orthogonal array, $L_{MN}(q \times s^{kk_1})$, where $q = Mu$ and C' is the transpose of C.

Observe that the kind of partitioning required to be satisfied by A for the method to work is always possible by sacrificing one factor in A (see e.g. Dey and Agrawal, 1985, p. 59). This observation enables one to generalize the procedure of Theorem 1.

Suppose A is an orthogonal array, $L_N(s_1^{n_1} \times s_2^{n_2} \times \cdots \times s_t^{n_t})$. Clearly, the *columns* of A can be partitioned as

$$A = L_N(s_1^{n_1} \times s_2^{n_2} \times \cdots \times s_k^{n_t}) = [L_N(s_1^{n_1}) \vdots L_N(s_2^{n_2}) \vdots \cdots \vdots L_N(s_t^{n_t})].$$

Permuting the rows of A according to the levels of one of the factors at s_1 levels (say) and deleting the column corresponding to this factor, we get an orthogonal array B with the partitioning

$$B = [L_N(s_1^{n_1-1}) \vdots L_N(s_2^{n_2}) \vdots \cdots \vdots L_N(s_t^{n_t})].$$

Also, the runs of B can be partitioned into s_1 sets, $B_0, B_1, \ldots, B_{s_1-1}$, each set containing N/s_1 runs, such that for $i = 0,1,2,\ldots,s_1-1$,

$$B_i = [A_{r1}^{(i)} \vdots A_{r2}^{(i)} \vdots \cdots \vdots A_{rt}^{(i)}],$$

where, for $j = 2,3,\ldots,t$, $A_{rj}^{(i)}$ is an $r \times n_j$ orthogonal array of strength unity and s_j levels, while $A_{r1}^{(i)}$ is an $r \times (n_1 - 1)$ orthogonal array of strength unity and s_1 levels.

Suppose now there exist difference matrices $D(M,k_i;s_i), i = 1,2,\ldots,t$. Then, using the generalized Kronecker sum, as in Wang and Wu (1991), one can show that

$$D = \left[D(M,k_1;s_1) * L_N(s_1^{n_1-1}) \vdots D(M,k_2;s_2) * L_N(s_2^{n_2}) \vdots \cdots \vdots D(M,k_t;s_t) * L_N(s_t^{n_t})\right]$$

is an orthogonal array, $L_{MN}(s_1^{k_1(n_1-1)} \times s_2^{k_2n_2} \times \cdots \times s_t^{k_tn_t})$. Also, the runs of D can be partitioned into Ms_1 sets of rows, such that, in each set, any column representing a factor at s_i levels $(i = 1,2,\ldots,t)$ has each of the s_i levels occurring equally often. Hence, utilizing D and Theorem A, we have

Theorem 2. *The existence of an orthogonal array $L_N(s_1^{n_1} \times s_2^{n_2} \times \cdots \times s_t^{n_t})$ and difference matrices $D(M,k_i;s_i)$, $i = 1,2,\ldots,t$, imply the existence of an orthogonal array $L_{MN}(Ms_1 \times s_1^{k_1(n_1-1)} \times s_2^{k_2n_2} \times \cdots \times s_t^{k_tn_t})$.*

For $n_1 = 1$, we get the following result, due to Jacroux (1993).

Corollary 1. *The existence of an orthogonal array $L_N(s_1 \times s_2^{n_2} \times \cdots \times s_t^{n_t})$ and difference matrices $D(M,k_j;s_j)$, $j = 2,3,\ldots,t$, imply the existence of an orthogonal array $L_{MN}(Ms_1 \times s_2^{k_2n_2} \times \cdots \times s_t^{k_tn_t})$.*

3. Examples of some arrays

Application of Theorems 1 and 2 can be made to arrive at several families of asymmetrical orthogonal arrays. We give some such examples.

(i) Suppose $s = p^u$, $\lambda = p^v$, where p is a prime and u and v are positive integers. Bose and Bush (1952) constructed an orthogonal array $L_N(s^{\lambda s})$ with $N = \lambda s^2$. This array can be partitioned into λs subarrays, each containing s runs such that each subarray is an orthogonal array of strength unity. Hence from Theorem 1, we have an orthogonal array $L_{MN}(Ms \times \lambda s^{k\lambda s})$, *provided* there exists a difference matrix $D(M,k;s)$.

(ii) Addelman and Kempthorne (1961b) constructed an orthogonal array $L_{N_0}(s^m)$ where $N_0 = 2s^n, m = 2(s^n-1)/(s-1)-1$, s is a prime or a prime power and n, a positive integer. This array can be partitioned into $2s$ sets of runs, each set containing s^{n-1} runs, such that, within each set, one factor has the same level while for every other factor each of the s levels occurs exactly s^{n-2} times. Omitting the factor that has the same level within a set, we get an orthogonal array $L_{N_0}(s^{m-1})$ with the desired property. Hence, from Theorem 1, we have an array $L_N(2Ms \times s^{(m-1)k})$, provided a difference matrix $D(M,k;s)$ exists. Here $N = 2Ms^n$.

For example, using the array $L_{18}(3^7)$ and the difference matrix $D(6,6;3)$ Masuyama (1957) gives an orthogonal array $L_{108}(36 \times 3^{36})$. Replacing the 36-level factor by the runs of $L_{36}(3^{12} \times 2^{11})$ (see e.g. Dey, 1985, p. 64), we get an array $L_{108}(3^{48} \times 2^{11})$, which was also constructed by J.C. Wang (1989).

(iii) Cheng (1989) constructed an orthogonal array $L_{nt}(4^{t-1} \times 2^{nt-3t+2})$, where n ($\geqslant 4$) and t ($\leqslant n$) are Hadamard numbers (a positive integer u is said to be a Hadamard number if a Hadamard matrix of order u exists). Since the difference matrices $D(4,4;4)$ and $D(4,4;2)$ exist, we have from Theorem 2 an orthogonal array $L_{4nt}(16 \times 4^{4t-8} \times 2^{4nt-12t+8})$.

In particular, for $t = 2$ and $n = 12$, we have the array $L_{96}(16 \times 2^{80})$. Replacing the 16-level factor by the runs of a 16-run orthogonal array, several other arrays can be obtained.

(iv) Cheng (1989) obtained an orthogonal array $L_{nt}(n \times 4^{h-1} \times 2^{n(t-1)-3(h-1)})$ where $t, n \geqslant 4$ are Hadamard numbers, $t/2$ is also a Hadamard number and $h = \min(n,t)$. Utilizing this array and the difference matrices $D(4,4;4)$ and $D(4,4;2)$, we get an orthogonal array $L_{4nt}(4n \times 4^{4(h-1)} \times 2^{4n(t-1)-12(h-1)})$, t, n, h being as defined above.

(v) An orthogonal array $L_{2n}(t \times 4 \times 2^{n-1})$, where $t = n/2$ and n is a Hadamard number, was constructed by Agrawal and Dey (1982). Since the difference matrices $D(4,4;4)$ and $D(4,4;2)$ exist, we have an orthogonal array $L_{8n}(4t \times 4^4 \times 2^{4(n-1)})$. For $n = 12$, we have a *new* array $L_{96}(24 \times 4^4 \times 2^{44})$. Replacing the 24-level factor by the runs of $L_{24}(4 \times 2^{20}), L_{24}(12 \times 2^{12}), L_{24}(6 \times 4 \times 2^{11})$, we get the arrays $L_{96}(4^5 \times 2^{64}), L_{96}(12 \times 4^4 \times 2^{56}), L_{96}(6 \times 4^5 \times 2^{55})$ respectively. Also, replacing the 12-level factor in $L_{96}(12 \times 4^4 \times 2^{56})$ by the runs of $L_{12}(3 \times 2^4)$ and $L_{12}(6 \times 2^2)$ (Wang and Wu, 1991), we get respectively the arrays $L_{96}(4^4 \times 3 \times 2^{60})$ and $L_{96}(6 \times 4^4 \times 2^{58})$. Again, using $D(12,12;4)$ and $D(12,12;2)$ in place of $D(4,4;4)$ and $D(4,4;2)$ respectively, one obtains the array $L_{24n}(6n \times 4^{12} \times 2^{12n-12})$, where n is a Hadamard number. In particular, for $n = 4$, we get a *new* array, $L_{96}(24 \times 4^{12} \times 2^{36})$. Replacing the 24-level factor by the runs of $L_{24}(4 \times 2^{20}), L_{24}(6 \times 4 \times 2^{11}), L_{24}(3 \times 4 \times 2^{13})$ and $L_{24}(12 \times 2^{12})$, we get the arrays $L_{96}(4^{13} \times 2^{56}), L_{96}(4^{13} \times 3 \times 2^{49}), L_{96}(6 \times 4^{13} \times 2^{47})$ and $L_{96}(12 \times 4^{12} \times 2^{48})$.

(vi) P.C. Wang (1990) constructed a family of arrays, $L_{nt}(t \times 4^t \times 2^{(nt-4t)})$, where $n > 4, n \neq t$ are Hadamard numbers such that $n/2$ is also a Hadamard number. Combining this array with the difference matrices $D(4,4;4)$ and $D(4,4;2)$, we get a family of arrays, $L_{4nt}(4t \times 4^{4t} \times 2^{4t(n-4)})$.

4. An open problem and a conjecture

In this section, we mention an open problem related to the construction of asymmetrical orthogonal arrays and comment on partial solutions to the problem. Recall that a symmetrical orthogonal array of strength two, $L_{\lambda s^2}(s^k)$ with $\lambda = \alpha\beta$, is called β-resolvable if it is the juxtaposition of αs different arrays, each with βs runs.

k constraints, s levels and strength *unity*. A 1-resolvable array is called *completely resolvable*. We then have the following result.

Lemma 1. *The orthogonal array, $L_{nt}(t \times s^k)$, where $n = ms$, exists if and only if the last k columns form an m-resolvable orthogonal array, $L_{nt}(s^k)$.*

Proof. The "if" part is trivial. To see the "only if" part, permute the rows of $L_{nt}(t \times s^k)$, so that the first n runs correspond to the first level, the next n runs to the second level,..., the last n runs correspond to the tth level of the first factor. It then follows from the definition of an orthogonal array that the last k columns of $L_{nt}(t \times s^k)$ must form an m-resolvable array, $L_{nt}(s^k)$. □

We now state a problem that has not yet been solved completely.

Problem. Given n, t, s, find an orthogonal array of strength two, $L_{nt}(t \times s^k)$, with the maximum value of k.

From Lemma 1, it follows that the above stated problem is equivalent to: Given n, t, s, find an m-resolvable array, $L_{nt}(s^k)$, where $m = n/s$, with the maximum value of k.

A trivial upper bound for k is

$$k \leqslant t(n-1)/(s-1). \tag{4.1}$$

We call an orthogonal array *saturated* if the upper bound in (4.1) is attained.

The problem stated above has not been solved completely. Even in the special case $s = 2$, only partial solutions are available.

In the remainder of this section, we specialize to the case $s = 2$. We first have the following result, which generalizes some of the results of Suen (1989).

Theorem 3. *Suppose n, t are Hadamard numbers. Then the array $L_{nt}(t \times 2^{t(n-1)})$ exists.*

Proof. Let H_n, H_t denote Hadamard matrices of orders n, t respectively. Write H_n as

$$H_n = [\mathbf{1}_n \,\vdots\, B].$$

Then,

$$D = [A \,\vdots\, H_t \otimes B]$$

is the required array, where $A = (1, 2, \ldots, t)' \otimes \mathbf{1}_n$, $\mathbf{1}_n$ is an n-component column vector of all ones and $\otimes$ denotes the Kronecker product of matrices. □

From (4.1), it follows that $k = t(n-1)$ is the maximum number of two-level factors that can be accommodated in $L_{nt}(t \times 2^k)$ if n, t are fixed, i.e., the arrays of Theorem 3 are saturated. For particular choices of n, t, we get the following corollaries to Theorem 3, due to Suen (1989).

Corollary 2. *If $H_{2\lambda}$ exists then we can construct a completely resolvable array $L_{4\lambda}(2^{2\lambda})$ and hence the array $L_{4\lambda}(2\lambda \times 2^{2\lambda})$.*

Corollary 3. *If H_t exists, then we can construct a 2-resolvable array $L_{4t}(2^{3t})$ and hence the array $L_{4t}(t \times 2^{3t})$.*

It is of interest to find such arrays when at least one of n,t is not a Hadamard number (e.g., t is odd and n is a Hadamard number). In this context the following results, due to Suen (1989), are known.

Theorem B. *If $\lambda > 1$ is* odd, *a completely resolvable array $L_{4\lambda}(2^2)$ exists, and hence the array $L_{4\lambda}(2\lambda \times 2^2)$ can be constructed. Also, $k = 2$ is the maximum number of 2-level columns that can be accommodated in $L_{4\lambda}(2\lambda \times 2^k)$.*

Theorem C. *If $n = 4u + 2$, then a 2-resolvable array $L_{4n}(2^{2n+2})$ exists, and hence one can construct the array $L_{4n}(n \times 2^{2n+2})$.*

When $n > 1$ is *odd*, no general result is known for constructing a 2-resolvable array, $L_{4n}(2^k)$. However, in specific cases, some arrays are known to exist. These are listed below:

(a) $L_{12}(2^4)$. Here $k = 4$ is the maximum number of 2-level factors, as shown by Wang and Wu (1992), using an exhaustive computer search.

(b) $L_{20}(2^8)$, and $k = 8$ is the maximum number of 2-level factors (Wang and Wu, 1992).

(c) $L_{28}(2^{12})$.

(d) $L_{36}(2^{13})$.

The above arrays were first constructed by Suen (1989). The arrays listed above therefore give rise to the following asymmetrical arrays:

$$L_{12}(3 \times 2^4),\ \ L_{20}(5 \times 2^8),\ \ L_{28}(7 \times 2^{12}),\ \ L_{36}(9 \times 2^{13}).$$

With 12 and 20 runs, we cannot get arrays with more 2-level factors than 4 and 8 respectively. It is not known whether arrays with 28 and 36 runs can accommodate more 2-level factors than 12 and 13 respectively. Combining the arrays $L_{4\lambda}(2^2)$, $\lambda > 1$ *odd* and $L_{4\lambda}(2^{2\lambda+2})$, $\lambda = 4u + 2$ with a difference matrix $D(t,t,2)$, where t is a Hadamard number, we get, from Theorem 2, asymmetrical arrays:

(i) $L_{4\lambda t}(2\lambda t \times 2^{2t})$, $\lambda (> 1)$ *odd*;

(ii) $L_{4\lambda t}(t\lambda \times 2^{t(2\lambda+2)})$, $\lambda = 4u + 2$.

Similarly, combining the 2-resolvable array $L_{4n}(2^{3n})$ with $D(t,t;2)$, n,t being Hadamard numbers, we get an array $L_{4nt}(nt \times 2^{3nt})$. This array is clearly saturated.

Finally, based on the above discussions and the numerical results of Wang and Wu (1992) regarding the attainable upper bounds to k in arrays $L_{nt}(t \times 2^k)$, we are prompted to make the following conjecture (suggested by Rahul Mukerjee):

Conjecture 4. *Consider the array $L_{4\lambda s}(s \times 2^k)$, where $s \geqslant 3$ is an* odd *integer and λ is an integer. Then $k \leqslant 4\lambda s - 2s - 2$.*

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