Two infinite series of E-optimal nested row–column designs

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Abstract

We construct two infinite series of nested row-column designs having 2×4 arrays as blocks and with treatments satisfying a rectangular association scheme with two rows and an odd number (n) of columns. The designs have the following parameters (v = number of treatments, b = number of blocks).

Series 1: $n = 1 \pmod 4$, $n \geqslant 5$, $v = 2n$, $b = n(n-1)/4$.

Series 2: $n = 3 \pmod 4$, $n \geqslant 3$, $v = 2n$, $b = n(n+1)/4$.

These designs are shown to be E-optimal and highly efficient with respect to A-optimality criterion. Equally important, these optimal designs are very economical in as much as they require as few observations as possible in the set-up.

1. Introduction

By a nested row–column set-up we mean a set-up with three nuisance factors, of which two (row and column) are crossed within one another, but both nested within the third (block) factor. The number of rows as well as the number of columns within each block is assumed to be constant. The analysis of such a design is presented in Singh and Dey (1979). In the same paper a few series of variance-balanced binary designs were constructed. Designs with the same property were also constructed in Agarwal and Prasad (1982).

The first optimality results on the nested set-up were obtained in Bagchi et al. (1990). A surprising fact is that the optimal designs are necessarily non-binary and they were not available in the literature. Several series of optimal designs were also constructed in this paper. Srivastava (1981) pointed out that for a nested row–column design it is desirable that the sizes of the rows as well as the columns be as small as possible so that the additive model is valid.

Now, the smallest possible designs in a nested set-up would have 2×2 or 2×3 rectangles as blocks. The construction of an optimal design is not really a new problem in these set-ups. This is because we can construct such a design from an optimal one-way design trivially as follows. For every block (ab) (or (abc)) of the one-way design d^* we take a block

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \left(\text{respectively} \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}\right)$$

of the nested design. By the main result of our earlier paper (see Theorem 3.1 below) such a nested design will have the same optimality property as d^*.

For the above reason, we attempt to construct optimal nested row–column designs with 2×4 rectangles as blocks. In our earlier paper, a series of such designs were presented with $v \equiv 0 \pmod 4$. Here we construct designs with $v \equiv 2 \pmod 4$, thus exhausting all even values of v. These designs are E-optimal and highly efficient with regard to A-criterion by the results of Bagchi and Cheng (1993) and Bagchi (1994).

The problem of constructing optimal designs in our set-up with minimum possible number of observations remains open for odd values of v.

2. Construction

2.1. Notation

Let p and q denote, respectively, the number of rows and columns of a block of a nested row–column set-up, b the number of blocks and v the number of treatments. Such a design will be referred to as $\mathrm{ND}(v, b, p, q)$. Let $L(v \times b)$ denote the treatment-block, $M(v \times bq)$ the treatment-column and $N(v \times bp)$ the treatment-row incidence matrices. Also, let

$$K = NN^{\mathrm{T}} - p^{-1}LL^{\mathrm{T}}. \tag{1}$$

2.2. Definition

We recall that a rectangular (m, n) association scheme is a three-class association scheme with mn treatments (i, j), $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. For $i \neq l$ and $j \neq k$, the treatment pair (i, j) and (i, k) are mutually first associates, (i, j) and (l, j) are second associates and (i, j) and (l, k) are third associates. A PBIBD with rectangular association scheme will be called a rectangular design.

2.3. Notation

S is the infinite sequence of which the ith term ($i \geqslant 1$) is given by

$$s_i = \begin{cases} 2i & \text{if } i \text{ is odd,} \\ 2i-1 & \text{otherwise.} \end{cases} \tag{2}$$

We shall now describe our method of construction.

2.4. Further notations

Fix an odd integer $n \geqslant 3$. Let I denote the set of integers modulo n. Let $V = I \cup I^*$ be our treatment set, where I^* is a copy of I disjoint from I. Let $i \mapsto i^*$, $i \in I$, be a bijection from I to I^*.

Let

$$t = [(n-1)/4]. \tag{3}$$

Here [ ] denotes the integral part function.

Let S_t denote the finite sequence consisting of the first t terms of S, i.e.

$$S_t = \{s_i,\ 1 \leqslant i \leqslant t\}. \tag{4}$$

For $j \in S_t$, let R_j denote the following rectangle with entries from V:

$$R_j = \begin{bmatrix} 1 & -j^* & -1 & j^* \\ j^* & 1 & -j^* & -1 \end{bmatrix}. \tag{5}$$

Also, let

$$R_0 = \begin{bmatrix} x & x^* & -x & -x^* \\ x^* & -x & -x^* & x \end{bmatrix}, \tag{6}$$

where

$$x = \begin{cases} t & \text{if } t \text{ is odd,} \\ t+1 & \text{otherwise.} \end{cases}$$

2.5. The designs

Let B_{ij} denote the 2×4 rectangle obtained by adding $i \pmod n$ to each entry of R_j, $j \in \{0\} \cup S_t$ (see (4)).

When $n \equiv 1 \pmod 4$, the tn arrays B_{ij}, $j \in S_t$, $0 \leqslant i \leqslant n-1$, constitute a nested design ND$(2n, tn, 2, 4)$. When $n \equiv 3 \pmod 4$, the $(t+1)n$ arrays B_{ij}, $j \in \{0\} \cup S_t$, $0 \leqslant i \leqslant n-1$, constitute a nested design ND$(2n, (t+1)n, 2, 4)$.

Theorem 1. *The designs constructed above have the following properties.*

(a) $K = 0$. (See Notation 2.1),

(b) *M is the incidence matrix of a PBIBD with rectangular association scheme with*
(i) $\lambda_1 = 0 = \lambda_2$, $\lambda_3 = 1$ *if* $n \equiv 1 \pmod 4$,
(ii) $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 1$ *if* $n \equiv 3 \pmod 4$ (See Definition 2.2).

Proof. (a) is obvious.

(b) Since each column of each of the generating blocks (and hence of all the blocks) has a unique treatment from I and a unique treatment from I^*, we have $\lambda_1 = 0$ for both the series. Hence, an unordered pair of the form (a, a^*) occurs as a column of R_0 only, and occurs twice there. Note that $\pm 1 \notin S_t$, as $t \leqslant (n-1)/4$. Thus, $\lambda_2 = 0$ or 2 according as $n \equiv 1$ or $3 \pmod 4$, respectively. It remains to show that $\lambda_3 = 1$ for both the series.

Case 1: $n \equiv 1 \pmod 4$. It is enough to verify that for each $a \in I$, $a \not\equiv 0 \pmod n$, a has a unique representation of the form

$$a = \pm b \pm 1, \quad b \in S_t. \tag{7}$$

Since there are $n-1$ such a's and S_t contains $(n-1)/4$ terms, the above will follow if we show that the four subsets $\pm 1 \pm S_t$ of $I - 0$ (addition elementwise modulo n) are pairwise disjoint. This follows from the following observations on S_t.

(i) If $i \in S_t$, $n - i \notin S_t$.

(ii) No two elements of S_t differ by $2 \pmod n$.

(iii) No two elements of S_t add up to $n - 2 \pmod n$.

(i) and (ii) are obvious. To prove (iii), we note that the elements of S_t are at most $(n-1)/2$. So if two elements add up to $n - 2 \pmod n$, they must be $(n-1)/2$ and $(n-1)/2 - 1$. But $(n-1)/2$ is even and the number preceeding any even member of S_t does not belong to S_t.

Now it is easy to see that observations (i), (ii) and (iii), respectively, imply that $1 + S_t$ is disjoint from $1 - S_t$, $-1 + S_t$ and $-1 - S_t$. The remaining part follows from symmetry.

Case 2: $n \equiv 3 \pmod 4$. Here we are to show that for every $a \in I$, a has a unique expression of the form $a = \pm b \pm 1$, $b \in S_t$ or $a = \pm 2x$, where

$$x = \begin{cases} t & \text{if } t \text{ is odd,} \\ t + 1 & \text{otherwise.} \end{cases}$$

But this amounts to showing that the five subsets $\pm 1 \pm S_t$ and $\{\pm 2x\}$ of $I - \{0\}$ are pairwise disjoint. Now it is easy to see that the observations (i), (ii) and (iii) hold in this case also and therefore the four sets $\pm 1 \pm S_t$ are disjoint. So it remains to show that none of $\pm 2x \pm 1$ belongs to S_t. This can be checked by a straightforward calculation. Hence, the theorem is proved. □

In order to clarify the construction, we provide two examples of the designs, one each for the two cases $n \equiv \pm 1 \pmod 4$.

Example. $n = 3$: Here $t = 0$ and $x = 1$. So the generating block is

$$\begin{bmatrix} 1 & 1^* & 2 & 2^* \\ 1^* & 2 & 2^* & 1 \end{bmatrix}.$$

$n = 5$: Here $t = 1$, so $S_1 = \{2\}$. Hence, the generating block is

$$\begin{bmatrix} 1 & -2^* & -1 & 2^* \\ 2^* & 1 & -2^* & -1 \end{bmatrix}.$$

3. Optimality

Let us note a few known results.

Theorem 3.1 (Bagchi, et al., 1990). *Suppose a nested design has $K = 0$ (see Notation 2.1) and M is the incidence matrix of a block design d^*. Then the nested design has the same optimality property as d^*.*

Theorem 3.2 (Bagchi and Cheng, 1993). *A rectangular design with two rows and $\lambda_1 = \lambda_2 = \lambda_3 - 1$ is E-optimal over the class $\mathscr{D}(b, v, 2)$ of all connected designs with v treatments, b blocks, each of size 2.*

Theorem 3.3 (Bagchi, 1994). *A rectangular design with two rows and $\lambda_1 = \lambda_2 - 2 = \lambda_3 - 1$ is E-optimal in $\mathscr{D}(b, v, 2)$, whenever $v \geqslant 10$.*

In view of the above theorems, it is clear that each of the designs constructed here is E-optimal in its respective class, except when $v = 6$.

Further, by Theorem 3.7 of Bagchi (1991) and Theorem 3.1 above, we see that these nested designs are highly efficient with respect to the A-criterion, the efficiency tending to 1 as $v \to \infty$.

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