STUDIES IN THE THEORY OF MEASURABLE MULTIFUNCTIONS

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CONTENTS

P.T.O.

INTRODUCTION AND SUMMARY

During the last fifteen years a large number of papers have been devoted to the study of closed set valued multifunctions. The studies were motivated by both the theoretical and applicational interests that such multifunctions have. From the applicational point of view it is worth noting that such multifunctions arise in various problems of control theory, dynamic programming etc. The theoretical aspects of these studies belong properly to classical descriptive set theory. Classical descriptive set theory asks questions about how sets are constructed and about other definability properties of sets. The results on closed set valued multifunctions asserting the existence of measurable selectors or asserting that closed set valued multifunctions can be expressed as images under Carathéodory maps of suitable spaces, whose descriptive nature is in some sense simple, would fall in this category.

It is known that most of the pleasant descriptive properties of closed set valued multifunctions fail to hold for more general multifunctions. Indeed there are many examples and some will be given in this thesis to show that F_σ valued multifunctions do not have these pleasant properties. A certain number of positive results are known about σ-compact valued multifunctions. These go back to late 30's and early 40's and can be found in the work of Kunugui, Novikov, Arsenin and Shchegolkov. Some further positive results about σ-compact valued multifunctions are included in this thesis.

The case of G_δ valued multifunctions is essentially an unexplored territory and in this thesis we initiate a detailed study of the structure of such multifunctions.

The main problems that are considered in this thesis are (i) the existence of a measurable selector for G_δ valued multifunctions, (ii) the existence of measurable cross sections for partitions into G_δ sets, (iii) the representation of G_δ valued multifunctions as images under special Carathéodory maps of particularly simple spaces, (iv) the Borel parametrization of G_δ valued multifunctions, (v) the representation of G_δ valued multifunctions as continuous images of closed set valued multifunctions.

The thesis is organised as follows :

In chapter 0 we fix some terminology and notation and state some known results. These are used throughout the thesis.

Chapter 1 is mainly concerned with the problem of representation of closed valued multifunctions as images under Carathéodory maps of simple spaces. Here we are able to give various refinements of a result of Ioffe [8]. One such theorem reads as follows :

Let T be a non-empty set, $\mathcal{L}$ a field of subsets of T and X a Polish space. If $F: T \to X$ is a closed valued, $\mathcal{L}_\sigma$-measurable multifunction then there is a map $f: T \times \Sigma \to X$ such that for each $t \in T$, $f(t,.)$ is a continuous map from Σ onto $F(t)$ and for each $\sigma \in \Sigma$, $f(.,\sigma)$ is $\mathcal{L}_\sigma$-measurable, where $\mathcal{L}_\sigma$ denotes the countably additive family of subsets of T generated by $\mathcal{L}$ and Σ denotes the space of irrationals.

One of the interesting corollaries to this result is the following :

Let T be a metric space and X a Polish space. If $F: T \to X$ is a closed valued, α^--multifunction, $\alpha > 0$, then there is a map $f: T \times \Sigma \to X$ such that for every $t \in T$, $f(t,.)$ is a continuous function from Σ onto $F(t)$ and for each $\sigma \in \Sigma$, $f(.,\sigma)$ is of class α.

We also show that if F is compact valued then under slightly stronger measurability conditions we can replace Σ in the general result stated above by a compact metric space. A similar representation for σ-compact valued multifunctions is also given.

Chapter 2 initiates the study of the existence of measurable selectors for multifunctions taking G_δ values in a Polish space. Our main result asserts that if T, X are Polish spaces if $\underline{\underline{A}}$ is a countably generated sub σ-field of the Borel σ-field $\underline{\underline{B}}_T$ and if $F : T \to X$ is a multifunction such that F is $\underline{\underline{A}}$-measurable, $Gr(F) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ and $F(t)$ is a G_δ in X for each $t \in T$ then there is a $\underline{\underline{A}}$-measurable selector for F. Here $Gr(F)$ denotes the set $\{(t,x) \in T \times X : x \in F(t)\}$.

In chapter 3 we study partitions of Polish spaces into G_δ sets. A partition of a Polish space X is called measurable if the saturation of every open set is Borel in X. We prove that a measurable partition of a Polish space into G_δ sets admits a Borel cross section. This answers a question raised by Kallman and Mauldin [10]. This result seems to have an interesting application in C*-algebras [10], [20]. Here we also study partitions of Polish spaces into σ-compact sets.

In chapter 4 we pursue problems (iii), (iv) and (v) as mentioned earlier. We take T, X to be Polish spaces, $\underline{\underline{A}}$ a countably generated sub σ-field of the Borel σ-field $\underline{\underline{B}}_T$ and $F : T \to X$ a multifunction such that F is $\underline{\underline{A}}$-measurable, $Gr(F) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ and $F(t)$ is a G_δ in X for each $t \in T$. We show the following :

(a) There is a map $f : T \times \Sigma \to X$ such that for each $t \in T$, $f(t,.)$ is a continuous, open map from Σ onto $F(t)$ and for each $\sigma \in \Sigma$, $f(.,\sigma)$ is $\underline{\underline{A}}$-measurable.

(b) If X is uncountable and $F(t)$ dense in itself for each $t \in T$ then there is a $\underline{\underline{A}} \times \underline{\underline{B}}_X$ -measurable map $f : T \times X \to X$ such that for each $t \in T$, $f(t,.)$ is a Borel isomorphism of X onto $F(t)$.

(c) There is a $\underline{\underline{A}}$-measurable, closed valued multifunction $W : T \to \Sigma$ and a continuous, open and onto map $\beta : \Sigma \to X$ such that $F(t) = \beta(W(t))$ for each $t \in T$.

In chapter 5 we give a complete characterization of G_δ valued multifunctions as follows :

Let T, X be Polish spaces, let $\underline{\underline{A}}$ be a countably generated sub σ-field of $\underline{\underline{B}}_T$ and let $F : T \to X$ be a multifunction. Then the following are equivalent :

(A) F is $\underline{\underline{A}}$-measurable, $Gr(F) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ and $F(t)$ is a G_δ in X for each $t \in T$.

(B) There is a map $f : T \times \Sigma \to X$ such that for each $t \in T$, $f(t,.)$ is a continuous, closed map from Σ onto $F(t)$ and for each $\sigma \in \Sigma$, $f(.,\sigma)$ is $\underline{\underline{A}}$-measurable.

CHAPTER 0

PRELIMINARIES

In this chapter we shall introduce basic definitions and notation that will be used in this thesis. We shall also state some known results. These will be used frequently, some without explicit mention.

THE TERMINOLOGY NOT DEFINED IN THIS THESIS IS FROM KURATOWSKI [11].

The set of positive integers will be denoted by N. S will denote the set of all finite sequences of positive integers, including the empty sequences e. For each $k \geq 0$, we denote by S_k the set of elements of S of length k. For $s \in S$, $|s|$ will denote the length of s and if $i \leq |s|$ is a positive integer, s_i will denote the i-th co-ordinate of s. If $n \in N$, sn will denote the catenation of s and n. We put $\Sigma = N^N$. Endowed with the product of discrete topologies on N, Σ becomes a homeomorph of irrationals. For $\sigma \in \Sigma$ and $k \in N$, σ_k will denote the k-th co-ordinate of σ and $\sigma|k = (\sigma_1, \ldots, \sigma_k)$. If $k = 0$, $\sigma|k = e$. If $s \in S_k$, $k \geq 0$, Σ_s will denote the set $\{\sigma \in \Sigma : \sigma|k = s\}$.

D will denote the set of all finite sequences of 0's and 1's, including the empty sequence e. C will denote the set $\{0,1\}^N$. Endowed with the product of discrete topologies on $\{0,1\}$, it becomes a homeomorph of the Cantor set. For $k \geq 0$, D_k will denote the set of elements of D of length k. For $d \in D_k$, $k \geq 0$, $i \in \{0,1\}$ and $\delta \in C$, d_j, $j \leq k$, $|d|$, di, δ_k and $\delta|k$ are similarly defined.

Let X and Y be non-empty sets. If $E \subset X \times Y$ and $x \in X$, E^x will denote the set $\{y \in Y : (x,y) \in E\}$ and will be called <u>the section of E at x</u>. We use $\prod_X$ and $\prod_Y$ to denote the projections from $X \times Y$ to X and from $X \times Y$ to Y respectively. We say that a set $B \subset E$ <u>uniformizes</u> E if sections of B are at most singletons and $\prod_X(B) = \prod_X(E)$.

Let $(X,\underline{\underline{A}})$ and $(Y,\underline{\underline{B}})$ be measurable spaces. We denote by $\underline{\underline{A}} \times \underline{\underline{B}}$ the product of the σ-fields $\underline{\underline{A}}$ and $\underline{\underline{B}}$. If f is a function from X into Y, the σ-field $\{f^{-1}(B) : B \in \underline{\underline{B}}\}$ on X will be denoted by $f^{-1}(\underline{\underline{B}})$. Let Z be a subset of X. Then the σ-field $\{A \cap Z : A \in \underline{\underline{A}}\}$ of subsets of Z will be denoted by $\underline{\underline{A}}|Z$ and will be referred to as <u>the trace of $\underline{\underline{A}}$ on Z</u>. A non-empty set $A \in \underline{\underline{A}}$ is called a <u>$\underline{\underline{A}}$ - atom</u> if no proper non-empty subset of A is in $\underline{\underline{A}}$. The measurable space $(X,\underline{\underline{A}})$ is said to be <u>atomic</u> if X is the union of $\underline{\underline{A}}$ - atoms. We say that the σ-field $\underline{\underline{A}}$ is <u>countably generated</u> if there exist subsets A_n, $n \geq 1$, of X such that $\underline{\underline{A}}$ is the smallest σ-field of subsets of X containing A_n, $n \geq 1$. In this case, we say that

$\underline{\underline{A}}$ is generated by $\{A_n : n \geq 1\}$. It is well known that a countably generated σ-field is atomic. If $(X, \underline{\underline{A}})$, $(Y, \underline{\underline{B}})$ are measurable spaces and f is a one-one map from X onto Y such that $\underline{\underline{A}} = f^{-1}(\underline{\underline{B}})$ then f is called a Borel isomorphism and $(X, \underline{\underline{A}})$ and $(Y, \underline{\underline{B}})$ are said to be isomorphic. If the underlying σ-fields are understood, we simply say that X and Y are isomorphic.

For a metric space X, $\underline{\underline{B}}_X$ will denote the Borel σ-field of X, that is, $\underline{\underline{B}}_X$ is the σ-field on X generated by all open sets in X. A second countable, completely metrizable topological space is called a Polish space. Metric spaces, unless otherwise stated, will be equipped with their Borel σ-fields.

If A_n, $n \geq 1$, is a sequence of subsets of X, then by the characteristic function of the sequence $\{A_n : n \geq 1\}$ is meant the function $f : X \rightarrow [0,1]$ defined by

$f(x) = \sum_{n=1}^{\infty} \frac{2}{3^n} I_{A_n}(x)$, $x \in X$, where I_{A_n} denotes the indicator function of A_n. Let $(X, \underline{\underline{A}})$ be a measurable space and let $\underline{\underline{A}}$ be generated by $\{A_n : n \geq 1\}$. If f is the characteristic function of $\{A_n : n \geq 1\}$ then f is a Borel isomorphism of X and $f(X)$. In particular, it follows that if x_1, x_2 belong to different $\underline{\underline{A}}$-atoms then $f(x_1) \neq f(x_2)$. These facts are well known and will be used without explicit mention. A measurable space $(X, \underline{\underline{A}})$ is said to be standard Borel if there is a function g on X onto a Borel subset Z of $[0,1]$ such that $\underline{\underline{A}} = g^{-1}(\underline{\underline{B}}_Z)$.

Let X and Y be topological spaces and $A \subset X$. We say that A is a retract of X if there is a continuous function $f : X \to A$ such that $f(x) = x$ for each $x \in A$. The map f is called a retraction of X onto A. A function $g : X \to$ is called open (resp. closed) if for every open (resp. closed) set W in X, $g(W)$ is open (resp. closed) in $g(X)$. Our definition of open and closed maps is slightly different from the ones in general usage. But for our purposes it is convenie to adopt the definitions given above. Say that a map f from X onto Y is perfect if f is continuous, closed and the set $f^{-1}(\{y\})$ is compact for each $y \in Y$.

Let T and X be non-empty sets. A multifunction $F : T \to X$ is a function whose domain is T and whose values are non-empty subsets of X. A function $f : T \to X$ is called a selector for F if $f(t) \in F(t)$ for each $t \in T$. The set $\{(t,x) \in T \times X : x \in F(t)\}$ is denoted by $Gr(F)$ and is called the graph of F. If $E \subset X$, we denote the set $\{t \in T : F(t) \cap E \neq \phi\}$ by $F^{-1}(E)$.

If $\underline{A}$ is a family of subsets of T and if X is a metric space, we say that F is <u>$\underline{A}$-measurable</u> (<u>strongly $\underline{A}$-measurable</u>) if for every open (closed) set V in X, $F^{-1}(V) \varepsilon \underline{A}$. In particular, a function $f : T \to X$ is $\underline{A}$-measurable if $f^{-1}(V) \varepsilon \underline{A}$ for every open set V in X. Multifunctions $\underline{A}$-measurable (strongly $\underline{A}$-measurable) in our sense are called weakly $\underline{A}$-measurable ($\underline{A}$-measurable) in the literature. However, for the purpose of this thesis, it will be convenient to adopt the terminology introduced above. If the family $\underline{A}$ is closed under countable unions then strongly $\underline{A}$-measurable multifunctions are necessarily $\underline{A}$-measurable.

If T is a metric space and $\underline{A}$ the family of Borel sets of additive class α (multiplicative class α), where α is a countable ordinal, then $\underline{A}$-measurable (strongly $\underline{A}$-measurable) multifunctions are called <u>α^--multifunctions</u> (<u>α^+-multifunctions</u>). In literature, 0^--multifunctions (0^+-multifunctions) are also called <u>lower semi-continuous</u> (<u>upper semi-continuous</u>).

Now we state a very important selection theorem due to Kuratowski and Ryll-Nardzewski [13].

<u>Theorem 0.1</u> Let T be a non-empty set, $\mathcal{L}$ a field of subsets of T and X a Polish space. Suppose $F : T \to X$ is a $\mathcal{L}_\sigma$-measurable multifunction such that $F(t)$ is closed in X for

each $t \in T$, where $\mathcal{L}_\sigma$ is the smallest family containing $\mathcal{L}$ and closed under countable unions. Then F admits a $\mathcal{L}_\sigma$-measurable selector.

The next lemma is proved by Kuratowski and Ryll-Nardzewski to establish their selection theorem. We shall also find it useful in the sequel.

Lemma 0.2 Let T, $\mathcal{L}$ and X be as in Theorem 0.1. Suppose f_n, $n \geq 1$, are $\mathcal{L}_\sigma$-measurable functions from T into X. If f_n converges uniformly to a function $f : T \to X$, then f is $\mathcal{L}_\sigma$-measurable.

An interesting corollary to this selection theorem is the following:

Corollary 0.3 Let T and X be Polish spaces. If $F : T \to$ is a closed valued, α^--multifunction, where $\alpha > 0$, then F admits a selector of class α.

The proofs of these results are omitted.

Let X be a non-empty set. By a partition $\underset{=}{Q}$ of X is meant a family of non-empty, disjoint subsets of X whose union is X. If $\underset{=}{Q}$ is a partition of X, $R(\underset{=}{Q})$ will denote the equivalence relation on X which induces $\underset{=}{Q}$, that is, $R(\underset{=}{Q}) = \bigcup\{E \times E : E \in \underset{=}{Q}\}$. A subset B of X is called a cross section for a partition $\underline{\underline{Q}}$ of X if B meets each element of $\underline{\underline{Q}}$ in exactly one point. If $A \subset X$ then the set $\cup\{E \in \underline{\underline{Q}} : A \cap E \neq \phi\}$ is denoted by $A^{*\underline{\underline{Q}}}$, is simply by A^* if there is no ambiguity, and is called the $\underline{\underline{Q}}$-saturation of A. A set $A \subset X$ is said to be $\underline{\underline{Q}}$-invariant if $A = A^*$.

Let X be a metric space, $\underline{\underline{Q}}$ a partition of X and $\underline{\underline{A}}$ a family of subsets of X. We say that $\underline{\underline{Q}}$ is $\underline{\underline{A}}$-measurable if $V^* \in \underline{\underline{A}}$ for every open set V in X. If $\underline{\underline{A}}$ is the family of Borel sets of additive class α and if $\underline{\underline{Q}}$ is $\underline{\underline{A}}$-measurable then $\underline{\underline{Q}}$ is called an α-partition. If $\underline{\underline{A}} = \underline{\underline{B}}_X$, the Borel σ-field of X, and $\underline{\underline{Q}}$ is $\underline{\underline{A}}$-measurable then we simply say that $\underline{\underline{Q}}$ is measurable. We denote by $\underline{\underline{A}}(\underline{\underline{Q}})$ the σ-field of $\underline{\underline{Q}}$-invariant Borel subsets of X, which will be called the σ-field induced by $\underline{\underline{Q}}$. If $\underline{\underline{A}}$ is an atomic σ-field on X and $\underline{\underline{Q}}$ is the set of atoms of $\underline{\underline{A}}$, we say $\underline{\underline{Q}}$ is the partition of X induced by $\underline{\underline{A}}$.

We now state some more results which are used frequently in this thesis.

Lemma 0.4 Let $(T,\underline{\underline{A}})$ and $(X,\underline{\underline{B}})$ be measurable spaces and let $\underline{\underline{A}}$ be atomic. If $G \in \underline{\underline{A}} \times \underline{\underline{B}}$ then $G^{t_1} = G^{t_2}$ whenever t_1 and t_2 belong to the same $\underline{\underline{A}}$-atom.

Proof : Let $x \in X$. As $G \in \underline{\underline{A}} \times \underline{\underline{B}}$, the set $\{t \in T : (t,x) \in G\} \in \underline{\underline{A}}$. As t_1, t_2 belong to the same $\underline{\underline{A}}$-atom, we get $(t_1,x) \in G \Longleftrightarrow (t_2,x) \in G$. It follows that $x \in G^{t_1} \Longleftrightarrow x \in G^{t_2}$. As $x \in X$ was arbitrary, this completes the proof.

The next result is due to Blackwell [2].

Lemma 0.5 Let T be a Polish space and $\underline{\underline{A}}$ a countably generated sub σ-field of $\underline{\underline{B}}_T$. If $A \in \underline{\underline{B}}_T$ is a union of $\underline{\underline{A}}$ - atoms, then $A \in \underline{\underline{A}}$.

Proof : Let $f : T \rightarrow [0,1]$ be the characteristic function of a countable generator of $\underline{\underline{A}}$. Then $f(A)$ and $f(T - A)$ are disjoint analytic sets in $[0,1]$. Let B be a Borel subset of $[0,1]$ such that $f(A) \subset B$ and $f(T - A) \cap B = \phi$ [11, pp 485]. We now have $A = f^{-1}(B)$ and $f^{-1}(B) \in \underline{\underline{A}}$.

The following result is an easy consequence of the above result of Blackwell and a result of Arsenin and Kunugui [1] which states that if T and X are Polish spaces and if $G \in \underline{\underline{B}}_{T \times X}$ such that G^t is σ-compact for each $t \in T$ then $\prod_T(G)$ is a Borel set in T. An interesting proof of this has been given recently by Saint-Raymond [24].

Lemma 0.6 Let T and X be Polish spaces and $\underline{\underline{A}}$ a countably generated sub σ-field of $\underline{\underline{B}}_T$. If $G \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ and G^t is σ-compact for each $t \in T$, then $\prod_T(G) \in \underline{\underline{A}}$.

Proof : By the result of Arsenin and Kunugui mentioned above, $\prod_T(G)$ is Borel in T. By Lemma 0.4, $\prod_T(G)$ is a union of $\underline{\underline{A}}$ - atoms. Hence, by Lemma 0.5, $\prod_T(G) \varepsilon \underline{\underline{A}}$.

Let $(T,\underline{\underline{A}})$ be a measurable space and let X and Y be two metric spaces. By a Carathéodory map we shall mean a function $f : T \times X \to Y$ such that for each $t \varepsilon T$, $f(t,.)$ is a continuous map from X into Y and for each $x \varepsilon X$, the map $f(.,x)$ defined on T is $\underline{\underline{A}}$ - measurable. If $f : T \times X \to Y$ is a Carathéodory map then the multifunction $F : T \to Y$ defined by $F(t) = f(t,X)$, $t \varepsilon T$, is said to be induced by f. Now we state some important properties of the multifunction F induced by the Carathéodory map $f : T \times X \to Y$.

Clearly, for any $x \varepsilon X$, $f(.,x)$ is a $\underline{\underline{A}}$ - measurable selector for F. Further, assume that X is separable and let $\{x_n : n \geq 1\}$ be a countable dense set in X. Then $\{f(t,x_n) : n \geq 1\}$ is dense in F(t) for each $t \varepsilon T$. It follows that there exist $\underline{\underline{A}}$- measurable selectors $f_n : T \to Y$ for F such that for each $t \varepsilon T$, $\{f_n(t) : n \geq 1\}$ is dense F(t). Also for any open set W in Y

$$F^{-1}(W) = \bigcup_{n=1}^{\infty} (f(.,x_n)^{-1}(W)).$$

Thus, F is $\underline{\underline{A}}$-measurable.

An extensive bibliography of selection theorems is to be found in Wagner [31].

CHAPTER 1

REPRESENTATIONS OF CLOSED VALUED MULTIFUNCTIONS

1 Introduction The motivation for the results proved in this chapter comes from the following theorem of Ioffe :

Theorem 1.1.1 Let X be a Polish space and $(T, \underline{A})$ a measurable space. Let $F : T \to X$ be a closed valued, strongly $\underline{A}$-measurable multifunction. Then there is a Polish space Y and a Carathéodory map $f : T \times Y \to X$ which induces F.

This result has been announced without proof in [8]. We give various refinements of this result. In section 2 we prove the above result for measurable F in the same framework as the one in which the selection theorem of Kuratowski and Ryll-Nardzewski (Theorem 0.1) is proved. A non-separable version of this result is also given. In section 3 we prove similar results for compact valued multifunctions. We show that in this case under slightly stronger measurability conditions Y can be taken to be compact. In section 4, using a recent result of Saint-Raymond, we then deduce a similar representation theorem for σ-compact valued multifunctions. In section 5 we prove some more such representation theorems for closed valued multifunctions.

2. Representation theorems for closed valued multifunctions-I We first state and prove a result of Maitra and Rao [18].

Lemma 1.2.1 Let T be a non-empty set, $\mathcal{L}$ a field of subsets of T and let X be a Polish space. If $F : T \to X$ is a $\mathcal{L}_\sigma$-measurable, closed valued multifunction then there exist $\mathcal{L}_\sigma$-measurable selectors $f_i : T \to X$ for F, $i \geq 1$, such that for each $t \in T$, $\{f_i(t) : i \geq 1\}$ is dense in $F(t)$.

Proof : Let $V_1, V_2, \ldots$ be a countable base for X. By Theorem 0.1, we get a $\mathcal{L}_\sigma$- measurable selector $h : T \to X$ for F. As F is $\mathcal{L}_\sigma$ - measurable, the set $T_n = F^{-1}(V_n) \in \mathcal{L}_\sigma$, $n \geq 1$. Suppose T_{nm}, $m \geq 1$, is a sequence of sets in $\mathcal{L}$ such that $T_n = \bigcup_{m=1}^{\infty} T_{nm}$. Let $F_{nm} : T_{nm} \to V_n$ be defined by $F_{nm}(t) = F(t) \cap V_n$, $t \in T_{nm}$, and let $\mathcal{L}_{nm} = \{L \cap T_{nm} : L \in \mathcal{L}\}$. Then $\mathcal{L}_{nm}$ is a field on T_{nm}, F_{nm} is closed valued and $(\mathcal{L}_{nm})_\sigma$ - measurable. So, since V_n is a Polish space, by Theorem 0.1, there is a $(\mathcal{L}_{nm})_\sigma$ - measurable selector $h_{nm} : T_{nm} \to V_n \subset X$ for F_{nm}. We define $g_{nm} : T \to X$ by

$$g_{nm}(t) = h_{nm}(t) \quad \text{if } t \in T_{nm}$$

$$= h(t) \quad \text{if } t \in T - T_{nm}.$$

As $T_{nm} \in \mathcal{L}$, so does $T - T_{nm}$. Further h_{nm} is $(\mathcal{L}_{nm})_\sigma$ - measurable and h is $\mathcal{F}_\sigma$ - measurable. Therefore, g_{nm} is $\mathcal{L}_\sigma$-measurable. Now, if $t \in T$ and $F(t) \cap V_n \neq \phi$ then $t \in T_{nm}$ for some $m \geq 1$. Hence $g_{nm}(t) \in F(t) \cap V_n$. Thus the sequence of functions g_{nm}, $n \geq 1$, $m \geq 1$, has the desired properties.

The proof given above is a slight modification of the arguments of Maitra and Rao.

<u>Lemma 1.2.2</u> Let $T, \mathcal{L}$, X and F be as in Lemma 1.2.1. Suppose $f : T \to X$ is a $\mathcal{L}_\sigma$ - measurable selector for F and ε is a positive real number. If the multifunction $G : T \to X$ is defined by $G(t) = \overline{S_\varepsilon(f(t)) \cap F(t)}$, $t \in T$, where $S_\varepsilon(f(t)) = \{x \in X : d(x, f(t)) < \varepsilon\}$, d being a complete metric on X, then G is $\mathcal{L}_\sigma$ - measurable.

<u>Proof</u> : By Lemma 1.2.1, let $f_i : T \to X$, $i \geq 1$, be $\mathcal{L}_\sigma$ - measurable functions such that $F(t) = \overline{\{f_i(t) : i \geq 1\}}$ for each $t \in T$. Let W be an open set in X and $t \in T$. Then

$$G(t) \cap W \neq \phi \iff (\exists n \geq 1)\ (f_n(t) \in W \text{ and } d(f_n(t), f(t)) < \varepsilon)$$

Therefore

$$\{t \in T : G(t) \cap W \neq \phi\} = \bigcup_{n=1}^{\infty} [f_n^{-1}(W) \cap \{t \in T : d(f_n(t), f(t)) < \varepsilon\}].$$

It is easily seen that for each $n \geq 1$, the function

$t \to (f_n(t), f(t))$, and hence also the function $t \to d(f_n(t),$ $f(t))$, are $\mathcal{L}_\sigma$ - measurable. Further, functions f_n, $n \geq 1$, are $\mathcal{L}_\sigma$ - measurable. Therefore, G is $\mathcal{L}_\sigma$ - measurable.

Theorem 1.2.3 Let $T, \mathcal{L}, X$ and F be as in Lemma 1.2.1. Then there is a function $f : T \times \Sigma \to X$ such that for each $t \in T$, $f(t,.)$ is a continuous function from Σ onto $F(t)$ and for each $\sigma \in \Sigma$, $f(.,\sigma)$ is $\mathcal{L}_\sigma$ - measurable.

Proof : We give X a complete metric d such that diameter $\delta(X)$ of X is less than 1. We now prove that there exists a system $\{g_s : s \in S\}$ of $\mathcal{L}_\sigma$ - measurable selectors for F such that for every $s \in S$ and $t \in T$,

$$\overline{\{g_{sn}(t) : n \geq 1\}} = \overline{F(t) \cap S_{2^{-|s|}}(g_s(t))}.$$

To see that such a system exists we proceed by induction on $|s|$. We define g_e to be an arbitrary $\mathcal{L}_\sigma$ - measurable selector for F. That such a selector exists follows from Theorem 0.1. Suppose for some $k \geq 0$, $g_s : T \to X$ have been defined for all $s \in S_i$, $i \leq k$, satisfying above conditions. Fix a $s \in S_k$. Let $F_s : T \to X$ be defined by $F_s(t) = \overline{F(t) \cap S_{2^{-k}}(g_s(t))}$, $t \in T$. By Lemma 1.2.2, F_s is $\mathcal{L}_\sigma$ - measurable. By Lemma 1.2.1, let $g_{sn} : T \to X$, $n \geq 1$, be $\mathcal{L}_\sigma$ - measurable selectors for F_s such that for every $t \in T$, $F_s(t) = \overline{\{g_{sn}(t) : n \geq 1\}}$. Since $s \in S$

was arbitrary, this completes the definition of g_s, $s \in S_{k+1}$.
Now we have

(a) $d(g_s(t), g_{sn}(t)) \leq 2^{-|s|}$, $s \in S$, $t \in T$ and $n \geq 1$

and (b) for each $t \in T$ and $x \in F(t)$, there is a $\sigma \in \Sigma$ such that $\lim_k g_{\sigma|k}(t) = x$.

We define $f : T \times \Sigma \to X$ by

$$f(t,\sigma) = \lim_k g_{\sigma|k}(t), \quad t \in T, \sigma \in \Sigma.$$

That the above limit exists follows from (a) and the fact that d is complete. Now let $t \in T$. From (a) and by the definition of f it follows that whenever $\sigma, \sigma' \in \Sigma$ and $\sigma|(k+2)=\sigma'|(k+2)$, $d(f(t,\sigma), f(t,\sigma')) \leq 2^{-k}$. Therefore $f(t,\cdot)$ is continuous. From (b) and the fact that $F(t)$ is closed, we get that the range of $f(t,\cdot)$ is precisely $F(t)$. Finally it follows from (a) that for every $\sigma \in \Sigma$, $g_{\sigma|k}$ converges uniformly to $f(\cdot,\sigma)$ as $k \to \infty$. Hence by Lemma 0.2, $f(\cdot,\sigma)$ is $\mathcal{L}_\sigma$ - measurable.

Corollary 1.2.4 Let T be a metric space and X a Polish space. Let $F : T \to X$ be a closed valued, α^- - multifunction, $\alpha > 0$. Then there is a function $f : T \times \Sigma \to X$ such that for every $t \in T$, $f(t,\cdot)$ is a continuous function from Σ onto $F(t)$ and for each $\sigma \in \Sigma$, $f(\cdot,\sigma)$ is of class α.

Proof : The result follows immediately from Theorem 1.2.3, by ta[king] $\mathcal{L}$ to be the family of all Borel sets of ambiguous class α in T.

We now extend Theorem 1.2.3 for non-separable X. In the rest of this section, α, β will denote ordinal numbers and λ an infinite cardinal number. Cardinal numbers are identified with initial ordinals. λ^+ will denote the successor cardinal to λ. A family $\mathcal{J}$ of subsets of T is said to be <u>λ - additive</u> if whenever $A_\alpha \in \mathcal{J}$, $\alpha < \beta$, $\beta < \lambda$, $\bigcup_{\alpha<\beta} A_\alpha \in \mathcal{J}$. $\mathcal{J}_\lambda$ will denote the smallest λ - additive family of subsets of T containing $\mathcal{J}$. $\mathcal{J}$ is called a <u>λ-field</u> if $\phi \in \mathcal{J}$, $\mathcal{J}$ is closed under complementation and is λ-additive. $B(\lambda)$ will denote <u>the Baire space of weight λ</u>, that is, $B(\lambda) = \lambda^N$ endowed with the product of discrete topologies on λ.

The next result is similar to Lemma 1.2.1. It is proved by Maitra and Rao [18] using the ideas contained in the proof of Lemma 1.2.1.

<u>Lemma 1.2.5</u> Let T be a non-empty set and $\mathcal{J}$ a λ - field on T. Let $F : T \to X$ be a closed valued, $\mathcal{J}_{\lambda^+}$ - measurable multifunction, where X is a complete metric space of topological weight $\leq \lambda$. Then there exist $\mathcal{J}_{\lambda^+}$ - measurable functions $f_\alpha : T \to X$, $\alpha < \lambda$, such that for every $t \in T$, $F(t)=\{f_\alpha(t) : \alpha <$

Using Lemma 1.2.5 and the ideas contained in the proof of Theorem 1.2.3, we now have

Theorem 1.2.6 Let T, $\mathfrak{I}$, X and F be as in Lemma 1.2.5. Then there is a function $f: T \times B(\lambda) \to X$ such that for every $t \in T$, $f(t,.)$ is a continuous function from $B(\lambda)$ onto $F(t)$ and for each $\sigma \in B(\lambda)$, $f(.,\sigma)$ is $\mathfrak{I}_{\lambda^+}$ - measurable.

3. Representation theorems for compact valued multifunctions. The following is the main theorem of this section.

Theorem 1.3.1 Let X be a second countable, metrizable space and let $(T,\underline{A})$ be a measurable space. If $F: T \to X$ is a compact valued, $\underline{A}$ - measurable multifunction then there is a Carathéodory map $f: T \times C \to X$ which induces F, where C denotes the Cantor set.

The above has been announced without proof by Ioffe [9]. We first prove an auxiliary lemma.

Lemma 1.3.2 Let X be a compact, metric space and $(T,\underline{A})$ a measurable space. Then, for every $\varepsilon > 0$, there is a positive integer n such that for every compact valued, $\underline{A}$ - measurable multifunction $F: T \to X$ there exist $\underline{A}$ - measurable selectors $f_1,\ldots,f_n$ for F such that $\{f_1(t),\ldots,f_n(t)\}$ is an ε - net in $F(t)$ for each $t \in T$.

Proof : We take n to be a positive integer such that there exist n open sets, say $W_1, \ldots, W_n$, of diameters $< \varepsilon$ which cover X.

Now let $F : T \to X$ be a compact valued, $\underline{\underline{A}}$ - measurable multifunction. Let $T_i = F^{-1}(W_i)$, $i \leq n$. Then $T_i \in \underline{\underline{A}}$ and $\bigcup_{i \leq n} T_i = T$. We define a multifunction F_i from T_i into the Polish space W_i as follows : $F_i(t) = F(t) \cap W_i$, $t \in T_i$. Then F_i is closed valued and $\underline{\underline{A}}|T_i$ - measurable. We get a $\underline{\underline{A}}|T_i$ - measurable selector $g_i : T_i \to W_i \subset X$ for F_i. Next fix a $\underline{\underline{A}}$ - measurable selector $g : T \to X$ for F. The existence of these selectors follow from Theorem 0.1.

For any positive integer $i \leq n$, we define a map $f_i : T \to X$ by

$$\begin{aligned} f_i(t) &= g_i(t) \quad \text{if } t \in T_i \\ &= g(t) \quad \text{if } t \in T - T_i . \end{aligned}$$

Plainly the functions f_i, $i \leq n$, are $\underline{\underline{A}}$ - measurable selectors for F such that $\{f_i(t) : i \leq n\}$ is an ε - net in $F(t)$ for each $t \in T$.

Proof of Theorem 1.3.1 Let Z be a metric compactification of X and d a metric on Z such that the diameter $\delta(Z) < 1$

We consider F as a multifunction into Z. Then F is compact valued and $\underline{\underline{A}}$ - measurable.

We now show that there exist positive integers $n_1, n_2, \ldots$ and for each $s \in S$ with $s_i \leq n_i$, $i \leq |s|$, an $\underline{\underline{A}}$ - measurable selector $g_s : T \to Y$ for F such that for each $t \in T$, $\{g_{si}(t) : i \leq n_{k+1}\}$ is a $2^{-(k+1)}$ - net in $\overline{F(t) \cap S_{2^{-|s|}}(g_s(t))}$.

We define g_e to be an arbitrary $\underline{\underline{A}}$ - measurable selector for F. We apply Lemma 1.3.2 to Z, $(T, \underline{\underline{A}})$ and $\varepsilon = 2^{-1}$ to get a positive integer n_1 and $\underline{\underline{A}}$ - measurable selectors $g_1, \ldots, g_{n_1}$ for F such that for each $t \in T$, $\{g_1(t), \ldots, g_{n_1}(t)\}$ is a 2^{-1} - net in $F(t)$. Suppose for some $k \in N$, positive integers n_i, $i \leq k$, and functions g_s for $s \in \bigcup_{i \leq k} S_i$ with $s_j \leq n_j$ for every $j \leq |s|$ have been defined satisfying the above conditions. We apply Lemma 1.3.2 to Z, $(T, \underline{\underline{A}})$ and $\varepsilon = 2^{-(k+1)}$ and get a positive integer n_{k+1}. We choose an arbitrary $s \in S_k$ such that $s_i \leq n_i$ for every $i \leq k$. Let the multifunction $F_s : T \to Z$ be defined by $F_s(t) = \overline{F(t) \cap S_{2^{-k}}(g_s(t))}$, $t \in T$. Then F_s is compact valued and, by Lemma 1.2.2, it is $\underline{\underline{A}}$ - measurable. By Lemma 1.3.2, let g_{si}, $i \leq n_{k+1}$, be $\underline{\underline{A}}$ - measurable selectors such that for each $t \in T$, $\{g_{si}(t) : i \leq n_{k+1}\}$ is a $2^{-(k+1)}$ - net in $F_s(t)$.

We take $Y = \overset{\infty}{\underset{i=1}{X}} (\{1,2,\ldots,n_i\})$. Endowed with the product of discrete topologies on $\{1,2,\ldots,n_i\}$ Y is a homeomorph of the Cantor set C. For any $\delta \varepsilon Y$ and $t \varepsilon T$, $\{g_{\delta|n}(t) : n \geq 1\}$ is a Cauchy sequence in Z. We define $f(t,\delta) = \lim_n g_{\delta|n}(t)$, $t \varepsilon T$, $\delta \varepsilon Y$. As in the proof of Theorem 1.2.3 we check that f has the desired properties.

Remark We do not know whether Theorem 1.3.1 is true when $\underline{\underline{A}}$ is replaced by a $\mathcal{L}_\sigma$, where $\mathcal{L}$ is a field on T. (P 1).

4. A representation theorem for σ-compact valued multifunctions.

The first result of this section, which plays a very important role in this thesis, seems to be of independent interest. It is closely related to and will be deduced from the following result of Saint-Raymond :

Lemma 1.4.1 Let X and Y be compact metric spaces. Suppose A and B are disjoint analytic subsets of $X \times Y$ such that A^x is σ-compact for each $x \varepsilon X$. Then there exist Borel sets B_n, $n \geq 1$, in $X \times Y$ such that B_n^x is compact for each $x \varepsilon X$ and $n \geq 1$, $A \subset \bigcup_{n=1}^{\infty} B_n$ and $B \cap (\bigcup_{n=1}^{\infty} B_n) = \emptyset$.

A proof of this can be found in [24].

Lemma 1.4.2 Let T and X be Polish spaces and $\underline{\underline{A}}$ a countably generated sub σ-field of $\underline{\underline{B}}_T$. Suppose $G \varepsilon \underline{\underline{A}} \times \underline{\underline{B}}_X$ and G^t is a G_δ in

X for each $t \in T$. Then there exist sets $G_n \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ such that G_n^t is open in X for each $t \in T$ and $n \geq 1$ and

$G = \bigcap_{n=1}^{\infty} G_n$.

Proof : Let Y be a metric compactification of X. By a well known result [11, pp. 430], X is a G_δ in Y. Let f be the characteristic function of a countable generator of $\underline{\underline{A}}$. Let $g : T \times X \to [0,1] \times Y$ be defined as follows :

$g(t,x) = (f(t), x)$. Let $Z = f(T)$, so that Z is analytic. Further, $g^{-1}(\underline{\underline{B}}_Z \times \underline{\underline{B}}_X) = \underline{\underline{A}} \times \underline{\underline{B}}_X$. Let $H = g(G)$. Since $G \in \underline{\underline{A}} \times \underline{\underline{B}}_X$, it follows that $H \in \underline{\underline{B}}_Z \times \underline{\underline{B}}_X$. As X is a G_δ in Y, we have $\underline{\underline{B}}_Z \times \underline{\underline{B}}_X \subset \underline{\underline{B}}_Z \times \underline{\underline{B}}_Y$. Hence $H \in \underline{\underline{B}}_Z \times \underline{\underline{B}}_Y$. Set $M = (Z \times Y) - H$, so $M \in \underline{\underline{B}}_Z \times \underline{\underline{B}}_Y$. Since H and M are relatively Borel subsets of the analytic set $Z \times Y$, it follows that H and M are disjoint analytic subsets of $[0,1] \times Y$. Again, since X is a G_δ in Y, by Lemma 0.4, it follows that H^z is a G_δ in Y for each $z \in [0,1]$. So M^z is σ-compact in Y for each $z \in [0,1]$. By Lemma 1.4.1, we get Borel sets B_n, $n \geq 1$, of $[0,1] \times Y$ such that B_n^z is compact for each $z \in [0,1]$ and $n \geq 1$, $M \subset \bigcup_{n=1}^{\infty} B_n$ and $H \cap (\bigcup_{n=1}^{\infty} B_n) = \phi$.

To complete the proof, let $H_n = (Z \times X) - B_n$, $n \geq 1$. Then each $H_n \in \underline{\underline{B}}_Z \times \underline{\underline{B}}_X$ and H_n^z is open in X for each $z \in Z$.

Furthermore, since $H \subseteq ([0,1] \times Y) - \bigcup_{n=1}^{\infty} B_n$ and $H \subseteq Z \times X$, we have $H \subseteq (Z \times X) - \bigcup_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} H_n$. To go the other way, we observe that $H = (Z \times Y) - M \supseteq (Z \times Y) - \bigcup_{n=1}^{\infty} B_n \supseteq (Z \times X) - \bigcup_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} H_n$. Thus, we have proved that $H = \bigcap_{n=1}^{\infty} H_n$. Finally, we put $G_n = g^{-1}(H_n)$, $n \geq 1$. The sets G_n have the desired properties.

<u>Lemma 1.4.3</u> Let T, X be Polish spaces and $\underline{\underline{A}}$ a countably generated sub σ - field of $\underline{\underline{B}}_T$. Suppose $G \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ and G^t is σ - compact for each $t \in T$. Then there exist sets $G_n \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ such that G_n^t is compact for each $t \in T$ and $n \geq 1$ and $G = \bigcup_{n=1}^{\infty} G_n$.

<u>Proof</u>: Let Y be a metric compactification of X. Since X is a G_δ in Y, $G \in \underline{\underline{A}} \times \underline{\underline{B}}_Y$. Let $H = (T \times Y) - G$. Then $H \in \underline{\underline{A}} \times \underline{\underline{B}}_Y$ and H^t is a G_δ in Y for each $t \in T$. So by Lemma 1.4.2, there exist sets $H_n \in \underline{\underline{A}} \times \underline{\underline{B}}_Y$ such that H_n^t is open in Y for each $t \in T$ and $n \geq 1$ and $H = \bigcap_{n=1}^{\infty} H_n$. Let $G_n = (T \times Y) - H_n$, $n \geq 1$. The sets G_n, $n \geq 1$ have the desired properties.

<u>Theorem 1.4.4</u> Let T, X be Polish spaces and $\underline{\underline{A}}$ a countably generated sub σ - field of $\underline{\underline{B}}_T$. Let $F : T \to X$ be a σ-compact valued multifunction such that its graph $Gr(F) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$. Then there is a locally compact, second countable, metrizable space Y and a Carathéodory map $f : T \times Y \to X$ which induces F.

<u>Proof</u> : It is sufficient to prove the result when X is a compact metric space. So we assume that X is compact. We put $G = Gr(F)$. By Lemma 1.4.3, we get sets $G'_n \in \underline{\underline{A}} \times \underline{\underline{B}}_X$, $n \geq 1$, with compact sections such that $G = \bigcup_{n=1}^{\infty} G'_n$. For any $n \geq 1$, let $n_1, n_2, \ldots$ be an enumeration of positive integers such that $n_1 = n$. Let

$$
\begin{aligned}
T^n_i &= \prod\nolimits_T (G'_{n_i}) && \text{if } i = 1 \\
&= \prod\nolimits_T (G'_{n_i}) - \bigcup_{j<i} \prod\nolimits_T (G'_{n_j}) && \text{if } i > 1 .
\end{aligned}
$$

By Lemma 0.6, $T^n_i \in \underline{\underline{A}}$, $i \geq 1$. Further $T^n_i \cap T^n_j = \phi$ for $i \neq j$ and $T = \bigcup_{i=1}^{\infty} T^n_i$. We put $G_n = \bigcup_{i=1}^{\infty} ((T^n_i \times X) \cap G'_{n_i})$. Then G_n, $n \geq 1$ belong to $\underline{\underline{A}} \times \underline{\underline{B}}_X$ with all sections non-empty and compact and $G = \bigcup_{n=1}^{\infty} G_n$. We define $F_n : T \to X$ by $F_n(t) = G^t_n$, $t \in T$, $n \geq 1$. Then F_n is compact valued and, by Lemma 0.6, $\underline{\underline{A}}$ - measurable.

We use Theorem 1.3.1 to get a Carathéodory map $f_n : T \times C \to X$ which induces F_n, where C denotes the Cantor set.

Let $Y = N \times C$ and let $f : T \times (N \times C) \to X$ be defined as follows : $f(t, (n, \delta)) = f_n(t, \delta)$. The function f and the space Y have the desired properties.

Note By Lemma 0.6, it is easily seen that a multifunction F satisfying the hypotheses of the above theorem is strongly $\underline{A}$ - measurable.

5. Representation theorems for closed valued multifunctions-II

We first prove the following well known result. The particular proof given below will be of some importance to us.

Lemma 1.5.1 Let E be a non-empty closed subset of Σ. Then there is a closed retraction f of Σ onto E.

Proof : For $s \in S$ such that $\Sigma_s \cap E \neq \phi$ we choose a point $x_s \in \Sigma_s \cap E$, for instance, x_s could be the lexicographic minimum of $\Sigma_s \cap E$. If $\sigma \in \Sigma$ we put $f(\sigma) = \sigma$. If $\sigma \in \Sigma$ since E is closed, there is a positive integer k such that $\Sigma_{\sigma|k} \cap E = \phi$. Let n be the first such integer. We define $f(\sigma) = x_{\sigma|n-1}$. It is clear that $f(\Sigma) \subset E$ and $f(\sigma) = \sigma$ for every $\sigma \in E$.

We now check that f is continuous at each $\sigma \in \Sigma$. First let $\sigma \in \Sigma - E$ and let $k \in N$ be such that $\Sigma_{\sigma|k} \cap E = \phi$ Then f is constant on $\Sigma_{\sigma|k}$. It follows that f is continuous at σ. Next, assume $\sigma \in E$. Then for any positive integer k, $\Sigma_{\sigma|k} \cap E \neq \phi$ and $f(\Sigma_{\sigma|k}) \subseteq \Sigma_{\sigma|k} \cap E$. This implies that f is continuous at σ in this case also.

To check that f is closed, let $W \subseteq \Sigma$ be closed and $x_n \to x$, where $x_n \in f(W)$, $n \geq 1$. We get $\sigma^n \in W$ such that $x_n = f(\sigma^n)$, $n \geq 1$. If $\sigma^n \in E$ for infinitely many n then $x \in W$ and $f(x) = x$, so that $x \in f(W)$. It now suffices to prove that $x \in f(W)$ in case $\sigma^n \notin E$ for every n. We, then, get $k_n \geq 0$ such that $x_n = x_{\sigma^n|k_n}$. If infinitely many k_n are equal then, as x_n converges, infinitely many $\sigma^n|k_n$ are the same. It follows that infinitely many x_n are equal. From this we deduce that $x \in f(W)$. Now we consider the case when $k_n \to \infty$. Let $x = (m_1, m_2, \ldots)$. For any positive integer ℓ, we get a large enough n such that $x_n \in \Sigma_{m_1 \ldots m_\ell}$ and $k_n > \ell$. Then $\sigma^n|\ell \in \Sigma_{m_1 \ldots m_\ell}$. Thus $W \cap \Sigma_{m_1 \ldots m_\ell} \neq \phi$. Since $\ell \in N$ was arbitrary and W is closed it follows that $x \in W$. Since E is closed, $x \in E$. Therefore $f(x) = x \in f(W)$. The proof is complete.

The proof given above is due to Frolik [6].

<u>Proposition 1.5.2</u> Let T be a non-empty set and $\mathcal{L}$ a field on T. Let $F : T \Rightarrow \Sigma$ be a closed valued multifunction such that $F^{-1}(\Sigma_s) \in \mathcal{L}$ for each $s \in S$. Then there is a map $g : T \times \Sigma \to \Sigma$ such that for each $t \in T$, $g(t,.)$ is a closed retraction of Σ onto $F(t)$; and for each $\sigma \in \Sigma$, $g(.,\sigma)$ is $\mathcal{L}_{\delta\sigma}$-measurable.

<u>Proof</u> : Let $s \in S$ and let $T_s = \{t \in T : F(t) \cap \Sigma_s \neq \phi\}$.
Then $T_s \in \mathcal{L}$. We define a closed valued multifunction $F_s : T_s \to \Sigma_s$ by $F_s(t) = F(t) \cap \Sigma_s$, $t \in T_s$. Then F_s is $\mathcal{L}_\sigma | T_s$ - measurable, where $\mathcal{L}_\sigma | T_s = \{L \cap T_s : L \in \mathcal{L}_\sigma\}$. By Theorem 0.1, let $f_s : T_s \to \Sigma$ be a $\mathcal{L}_\sigma | T_s$ - measurable selector for F_s.

We now define $g : T \times \Sigma \to \Sigma$ by

$$g(t,\sigma) = \sigma \qquad \text{if } \sigma \in F(t)$$

$$= f_{\sigma|n-1}(t) \qquad \text{if } \sigma \notin F(t) \text{ and } n \text{ is the first positive } k \text{ such that } F(t) \cap \Sigma_{\sigma|k} = \phi .$$

As F is closed valued, g is defined on the whole of $T \times \Sigma$. Let $t \in T$. Arguments in the proof of Lemma 1.5.1 show that $g(t,.)$ is a closed retraction of Σ onto $F(t)$.

To check the final conclusion, we fix a $\sigma \in \Sigma$ and define

$$T^n = \left(\bigcap_{m<n} T_{\sigma|m} \right) - T_{\sigma|n} , \quad n \geq 1.$$

Then the sets $T^n \in \mathcal{L}$ and

$$g(t,\sigma) = f_{\sigma|n-1}(t) \quad \text{if } t \in T^n$$

$$= \sigma \quad \text{if } t \in T - \left(\bigcup_{n=1}^{\infty} T^n \right).$$

It follows that $g(\cdot,\sigma)$ is $\mathcal{L}_{\delta\sigma}$ - measurable.

The next result also improves the result of Ioffe (Theorem 1.1.1)

Theorem 1.5.3 Let T be a non-empty set and $\mathcal{L}$ a field on T. Let X be a Polish space and $F : T \to X$ a closed valued, strongly $\mathcal{L}$ - measurable multifunction. Then there is a map $f : T \times \Sigma \to X$ such that for each $t \in T$, $f(t,\cdot)$ is a continuous closed map from Σ onto $F(t)$ and for each $\sigma \in \Sigma$, $f(\cdot,\sigma)$ is $\mathcal{L}_{\delta\sigma}$ - measurable.

Proof : A result of Engelking [5] states that every Polish space is the image of Σ under a continuous and closed map. So let $h : \Sigma \to X$ be a continuous, closed and onto map. We define $H : T \to \Sigma$ by $H(t) = h^{-1}(F(t))$, $t \in T$. Then for any $s \in S$

$$\{t \in T : H(t) \cap \Sigma_s \neq \phi\} = \{t \in T : F(t) \cap h(\Sigma_s) \neq \phi\}$$

Since h is closed it follows that $H^{-1}(\Sigma_s) \in \mathcal{L}$.

By Proposition 1.5.2, let $g : T \times \Sigma \to \Sigma$ be such that for each $t \varepsilon T$, $g(t,.)$ is continuous, closed and onto $H(t)$ and for each $\sigma \varepsilon \Sigma$, $g(.,\sigma)$ is $\mathcal{L}_{\delta\sigma}$ - measurable. Put $f = h \circ g$. Plainly f has the desired properties.

<u>Corollary 1.5.4</u> Let T and X be Polish spaces and let $F : T \to X$ be a closed valued, α^{+} - multifunction. Then there is a map $f : T \times \Sigma \to X$ such that for each $t \varepsilon T$, $f(t,.)$ is continuous, closed and onto $F(t)$ and for each $\sigma \varepsilon \Sigma$, $f(.,\sigma)$ is of class $(\alpha + 2)$.

<u>Proof</u> : The result follows from Theorem 1.5.3 by taking to be the family of Borel sets of ambiguous class $(\alpha + 1)$.

CHAPTER 2

SELECTION THEOREMS FOR G_δ VALUED MULTIFUNCTIONS

1. Introduction   In recent years a large number of selection and representation theorems for multifunctions taking closed values in a Polish space have been proved. We shall now consider multifunctions taking values in a fixed Borel class of a Polish space. The following example, due to Kallman and Mauldin [10], shows that multifunctions with values in additive class 1, a fortiori in any higher class, need not admit even a measurable selector :

Example 2.1.1   Let M be a closed set in $\Sigma \times \Sigma$ such that for each $\sigma \in \Sigma$, M^σ is non-empty, and M is not Borel uniformizable. The existence of such a set has been shown by several authors, most recently by Blackwell [3]. For $s \in S$, let h_s be a homeomorphism of Σ onto Σ_s and let $T_s : \Sigma \times \Sigma \to \Sigma \times \Sigma_s$ be defined as follows : $T_s(\sigma, \sigma') = (\sigma, h_s(\sigma'))$. We now put $H = \bigcup_{s \in S} T_s(M)$. Then H is an F_σ subset of $\Sigma \times \Sigma$ such that the projection map from H onto the horizontal axis is open. If H admits a Borel uniformization, say E, then $\bigcup_{s \in S} T_s^{-1}(E \cap T_s(M))$ is a Borel subset of M with all sections non-empty and countable. It is well known that such sets are Borel uniformizable [16, pp. 244]. It follows that M is Borel uniformizable. But by assumption this is impossible. So H does not admit a Borel uniformization.

We now take $T = X = \Sigma$, $\underline{\underline{A}} = \underline{\underline{B}}_T$ and $F(t) = H^t$, $t \varepsilon T$. Then the multifunction F is lower semi-continuous, in parti $\underline{\underline{A}}$ - measurable and $GR(F) \varepsilon \underline{\underline{A}} \times \underline{\underline{B}}_X$. In fact, its graph is an F_σ in $T \times X$ and it takes values in additive class 1. But F does not admit a measurable selector.

Subsequent results proved in this thesis show that under fairly mild restrictions G_δ valued multifunctions behave wel and this chapter is devoted towards proving the existence of measurable selectors for such multifunctions.

In section 2 we prove some auxiliary results. An interes ing result proved is an invariant version of Novikov's first multiple separation principle for analytic sets. In section 3 we prove the main result of this chapter. A uniformization result is established in section 4.

2. <u>Auxiliary Results</u> In this section we set down some resul which will be found useful in the sequel.

<u>Lemma 2.2.1</u> Let $\underline{\underline{A}}$ be a countably generated sub σ - field o the Borel σ - field of a Polish space E. Let $\underline{\underline{Q}}$ be the part tion of E induced by $\underline{\underline{A}}$. Then the equivalence relation $R(\underline{\underline{Q}}) \varepsilon \underline{\underline{A}} \times \underline{\underline{A}}$ and consequently, $R(\underline{\underline{Q}})$ is a Borel subset of E

<u>Proof</u> : Let A_n, $n \geq 1$, generate the σ - field $\underline{\underline{A}}$ and let f be the characteristic function of the sequence $\{A_n\}$. Then

$$R(\underline{\underline{Q}}) = \{(x,y) \in E \times E : f(x) = f(y)\}$$

so that $R(\underline{\underline{Q}}) \in \underline{\underline{A}} \times \underline{\underline{A}}$.

We shall now prove an invariant version of Novikov's first multiple separation principle [11; pp. 510] for analytic sets. This is proved rather easily using Novikov's first multiple separation principle and an invariant version of first separation principle for a pair of disjoint analytic sets. A simple proof of Novikov's separation principle mentioned above has been given recently by Mokobodzki [22]. We give below a proof of an invariant version of first separation principle for two disjoint analytic sets. The proof is due to C. Ryll-Nardzewski.

<u>Lemma 2.2.2</u> Let $\underline{\underline{Q}}$ be a partition of a Polish space E such that $R(\underline{\underline{Q}})$ is an analytic subset of $E \times E$. Suppose A and B are two disjoint analytic subsets of E such that A is $\underline{\underline{Q}}$ - invariant. Then there is a $\underline{\underline{Q}}$ - invariant Borel set W such that $B \subset W$ and $W \cap A = \phi$.

<u>Proof</u> : We first note that if V is an analytic subset of E so is its $\underline{\underline{Q}}$ - saturation V^*. Since $R(\underline{\underline{Q}})$ is analytic this follows immediately from the following equivalence :

$$x \in V^* \iff (\exists y \in E)\ (y \in V \text{ and } (x,y) \in R(\underset{=}{Q}))\,.$$

Now we define Borel sets C_n, $n \geq 1$, and $\underset{=}{Q}$ - invariant analytic sets D_n, $n \geq 1$, such that $B \subseteq C_n$, $C_n \subseteq D_n \subseteq C_{n+1}$ and $D_n \cap A = \phi$. We define these by induction on n. We take C_1 to be a Borel set such that $B \subseteq C_1$ and $C_1 \cap A = \phi$. Let $D_1 = C_1^*$. By the observation made above D_1 is analytic in E. Since A is $\underset{=}{Q}$ - invariant, $D_1 \cap A = \phi$. Suppose for some n, C_i, D_i, $i \leq n$, have been defined satisfying above conditions. We take C_{n+1} to be a Borel set such that $D_n \subseteq C_{n+1}$ and $C_{n+1} \cap A = \phi$. We then put $D_{n+1} = C^*_{n+1}$.

Now let $W = \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} D_n$. It is easily seen that W has the desired properties.

Lemma 2.2.3 Let $\underset{=}{Q}$ be a partition of a Polish space E such that $R(\underset{=}{Q})$ is an analytic subset of $E \times E$. If A_n, $n \geq 1$, are $\underset{=}{Q}$ - invariant analytic subsets of E such that $\bigcap_{n=1}^{\infty} A_n = \phi$, then there exist $\underset{=}{Q}$ - invariant Borel sets B_n, $n \geq 1$, such that $A_n \subseteq B_n$ for each $n \geq 1$ and $\bigcap_{n=1}^{\infty} B_n = \phi$.

Proof: By Novikov's first multiple separation principle for analytic sets, there exist Borel sets C_n in E such that

$A_n \subset C_n$ and $\bigcap_{n=1}^{\infty} C_n = \phi$. Since A_n and $(E - C_n)$ are disjoint analytic sets in E such that A_n is $\underset{=}{Q}$ - invariant, by the last Lemma, we get a $\underset{=}{Q}$ - invariant Borel set B_n in E such that $A_n \subset B_n$ and $B_n \cap (E - C_n) = \phi$. But then $B_n \subset C_n$. Since $\bigcap_{n=1}^{\infty} C_n = \phi$, it follows that $\bigcap_{n=1}^{\infty} B_n = \phi$.

<u>Lemma 2.2.4</u> Let $\underset{=}{Q}$ be a partition of a Polish space E such that $R(\underset{=}{Q})$ is analytic in $E \times E$. If Z_n, $n \geq 1$, are $\underset{=}{Q}$ - invariant coanalytic subsets of E such that $\bigcup_{n=1}^{\infty} Z_n$ is Borel in E, then there exist $\underset{=}{Q}$ - invariant Borel subsets D_n, $n \geq 1$, of E such that $D_n \subset Z_n$ for each $n \geq 1$, $D_n \cap D_m = \phi$ for $n \neq m$ and $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} Z_n$.

<u>Proof</u>: Set $B = \bigcup_{n=1}^{\infty} Z_n$ and $A_n = B - Z_n$, $n \geq 1$. Then the sets A_n are $\underset{=}{Q}$ - invariant and analytic such that $\bigcap_{n=1}^{\infty} A_n = \phi$. So by Lemma 2.2.3, there exist $\underset{=}{Q}$ - invariant Borel sets B_n in X such that $A_n \subset B_n$ and $\bigcap_{n=1}^{\infty} B_n = \phi$. We define

$$D_n = B - B_n \qquad \text{if } n = 1$$

$$= (B - B_n) \cap \bigcap_{i<n} B_i \qquad \text{if } n > 1$$

The sets D_n have the desired properties.

3. Selection Theorems Our main selection theorem for G_δ valued multifunctions reads as follows :

Theorem 2.3.1 Let T, X be Polish spaces and let $\underline{\underline{A}}$ be a countably generated sub σ - field of $\underline{\underline{B}}_T$. Suppose $F : T \to X$ is a multifunction such that F is $\underline{\underline{A}}$ - measurable, $Gr(F) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ and F(t) is a G_δ in X for each $t \in T$. Then there is a $\underline{\underline{A}}$ - measurable selector f for F.

Proof : The idea of the proof is as follows : Given a non-empty G_δ in a Polish space, there is an effective procedure for selecting a point from it [11, pp. 418]. We apply this procedure to each F(t) uniformly.

We fix a system $\{V_s : s \in S\}$ of non-empty open subsets of X such that

(a) $V_e = X$,

(b) $\delta(V_s) < 2^{-k}$, $s \in S_k$, $k \geq 0$,

(c) $\{V_{sm} : m \geq 1\}$ is an open base for V_s , $s \in S$, and

(d) $\overline{V}_{sm} \subset V_s$, $s \in S$, $m \geq 1$.

Put $G_o = T \times X$, and let G_k, $k \geq 1$, be a sequence of sets in $\underline{\underline{A}} \times \underline{\underline{B}}_X$ such that G_k^t is open for each $t \in T$ and $k \geq 1$ and $G = \bigcap_{k=1}^{\infty} G_k$, where G denotes the graph of

F. The existence of such a sequence of sets is ensured by Lemma 1.4.2. We denote by $\underline{\underline{Q}}$ the partition of T induced by $\underline{\underline{A}}$. By Lemma 0.5, $\underline{\underline{A}} = \underline{\underline{A}}(\underline{\underline{Q}})$. We shall now prove that there is a system $\{B_s : s \in S\}$ of subsets of T satisfying the following conditions :

(i) $B_e = T$,

(ii) $B_s = \bigcup_{m=1}^{\infty} B_{sm}$, $s \in S$,

(iii) $s, s' \in S_k$, $s \neq s' \Rightarrow B_s \cap B_{s'} = \phi$,

(iv) $s \in S_k$ and $t \in B_s \Rightarrow G^t \cap V_s \neq \phi$ and $\overline{V}_s \subset G^t_k$,

and (v) $B_s \in \underline{\underline{A}}(\underline{\underline{Q}})$, $s \in S$.

To see that such a system can be defined, we proceed inductively. First define $B_e = T$. Next suppose that B_s, $s \in S_i$, $i \leq k$, have been defined in such a way that the above conditions are satisfied. We shall now define $B_{s'}$, $s' \in S_{k+1}$. Fix a $s \in S_k$ and set

$$Z_m = \{t \in B_s : G^t \cap V_{sm} \neq \phi \text{ and } \overline{V}_{sm} \subset G^t_{k+1}\}, \quad m \geq 1.$$

Then

$$Z_m = B_s \cap F^{-1}(V_{sm}) \cap (T - \prod_T ((T \times \overline{V}_{sm}) - G_{k+1})).$$

It follows that the sets Z_m are coanalytic in T. Moreover, since $B_s \in \underline{\underline{A}}(\underline{\underline{Q}})$, $F^{-1}(V_{sm}) \in \underline{\underline{A}} = \underline{\underline{A}}(\underline{\underline{Q}})$ and $G_{k+1} \in \underline{\underline{A}} \times \underline{\underline{B}}_X$, by Lemma 0.4, the sets Z_m are $\underset{=}{Q}$ - invariant. Now we check that $\bigcup_{m=1}^{\infty} Z_m = B_s$. To see this, let $t \in B_s$, so that $G^t \cap V_s \neq \phi$. Choose $x \in G^t \cap V_s \subset G^t_{k+1} \cap V_s$. Since G^t_{k+1} is open in X, we can find m such that $x \in V_{sm} \subset \overline{V}_{sm} \subset G^t_{k+1} \cap V_s$. It now follows that $t \in Z_m$, which proves the inclusion $B_s \subset \bigcup_{m=1}^{\infty} Z_m$. The reverse inclusion is obvious. By Lemma 2.2.1, $R(\underset{=}{Q})$ is a Borel, and therefore analytic, subset of $T \times T$. So Lemma 2.2.4 can be applied to the sets Z_m. We will then get sets $D_m \in \underset{=}{A}(\underset{=}{Q})$, $m \geq 1$, such that $D_m \subset Z_m$, $D_n \cap D_m = \phi$ for $n \neq m$ and $\bigcup_{m=1}^{\infty} D_m = B_s$. Define $B_{sm} = D_m$, $m \geq 1$. Since $s \in S_k$ was fixed but arbitrary, this completes the definition of sets $B_{s'}$, $s' \in S_{k+1}$. It is now an easy matter to verify that these sets satisfy the required conditions.

For the final step in the proof, we define

$$E_k = \bigcup_{s \in S_k} (B_s \times \overline{V}_s), \quad k \geq 0$$

and

$$E = \bigcap_{k \geq 0} E_k.$$

Using conditions (a) - (d) and (i) - (v), we check that each $E_k \in \underset{=}{A}(\underset{=}{Q}) \times \underset{=}{B}_X$ and so $E \in \underset{=}{A}(\underset{=}{Q}) \times \underset{=}{B}_X$, that E^t contains exactly one point for each $t \in T$ and that $E \subset G$. The set E defines uniquely a function f on T to X

whose graph is E. Since E is a Borel subset of $T \times X$, f is Borel measurable [11, pp. 489]. It follows from this, the fact that $E \in \underline{\underline{A}}(\underline{\underline{Q}}) \times \underline{\underline{B}}_X$, $\underline{\underline{A}} = \underline{\underline{A}}(\underline{\underline{Q}})$ and Lemmas 0.4 and 0.5 that f is an $\underline{\underline{A}}$ - measurable selector for F. This completes the proof.

Next we relax somewhat the requirement in Theorem 2.3.1 that T be a Polish space.

<u>Theorem 2.3.2</u> Let T be an analytic set and let $\underline{\underline{A}}$ be a countably generated sub σ - field of $\underline{\underline{B}}_T$. Let X be a Polish space. Suppose $F : T \to X$ is a multifunction such that F is $\underline{\underline{A}}$ - measurable, $Gr(F) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ and F(t) is a G_δ in X for each $t \in T$. Then there is a $\underline{\underline{A}}$ - measurable selector f for F.

<u>Proof</u>: Let h be a continuous function on Σ onto T. Set $\underline{\underline{A}}' = h^{-1}(\underline{\underline{A}})$, so $\underline{\underline{A}}'$ is a countably generated sub σ - field of the Borel σ - field of Σ. Define a multifunction $F' : \Sigma \to X$ by $F'(\sigma) = F(h(\sigma))$. It is easy to check that F' is $\underline{\underline{A}}'$- measurable, $Gr(F') \in \underline{\underline{A}}' \times \underline{\underline{B}}_X$ and $F'(\sigma)$ is a G_δ in X for each $\sigma \in \Sigma$. So, by Theorem 2.3.1, there exists a $\underline{\underline{A}}'$- measurable selector g for F'. Finally, define a function f on T to X by the formula: $f(h(\sigma)) = g(\sigma)$, $\sigma \in \Sigma$. As g is $\underline{\underline{A}}'$ - measurable, g is constant on atoms of $\underline{\underline{A}}'$. Further, for each $t \in T$, $h^{-1}(t)$ is a subset of an atom of $\underline{\underline{A}}'$. For if

A is the atom of $\underline{\underline{A}}$ containing t, then $h^{-1}(A)$ must be an atom of $\underline{\underline{A}}'$. It follows that if $h(\sigma) = h(\sigma')$, then $g(\sigma) = g(\sigma')$. Hence f is well defined. Plainly f is a selector for F. That f is $\underline{\underline{A}}$ - measurable follows now from the fact that g is $\underline{\underline{A}}'$ - measurable and $\underline{\underline{A}}' = h^{-1}(\underline{\underline{A}})$.

Remark In Theorem 2.3.2 the condition that $\underline{\underline{A}}$ be countably generated can be dropped. Indeed, let $\underline{\underline{A}}$ be any sub σ - field of $\underline{\underline{B}}_T$ and F be as in Theorem 2.3.2. Since $Gr(F) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$, there exist rectangles $A_i \times B_i \in \underline{\underline{A}} \times \underline{\underline{B}}_X$, $i \geq 1$, such that $Gr(F)$ is already in the σ - field generated by $A_i \times B_i$, $i \geq 1$. Next, set $C_n = F^{-1}(V_n)$, $n \geq 1$, where V_n, $n \geq 1$, is an open base for X. Now let $\underline{\underline{A}}_o$ be the σ - field on T generated by the sets A_i, C_i, $i \geq 1$. Then, as is easy to verify, $\underline{\underline{A}}_o$ is a countably generated sub σ - field of $\underline{\underline{A}}$, F is $\underline{\underline{A}}_o$ - measurable and $Gr(F) \in \underline{\underline{A}}_o \times \underline{\underline{B}}_X$. By Theorem 2.3.2, there is a $\underline{\underline{A}}_o$ - measurable selector f for F. Clearly f is $\underline{\underline{A}}$ - measurable.

If we repeat the arguments made in the proof of Lemma 1.2.1 and use Theorem 2.3.1 instead of the selection theorem of Kuratowski and Ryll-Nardzewski we get the following result :

Corollary 2.3.3 Let T, $\underline{\underline{A}}$, X and F be as in Theorem 2.3.1. Then there is a sequence f_i, $i \geq 1$, of $\underline{\underline{A}}$ - measurable selectors for F such that for each $t \in T$, $\{f_i(t) : i \geq 1\}$ is dense in $F(t)$.

We conclude this section by showing that the condition in Theorem 2.3.1 that $G \in \underline{A} \times \underline{\underline{B}}_X$ cannot be dropped. Let $T = X = [0,1]$ and $\underline{A} = \underline{\underline{B}}_T$. Let $\underline{\Phi}$ be the set of all Borel measurable functions of T to X and let φ map T onto $\underline{\Phi}$. Define a multifunction $F : T \to X$ by $F(t) = [0,1] - \{\varphi(t)\}$. Then $F(t)$ is open in X for each $t \in T$. As $F(t)$ is dense in X for each $t \in T$, F is $\underline{A}$ - measurable. Now suppose f is a $\underline{A}$ - measurable selector for F. Then $f \in \underline{\Phi}$, so $f = \varphi(t_o)$ for some $t_o \in T$. It now follows that $\varphi(t_o)(t_o) \neq \varphi(t_o)(t_o)$, which is a contradiction. So F does not admit a $\underline{A}$ - measurable selector.

4. <u>A uniformization result</u> An important consequence of Theorem 2.3.2 is the following result on the uniformization of Borel sets.

<u>Theorem 2.4.1</u> Let L and M be Polish spaces. Suppose B is a Borel subset of $L \times M$ such that B^t is a G_δ in M for each $t \in L$ and $\Pi_L(B \cap (L \times V))$ is relatively Borel in $\Pi_L(B)$ for every open set V in M. Then B can be uniformized by a Borel subset of $L \times M$.

<u>Proof</u> : An application of Theorem 2.3.2 yields a Borel measurable function f on $\Pi_L(B)$ to M such that I is a uniformization of B, where I is the graph of f. By a result of Kuratowski [11, pp. 434], there is a Borel measurable function g on L

to M which extends f. Let J be the graph of g. Then J is a Borel subset of $L \times M$ and $I = J \cap B$, which proves that I is a Borel subset of $L \times M$. This completes the proof.

Corollary 2.4.2 Let L and M be Polish spaces. Suppose B is a Borel set in $L \times M$ satisfying the hypotheses of Theorem 2.4.1. Then $\Pi_L(B)$ is Borel in L.

Corollary 2.4.3 Let L and M be Polish spaces. Suppose B is a Borel set in $L \times M$ such that for each $t \in \Pi_L(B)$, B^t is a dense G_δ in M. Then B is Borel uniformizable. In particular, $\Pi_L(B)$ is Borel in L.

The last corollary is a particular case of a uniformization result of H. Sarbadhikari [25]. It should be noted that in the proof of the last uniformization result we do not use the fact that '$\Pi_L(B)$ is Borel in L'.

We note that the above Theorem on the uniformization of Borel sets cannot be improved upon. Indeed, if B is a G_δ in the plane whose projection to the first coordinate is not Borel then B cannot be uniformized by a Borel subset of the plane. This shows that the condition requiring $\Pi_L(B \cap (L \times V))$ to be relatively Borel in $\Pi_L(B)$ cannot be dropped from Theorem 2.4.1. On the other hand, Example 2.1.1 shows that the condition requiring B^t to be a G_δ for $t \in L$ cannot be dropped.

CHAPTER 3

PARTITIONS OF POLISH SPACES

1. Introduction   In this chapter we investigate the problem of the existence of a Borel cross section for a measurable partition $\underline{\underline{Q}}$ of a Polish space into Borel sets of a fixed class. A large number of results are already known in case members of $\underline{\underline{Q}}$ are closed. An extensive bibliography of such results is to be found in Wagner [31]. The well known partition, given by Vitali, of the real line $\mathbb{R}$ induced by the equivalence relation $\{(x,y) \in \mathbb{R} \times \mathbb{R} : x - y$ is a rational$\}$ is lower semi-continuous, that is, saturation of every open set is open in $\mathbb{R}$, and yet does not admit a Lebesgue measurable and, a fortiori, a Borel cross section. Thus it remains to investigate measurable partitions of Polish spaces into G_δ sets. In section 2 we establish the existence of a Borel cross section for such partitions. A cross section theorem for partitions into σ - compact sets is proved in section 3.

2. Cross section theorems for partitions into G_δ sets   We first prove a result on σ - fields induced by partitions. We shall use the following lemma of Kallman and Mauldin [10].

Lemma 3.2.1   Let X be a Polish space and let V_n, $n \geq 1$, be an open base for X. If E is non-empty and simultaneously a F_σ and a G_δ in X then $E \cap V_n = \bar{E} \cap V_n \neq \phi$ for some $n \geq 1$.

Proof : Let $E = \bigcup_{m=1}^{\infty} F_m$, where F_m, $m \geq 1$, are closed in X. Since E is a dense G_δ in $\bar{E}$, the Baire Category theorem implies that E is non-meager relative to $\bar{E}$. It follows that there is a F_m with non-empty interior relative to $\bar{E}$. Consequently, there is a V_n such that $\phi \neq \bar{E} \cap V_n \subset F_m \subset E$, from which the desired conclusion follows.

Proposition 3.2.2 Suppose that $\underline{\underline{Q}}$ is a measurable partition of a Polish space X such that each element of $\underline{\underline{Q}}$ is a G_δ in X. Then the σ - field $\underline{\underline{A}}(\underline{\underline{Q}})$ induced by $\underline{\underline{Q}}$ is countably generated.

Proof : Let V_n, $n \geq 1$, be an open base for X. We shall now show that if E_1, E_2 are distinct elements of $\underline{\underline{Q}}$, then there is a V_i such that $E_1 \subset V_i^* \subset (X - E_2)$ or $E_2 \subset V_i^* \subset (X - E$ Two cases arise. First suppose that $E_2 - \bar{E}_1 \neq \phi$. So if $x \in E_2 - \bar{E}_1$, we can find V_i such that $x \in V_i$ and $V_i \cap E_1 = \phi$ It follows that $E_2 \subset V_i^* \subset (X - E_1)$. Next assume that $E_2 \subset \bar{E}_1$. By the first principle of separation for G_δ sets [11, pp. 350], there is a set B, which is simultaneously a F_σ and a G_δ in X, such that $E_1 \subset B$ and $E_2 \cap B = \phi$. We can assume $B \subset \bar{E}_1$ by replacing B by $B \cap \bar{E}_1$ if necessary. By Lemma 3.2.1, we can find V_i such that $B \cap V_i = \bar{B} \cap V_i \neq \phi$

Since $\overline{B} = \overline{E}_1$, it follows that $\overline{E}_1 \cap V_i \neq \phi$ and so $E_1 \cap V_i \neq \phi$, as V_i is open. Also $E_2 \subset \overline{E}_1 = \overline{B}$, so $E_2 \cap V_i \subset (\overline{B} - B) \cap V_i = \phi$. It now follows that $E_1 \subset V_i^* \subset (X - E_2)$.

To complete the proof, let $\underline{\underline{A}}$ be the σ-field on X generated by the sets V_n^*, $n \geq 1$. Since $\underline{\underline{Q}}$ is a measurable partition, it follows that $\underline{\underline{A}}$ is a countably generated sub σ-field of $\underline{\underline{B}}_X$. Moreover, the assertion made at the beginning of the previous paragraph implies that $\underline{\underline{Q}}$ is just the partition of X induced by $\underline{\underline{A}}$. So by Lemma 0.5, $\underline{\underline{A}} = \underline{\underline{A}}(\underline{\underline{Q}})$ and hence, $\underline{\underline{A}}(\underline{\underline{Q}})$ is countably generated.

Miller [20] has given an alternative proof of this proposition.

It is well known that the σ - field on $\mathbb{R}$ induced by the Vitali's partition is not countably generated. Therefore the above proposition need not be true for measurable (even lower semi-continuous) partitions into F_σ sets.

We now prove the following interesting result.

Theorem 3.2.3 Let $\underline{\underline{Q}}$ be a measurable partition of a Polish space X such that each element of $\underline{\underline{Q}}$ is a G_δ in X. Then there is a Borel cross section for $\underline{\underline{Q}}$.

Proof : Define a multifunction $F : X \to X$ as follows :

$F(x)$ = the element of $\underline{\underline{Q}}$ containing x. Then $F(x)$ is a G_δ in X for each $x \in X$. Since $\underline{\underline{Q}}$ is a measurable partition, it follows that F is $\underline{\underline{A}}(\underline{\underline{Q}})$ - measurable.

By Proposition 3.2.2, $\underline{\underline{A}}(\underline{\underline{Q}})$ is countably generated. Also the partition of X induced by $\underline{\underline{A}}(\underline{\underline{Q}})$ is just $\underline{\underline{Q}}$. So by Lemma 2.2.1, $R(\underline{\underline{Q}}) \in \underline{\underline{A}}(\underline{\underline{Q}}) \times \underline{\underline{A}}(\underline{\underline{Q}})$. But $R(\underline{\underline{Q}}) = Gr(F)$, so $Gr(F) \in \underline{\underline{A}}(\underline{\underline{Q}}) \times \underline{\underline{B}}_X$. We apply Theorem 2.3.1 to the multifunction F. This will yield a $\underline{\underline{A}}(\underline{\underline{Q}})$ - measurable selector f for F. Now let $B = \{x \in X : f(x) = x\}$. Then B is a Borel cross section for $\underline{\underline{Q}}$.

It is pointed out by Kallman and Mauldin [10] (also by Miller [20]) that the above cross section theorem has an interesting application in C^* - algebras.

Miller [21] has also studied partitions of Polish spaces into G_δ sets of determined complexity and has established the following:

<u>Theorem 3.2.4</u> Let $\underline{\underline{Q}}$ be a partition of a Polish space such that each element of $\underline{\underline{Q}}$ is a G_δ in X. If the saturation of each basic open set is of ambiguous class $\alpha > 0$ then there is a Borel cross section of class γ for $\underline{\underline{Q}}$, where $\gamma = \mathrm{Sup}\{\alpha + \beta : \beta < \alpha\}$.

Miller established the above result some months after the author had proved Theorem 3.2.3. It should also be mentioned that Miller's methods are quite different from that of ours.

Corollary 3.2.5 Let T be a Borel subset of a Polish space X. Suppose $\underline{\underline{Q}}$ is a measurable partition of T such that each element of $\underline{\underline{Q}}$ is a G_δ in X. Then there is a Borel cross section for $\underline{\underline{Q}}$.

Proof: Define a partition $\underline{\underline{Q}}'$ of X as follows:

$\underline{\underline{Q}}' = \underline{\underline{Q}} \cup \{\{x\} : x \in X - T\}$. Then, as is easy to check, $\underline{\underline{Q}}'$ is a measurable partition of X. Also each member of $\underline{\underline{Q}}'$ is a G_δ in X. So by Theorem 3.2.3, there is a Borel cross section B for $\underline{\underline{Q}}'$. Plainly $B \cap T$ is a Borel cross section for $\underline{\underline{Q}}$.

We mention here that the condition " each element of $\underline{\underline{Q}}$ is a G_δ in X " cannot be replaced by the condition " each element of $\underline{\underline{Q}}$ is a G_δ in T " from the last corollary. In fact, if we take X to be the space $\Sigma \times \Sigma$, T to be the Borel set H defined in Example 2.1.1 of Kallman and Mauldin and $\underline{\underline{Q}}$ to be the partition $\{T^\sigma : \sigma \in \Sigma\}$ then $\underline{\underline{Q}}$ is lower semi-continuous, each element of $\underline{\underline{Q}}$ is closed in T and $\underline{\underline{Q}}$ does not admit a Borel cross section.

Corollary 3.2.6 Let T be a Borel subset of a Polish space X. Suppose $\underline{\underline{Q}}$ is a measurable partition of T such that each element of $\underline{\underline{Q}}$ is a G_δ in X. Then $(T, \underline{\underline{A}}(\underline{\underline{Q}}))$ is a standard Borel space.

Proof: If $\underline{\underline{Q}}'$ is as in the proof of Corollary 3.2.5, then $\underline{\underline{A}}(\underline{\underline{Q}}) = \underline{\underline{A}}(\underline{\underline{Q}}')|T$. So, by Proposition 3.2.2, $\underline{\underline{A}}(\underline{\underline{Q}}')$ is countably generated and hence so is $\underline{\underline{A}}(\underline{\underline{Q}})$. Now let $f : T \to [0,1]$ be the characteristic function of a sequence of subsets of T which generates $\underline{\underline{A}}(\underline{\underline{Q}})$. By Corollary 3.2.5 there is a Borel cross section B for $\underline{\underline{Q}}$. So f restricted to B is one-one and $f(B) = f(T)$. A well known result of Lusin states that the image of a Borel set under a one-one, Borel measurable function on a Polish space into another Polish space is Borel [11, pp. 489]. Hence $f(B)$ is a Borel subset of $[0,1]$. So $f(T)$ is Borel in $[0,1]$, which completes the proof.

We close this section by proving the following consequence of Theorem 3.2.3.

Corollary 3.2.7 Let X, Y be Polish spaces and $f : X \to Y$ be a function of class 1. If $f(V)$ is relatively Borel in $f(X)$ for every open set V in X then $f(X)$ is a Borel subset of Y. Moreover, there is a Borel measurable function g on $f(X)$ into X such that $f(g(y)) = y$ for every y in $f(X)$.

Proof: Let $\underline{\underline{Q}} = \{f^{-1}(y) : y \in f(X)\}$. Then $\underline{\underline{Q}}$ is a measurable partition of X and each element of $\underline{\underline{Q}}$ is a G_δ in X. So by Theorem 3.2.3, there is a Borel cross section B

$\underset{=}{Q}$. Now f restricted to B is one-one and $f(B) = f(X)$. By the result of Lusin mentioned earlier, $f(X)$ is a Borel subset of Y. Now define g on $f(X)$ into X by $g(y) =$ the unique point of $B \cap f^{-1}(y)$. To see that g is Borel measurable, we observe that for a Borel set E in X, $g^{-1}(E) = f(E \cap B)$, which by Lusin's theorem is a Borel set in Y. This completes the proof.

3. A cross section theorem for partitions into σ - compact sets

Vitali's example discussed in the introduction of this chapter shows that a measurable partition of a Polish space X into σ - compact (in fact, into countable) sets need not admit a Borel cross section. However, we can show the following :

Theorem 3.3.1 Let $\underset{=}{Q}$ be a partition of a Polish space X such that each member of $\underset{=}{Q}$ is σ - compact. If $\underset{=}{A}(\underset{=}{Q})$ is countably generated then $\underset{=}{Q}$ admits a Borel cross section.

Proof : By Lemma 2.2.1, $R(\underset{=}{Q}) \in \underset{=}{A}(\underset{=}{Q}) \times \underset{=}{B}_X$. Since elements of $\underset{=}{Q}$ are σ - compact and $\underset{=}{A}(\underset{=}{Q})$ is countably generated, by Lemma 1.4.3, there exist sets G_n in $\underset{=}{A}(\underset{=}{Q}) \times \underset{=}{B}_X$ such that G_n^x is compact for each $x \in X$ and $n \geq 1$ and $R(\underset{=}{Q}) = \bigcup_{n=1}^{\infty} G_n$.

We define

$$T_n = \prod_X(G_n) \qquad \text{if } n = 1$$

$$= \prod_X(G_n) - \bigcup_{m<n} \prod_X(G_m) \qquad \text{if } n > 1 .$$

By Lemma 0.6, the sets T_n, $n \geq 1$, belong to $\underline{\underline{A}}$. Also $T_n \cap T_m = \phi$ for $n \neq m$ and $X = \bigcup_{n=1}^{\infty} T_n$. Let $G = \bigcup_{n=1}^{\infty} ((T_n \times X) \cap G_n)$. The set G is the graph of a compact valued multifunction $F : X \to X$ defined by $F(x) = G^x$, $x \in X$. Since $G \in \underline{\underline{A}}(\underline{\underline{Q}}) \times \underline{\underline{B}}_X$ and G^x is compact for each $x \in X$, by Lemma 0.6, the multifunction F is $\underline{\underline{A}}(\underline{\underline{Q}})$ - measurable. By the selection theorem of Kuratowski and Ryll-Nardzewski, let $f : X \to X$ be a $\underline{\underline{A}}(\underline{\underline{Q}})$ - measurable selector for F. Then the set $B = \{x \in X : f(x) = x\}$ is a Borel cross section for $\underline{\underline{Q}}$.

Remark 3.3.2 It may be noted that we can prove Theorem 3.3.1 by applying our selection theorem for σ - compact valued multifunctions (Theorem 1.4.4) to the multifunction $H : X \to X$ defined as follows : $H(x) = R(\underline{\underline{Q}})^x$, $x \in X$.

Remark 3.3.3 If X and $\underline{\underline{Q}}$ satisfy the hypotheses of Theorem 3.3.1 then $(X, \underline{\underline{A}}(\underline{\underline{Q}}))$ is a standard Borel space.

Remark 3.3.4 If $\underline{\underline{Q}}$ is a partition of a Polish space satisfying the hypotheses of Theorem 3.3.1 then $\underline{\underline{Q}}$ is measurable.

Remark 3.3.5 Theorem 3.3.1 remains true also when X is an absolute Borel set, that is, when X is a Borel set in its (metric) completion.

CHAPTER 4

REPRESENTATIONS OF G_δ VALUED MULTIFUNCTIONS

1. Introduction In this chapter we study various aspects of the structure of G_δ valued multifunctions satisfying the hypotheses of Theorem 2.3.1.

The chapter is organised as follows : In section 2 we fix some notation which will be followed in the rest of the thesis. In section 3 we prove a representation theorem for such multifunctions. This, in particular, shows that the graph of such multifunctions can be expressed as the union of graphs of measurable selectors. In section 4 we assume that values of the multifunctions are of the same cardinality and examine the problem of decomposing its graph as the disjoint union of graphs of measurable selectors. In section 5 we relate G_δ valued multifunctions to closed valued multifunctions. This enables us to conclude most of the results proved so far in this thesis on G_δ valued multifunctions from the corresponding results on closed valued multifunctions.

2. Notation Now on throughout this thesis our object of study would be a specific class of G_δ valued multifunctions. In this section we specify our set up and fix some notation. This will be followed throughout the rest of the thesis without any specific mention. We use T and X to denote two arbitrary but

fixed Polish spaces. We shall only state extra hypotheses on T or X as and when necessary. By $\underline{\underline{A}}$ we shall mean a countably generated sub σ - field of the Borel σ - field $\underline{\underline{B}}_T$. We fix a countable base $\{V_n : n \geq 1\}$ closed under finite inter-sections and finite unions for X such that $V_1 = \phi$ and $V_2 =$ We give X a complete metric d such that the diameter $\delta(X)$ By $F : T \to X$ we shall mean a multifunction satisfying the following conditions: F is $\underline{\underline{A}}$ - measurable, $Gr(F) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ and $F(t)$ is a G_δ in X for each $t \in T$. We put $G = Gr(F)$ and fix a non-increasing sequence of sets G_n, $n \geq 1$, in $\underline{\underline{A}} \times \underline{\underline{B}}_X$ such that G_n^t is open in X for $t \in T$ and $n \geq 1$ and $G = \bigcap_{n \geq 1} G_n$. The existence of such a sequence is ensured by Lemma 1.4.2.

3. A representation theorem We first state and prove a stand result [11, pp. 440]. The proof is sketched for the sake of completeness.

Lemma 4.3.1 Every non-empty G_δ subset of X is the image of the space of irrationals under a continuous, open map.

Proof : Let A be a non-empty G_δ set in X and let $U_1, U_2, \ldots$ be a non-increasing sequence of open sets in X such that $A = \bigcap_{n \geq 1} U_n$. Let $\{n_s : s \in S\}$ be a system of positive integers such that for $s \in S_k$, $k \geq 0$, the following hold :

(i) $\delta(V_{n_s}) < 2^{-k}$,

(ii) $A \subset V_{n_e}$,

(iii) $\phi \neq A \cap V_{n_s} \subset \bigcup_{m \geq 1} V_{n_{sm}}$,

(iv) $\overline{V}_{n_{sm}} \subset U_{k+1} \cap V_{n_s}$, $m \geq 1$.

That such a system exists is easily seen. Let $\sigma \in \Sigma$. As d is complete, $\bigcap_{k=1}^{\infty} \overline{V}_{n_{\sigma|k}}$ is a singleton. We define $f(\sigma)$ to be the unique point of $\bigcap_{k=1}^{\infty} \overline{V}_{n_{\sigma|k}}$. By conditions (ii), (iii) and (iv), it follows that $f(\Sigma) = A$. Further, $f(\Sigma_s) = A \cap V_{n_s}$, $s \in S$. This implies that f is open. Now we check the continuity of f . We take an arbitrary $\sigma \in \Sigma$ and a positive integer k . By (i), $f(\Sigma_{\sigma|k}) = A \cap V_{n_{\sigma|k}} \subset S_{2^{-k}}(f(\sigma))$. From this we conclude that f is continuous.

We shall now prove the following theorem for F :

<u>Theorem 4.3.2</u> There is a map $f : T \times \Sigma \to X$ such that for each $t \in T$, $f(t,.)$ is a continuous open map from Σ onto $F(t)$ and $f(.,\sigma)$ is $\underline{A}$ - measurable for each $\sigma \in \Sigma$.

This interesting result is established by applying the procedure of Lemma 4.3.1 to each $F(t)$ uniformly. We first prove an auxiliary Lemma.

Lemma 4.3.3 Let X be compact. Then for each $t \in T$, there is a system $\{n_s^t : s \in S\}$ of positive integers such that for $s \in S_k$, $k \geq 0$, and $t \in T$

(i) the map $t' \to n_s^{t'}$, defined on T, is $\underline{A}$ - measurab[le]

(ii) $\delta(V_{n_s^t}) < 2^{-k}$,

(iii) $G^t \subset V_{n_e^t}$,

(iv) $\phi \neq G^t \cap V_{n_s^t} \subset \bigcup_{m \geq 1} V_{n_{sm}^t}$,

(v) $\overline{V}_{n_{sm}^t} \subset V_{n_s^t} \cap G_{k+1}^t$, $m \geq 1$.

Proof : We define maps $t \to n_s^t$, $s \in S$, by induction on $|s|$. We put $n_e^t = 2$ for each $t \in T$. The above conditions are clearly satisfied for $s = e$. Suppose for some $k \geq 0$, maps $t \to n_s^t$ have been defined satisfying the above conditions for all $s \in S_i$, $i \leq k$. Fix a $s \in S_k$. We define $t \to n_{sm}^t$, $m \geq 1$, by induction on m.

We first make a simple observation. Let $W \subset X$ be closed and $t \in T$. Then

$$W \subset G_{k+1}^t \cap V_{n_s^t} \iff (\exists \lambda \geq 1)\ (n_s^t = \lambda \text{ and } W \subset G_{k+1}^t \cap V_\lambda)$$

Hence

$$\{t \in T : W \subset G^t_{k+1} \cap V_{n^t_s}\}$$

$$= \bigcup_{\lambda \geq 1} [\{t \in T : n^t_s = \lambda\} \cap (T - \prod_T((T \times W) \cap (((T \times X) - G_{k+1}) \cup (T \times (X - V_\lambda)))))].$$

By induction hypotheses and by Lemma 0.6, it follows that the set $\{t \in T : W \subset G^t_{k+1} \cap V_{n^t_s}\} \in \underline{\underline{A}}$.

For $m \geq 1$, we define

$T^o_m = \phi$ if $\delta(V_m) \geq 2^{-(k+1)}$.

Otherwise, let

$$T^o_m = \{t \in T : \quad G^t \cap V_m \neq \phi \ , \ \bar{V}_m \subset G^t_{k+1} \cap V_{n^t_s}$$

and

$$(\forall \lambda < m) \ (\delta(V_\lambda) < 2^{-(k+1)} \Rightarrow (G^t \cap V_\lambda = \phi$$

$$\text{or } \bar{V}_\lambda \not\subset G^t_{k+1} \cap V_{n^t_s}))\}.$$

As F is $\underline{\underline{A}}$ - measurable, by the above observation, $T^o_m \in \underline{\underline{A}}$ for each $m \geq 1$. Also, the sets T^o_m, $m \geq 1$, are pairwise disjoint and $T = \bigcup_{m \geq 1} T^o_m$. We define $n^t_{s1} = m$ if $t \in T^o_m$. Clearly the map $t \rightarrow n^t_{s1}$ is $\underline{\underline{A}}$ - measurable.

Now suppose for some $p \geq 1$, maps $t \to n^t_{si}$, $i \leq p$, have been defined to be $\underline{\underline{A}}$ - measurable. For $m \geq 1$, let

$T^p_m = \phi$ if $\delta(V_m) \geq 2^{-(k+1)}$.

Otherwise, let

$$T^p_m = \{t \in T : n^t_{sp} < m, G^t \cap V_m \neq \phi, \bar{V}_m \subset G^t_{k+1} \cap V_{n^t_s}$$

and

$$(\forall \lambda < m)\ (\delta(V_\lambda) < 2^{-(k+1)} \Rightarrow (n^t_{sp} \geq \lambda \text{ or } G^t \cap V_\lambda =$$

$$\text{or } \bar{V}_\lambda \not\subset G^t_{k+1} \cap V_{n^t_s}))\}$$

It is easily checked that $T^p_m \in \underline{\underline{A}}$ for each $m \geq 1$ and $T^p_m \cap T^p_n = \phi$ for $m \neq n$. We define

$$n^t_{s,p+1} = m \quad \text{if } t \in T^p_m$$

$$= n^t_{sp} \quad \text{if } t \in T - \bigcup_{m=1}^{\infty} T^p_m .$$

The definition of n^t_s, $s \in S$, is now complete. Conditions (i) - (v) are easily verified.

<u>Proof of Theorem 4.3.2</u> Let Y be a metric compactification of X. By a well known result X is a G_δ in Y. Hence, Lemma 4.3.2 is applicable to T, Y and F (considered as a multifunction into Y). So, for each $t \in T$, we get a system $\{n^t_s : s \in S\}$ of positive integers satisfying conditions (i) - (v) of Lemma 4.3.2. Let $t \in T$ and $\sigma \in \Sigma$. Then

$\bigcap_{k=1}^{\infty} \overline{V}_{n^t_{\sigma|k}}$ is a singleton. We define $f(t,\sigma)$ to be the unique point belonging to this intersection. By the arguments made in the proof of Lemma 4.3.1, we check that $f(t,.)$ is a continuous, open map from Σ onto $F(t)$ for each $t \in T$. In particular, f is a map from $T \times \Sigma$ into X. Finally, let $U \subset Y$ be open, $t \in T$ and $\sigma \in \Sigma$. Then

$$f(t,\sigma) \in U \iff \bigcap_k \overline{V}_{n^t_{\sigma|k}} \subset U$$

$$\iff (\exists k \geq 1)\,(\exists \lambda \geq 1)\,(\overline{V}_\lambda \subset U \text{ and } n^t_{\sigma|k} = \lambda)$$

Therefore

$$f(.,\sigma)^{-1}(U) = \bigcup \bigcup \{t \in T : n^t_{\sigma|k} = \lambda\},$$

where the inner union is taken over all $\lambda \in N$ such that $\overline{V}_\lambda \subset U$ and the outer union is over all $k \in N$. By condition (i) of Lemma 4.3.1, it follows that $f(.,\sigma)$ is $\underline{\underline{A}}$ - measurable for each $\sigma \in \Sigma$.

<u>An open problem</u> Let Y be a Polish space and let $f : T \times Y \to X$ be a Carathéodory map such that for each $t \in T$, $f(t,.)$ is continuous, open and for each $y \in Y$, $f(.,y)$ is $\underline{\underline{A}}$ - measurable. Let $H : T \to X$ be the multifunction induced by f. So that, $H(t) = f(t,Y)$, $t \in T$. By a well known result of Hausdorff [7], $H(t)$ is completely metrizable and hence a G_δ in X for each

$t \in T$. Let $\{y_n\}_{n\geq 1}$ be a dense sequence in Y. Then for each $t \in T$, $\{f(t, y_n)\}_{n\geq 1}$ is dense in $H(t)$. Thus, if U is an open set in X then

$$\{t \in T : H(t) \cap U \neq \phi\} = \bigcup_{n\geq 1} \{t \in T : f(t, y_n) \in U\}.$$

It follows that H is $\underline{\underline{A}}$ - measurable. Question arises : Does $Gr(H) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$? (P 2). We do not know the answer. An affirmative answer to this question will show that the class of G_δ valued multifunctions under consideration coincides with the class of multifunctions induced by the special Carathéodory maps of the above kind.

4. Decomposition of Gr(F) into graphs of measurable selectors

In this section we assume that X is uncountable and $F(t)$, $t \in T$, are all of the same cardinality. We ask ourselves : Can we express $Gr(F)$ as the disjoint union of the graphs of measurable selectors for F ? Luzin [16] has proved that an analytic set in the product of two Polish spaces having countable sections can be covered by countably many Borel graphs. A new proof of this result is recently given by Maitra [17]. A simple application of this result shows that the answer to the above question is in the affirmative if we moreover assume that $F(t)$, $t \in T$, are all countable (finite or infinite). The details are omitted.

We do not know the answer to this question when $F(t)$, $t \in T$, are all uncountable. (P 3). We also do not know whether $Gr(F)$ contains uncountably many pairwise disjoint graphs of $\underline{\underline{A}}$ - measurable selectors for F. (P 4). However, we can prove the following :

Theorem 4.4.1 If $F(t)$ is dense-in-itself for each $t \in T$ then there is a $\underline{\underline{A}} \times \underline{\underline{B}}_X$ - measurable map $f : T \times X \to X$ such that for each $t \in T$, $f(t,.)$ is a Borel isomorphism of X onto $F(t)$.

This result is analogous to a result of Mauldin [19] and we follow some of his ideas.

Lemma 4.4.2 Let X be compact and $F(t)$, $t \in T$, be dense-in -itself. Then for each $t \in T$ and $d \in D$, the set of all finite sequences of 0's and 1's, there is a positive integer n_d^t such that for $t \in T$ and $d \in D_k$, $k \geq 0$, the following hold :

(i) the map $t' \to n_d^{t'}$, defined on T, is $\underline{\underline{A}}$ - measurable,

(ii) $\delta(V_{n_d^t}) < 2^{-k}$,

(iii) $d' \in D_k$, $d \neq d' \Rightarrow \overline{V}_{n_d^t} \cap \overline{V}_{n_{d'}^t} = \phi$,

(iv) $F(t) \cap V_{n_d^t} \neq \phi$,

(v) $\overline{V}_{n_{di}^t} \subset G_{k+1}^t \cap V_{n_d^t}$, $i = 0$ or $i = 1$.

Proof : We define maps $t \to n_d^t$, $d \in D$, by induction on $|d|$. We put $n_e^t = 2$ for all $t \in T$. Suppose for some $k \geq 0$, maps $t \to n_d^t$ have been defined for all $d \in D_\lambda$, $\lambda \leq k$, satisfying the above conditions. Fix a $d \in D_k$ and let $T^m = \{ t \in T : n_d^t = m \}$, $m \geq 1$. By induction hypotheses $T^m \in \underline{\underline{A}}$, $m \neq n \Rightarrow T^m \cap T^n = \phi$ and $T = \bigcup_{m \geq 1} T^m$.

Now, for any pair (u,v) of positive integers we define T^m_{uv}, $m \geq 1$, as follows : If $\delta(V_u) < 2^{-(k+1)}$, $\delta(V_v) < 2^{-(k+1)}$, $\overline{V}_u \subset V_m$, $\overline{V}_v \subset V_m$ and $\overline{V}_u \cap \overline{V}_v = \phi$ then we put

$$T^m_{uv} = \{ t \in T : \overline{V}_u \subset G^t_{k+1},\ \overline{V}_v \subset G^t_{k+1},$$

$$V_u \cap F(t) \neq \phi \text{ and } V_v \cap F(t) \neq \phi \},$$

otherwise, we put $T^m_{uv} = \phi$.

By Lemma 0.6 and $\underline{\underline{A}}$ - measurability of F, $T^m_{uv} \in \underline{\underline{A}}$. Also as is easy to check, $T^m = \bigcup_{(u,v)} T^m_{uv}$.

Let $\alpha : N \to N \times N$ be a one-one, onto function. Put

$$S^m_i = T^m_{\alpha(i)} \qquad \text{if } i = 1$$

$$= T^m_{\alpha(i)} - \bigcup_{j<i} T^m_{\alpha(j)} \qquad \text{if } i > 1$$

We get $S_i^m \in \underline{\underline{A}}$, $i \neq j \Rightarrow S_i^m \cap S_j^m = \phi$ and $T^m = \bigcup_{i \geq 1} S_i^m$.

We define

$$\left.\begin{aligned} n_{d0}^t &= (\alpha(j))_1 \\ n_{d1}^t &= (\alpha(j))_2 \end{aligned}\right\} \text{ if } t \in S_j^m \text{ for any } j \text{ and } m .$$

This completes the definition of n_d^t, $d \in D_{k+1}$, $t \in T$. It is easily checked that (i) - (v) are satisfied.

<u>Lemma 4.4.3</u> If $F(t)$ is dense-in-itself for each $t \in T$, then there is a map $g : T \times C \to X$ such that for each $t \in T$, $g(t,.)$ is a homeomorphism of C into $F(t)$ and for each $\delta \in C$, $g(.,\delta)$ is $\underline{\underline{A}}$ - measurable, where C denotes the Cantor set.

<u>Proof</u> : Let Y be a metric compactification of X. Lemma 4.4.2 is applicable to T, Y and F (considered as a multifunction into Y). We get positive integers n_d^t, $d \in D$, $t \in T$, satisfying conditions (i) - (v) of Lemma 4.4.2. Let $t \in T$ and $\delta \in C$. Then $\bigcap_{k \geq 1} \overline{V}_{n^t_{\delta|k}}$ is a singleton. We define $g(t,\delta)$ to be the unique point in $\bigcap_{k \geq 1} \overline{V}_{n^t_{\delta|k}}$.

It is clear that for each $t \in T$, $g(t,.)$ is a one-one continuous map and hence a homeomorphism from C into Y. By (v), $g(t,.)$ is into $F(t)$. In particular, g is a map into X. Let $\delta \in C$. To check that $g(.,\delta)$ is $\underline{\underline{A}}$ - measurable, we first notice that if $t \in T$ and U is an open set in X then

$$g(t,\delta) \in U \iff \bigcap_{k\geq 1} \overline{V}_{n^t_{\delta|k}} \subset U$$

$$\iff (\exists k \geq 1)(\exists \lambda \geq 1)(\overline{V}_\lambda \subset U)(n^t_{\delta|k} = \lambda).$$

Therefore

$$g(\cdot,\delta)^{-1}(U) = \cup \cup (\{t \in T : n^t_{\delta|k} = \lambda\}),$$ where

the inner union is taken over all $\lambda \in N$ such that $\overline{V}_\lambda \subset U$ and the outer union is over all k. It follows that $g(\cdot,\delta)$ is $\underline{\underline{A}}$ - measurable.

<u>Proof of Theorem 4.4.1</u> By Lemma 4.4.3, we get a map $g : T \times C \to X$ such that for each $t \in T$, $g(t, \cdot)$ is a homeomorphism of C into $F(t)$ and for all $\delta \in C$, $g(\cdot,\delta)$ is $\underline{\underline{A}}$ - measurable. Hence g is $\underline{\underline{A}} \times \underline{\underline{B}}_C$ - measurable [11, pp.378]. Now we use the fact that there is a Borel isomorphism from X onto C [11, pp.450] to get a $\underline{\underline{A}} \times \underline{\underline{B}}_X$ - measurable map $h : T \times X \to X$ such that for each $t \in T$, $h(t,\cdot)$ is a Borel isomorphism from X into $F(t)$.

Let $k : T \times X \to T \times X$ be defined by

$$k(t,x) = (t, h(t,x)), \quad t \in T, x \in X,$$

and let

$$B = \{(t,x) \in T \times X : x \in h(t,X)\}.$$

Then B is a subset of G (the graph of F) and, as k is one-one and Borel measurable, B is Borel in $T \times X$ [11, pp. 489]. Further, B is a union of $\underline{\underline{A}} \times \underline{\underline{B}}_X$ - atoms. So, by Lemma 0.5, $B \in \underline{\underline{A}} \times \underline{\underline{B}}_X$. Also, $k : T \times X \to T \times X$ is measurable when both its range and domain spaces are equipped with the σ - field $\underline{\underline{A}} \times \underline{\underline{B}}_X$.

Now, we sketch a Cantor-Bernstein type argument to define a map $\alpha : T \times X \to T \times X$ such that α is measurable when both its range and domain spaces are equipped with $\underline{\underline{A}} \times \underline{\underline{B}}_X$ and such that for each $t \in T$, $\alpha(t, \cdot)$ is a Borel isomorphism from X onto $\{t\} \times F(t)$. First, for $A \subseteq T \times X$, let

$$\beta(A) = T \times X - (G - k(A)).$$

Note that if $A \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ then so does $\beta(A)$. Now, we choose a set $A_o \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ such that $\beta(A_o) = A_o$, for instance, A_o could be the set $\bigcup_{n \geq o} \beta^n(\phi)$. We define

$$\alpha(t,x) = k(t,x) \quad \text{if } (t,x) \in A_o$$

$$= (t,x) \quad \text{if } (t,x) \notin A_o .$$

We put $f = \Pi_X \circ \alpha$. It is easy to verify that f has the desired properties.

When $\underline{\underline{A}} = \underline{\underline{B}}_T$, the function f whose existence is asserted in the above theorem is called a Borel Parametrization of the multifunction F. The notion of Borel parametrization was introduced by Mauldin [19], who found necessary and sufficient conditions for a multifunction to admit Borel parametrizations.

The following example shows that the condition ' F(t) is dense-in-itself ' cannot be replaced by the condition ' F(t) is uncountable ' in Theorem 4.4.1.

<u>Example 4.4.4</u> Let X be an uncountable Polish space containing a countable, dense, open set W. (The union of the Cantor set and the mid-points of the ternary intervals is such a Polish space. Let $T = \Sigma$, $\underline{\underline{A}} = \underline{\underline{B}}_T$ and let $Y = X - W$. Let B be a G_δ set in $\Sigma \times Y$ such that every section of B is uncountable and B does not admit a Borel uniformization. Let $F : T \to X$ be a multifunction defined by $F(t) = B^t \cup W$, $t \in T$. It is clear that F(t) is a G_δ in X for each $t \in T$ and $Gr(F) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$. As W is dense in X and as $W \subseteq F(t)$ for all t, F is $\underline{\underline{A}}$ - measurable. If F satisfies the conclusions of Theorem 4.4.1 then by the Borel Parametrization Theorem [19, Theorem A] of Mauldin, there exists a Borel set $M \subseteq Gr(F)$ such that M^t is non-empty, compact and dense-in-itself for each $t \in T$. It follows that $M \subseteq B$ and hence B admits a Borel uniformization. This is impossible.

We point out here that there is a closed set M in $\Sigma \times$ such that for each $\sigma \in \Sigma$, M^σ is uncountable and M is not

Borel uniformizable. Let F be the multifunction defined as in Example 2.1.1 with M having the above properties. It has been already concluded that F does not admit a measurable selector. In particular, we conclude that the condition 'F(t) is a G_δ in X' cannot be replaced by 'F(t) is uncountable and an F_σ in X' in the last theorem.

It is also worth noting that Mauldin has an example of a $\underline{\underline{A}}$ - measurable, compact valued multifunction F such that $F(t)$ is uncountable for each $t \in T$, but F does not admit a Borel parametrization.

We close this section by generalizing a result of Larman [14].

Corollary 4.4.5 Let $M \subset T \times X$ be a Borel set such that M^t is dense-in-itself and both a K_σ and a G_δ set in X for each $t \in \prod_T(M)$. Then M is a union of $2^{\aleph_0}$ disjoint, Borel uniformizations.

Proof : Without loss of generality we assume that X is-dense-in-itself. We take $\underline{\underline{A}} = \underline{\underline{B}}_T$ and define $F : T \to X$ by

$$F(t) = M^t \quad \text{if} \quad t \in \prod_T(M)$$

$$= X \quad \text{if} \quad t \in T - \prod_T(M) .$$

In view of Lemma 0.6, T, $\underline{A}$, X and F satisfy the hypotheses of Theorem 4.4.1. Hence, $Gr(F^*)$ is a union of $2^{\aleph_0}$ many disjoint Borel uniformizations. From this the result follows.

Under the hypotheses of the above corollary, Larman had proved that M contains $2^{\aleph_0}$ many disjoint Borel uniformizations. We do not know whether the above corollary remains true if the condition ' M^t is dense-in-itself ' is replaced by the condition ' M^t is uncountable '. (P.5)

5. Relationship to closed valued multifunctions Now we give a simple relationship between G_δ valued multifunctions and closed valued multifunctions. We shall follow some of the ideas of Leese [15].

Definition 4.5.1 Let $(L, \underline{L})$ be a measurable space and Z a metric space. A multifunction $H : L \to Z$ is said to be of Souslin Type if there is a Polish space P, a continuous map $\beta : P \to Z$ and a $\underline{L}$ - measurable, closed valued multifunction $W : L \to P$ such that $H(t) = \beta(W(t))$ for each $t \in L$.

The following facts regarding a multifunction $H : L \to Z$ of Souslin Type defined above are easy to deduce from the corresponding properties of the closed valued multifunction $W : L \to P$.

Fact 1 H is $\underline{\underline{L}}$ - measurable.

Fact 2 H admits a $\underline{\underline{L}}$ - measurable selector.

Fact 3 There exist $\underline{\underline{L}}$ - measurable selectors h_n, $n \geq 1$, for H such that $\{h_n(t) : n \geq 1\}$ is dense in $H(t)$ for each $t \in L$.

Fact 4 There is a Carathéodory map $h : L \times \Sigma \to Z$ which induces H.

We omit the details. The following result is proved jointly with H. Sarbadhikari.

Theorem 4.5.2 The multifunction $F : T \to X$ is of Souslin type.

Proof : We first assume that X is also compact and zero - dimensional and that the basic open sets $V_1, V_2, \ldots$ are clopen. Let

$$T_{nm} = \{t \in T : V_m \subset G_n^t\}, \quad m \geq 1, \quad n \geq 1.$$

By Lemma 0.6, $T_{nm} \in \underline{\underline{A}}$ for each m and n. Further, as sections of G_n are open

$$G_n = \bigcup_m (T_{nm} \times V_m), \quad n \geq 1.$$

We put $P = X \times \Sigma$ and $\beta = \prod_X$. Let

$$B = \bigcap_n \bigcup_m (T_{nm} \times V_m \times \Sigma_m^n),$$

where

$$\Sigma_m^n = \{\sigma \in \Sigma : \sigma_n = m\}, \quad n \geq 1, \quad m \geq 1 .$$

We define $W : T \to P$ by

$$W(t) = B^t , \quad t \in T .$$

Then, for $t \in T$

$$W(t) = \cap \cup (V_m \times \Sigma_m^n) ,$$

where the inner union is taken over all $m \in N$ such that $t \in T_{nm}$ and the outer intersection is over all n.

For each $n \geq 1$, the family $\{V_m \times \Sigma_m^n : m \geq 1\}$ is a discrete family of closed sets in P. Therefore, $W(t)$ is closed in P. Also, for any $t \in T$,

$\beta(W(t)) = \cap \cup V_m$, where the inner union is taken over all m such that $t \in T_{nm}$ and the outer intersection is over all n,

$= F(t)$.

It remains to check that W is $\underline{\underline{A}}$ - measurable. Let $i \in N$ and $s \in S_k$, $k \geq 0$. It is enough to show that

$$\Pi_T ((T \times V_i \times \Sigma_s) \cap B) \in \underline{\underline{A}} .$$

Notice that

$$(T \times V_i \times \Sigma_s) \cap B$$

$$= \bigcap_n \bigcup_m (T_{nm} \times (V_m \cap V_i) \times (\Sigma_m^n \cap \Sigma_s))$$

$$= \bigcap_{j=1}^{k} (T_{js_j} \times (V_{s_j} \cap V_i) \times \Sigma_s) \cap (\bigcap_{n>k} \bigcup_m (T_{nm} \times (V_m \cap V_i) \times (\Sigma_m^n \cap \Sigma_s)))$$

$$= \bigcap_{j=1}^{k} (T_{js_j} \times P) \cap (\bigcap_{n>k} \bigcup_{m\geq 1} (T_{nm} \times (V_m \cap V_i \cap \bigcap_{j=1}^{k} V_{s_j}) \times (\Sigma_m^n \cap \Sigma_s)))$$

Hence,

$$\prod_T ((T \times V_i \times \Sigma_s) \cap B)$$

$$= \bigcap_{j=1}^{k} T_{js_j} \cap \prod_T (\bigcap_{n>k} \bigcup_{m\geq 1} (T_{nm} \times (V_m \cap V_i \cap \bigcap_{j=1}^{k} V_{s_j}) \times (\Sigma_m^n \cap \Sigma_s)))$$

$$= \bigcap_{j=1}^{k} T_{js_j} \cap \prod_T (\bigcap_{n>k} \bigcup_{m\geq 1} (T_{nm} \times (V_m \cap V_i \cap \bigcap_{j=1}^{k} V_{s_j})))$$

$$= \bigcap_{j=1}^{k} T_{js_j} \cap \prod_T ((T \times (V_i \cap \bigcap_{j=1}^{k} V_{s_j})) \cap (\bigcap_{n>k} \bigcup_{m\geq 1} (T_{nm} \times V_m)))$$

$$= \bigcap_{j=1}^{k} T_{js_j} \cap \prod_T ((T \times (V_i \cap \bigcap_{j=1}^{k} V_{s_j})) \cap G) .$$

To justify the last equality we use the fact that the sequence $\{G_n\}$ is non-increasing. Since $T_{nm} \in \underline{A}$ for each n and m and F is $\underline{A}$ - measurable, it follows that

$$\prod_T ((T \times V_i \times \Sigma_s) \cap B) \in \underline{A} .$$

Now, let X be a zero-dimensional Polish space. Let Y be a zero-dimensional, compact metric space containing (a homeomorph of) X. Then X is a G_δ in Y. We consider F as a multifunction into Y. By the previous case, we get a Polish space Q, a continuous map $g : Q \to Y$ and a $\underline{A}$ - measurable, closed valued multifunction $H : T \to Q$ such that $F(t) = g(H(t))$ for each $t \in T$. We take $P = g^{-1}(X)$ and β the restriction of g to P. As X is a G_δ in Y, by a well known result of Alexandrov [11, pp. 408], P is a Polish space. Note that $H(t) \subset P$, $t \in T$. We put $W = H$.

Finally, let X be an arbitrary Polish space. Let $g : \Sigma \to X$ be a continuous, open and onto map. We define a multifunction $H : T \to \Sigma$ by $H(t) = g^{-1}(F(t))$, $t \in T$. Let U be an open set in Σ. Then

$$\{t \in T : H(t) \cap U \neq \phi\} = \{t \in T : F(t) \cap g(U) \neq \phi\} .$$

As F is $\underline{A}$ - measurable and g open, H is $\underline{A}$ - measurable.

We now show that $Gr(H) \in \underline{\underline{A}} \times \underline{\underline{B}}_\Sigma$. Let $h : T \times \Sigma \to T \times X$ be defined as follows : $h(t,\sigma) = (t, g(\sigma))$. Then h is measurable when its domain and range spaces $T \times \Sigma$ and $T \times X$ are equipped with the σ - fields $\underline{\underline{A}} \times \underline{\underline{B}}_\Sigma$ and $\underline{\underline{A}} \times \underline{\underline{B}}_X$ respectively. Therefore, since $Gr(H) = h^{-1}(G)$ and $G \in \underline{\underline{A}} \times \underline{\underline{B}}_X$, it follows that $Gr(H) \in \underline{\underline{A}} \times \underline{\underline{B}}_\Sigma$. By the previous case, we get a Polish space P, a closed valued, $\underline{\underline{A}}$ - measurable multifunction $W : T \to P$ and a continuous map $f : P \to \Sigma$ such that $H(t) = f(W(t))$, $t \in T$. We take $\beta = g \circ f$. The desired properties are easy to verify.

A close examination of the various cases in the proof given above reveals that the map $\beta : P \to X$ is obtained to be continuous, open and onto. Since every Polish space is the image of Σ under a continuous open map, we now get the following :

<u>Theorem 4.5.3</u> There is a $\underline{\underline{A}}$ - measurable, closed valued multifunction $W : T \to \Sigma$ and a continuous, open and onto map $\beta : \Sigma \to X$ such that $F(t) = \beta(W(t))$ for each $t \in T$.

It should be noted here that the selection theorems, the existence of a dense sequence of measurable selectors (Corollary 2.3.4) etc. that were proved in Chapter 2 follow from Theorem 4.5.3. We are thus able to use Theorem 4.5.3 to reduce these problems for G_δ valued multifunctions to corresponding problems for closed valued multifunctions.

CHAPTER 5

A CHARACTERIZATION OF G_δ VALUED MULTIFUNCTIONS

1 Introduction In this chapter we give a complete characterization of multifunctions of the type considered in the previous chapter. Recall in the previous chapter we considered multifunctions $F : T \to X$ where T and X are Polish spaces, $F(t)$ is a G_δ in X for each $t \in T$, F is $\underline{\underline{A}}$ - measurable and $Gr(F) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$, where $\underline{\underline{A}}$ is a countably generated sub σ-field of the Borel σ - field $\underline{\underline{B}}_T$. Such multifunctions will now be characterized.

Let us now recall the following characterization of Polish spaces : A metrizable space is Polish if and only if it is the image of the space of irrationals under a closed continuous map. The 'if' part of this result was proved by Vaĭnšteĭn [30] ; Engelking [5] established the 'only if' part. The above characterization of Polish spaces yields a clue to a way that G_δ valued multifunctions of the type under consideration can be characterized. In other words, we prove the following :

Theorem 5.1.1 Let T, X be Polish spaces, $\underline{\underline{A}}$ a countably generated sub σ - field of the Borel σ - field $\underline{\underline{B}}_T$ and $F : T \to X$ a multifunction. Then the following are equivalent :

(A) F is $\underline{\underline{A}}$ - measurable, $Gr(F) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$ and $F(t)$ is a G_δ in X for each $t \in T$.

(B) There is a function $f : T \times \Sigma \to X$ such that for $t \in T$, $f(t,.)$ is a closed continuous map from Σ onto $F(t)$ and for $\sigma \in \Sigma$, $f(.,\sigma)$ is $\underline{\underline{A}}$ - measurable.

This result can also be viewed as a sectionwise version of the above characterization of Polish spaces. The proof of the proposed result is divided into three parts. In section 2 we prove (A) $\Rightarrow$ (B) when X is a zero-dimensional Polish space It is well known that every zero-dimensional Polish space is homeomorphic to a closed subset of irrationals. A proof of this is given in [11, pp. 441]. We carry over this proof sectionwise and relate F suitably to a measurable multifunction whose values are closed subsets of irrationals. We then invoke Proposition 1.5.2. We prove (A) $\Rightarrow$ (B) for an arbitrary Polish space X by reducing the problem to the zero-dimensional case using an idea of Ponomarev [23]. This is done in section 3. In section 4 we prove (B) $\Rightarrow$ (A).

2. The Zero-dimensional case In addition to assumptions and notation fixed in section 2 of the last chapter, in this section we assume that X is also zero-dimensional and that basic open sets $V_1, V_2, \ldots$ are clopen as well. (Recall $V_1 = \phi$ and $V_2 = X$).

<u>Lemma 5.2.1</u> Let X be compact. Then there is a system of $\underline{\underline{A}}$ - measurable functions $t \to n_s^t$, $s \in S$, from T into the set of all positive integers such that if for $s \in S_k$, $k \geq 0$, and $t \in T$ we define a clopen set $F_s^{(t)}$ in X as follows :

$$(*) \quad F_s^{(t)} = V_{n_s^t} \qquad \text{if } k = 0 \text{ or } k \geq 1 \text{ and } s_k = 1$$

$$= V_{n_s^t} - \bigcup_{i<s_k} V_{n^t_{s|k-1,i}} \qquad \text{if } k \geq 1 \text{ and } s_k > 1 ,$$

then the following hold :

(i) $\delta(F_s^{(t)}) < 2^{-k}$,

(ii) $G^t \subseteq F_e^{(t)}$ and $G^t \cap F_s^{(t)} \subseteq \bigcup_{\ell=1}^{\infty} F_{s\ell}^{(t)}$,

(iii) $F_{sm}^{(t)} \subseteq G_{k+1}^t \cap F_s^{(t)}$, $m \geq 1$.

In particular, $s' \in S_k$, $s \neq s' \Rightarrow F_s^{(t)} \cap F_{s'}^{(t)} = \phi$.

<u>Proof</u> : We define this system of functions by induction on $|s|$. Let $n_e^t = 2$ and $F_e^{(t)} = V_{n_e^t}$ for each $t \in T$. Suppose n_s^t and $F_s^{(t)}$ are defined for all $t \in T$ and $s \in S$ of length $\leq k$ satisfying the above conditions. Fix a $s \in S_k$.

We first observe that if U is an open set in X then $\{t \in T : \bar{U} \subseteq G_{k+1}^t \cap F_s^{(t)}\} \in \underline{\underline{A}}$. To see this note that for each $t \in T$ we have the following :

If $k = 0$ or $k \geq 1$ and $s_k = 1$ then

$$U \subset G^t_{k+1} \cap F^{(t)}_s$$

$$\Longleftrightarrow (\exists \lambda \in N)\ (n^t_s = \lambda \text{ and } U \subset G^t_{k+1} \cap V_\lambda),$$

whereas if $k \geq 1$ and $s_k > 1$ then

$$U \subset G^t_{k+1} \cap F^{(t)}_s$$

$$\Longleftrightarrow (\exists (\lambda_1, \ldots, \lambda_{s_k}) \in N^{s_k})((\forall i \leq s_k)\ (n^t_{s|k-1,i} = \lambda_i)$$

$$\text{and}$$

$$U \subset G^t_{k+1} \cap (V_{\lambda_{s_k}} - \bigcup_{i<s_k} V_{\lambda_i})).$$

By induction hypotheses and Lemma 0.6, the assertion now follows.

We shall now define the functions $t \to n^t_{sp}$, $p \in N$, by induction on p. We define

$$T^o_m = \phi \quad \text{if } m = 1 \text{ or } \delta(V_m) \geq 2^{-(k+1)}.$$

Otherwise, let

$$T^o_m = \{ t \in T : \quad V_m \subset G^t_{k+1} \cap F^{(t)}_s$$

$$\text{and}$$

$$(\forall \lambda)\ (1 < \lambda < m \text{ and } \delta(V_\lambda) < 2^{-(k+1)} \Rightarrow V_\lambda \not\subset G^t_{k+1} \cap F^{(t)}_s$$

By the above observation, the sets T_m^o, $m \geq 1$, belong to $\underline{\underline{A}}$ and are pairwise disjoint. We define

$$n_{s1}^t = m \quad \text{if} \quad t \in T_m^o$$

$$= 1 \quad \text{if} \quad t \in T - \bigcup_{m=1}^{\infty} T_m^o \; .$$

Clearly, the map $t \to n_{s1}^t$, thus defined, is $\underline{\underline{A}}$ - measurable. Suppose for some $p \in N$, the maps $t \to n_{si}^t$ are defined to be $\underline{\underline{A}}$ - measurable for all $i \leq p$. For $m \in N$, we now define

$T_m^p = \phi$ if $\delta(V_m) \geq 2^{-(k+1)}$.

Otherwise, we put

$$T_m^p = \{t \in T : \quad n_{sp}^t < m , \; V_m \subset G_{k+1}^t \cap F_s^{(t)}$$

and

$$(\forall \lambda < m)(\delta(V_\lambda) < 2^{-(k+1)} \Rightarrow (n_{sp}^t \geq \lambda \text{ or } V_\lambda \not\subset G_{k+1}^t \cap F_s^{(t)}))\} \; .$$

It is easily checked that the sets T_m^p, $m \geq 1$, belong to $\underline{\underline{A}}$ and are pairwise disjoint. We define

$$n_{s,p+1}^t = m \quad \text{if} \quad t \in T_m^p$$

$$= 1 \quad \text{if} \quad t \in T - \bigcup_{m=1}^{\infty} T_m^p \; .$$

The definition of the maps $t \to n_s^t$, $s \in S_{k+1}$, is complete. We define $F_s^{(t)}$, $t \in T$, $s \in S_{k+1}$, by the formula (*). It is routine to check that conditions (i) - (iii) are satisfied.

<u>Proof of (A) $\Rightarrow$ (B) when X is a zero-dimensional Polish space</u> : Without loss of generality we assume that X is also compact. So, now X is a compact, zero-dimensional metric space. We get a system of $\underline{\underline{A}}$ - measurable functions $t \to n_s^t$, $s \in S$ and a system of clopen sets $F_s^{(t)}$, $s \in S$, in X satisfying the conclusions of Lemma 5.2.1. Define a multifunction $H : T \to \Sigma$ by

$$H(t) = \{\sigma \in \Sigma : F_{\sigma|k}^{(t)} \neq \phi \text{ for all } k \in N\}, \quad t \in T.$$

Let $t \in T$ and $s \in S_k$, $k \geq 0$. We note that $\sigma \notin H(t)$ if and only if $\Sigma_{\sigma|\lambda} \cap H(t) = \phi$ for some $\lambda \in N$. It follows that $H(t)$ is closed in Σ. We now observe the following equivalences :

(a) $\quad H(t) \cap \Sigma_s \neq \phi \iff G^t \cap F_s^{(t)} \neq \phi$,

(b) $\quad$ If $k = 0$ or $k \geq 1$ and $s_k = 1$ then

$$G^t \cap F_s^{(t)} \neq \phi$$

$$\iff F(t) \cap V_{n_s^t} \neq \phi$$

$$\iff (\exists \lambda \in N)\ (n_s^t = \lambda \text{ and } F(t) \cap V_\lambda \neq \phi)$$

(c) If $k \geq 1$ and $s_k > 1$ then

$$G^t \cap F_s^{(t)} \neq \phi$$

$$\Longleftrightarrow \quad F(t) \cap (V_{n_s^t} - \bigcup_{i<s_k} V_{n^t_{s|k-1,i}}) \neq \phi$$

$$\Longleftrightarrow \quad (\exists(\lambda_1,\ldots,\lambda_{s_k}) \in N^{s_k})\ ((\forall\, i \leq s_k)(n^t_{s|k-1,i} = \lambda_i)$$

and

$$F(t) \cap (V_{\lambda_{s_k}} - \bigcup_{i<s_k} V_{\lambda_i}) \neq \phi).$$

By $\underline{\underline{A}}$ - measurability of F and of functions $t \to n_s^t$, $s \in S$, and by the above equivalences it follows that $\{t \in T : H(t) \cap \Sigma_s \neq \phi\} \in \underline{\underline{A}}$. Since $s \in S$ is arbitrary, this proves that F is $\underline{\underline{A}}$ - measurable.

By Proposition 1.5.2, we get a function $h : T \times \Sigma \to \Sigma$ such that for each $t \in T$, $h(t,.)$ is a closed retraction of Σ onto $H(t)$ and for each $\sigma \in \Sigma$, $h(.,\sigma)$ is $\underline{\underline{A}}$ - measurable.

Let $(t,\sigma) \in Gr(H)$. Then $\bigcap_{k=1}^{\infty} F^{(t)}_{\sigma|k}$ is a singleton. We put $g(t,\sigma)$ to be the unique point belonging to this intersection. We now show that the map $g : Gr(H) \to X$ thus defined is $\underline{\underline{A}} \times \underline{\underline{B}}_\Sigma | Gr(H)$ - measurable. If U is an open set in X and $(t,\sigma) \in Gr(H)$ then

$$g(t,\sigma) \in U$$

$$\iff \bigcap_k F^{(t)}_{\sigma|k} \subset U$$

$$\iff (\exists k)\ (F^{(t)}_{\sigma|k} \subset U)$$

$$\iff (\exists s \in S)\ (\sigma \in \Sigma_s \text{ and } F^{(t)}_s \subset U)$$

Thus,

$$g^{-1}(U) = Gr(H) \cap \bigcup_{s \in S} (\{t \in T : F^{(t)}_s \subset U\} \times \Sigma_s) .$$

We consider various cases in the definition of $F^{(t)}_s$ as before and show that for every $s \in S$, $\{t \in T : F^{(t)}_s \subset U\} \in \underline{\underline{A}}$. It follows that g is $\underline{\underline{A}} \times \underline{\underline{B}}_\Sigma \mid Gr(H)$ - measurable.

Let $f : T \times \Sigma \to X$ be defined by

$$f(t,\sigma) = g(t, h(t,\sigma)), \quad t \in T, \quad \sigma \in \Sigma .$$

It is easy to see that f has the desired properties.

3. <u>The General Case</u> In this section we prove the implication (A) => (B) of Theorem 5.1.1. Notation and assumptions are as fixed in section 2 of the last chapter.

<u>Lemma 5.3.1</u> Let X be compact. Then for each $t \in T$ and $i, j \in N$ there exist positive integers n^t_{ij} and n^t_i such that

(i) the maps $t \to n^t_{ij}$ and $t \to n^t_i$ are $\underline{\underline{A}}$ - measurable,

(ii) $\delta(V_{n^t_{ij}}) < 2^{-i}$,

(iii) $\overline{F(t)} \subset \bigcup_{m=1}^{\infty} V_{n^t_{im}}$,

(iv) $m > n^t_i \Rightarrow n^t_{im} = 1$.

<u>Proof</u> : Let $\tilde{G} = \{(t,x) \in T \times X : x \in \overline{F(t)}\}$. First note that if U is an open set in X then

$$\{t \in T : \tilde{G}^t \cap U \neq \phi\}$$

$$= \{t \in T : F(t) \cap U \neq \phi\} \in \underline{\underline{A}} .$$

Fix $i \in N$. We shall define maps $t \to n^t_{ij}$, $j \in N$, by induction on j . For $m \geq 1$, we put

$$T^o_m = \phi \quad \text{if} \quad \delta(V_m) \geq 2^{-i} ,$$

otherwise, let

$$T^o_m = \{t \in T : \tilde{G}^t \cap V_m \neq \phi$$

and

$$(\forall \lambda < m)(\delta(V_\lambda) < 2^{-i} \Rightarrow \tilde{G}^t \cap V_\lambda = \phi)\} .$$

By the above observation, $T^o_m \in \underline{\underline{A}}$, $m \geq 1$. Further,

$m \neq n \Rightarrow T_m^o \cap T_n^o = \phi$ and $T = \bigcup_{m=1}^{\infty} T_m^o$. We define $n_{i1}^t = m$ if $t \in T_m^o$. The map $t \to n_{i1}^t$ thus defined is clearly $\underline{\underline{A}}$ - measurable. Now, suppose for some $p \in N$ maps $t \to n_{ij}^t$, $j \leq p$, have been defined to be $\underline{\underline{A}}$ - measurable. Let us observe that for $t \in T$ and $x \in X$

$$(t,x) \notin \tilde{G} \iff x \notin \overline{F(t)}$$

$$\iff (\exists n \in N)\ (x \in V_n \text{ and } V_n \cap F(t) = \phi),$$

so that

$$(T \times X) - \tilde{G} = \bigcup_{n=1}^{\infty} (F^{-1}(V_n) \times V_n).$$

As F is $\underline{\underline{A}}$ - measurable, it follows that $\tilde{G} \in \underline{\underline{A}} \times \underline{\underline{B}}_X$. Now for an open set U in X

$$(\tilde{G}^t - \bigcup_{j \leq p} V_{n_{ij}^t}) \cap U \neq \phi$$

$$\iff (\exists (\lambda_1, \ldots, \lambda_p) \in N^p)\ ((\forall j \leq p)\ (n_{ij}^t = \lambda_j) \text{ and } (\tilde{G}^t - \bigcup_{j \leq p} V_{\lambda_j}) \cap U \neq \phi)$$

By the above observation, Lemma 0.6 and the induction hypothesis we get the following :

$$\{t \in T : (\tilde{G}^t - \bigcup_{j\le p} V_{n^t_{ij}}) \cap U \neq \phi\} \in \underline{\underline{A}}.$$

For $m \geq 1$, we now define

$T^p_m = \phi$ if $\delta(V_m) \geq 2^{-i}$.

In case $\delta(V_m) < 2^{-i}$, we define

$$T^p_m = \{t \in T : n^t_{ip} < m, (\tilde{G}^t - \bigcup_{j\le p} V_{n^t_{ij}}) \cap V_m \neq \phi$$

and

$$(\forall \lambda < m) (\delta(V_\lambda) < 2^{-i} \Rightarrow (n^t_{ip} \geq \lambda \quad \text{or} \quad (\tilde{G}^t - \bigcup_{j\le p} V_{n^t_{ij}}) \cap V_\lambda = \phi))\}.$$

By the observation made last and the induction hypothesis, $T^p_m \in \underline{\underline{A}}$, $m \geq 1$. Also, $m \neq n \Rightarrow T^p_m \cap T^p_n = \phi$. We define

$$n^t_{i,p+1} = m \quad \text{if } t \in T^p_m$$

$$= 1 \quad \text{if } t \in T - \bigcup_{m=1}^{\infty} T^p_m .$$

As $p \in N$ was arbitrary, this completes the definition of $t \to n^t_{ij}$, $j \in N$. To define n^t_i, $t \in T$, notice that $\tilde{G}^t$ is compact and so $(\exists\, m \in N) (\forall \lambda > m) (n^t_{i\lambda} = 1)$. We define

n_i^t to be the first such positive integer. It is clear that conditions (i) - (iv) are satisfied.

Lemma 5.3.2 Let X be compact. Then there is a set $B \subset T \times \Sigma$ and a map $g : B \to X$ such that for $t \in T$

(i) $B \in \underline{\underline{A}} \times \underline{\underline{B}}_\Sigma$,

(ii) B^t is non-empty and compact,

(iii) $g(t,.)$ is a continuous map from B^t on $\overline{F(t)}$,

(iv) D is a dense subset of $\overline{F(t)} \Rightarrow \{\sigma \in B^t : g(t,\sigma) \in D\}$ is dense in B^t,

(v) g is $\underline{\underline{A}} \times \underline{\underline{B}}_\Sigma \mid B$ - measurable .

Proof : For $t \in T$ and $i,j \in N$ we get positive integers n_i^t and n_{ij}^t satisfying conclusions (i) - (iv) of Lemma 5.3.1. Let $\tilde{G} = \{(t,x) \in T \times X : x \in \overline{F(t)}\}$. In the proof of Lemma 5.3.1 we showed that $\tilde{G} \in \underline{\underline{A}} \times \underline{\underline{B}}_X$. For $t \in T$ we define

$$U_{ij}^{(t)} = V_{n_{ij}^t} \cap \tilde{G}^t \qquad \text{if } j = 1$$

$$= (V_{n_{ij}^t} \cap \tilde{G}^t) - \bigcup_{\ell<j} (V_{n_{i\ell}^t} \cap \tilde{G}^t) \qquad \text{if } j > 1 .$$

We have

(1) $\delta(U_{ij}^{(t)}) < 2^{-i}$,

(2) $U_{ij}^{(t)}$ is relatively open in $\tilde{G}^t$,

(3) $m \neq n \Rightarrow U_{im}^{(t)} \cap U_{in}^{(t)} = \phi$,

(4) $m > n_i^t \Rightarrow U_{im}^{(t)} = \phi$,

(5) $\tilde{G}^t = \bigcup_{k=1}^{\infty} \overline{U_{ik}^{(t)}}$,

(6) for every open set U in X

$$\{t \in T : \tilde{G}^t \cap U \subset U_{ij}^{(t)}\} \in \underline{\underline{A}} ,$$

(7) if P is a finite subset of $N \times N$ and if U is open in X then

$$\{t \in T : \bigcap_{(m,n)\in P} U_{mn}^{(t)} \cap U \neq \phi\} \in \underline{\underline{A}} .$$

The facts (1) - (5) are clearly seen from (i) - (iv) of Lemma 5.3.1. To see (6), we note that if $j = 1$ then

$$\tilde{G}^t \cap U \subset U_{ij}^{(t)}$$

$$\Longleftrightarrow \quad (\exists \lambda \in N)\ (n_{ij}^t = \lambda \quad \text{and} \quad \tilde{G}^t \cap U \subset V_\lambda) ,$$

and if $j > 1$ then

$$\tilde{G}^t \cap U \subseteq U^{(t)}_{ij}$$

$$\Longleftrightarrow \quad \tilde{G}^t \cap U \subseteq V_{n^t_{ij}} - \overline{\bigcup_{k<j} (V_{n^t_{ik}} \cap \tilde{G}^t)}$$

$$\Longleftrightarrow \quad \tilde{G}^t \cap U \subseteq V_{n^t_{ij}} \text{ and } (\forall\, k < j)\,(\tilde{G}^t \cap U \cap V_{n^t_{ik}} = \phi)$$

$$\Longleftrightarrow \quad (\exists (\ell_1, \ldots, \ell_j) \in N^j)((\forall k \leq j)(n^t_{ik} = \ell_k),\ \tilde{G}^t \cap U \subseteq V_{\ell_j}$$

$$\text{and}$$

$$(\forall k < j)\,(\tilde{G}^t \cap U \cap V_{\ell_k} = \phi)).$$

Now, (6) follows from (i) of Lemma 5.3.1, the fact that $\tilde{G} \in \underline{A} \times \underline{B}_X$ and Lemma 0.6. The fact (7) follows from (6) and the following equivalence:

$$\bigcap_{(m,n)\in P} U^{(t)}_{mn} \cap U \neq \phi$$

$$\Longleftrightarrow \quad (\exists k \in N)\,(V_k \subseteq U \text{ and } (\forall (m,n) \in P)(\tilde{G}^t \cap V_k \subseteq U^{(t)}_{mn})).$$

For $t \in T$ and $i, j \geq 1$ we define the following by induction on i:

$$m^t_i = n^t_i \qquad \text{if } i = 1$$

$$= m^t_{i-1} \cdot n^t_i \qquad \text{if } i > 1;$$

and $W_{ij}^{(t)} = U_{ij}^{(t)}$ if $i = 1$

$= U_{i\lambda}^{(t)} \cap W_{i-1,k}^{(t)}$ if $i > 1,\ 1 \leq k \leq m_{i-1}^{t}$,

$1 \leq \lambda \leq n_i^t$ and

$j = (k-1)\, n_i^t + \lambda$

$= \phi$ if $j > m_i^t$.

We have

(a) the map $t \to m_i^t$ is $\underline{A}$ - measurable,

(b) $W_{ij}^{(t)}$ is relatively open in $\tilde{G}^t$,

(c) $\delta(W_{ij}^{(t)}) < 2^{-i}$,

(d) $m \neq n \Rightarrow W_{im}^{(t)} \cap W_{in}^{(t)} = \phi$,

(e) $k > m_i^t \Rightarrow W_{ik}^{(t)} = \phi$,

(f) $\tilde{G}^t = \bigcup_{\lambda=1}^{\infty} \overline{W_{i\lambda}^{(t)}}$,

(g) $(\forall (i,j) \in N \times N)\ (\exists k \in N)\ (W_{i+1,j}^{(t)} \subset W_{ik}^{(t)})$,

(h) $\overline{W_{i+1,j}^{(t)}} \subset \overline{W_{ik}^{(t)}} \Rightarrow W_{i+1,j}^{(t)} \subset W_{ik}^{(t)}$,

(u) $\{t \in T : W_{ij}^{(t)} \neq \phi\} \in \underline{A}$,

(v) U is open in $X \Rightarrow \{t \in T : \overline{W_{ij}^{(t)}} \subset U\} \in \underline{A}$,

(w) $\{t \in T : W_{k+1,j_2}^{(t)} \subset W_{kj_1}^{(t)}\} \in \underline{A}$.

Using (i) of Lemma 5.3.1, (a) follows by induction on i. (b) - (f) follow from facts (1) - (5) stated before. (g) is clear. We now check (h). It is clear when $W^{(t)}_{i+1,j} = \phi$. So, let $W^{(t)}_{i+1,j} \neq \phi$ and k' be a positive integer such that $W^{(t)}_{i+1,j} \subset W^{(t)}_{ik'}$. Therefore $\phi \neq W^{(t)}_{i+1,j} \subset \overline{W^{(t)}_{ik}} \cap W^{(t)}_{ik'}$. This forces $W^{(t)}_{ik} \cap W^{(t)}_{ik'} \neq \phi$ as $W^{(t)}_{ik'}$ is open in $\tilde{G}^{t}$. Hence by (d), $k = k'$. In other words $W^{(t)}_{i+1,j} \subset W^{(t)}_{ik}$. The fact (u) is a particular case of (7). To check (v) we first note that as the base $\{V_n : n \geq 1\}$ for X is assumed to be closed under finite unions and as $X - U$ is compact, we have

$$\overline{W^{(t)}_{ij}} \subset U$$

$$\Longleftrightarrow (\exists k)\ (X - U \subset V_k \text{ and } V_k \cap \overline{W^{(t)}_{ij}} = \phi)$$

$$\Longleftrightarrow (\exists k)\ (X - U \subset V_k \text{ and } V_k \cap W^{(t)}_{ij} = \phi)$$

Thus, $\{t \in T : \overline{W^{(t)}_{ij}} \subset U\}$

$$= \bigcup \{t \in T : W^{(t)}_{ij} \cap V_k = \phi\},$$

where the union is taken over all $k \geq 1$ such that $X - U \subset V_k$. Now, (v) follows from fact (7). To verify (w), we first notice that

$$\{t \in T : W^{(t)}_{k+1,j_2} \subset W^{(t)}_{kj_1}\}$$

$$= \{t \in T : W^{(t)}_{k+1,j_2} = \phi\} \cup \{t \in T : \phi \neq W^{(t)}_{k+1,j_2} \subset W^{(t)}_{kj_1}\}$$

$$= \{t \in T : W^{(t)}_{k+1,j_2} = \phi\} \cup \cup \{t \in T : j_1 \leq m^t_k \text{ and } n^t_{k+1} = p\},$$

where the last union is taken over all $(p,q) \in N \times N$ such that $q \leq p$ and $j_2 = (j_1 - 1)p + q$. Now (w) is easily seen from facts (a) and (u) stated above and from (i) of Lemma 5.3.1.

We now take

$$B = \{(t,\sigma) \in T \times \Sigma : (\forall k)\ (W^{(t)}_{k\sigma_k} \neq \phi \text{ and } \overline{W^{(t)}_{k+1,\sigma_{k+1}}} \subset \overline{W^{(t)}_{k,\sigma_k}})\}$$

By the fact (h) stated above, we have

$$B = \{(t,\sigma) \in T \times \Sigma : (\forall k)\ (W^{(t)}_{k\sigma_k} \neq \phi \text{ and } W^{(t)}_{k+1,\sigma_{k+1}} \subset W^{(t)}_{k\sigma_k})\}$$

$$= \{(t,\sigma) \in T \times \Sigma : (\forall k)\ (W^{(t)}_{1\sigma_1} \supset \ldots \supset W^{(t)}_{k\sigma_k} \neq \phi)\}$$

$$= \bigcap_{k=1}^{\infty} \bigcup_{s \in S_k} (\{t \in T : W^{(t)}_{1s_1} \supset \ldots \supset W^{(t)}_{ks_k} \neq \phi\} \times \Sigma_s)$$

By facts (u) and (w) it follows that $B \in \underline{\underline{A}} \times \underline{\underline{B}}_{\Sigma}$.

We now check the assertion (ii). Let $t \in T$. Then $B^t \subset \mathop{X}\limits_{i=1}^{\infty} (\{1,\ldots,m^t_i\})$. To show that $B^t \neq \phi$ we shall get positive integers p_n, $n \geq 1$, inductively such that for every k, $W^{(t)}_{kp_k} \neq \phi$ and $W^{(t)}_{kp_k} \supset W^{(t)}_{k+1,p_{k+1}}$. Let p_1 be a positive integer such that $W^{(t)}_{1,p_1} \neq \phi$. That such a positive integer exists follows from (f). Suppose for some $k \in N$ we have obtained positive integers $p_1, \ldots, p_k$ such that $W^{(t)}_{1p_1} \supset \ldots \quad \ldots \supset W^{(t)}_{kp_k} \neq \phi$. By facts (4),(5) and (b), there exists a positive integer λ such that $W^{(t)}_{kp_k} \cap U_{k+1,\lambda} \neq \phi$. It follows that there exists $p_{k+1} \geq 1$ such that $W^{(t)}_{kp_k} \supset W^{(t)}_{k+1,p_{k+1}} \neq \phi$. Finally,

$$\sigma \notin B^t \iff (t,\sigma) \notin B$$

$$\iff (\exists k)\ (W^{(t)}_{k\sigma_k} = \phi \text{ or } \overline{W^{(t)}_{k+1,\sigma_{k+1}}} \not\subset \overline{W^{(t)}_{k\sigma_k}})$$

$$\iff (\exists s \in S)\ (\sigma \in \Sigma_s \text{ and } \Sigma_s \cap B^t = \phi) .$$

It follows that B^t is closed. The assertion (ii) is established.

Now notice that if $(t,\sigma) \in B$ then $\bigcap_k \overline{W^{(t)}_{k\sigma_k}}$ is a singleton. We define $g(t,\sigma)$ to be the unique point belonging to this intersection.

We shall check assertions (iii) and (iv). We fix a $t \in T$ and $\sigma \in B^t$. Let U be an open set in X. Then

$$g(t,\sigma) \in U$$

$$\iff \bigcap_{k=1}^{\infty} \overline{W^{(t)}_{k\sigma_k}} \subset U$$

$$\Longleftrightarrow \quad (\exists k) \left(\overline{W^{(t)}_{k\sigma_k}} \subset U \right), \quad \text{as } X \text{ is complete and } \delta\left(\overline{W^{(t)}_{k\sigma_k}} \right) \longrightarrow 0 \text{ as } k \longrightarrow \infty$$

$$\Rightarrow \quad (\exists k) \left(g(t, \Sigma_{\sigma|k} \cap B^t) \subset U \right).$$

This observation shows that $g(t,\cdot)$ is continuous on B^t. To show that the range of $g(t,\cdot)$ is precisely $\overline{F(t)}$ we fix $x \in \overline{F(t)}$. Using facts (e), (f) and (g) we see that König's infinity Lemma [12, pp.326] is applicable to the collection.

$$\bigcup_{k=1}^{\infty} \left\{ s \in S_k : x \in \overline{W^{(t)}_{ks_k}} \subset \cdots \subset \overline{W^{(t)}_{1s_1}} \right\}.$$

Now it is easily seen that $g(t, B^t) = \overline{F(t)}$. To prove (iv), suppose for some $s \in S_k$, $k \geq 0$, $\Sigma_s \cap B^t \neq \phi$. Then $\bigcap_{i=1}^{k} W^{(t)}_{is_i}$ is a non-empty, relatively open set in $\widetilde{G}^t = \overline{F(t)}$. Since $g(t,\cdot)$ is onto $\overline{F(t)}$ and D is dense in $\overline{F(t)}$ we get a $\sigma \in B^t$ such that $g(t,\sigma) \in \bigcap_{i=1}^{k} W^{(t)}_{is_i} \cap D$. If for some $i \leq k$, $s_i \neq \sigma_i$ we get that $g(t,\sigma) \in W^{(t)}_{is_i} \cap \overline{W^{(t)}_{i\sigma_i}}$. As $W^{(t)}_{is_i}$ is relatively open in $\overline{F(t)}$ this implies that $W^{(t)}_{is_i} \cap W^{(t)}_{i\sigma_i} \neq \phi$. But this contradicts (d). Thus $\sigma \in B^t \cap \Sigma_s$ and $g(t,\sigma) \in D$.

To verify (v), let $(t,\sigma) \in B$ and U be open in X. Then

$$g(t,\sigma) \in U$$

$$\Longleftrightarrow (\exists k) \quad (\overline{W_{k\sigma_k}^{(t)}} \subset U)$$

$$\Longleftrightarrow (\exists k)\ (\exists m)\ (\overline{W_{km}^{(t)}} \subset U \text{ and } \sigma_k = m) .$$

Therefore,

$$g^{-1}(U) = B \cap \bigcup_k \bigcup_m (\{t \in T : \overline{W_{km}^{(t)}} \subset U\} \times \{\sigma \in \Sigma : \sigma_k = m\})$$

By (v), we conclude that g is $\underline{\underline{A}} \times \underline{\underline{B}}_\Sigma | B$ - measurable. This completes the proof of Lemma 5.3.2.

<u>Proof of (A) ⇒ (B)</u> : Without loss of generality we assume that X is a compact metric space. We get a set $B \subset T \times \Sigma$ and a map $g : B \to X$ satisfying conclusions (i) - (v) of Lemma 5.3.2. We define a multifunction $H : T \to \Sigma$ as follows :

$$H(t) = \{\sigma \in B^t : g(t,\sigma) \in F(t)\} .$$

Then $H(t)$ is a non-empty G_δ in Σ. By (iv) of Lemma 5.3.2, $H(t)$ is a dense subset of B^t for each $t \in T$. It follows that for every open set U in X

$$\{t \in T : H(t) \cap U \neq \phi\}$$

$$= \{t \in T : B^t \cap U \neq \phi\}$$

$$= \prod_T (B \cap (T \times U)) .$$

By assertions (i) and (ii) of Lemma 5.3.2 and Lemma 0.6, it follows that H is $\underline{\underline{A}}$ - measurable. Also, it is easily seen that the restriction of $g(t,\cdot)$ to $H(t)$ is closed, $t \in T$. Let $\varphi : B \to T \times X$ be defined as follows: $\varphi(t,\sigma) = (t, g(t,\sigma)), (t,\sigma) \in B$. Then φ is measurable when B is equipped with the σ - field $\underline{\underline{A}} \times \underline{\underline{B}}_{\Sigma}|B$ and $T \times X$ is equipped with $\underline{\underline{A}} \times \underline{\underline{B}}_X$. Since $B \in \underline{\underline{A}} \times \underline{\underline{B}}_{\Sigma}$, $Gr(H) = \varphi^{-1}(Gr(F)) \in \underline{\underline{A}} \times \underline{\underline{B}}_{\Sigma}$. By the previous case proved in the last section, we get a map $h : T \times \Sigma \to \Sigma$ such that for each $t \in T$, $h(t,\cdot)$ is a continuous closed map from Σ onto $H(t)$ and for each $\sigma \in \Sigma$, $h(\cdot,\sigma)$ is $\underline{\underline{A}}$ - measurable. We define $f : T \times \Sigma \to X$ by $f(t,\sigma) = g(t, h(t,\sigma))$, $t \in T$, $\sigma \in \Sigma$. It is routine to check that f satisfies (B).

4. <u>Proof of (B) $\Rightarrow$ (A)</u> We first check that F is $\underline{\underline{A}}$ - measurable. Let $\{\sigma^n : n \geq 1\}$ be a dense sequence in Σ and $t \in T$. Since $f(t,\cdot)$ is continuous, $\{f(t,\sigma^n) : n \geq 1\}$ is a dense subset of $F(t)$. Therefore, for any open set U in X

$$F^{-1}(U) = \bigcup_{n=1}^{\infty} (f(\cdot,\sigma^n)^{-1}(U)) \in \underline{\underline{A}} .$$

Thus F is $\underline{\underline{A}}$ - measurable.

Define a set $B \subset T \times \Sigma$ as follows:

$(t,\sigma) \in B \iff (\exists x \in X)(f(t,.)^{-1}(\{x\})$ is not open and σ is a point in the (topological) boundary of $f(t,.)^{-1}(\{x\})$ or $f(t,.)^{-1}(\{x\})$ is open and $\sigma = \sigma^n$ where n is the first positive integer m such that $f(t, \sigma^m) = x)$.

Let $t \in T$. It is clear that $f(t, B^t) = f(t,\Sigma) = F(t)$. Also it is easy to check that B^t is closed in Σ. Thus the restriction of $f(t,.)$ to B^t is closed. Let us now recall the following result of Vaĭnšteĭn [30] : Let L and M be metrizable spaces. If f is a closed continuous map of L onto M, then the topological boundary of $f^{-1}(\{z\})$ is compact for every $z \in M$. From this it is easily deduced that the restriction of $f(t,.)$ to B^t is perfect. In the paper referred above Vaĭnšteĭn has also shown that the image of a Polish space under a perfect map, if metrizable, is Polish. We conclude from this that $F(t)$ is a G_δ in X for each $t \in T$.

Now let $\{V_n : n \geq 1\}$ be a countable base for X. Then, using the given properties of f we check that

$$(t,\sigma) \in B \iff ((\forall s \in S)(\sigma \in \Sigma_s \Rightarrow (\exists k)(\sigma^k \in \Sigma_s)(f(t,\sigma^k) \neq f(t,\sigma)))$$

or $(\exists n)((\sigma = \sigma^n, (\forall \lambda < n)(f(t,\sigma^\lambda) \neq f(t,\sigma^n))$ and

$(\exists p)(f(t,\sigma^n) \in V_p$ and $(\forall \lambda)(f(t,\sigma^\lambda) \in V_p$

$\Rightarrow f(t,\sigma^\lambda) = f(t,\sigma^n)))))$.

By the given properties of f, it follows that $B \in \underline{\underline{A}} \times \underline{\underline{B}}_\Sigma$. Now we notice that

$$(t,x) \in Gr(F) \iff (\exists \sigma \in \Sigma)((t,\sigma) \in B \text{ and } f(t,\sigma) = x).$$

Therefore,

$$Gr(F) = \prod_{T\times X}(\{(t,x,\sigma) \in T \times X \times \Sigma : (t,\sigma) \in B \text{ and } f(t,\sigma) = x\}).$$

Now note that if $t \in T$ and $x \in X$ are fixed then the (t,x)-section of the set within braces is compact. This follows from the fact that the restriction of $f(t,\cdot)$ to B^t is perfect. Hence, it follows from the fact that the set within braces is in $\underline{\underline{A}} \times \underline{\underline{B}}_X \times \underline{\underline{B}}_\Sigma$ and Lemma 0.6 that $Gr(F) \in \underline{\underline{A}} \times \underline{\underline{B}}_X$. This completes the proof.

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INDEX TO UNSOLVED PROBLEMS

REFERENCES

[1] W. Arsenin and A. Ljapunov, Theory of A - sets, Uspehi Mat. Nauk 5(1950), no. 5 (39), 45 - 108. (Russian).

[2] D. Blackwell, On a class of probability spaces, Proc. 3rd. Berkeley Sympos. Math. Statist. and Prob. 2 (1956), 1 - 6.

[3] D. Blackwell, A Borel set not containing a graph, Ann. Math. Statist. 39 (1968), 1345 - 1347.

[4] R. Engelking, Outline of general topology, North Holland, Amsterdam, 1968.

[5] R. Engelking, On closed images of the space of irrationals Proc. Amer. Math. Soc. 21 (1969), 583 - 586.

[6] Z. Frolik, A contribution to the descriptive theory of sets and spaces, General topology and its relations to modern analysis and algebra, Proc. Sym. Prague (1961), 157 - 173.

[7] F. Hausdorff. Über Innere Abbildungen, Fund. Math. 23 (1934), 279 - 291.

[8] A.D. Ioffe, Representation theorems for multifunctions and analytic sets, Bull. Amer. Math. Soc. 84(1978), 142 - 144.

[9] A.D. Ioffe, Extremal representations of multifunctions and Souslin sets in product spaces, preprint.

[10] R.R. Kallman and R.D. Mauldin, A cross section theorem and an application to C* - algebras, Proc. Amer. Math. Soc. 69 (1978), 57 - 61.

[11] K. Kuratowski, Topology, Vol. 1, Academic Press, New York and London, Warsaw, PWN, 1966.

[12] K. Kuratowski and A. Mostowski, Set theory, North Holland Publishing Company, Amsterdam, New York, Oxford, PWN, 1976.

[13] K. Kuratowski and C. Ryll-Nardzewski, A general theorem on selectors, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys. 13 (1965), 397 - 403.

[14] D. G. Larman, Projecting and uniformizing Borel sets with K_σ sections II, Mathematatika 20 (1973), 233 - 246.

[15] S. J. Leese, Multifunctions of Souslin type, Bull. Austral. Math. Soc. 11 (1974), 395 - 411.

[16] N. Lusin, Lecons sur les ensembles analytiques et leurs applications, Gauthier - Villars, Paris, 1930.

[17] A. Maitra, Analytic sets with countable sections, Fund. Math., to appear.

[18] A. Maitra and B. V. Rao, Generalizations of Castaing's theorem on selectors, Colloq. Math., to appear.

[19] R. D. Mauldin, Borel Parametrizations, Trans. Amer. Math. Soc., to appear.

[20] D. E. Miller, A selector for equivalence relations with G_δ orbits, Proc. Amer. Math. Soc., to appear.

[21] D. E. Miller, Borel selectors for separated quotients, preprint.

[22] G. Mokobodzki, Démonstration élémentaire d'un théorème de Novikov, Séminaire de probabilités X, 539 - 543, Lecture notes no. 511, Springer, Berlin, 1974-75.

[23] V. Ponomarev, Normal spaces as images of zero-dimensional ones, Soviet Math. Doklady 1(1960), 774 - 777.

[24] J. Saint-Raymond, Boréliens a coupes K_σ, Bull. Soc. Math. France. 104(1976), 389 - 400.

[25] H. Sarbadhikari, Some uniformization results, Fund. Math. 97(1977), 209 - 214.

[26] H. Sarbadhikari and S.M. Srivastava, G_δ valued multifunctions, Trans. Amer. Math. Soc., submitted.

[27] S.M. Srivastava, A representation theorem for closed valued multifunctions, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys., submitted.

[28] S.M. Srivastava, Selection theorems for G_δ valued Multifunctions, Trans. Amer. Math. Soc., to appear.

[29] S.M. Srivastava, A representation theorem for G_δ valued multifunctions, Amer. J. of Math., submitted.

[30] I. A. Vaĭnšteĭn, On closed mappings of metric spaces, Doklady Akad. Nauk SSSR (NS) 57 (1947), 319 - 321.

[31] D.H. Wagner, Survey of measurable selection theorems, Siam J. Control and Optimization 15 (1977), 859-903.