THE LINEAR COMPLEMENTARITY PROBLEM WITH A Z-MATRIX

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PREFACE

A square matrix M whose off diagonal elements are nonpositive is known as a Z-matrix. Z-matrices and their generalisations known as L matrices have been used in interindustry models by Leontief and Gale. More recently Z-matrices have been considerered in the contexts of some operational research problems such as the minimum cost multifacility inventory systems and resource allocation.

Given a square matrix M of order n and a vector q in R^n, the problem of finding nonnegative solutions in the variables w_i's, $i = 1,2, \ldots, n$ and z_i's, $i = 1,2, \ldots, n$ to the system of equations

$$w - Mz = q, \quad w \in R^n, \quad z \in R^n,$$

$$\sum_{i=1}^{n} w_i z_i = 0$$

is known as the linear complementarity problem. This problem has been shown to be a unified form of many problems arising in mathematical programming, game theory, structural engineering and fluid mechanics.

In this dissertation we consider the above linear complementarity problem with M as a Z-matrix. A problem of fluid mechanics can be formulated as a linear complementarity problem with M as a Z-matrix. More generally such problems occur in the discretisation of elliptic partial differential equations.

Chapter 1 provides a general introduction to the linear complementarity problem and reviews the relevant results on this problem and on Z-matrices. Our results on the linear complementarity problem with a Z-matrix are presented in chapters 2,3 and 4. In section 1.7 we present a chapterwise summary of these results.

CONTENTS

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1. General introduction and review

1.1. The problem and its importance :

Let M be a given square matrix of order n and q a given vector in R^n. The problem of determining solutions $w \in R^n$ and $z \in R^n$ satisfying the system (M, q) of equations

$$w - Mz = q, \quad w \geqq 0, \quad z \geqq 0 \qquad \ldots \quad \ldots (1.1.1)$$

$$w^T z = 0 \qquad \ldots \quad \ldots (1.1.2)$$

where w^T is the transpose (row) of w is known as the linear complementarity problem. This problem arises naturally in many fields. Many mathematical programming problems such as the linear programming problem, convex quadratic programming problem and the problem of finding the Nash equilibrium points of bimatrix games can all be transformed into linear complementarity problems. We refer the readers to [7, pp.103-108] and [22, 23]. There are also examples of engineering problems [18, 38] and problems of structural mechanics [12, 24] which have been given the above complementarity formulation. In [11] it is shown that a problem of fluid mechanics can be formulated as linear complementarity problem.

We wish to point out here that there is one more class of programming problems which can be given the linear complementarity formulation. Let $c \in R^n$, $d \in R^n$, $\alpha \in R$, $\beta \in R$, $b \in R^m$ and A a matrix of order $m \times n$ be given. Consider the fractional linear programming problem

$$\text{maximise} \quad f(x) = \frac{c^T x + \alpha}{d^T x + \beta}$$

$$\text{subject to} \quad Ax \leqq b, \quad x \geqq 0$$

This problem can be directly cast as a linear complementarity problem. We believe that this example has not been observed in the literature so far. We therefore give the details of formulation in appendix 1.

In this work we consider the linear complementarity problem (M, q) where M is a square matrix whose off diagonal elements are nonpositive. In this chapter we give a general review of the literature on the linear complementarity problem which is relevant to our work. In section 1.2. we introduce our notations and basic concepts. In section 1.3. we present a general review of the various classes of matrices which have been considered in connection with the problem (M, q). The relationship among these classes and between these classes and the class of matrices with nonpositive off diagonal elements is also discussed. Section 1.4. summarises the known methods of computation of solutions to (M, q) applicable for various classes of matrices. Section 1.5. reviews the results on the number of solutions to (M, q) and the constant parity property. The problem of existence of rays of solutions to (M, q) and its importance is indicated in section 1.6. section 1.7. presents a summary of the results obtained in this dissertation.

1.2. Notations and basic definitions :

1.2.1. Matrices, vectors : Throughout this dissertation M stands for a square matrix of order n. m_{ij} denotes the ij^{th} element of M. For any matrix A, $A_{.j}$ stands for the j^{th} column of A and $A_{i.}$ for the i^{th} row. A^T denotes the transpose of A. Unless otherwise indicated all vectors $x \in R^n$ are column vectors, x^T's are row vectors. $e_n \in R^n$ is the column vector whose coordinates are all equal to 1. I stands for the identity matrix whose order is determined from the context. The symbol 0 is used both for the null vector in R^n and for $0 \in R$ depending on the context. For given $x, y \in R^n$ the symbol $x > y$ indicates that $x_i > y_i$, $1 \leqq i \leqq n$, the symbol $x \geq y$ indicates that $x_i \geqq y_i$, $1 \leqq i \leqq n$ and for at least one j, $1 \leqq j \leqq n$, $x_j > y_j$ (i.e. $x \neq y$) ; and the symbol $x \geqq y$ indicates that $x_i \geqq y_i$ for all $1 \leqq i \leqq n$ and the possibility $x = y$ is permitted. We write $x > 0 \in R^n$ for $x \in R^n$, $0 \in R^n$ and $x > 0$.

1.2.2. The system (M, q) : We use the symbol (M, q) as indicated before for the system of equations and inequalities given by (1.1.1) and (1.1.2). Note that if $\begin{bmatrix} w^* \\ z^* \end{bmatrix} \in R^{2n}$ is a solution to (M, q) atmost n coordinates of $\begin{bmatrix} w^* \\ z^* \end{bmatrix}$ are positive. We denote the solution by (w^*, z^*).

1.2.3. Complementary basis : Let B be a matrix of order n x n whose columns are columns of (I, -M). B is said to be a basis matrix if its columns are linearly independent. B is said to be a complementary basis matrix if .

i) $I_{.j}$ is a column of B $\Longrightarrow$ $-M_{.j}$ is not.

ii) $-M_{.j}$ is a column of B $\Longrightarrow$ $I_{.j}$ is not.

iii) All the n columns are linearly independent.

A solution (w^*, z^*) to (M, q) is said to be a complementary basic feasible solution if the set of columns $I_{.j}$, for j such that $w_j^* > 0$ and the set of columns $-M_{.k}$ for k such that $z_k^* > 0$ form a linearly independent set. We note that this set need not in general contain n columns.

1.2.4. Degeneracy : A solution (w^*, z^*) to (M, q) is said to be degenerate if less than n coordinates of (w^*, z^*) are positive. We say that q is nondegenerate with respect to M if either (M, q) is infeasible or if all solutions to (M, q) are nondegenerate.

We note that q is nondegenerate with respect to M if and only if all the complementary basic feasible solutions to (M, q) are nondegenerate.

1.2.5. Complementary cones : The concept of complementary cones was first introduced by K.G. Murty [31]. A complementary set of column vectors is a set of n column vectors $\{A_{\cdot j}, \ j = 1,2, \ldots n\}$ where $A_{\cdot j}$ is either $I_{\cdot j}$ or $-M_{\cdot j}$. A complementary cone is a convex cone generated by a complementary set of column vectors. (The convex cone generated by a set of column vectors is the set of all nonnegative linear combinations of these columns). There are thus 2^n complementary cones. We note that all complementary basis matrices generate complementary cones with nonempty interior in R^n. But there may be complementary sets of column vectors which are linearly dependent. Such complementary sets generate complementary cones whose interior in R^n is empty.

Suppose C_i, $i = 1,2, \ldots 2^n$ are the complementary cones of (I, -M). We use the symbol D(M) for $\bigcup_{i=1}^{2^n} C_i$. We note that

$$D(M) = \{q \mid q \in R^n, \ (M,q) \text{ has a solution}\}.$$

We also let,

$$D_1(M) = \{q \mid q \in R^n, \ q \text{ is nondegenerate with respect to } M\}$$

$$D_2(M) = \{q \mid q \in R^n, \ q \text{ is nondegenerate with respect to } M \text{ and } q \in D(M)\}.$$

We note that

$$D_2(M) \subseteq D_1(M)$$

We use the symbol Pos(B) for the convex cone generated by the columns of B.

1.2.6. Nondegenerate cones : A complementary cone Pos(B) where B forms a complementary set of column vectors is said to be nondegenerate if

$$\left\{ x \mid Bx = 0 , \quad x \geqq 0, \quad x \in R^n \right\} = \left\{ 0 \right\}$$

We note that according to this definition a complementary cone whose interior is empty can also be a nondegenerate cone. However it is easy to see that q is nondegenerate with respect to M only if it is not contained in any complementary cone whose interior is empty.

1.2.7. Principal rearrangement, principal submatrix :

Let $N = \left\{1,2, \ldots\ldots n\right\}$ and M be a matrix of order n. Suppose that

$$J_1 = \left\{ i_1 , \ldots\ldots i_r \right\} ; \quad J_2 = \left\{ j_1, \ldots\ldots\ldots j_s \right\} .$$

are two subsets of N and that $i_1 < i_2 < \ldots < i_r$, $j_1 < j_2 < \ldots < j_s$.
The symbol $M_{J_1 J_2}$ stands for the matrix formed by elements m_{ij}, $i \in J_1$, $j \in J_2$ taken in the order $i_1 < \ldots < i_r$ and $j_1 < \ldots\ldots < j_s$.

The symbol M_J is used for the submatrix M_{JJ}. It is called a principal submatrix of M. The determinant of a principal submatrix is called principal minor.

Let $\bar{M}$ be a matrix obtained from M by permuting its rows and columns, applying the same permutation rule both to the set of rows and to the set of columns. $\bar{M}$ is called a principal rearrangement of M. Let π be a permutation function defined on N to N. For any $x \in R^n$ let $\pi(x)$ denote the vector which is obtained from x by permuting its coordinates according to the rule π. Let $\pi(M)$ denote the principal rearrangement of M according to the rule π. We note that $(\bar{w}, \bar{z})$ is a solution to $(\pi(M), \pi(q))$ if and only if $(\pi^{-1}(\bar{w}), \pi^{-1}(\bar{z}))$ is a solution to (M, q), where π^{-1} is the inverse permutation. In particular therefore $D(M) = D(\pi(M))$, $D_1(M) = D_1(\pi(M))$ and $D_2(M) = D_2(\pi(M))$. Also the complementary cones of M correspond to those of $\pi(M)$ in an obvious manner.

1.2.8. E(M) and regular E(M) :

Let B be a complementary set of column vectors and let B_1 be a submatrix of some columns of B. The cone

$$\mathrm{Pos}(B_1) = \left\{ y \mid y = B_1 x, \text{ for some } x \geq 0, \ x \in R^m \text{ where m is the number of columns in } B_1 \right\}.$$

is called a k-face of the cone Pos(B) where k is the rank of B_1. Also, Pos(B) is called an m-cone if rank (B) = m.

Let $E(M) = \{ Pos(B)$ and their faces / columns of B form a complementary set, $rank(B) \geq n-1 \}$.

Let F be a (n-1) face of some cone in E(M). We say that the two complementary cones C_1, C_2 incident on it are properly situated if $C_1 \cap C_2 = F$; otherwise C_1 and C_2 are not properly situated.

Let F be an (n-1) face of some cone in E(M). We say that F is on the boundary of D(M) if $q \in F$ implies that q is a boundary point of D(M). We say that a (n-1) face F of some cone in E(M) is proper if either it is on the boundary of D(M), or if the two cones incident on it are properly situated.

We say that E(M) is regular if all the (n-1) faces in it are proper. We say that E(M) is nondegenerate if all the complementary cones in E(M) are nondegenerate.

In the above we have adopted the definitions given by Romesh Saigal in [36, pp.47-48]. These definitions will be useful to us in our discussion of Saigal's results in sections 1.3 and 1.5.

1.3. Classes of matrices :

Different classes of matrices have been considered in the literature on linear complementarity problems in the context of computational methods and applications. In this section we review the relevant results.

1.3.1. P, P_o, S, S_o matrices : We say that M is a P matrix if all the principal minors of M are positive. M is called a P_o matrix if all its principal minors are nonnegative.

M is said to be a $S(S_o)$ matrix if and only if there exists $x \geq 0 \in R^n$ such that $Mx > 0$ ($Mx \geqq 0$).

We have the following theorem.

Theorem 1.3.1 : Either M is a S_o matrix or $-M^T$ is a S-matrix, but never both.

This result follows from a theorem of the alternative due to Motzkin. See [27, pp.34].

1.3.2. Todd's classes of matrices : The following classes of matrices were introduced by M.J. Todd [42, pp.61] in the context of the applicability of Lemke-Howson algorithm which will be discussed in the next section.

Let $d \in R^n$ and let $(\bar{w}, \bar{z})$ be a solution to (M, d). Let J be any set such that

$$\{i \mid \bar{z}_i > 0\} \subseteq J \subseteq \{i \mid \bar{w}_i = 0\} \quad \ldots \quad (1.3.1)$$

Let

i) $\det(M_J) > 0$, if J satisfies (1.3.1) (If $J = \emptyset$ define $\det(M_J) = 1$)

ii) There exists $x \geq 0 \in R^n$ such that $y = -M^T x \geqq 0$,

$x \leqq \bar{z}, \quad y \leqq \bar{w}$.

$\bar{E}(d) = \{ M \mid \text{Either (i) or (ii) is satisfied} \}$.

$\bar{E}^*(d) = \{ M \mid \text{(i) above is satisfied} \}$.

Let $\bar{L}(d) = \bar{E}(d) \cap \bar{E}(0)$; $\bar{L}^*(d) = \bar{E}^*(d) \cap \bar{E}^*(0)$.

1.3.3. Garcia's classes : Todd's classes are generalisations of the following classes of matrices considered by C.B. Garcia [16, p.303].

Let $d \in R^n$ and consider the conditions

i) $(\bar{w}, \bar{z})$, $\bar{z} \neq 0$ is a solution to $(M, d) \Longrightarrow$ There exists $x \geq 0 \in R^n$ such that $y = -M^T x \geqq 0$, $x \leqq \bar{z}$, $y \leqq \bar{w}$.

ii) $(\bar{w}, \bar{z})$ is a solution to $(M, d) \Longrightarrow \bar{z} = 0$.

Let $E(d) = \{ M \mid \text{(i) above is satisfied} \}$.

$E^*(d) = \{ M \mid \text{(ii) above is satisfied} \}$.

$L(d) = E(d) \cap E(0)$; $L^*(d) = E^*(d) \cap E^*(0)$.

Garcia observes that matrices arising from polymatrix games are in $L(d)$ for a suitable $d \geq 0$. See [16, pp.307].

1.3.4. Eaves' classes : The following classes of matrices were introduced by B.C. Eaves [13, pp.619].

Consider the conditions

i) For any $x \geq 0 \in R^n$ there is a $1 \leqq k \leqq n$ such that $x_k > 0$ and $(Mx)_k \geqq 0$.

ii) If for some $x \geq 0 \in R^n$, $Mx \geqq 0$ and $x^T Mx = 0$ then there exist diagonal matrices A and B, $\geqq 0$ such that

$$(AM + M^T B)\, x = 0 \quad \text{and} \quad Bx \neq 0$$

$L = \{ M \mid M$ satisfies (i) and (ii) above $\}$

$L^* = \{ M \mid M$ satisfies (i) above and $x^T Mx > 0$ for all $x \geq 0 \in R^n$ for which $Mx \geqq 0 \}$

We note that $L = \bigcap_{d>0} L(d)$ and $L^* = \bigcap_{d>0} L^*(d)$.

The class L contains matrices arising from bimatrix games, certain P_o matrices known as adequate matrices, (defined below) and the copositive plus matrices introduced by C.E. Lemke [22, pp. 687].

Adequate matrices : A P_o - matrix is said to be an adequate matrix if det $(M_J) = 0$ for some $J \subsetneq N$ implies that the set of rows of M and the set of columns of M, whose indicies are in J are linearly dependent sets. See [13, pp.622].

1.3.5. Copositive plus matrices : These matrices were considered by Cottle and Dantzig in [7, pp.116].

A matrix M is said to be copositive plus if

i) For all $x \geqq 0 \in R^n$, $x^T Mx \geqq 0$

ii) $x^T Mx = 0$, $x \geqq 0 \in R^n \Longrightarrow (M + M^T)x = 0$

A matrix which satisfies (i) above alone is called a copositive matrix and if $x^T Mx > 0$ for all $x \geq 0 \in R^n$ then M is strictly copositive. Matrices arising from linear programming problems, convex quadratic programming problems, linear fractional programming problems are all positive semi-definite and therefore are also copositive plus. Let CP^+ denote the class of copositive plus matrices, CP the class of copositive matrices and SCP the class of all strictly copositive matrices.

We have,

$CP^+ \subseteq L \subseteq L(d) \subsetneq \bar{L}(d)$ and

$SCP \subseteq L^* \subseteq L^*(d) \subsetneq \bar{L}^*(d)$ for any $d > 0 \in R^n$.

We also note that positive semi-definite matrices belong to CP^+ and positive definite matrices are contained in SCP. P matrices are also in L^* However the class P_0 is not contained in any of the above classes, not even in $\bar{L}(d)$ for any $d > 0 \in R^n$.

A more detailed discussion on the relationship among the above classes of matrices is given by Karamardian in [21, pp.109].

1.3.6. Z-matrices : We say that M is a Z-matrix if $m_{ij} \leqq 0$ for all $i \neq j$. M is said to be a $\bar{Z}$-matrix if it is a Z-matrix and if $m_{ii} \geqq 0$ for all $i = 1,2, \ldots . n$.

The class of Z-matrices and its generalisation known as Leontief matrices have been used by Leontief in inter industry models. For such applications we refer the readers to [15]. The properties of Z-matrices have been studied by various authors. Most of these results appear in a survey article by Fiedler and Ptak [14]. The class Z also arises in some resource allocation problems [40] and in multifacility inventory problems [44]. See also [43]. The linear complementarity problems with Z-matrices arise in some problems of fluid mechanics [8,9,11,32]. Linear complementarity problems with Z-matrices have earlier been considered by R.Chandrasekaran, [1], Romesh Saigal [33, 34]. In this paragraph we summarise the properties of Z-matrices which will be useful to us in our study of the linear complementarity problems with Z-matrices.

Theorem 1.3.2 (Fiedler and Ptak) : Let $M \in Z$. The following statements are equivalent.

i) There exists $x \geq 0 \in R^n$ such that $Mx > 0$ (i.e. $M \in S$).

ii) There exists $x > 0 \in R^n$ such that $Mx > 0$.

iii) If $A \in Z$, $A \geq M$ then A^{-1} exists and $A^{-1} \geq 0$.

iv) All the principal minors of M are positive.

v) All the real eigen values of M are positive.

We denote the class of matrices satisfying any one of the above by K. See [14, pp.387].

Theorem 1.3.3 (Fiedler and Ptak) : Let $M \in Z$. The following statements are equivalent.

i) All the principal minors of M are nonnegative.

ii) $M + \theta I \in K$ for all $\theta > 0$, (θ a real number)

iii) All the real eigen values of M are nonnegative.

We denote by K_0 the class of matrices satisfying any one of the above conditions.

We note that $K \subsetneq K_0$. See [14, pp.391].

Theorem 1.3.4 (Fiedler and Ptak) : Let $M \in Z$. If there exists $x > 0 \in R^n$ such that $Mx \geqq 0$ then $M \in K_0$. See [14, pp.391].

Theorem 1.3.5 : Let $M \in K_0$ be singular and irreducible (i.e. there does not exist a principal rearrangement $\bar{M}$ of M of the form $\begin{bmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{bmatrix}$). Then M has rank (n-1) and there exists $y > 0 \in R^n$ such that $My = 0$. See [14, pp.391].

<u>Theorem 1.3.6</u> (Fiedler and Ptak) : Let $M \in K_0$ be irreducible then all the proper principal minors of M are positive. See [14, p.392].

<u>Example 1.3.1</u> : The following example shows that K_0 matrices and therefore Z-matrices are not contained in $\bar{L}(d)$ for any $d \in R^n$.

$$M = \begin{bmatrix} 2 & -1 & -3 & -1 \\ -2 & 1 & -2 & -4 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -1 & 4 \end{bmatrix}$$

It is easy to verify that this is a K_0-matrix. $\bar{z} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ and $\bar{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ solves (M, 0) and we also observe that the principal minor

$$\det \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = 0 .$$

We also note that the system of inequalities $x \geq 0$, $y = -M^T x \geqq 0$, $x \leqq \bar{z}$, $y \leqq \bar{w}$ has no solution because

$y \leqq \bar{w} \Longrightarrow y = 0$ and $x \geq 0$, $x^T M = 0 \Longrightarrow x_1 > 0, x_2 > 0, x_3 > 0, x_4 > 0$.

Thus $x \leqq \bar{z}$ is impossible.

Therefore $M \notin \bar{E}(0)$, and hence $M \notin \bar{L}(d)$ for any $d \in R^n$.

We also note the following trivial result.

Theorem 1.3.7 : Let $M \in K_o$ be symmetric. Then $M \in C P^+$. In [36, p.56] Romesh Saigal proves the following theorem.

Theorem 1.3.8 : Let $M \in \bar{Z}$. Then $E(M)$ is regular.

Remark 1.3.1 : We also note that $K_o \subsetneq \bar{Z}$.

Example 1.3.2 : The following example shows that if $M \in Z - \bar{Z}$, $E(M)$ need not be regular.

$$M = \begin{bmatrix} -5 & -2 \\ -3 & -3 \end{bmatrix}$$

For this M the two cones incident on the (n-1) face $-M_{.1}$, Pos $(-M_{.1}, -M_{.2})$ and Pos $(I_{.1}, I_{.2})$, are not properly situated. Also Pos$(-M_{.1})$ does not lie on the boundary of D(M). In fact the boundary of D(M) is Pos$(I_{.1}) \cup$ Pos$(I_{.2})$. Thus E(M) is not regular.

1.4. Computational methods :

In this section we summarise the computational methods available in the literature for solving (M, q).

1.4.1. Lemke-Howson algorithm : In connection with the problem of finding the Nash equilibrium points of bimatrix games Lemke and Howson [23], [22] have given an algorithm which can be used to solve (M, q). See also [7, pp. 108]. We discuss here a version of their algorithm.

The algorithm is based on pivot steps. The initial solution to (1.1.1) and (1.1.2) is taken as

$$w = Mz + q + e_n z_o \geqq 0$$

$$z = 0$$

where z_o is an artificial variable which takes a large enough initial value so that $w \geqq 0$. This is called the primary ray.

Step 1. Decrease z_o so that one of the variables w_i, $i = 1,2,\ldots n$, say w_α, is reduced to 0 and $w \geqq 0$ is satisfied. We now have a basic feasible solution with z_o in place of w_α and with atleast one pair of complementary variables (w_α, z_α) as nonbasic. If we assume that all solutions to

$$w - Mz - e_n z_o = q, \quad (w, z, z_o) \geq 0, \quad w^T z = 0$$

are nondegenerate ,the pair of nonbasic complementary variables is uniquely determined.

Step 2. At each iteration the complement of the variable which was removed in the previous iteration is to be increased. If nondegeneracy assumption holds the algorithm uniquely determines the variable to be increased at each iteration. (In the second iteration, for instance, z_α will be increased).

Step 3. If the variable selected at step 2 to enter the basis can be arbitrarily increased then the procedure is said to terminate in a secondary ray. If a new basic feasible solution is obtained with $z_o = 0$, we get a solution to (M, q). If in the new basic feasible solution $z_o > 0$, we obtain a new pair of complementary nonbasic variables (w_β, z_β). We repeat step 2.

The algorithm consists of the repeated applications of steps 1 and 2. If nondegeneracy assumption is made no basis repeats and the algorithm terminates either in a secondary ray or in a solution to (M, q) in a finite number of iterations. If degenerate solutions are generated by the algorithm the standard procedures as discussed by B.C. Eaves [13, pp.614] uniquely determines the variable to be increased at each iteration and ensures termination in a finite number of steps.

The above algorithm can also be applied with any $d > 0 \in R^n$ in place of e_n. We use the notation L(M, q, d) for Lemke-Howson procedure with $d > 0$ applied to (M, q).

1.4.2. Near complementary basis matrix : Let B be a matrix of order n whose columns are columns of (I, -M, -d). B is said to be a near complementary basis matrix if the following conditions hold.

i) One of the columns of B is -d.

ii) If $I_{\cdot j}$ is a column of B then $-M_{\cdot j}$ is not.

iii) If $-M_{\cdot j}$ is a column of B then $I_{\cdot j}$ is not.

iv) The columns of B are linearly independent.

We note that L(M, q, d) starts with a near complementary basis matrix and generates a sequence of near complementary basis matrices terminating in either a solution to (M, q) or in a secondary ray. Cottle and Dantzig [7, pp.111] proved the following theorem.

Theorem 1.4.1 : (Cottle and Dantzig) When L(M, q,d) is applied to (M, q) the algorithm never terminates in the primary ray. (i.e. the secondary ray and the primary ray are different).

1.4.3. The applicability of L(M, q, d) : It should be noted that in general when L(M, q, d) terminates in a secondary ray no conclusion can be reached about the existence of a solution to (M, q). The procedure terminates indeterminately. We then say that L(M, q, d) is not applicable to (M, q). There are many sufficient conditions on M so that L(M, q, d) either computes a solution to (M, q) or termination in a secondary ray implies that (1.1.1) does not have solution. We note the following theorem proved by M.J. Todd [42, pp.61].

Theorem 1.4.2 (M.J. Todd) : Suppose $M \in \bar{L}(d)$ for some $d > 0 \in R^n$. If L(M,q,d) terminates in a secondary ray (1.1.1) does not have a solution. Suppose $M \in \bar{L}^*(d)$. Then L(M,q,d) never terminates in a secondary ray.

There does not seem to be a simple algebraic proof of the above theorem. However when stated for $L^*(d) \subseteq \bar{L}^*(d)$ the above theorem can be easily proved as demonstrated below.

Theorem 1.4.3. (C.B. Garcia) : Suppose for some $d > 0 \in R^n$, $M \in L^*(d)$. Then $L(M,q,d)$ never terminates a in a secondary ray. See [16, pp.305].

Proof : Suppose for some $d > 0$, $M \in L^*(d)$. We note that (M,d) and $(M, 0)$ have unique solutions.

Suppose $L(M,q^*,d)$ terminates in a secondary ray for some $q^* \in R^n$. Let B be the near complementary basis matrix at termination and let (w_j, z_j) be the pair of nonbasic complementary variables.

Termination in a secondary ray implies that $y = B^{-1}(A_{.j}) \leq 0$ where $A_{.j}$ is either $I_{.j}$ or $-M_{.j}$, the column selected to enter the basis by $L(M,q,d)$ entry criterion. Or

$$\sum_{k=1}^{n} B_{.k}\, y_k = A_{.j} \qquad \dots \quad \dots \quad \dots \qquad (1.4.1).$$

Without loss of generality let us assume that $B_{.1} = -d$.

Case 1 : $y_1 = 0$. We have,

$$\sum_{k=2}^{n} B_{.k}\,(-y_k) = 0$$

which implies that there is a nonzero solution to $(M,0)$ contradicting our hypothesis that $M \in E^*(0)$.

Case 2 : $y_1 < 0$. We get from (1.4.1)

$$d = \sum_{k=2}^{n} \frac{y_k}{y_1} B_{.k} - \frac{1}{y_1} A_{.j}$$

Since the primary ray is different from the secondary ray by theorem 1.4.1. it follows from the above that (M,d) has a solution different from $w = d$, $z = 0$. This contradicts our hypothesis that $M \in E^*(d)$.

Remark 1.4.1 : If $M \in L(d)$ for some $d > 0$ then it is easy to see that (M,d) has unique solution. Therefore if L(M,q,d) terminates in a secondary ray for some $q \in R^n$ then this must imply, in view of theorem 1.4.3, that (M,0) has a nontrivial solution. The conditions imposed on M then ensure that (1.1.1) does not have a solution. It follows from here that L(M,q,d) is applicable to B.C. Eaves' class as well and hence also to matrices which arise in mathematical programming problems and game problems. The following two results are due to Romesh Saigal.

Theorem 1.4.4 : (Romesh Saigal) Let $d > 0$ and suppose that E(M) is regular and nondegenerate. L(M,q,d) is applicable to (M,q) if and only if for each $q \in D(M)$ and for all $0 < \theta < 1$, $\theta d + (1-\theta)q \in D(M)$. Moreover the artificial variable z_0 decreases from iteration to iteration. See [36, pp.52].

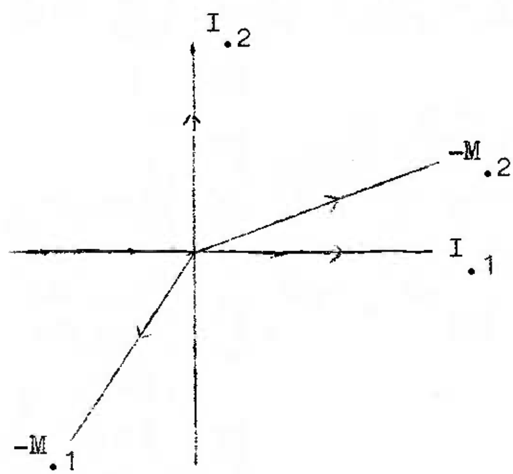
<u>Theorem 1.4.5</u> : (Romesh Saigal) Let $M \in Z$. $L(M,q,e_n)$ is applicable to (M,q). See [34, pp.206].

<u>Remark 1.4.3</u> : Theorem 1.4.4 covers certain matrices not in $\bar{L}$ (d). However not all matrices in $\bar{L}$ (d) possess regular and nondegenerate $E(M)$ as the following example shows.

<u>Example 1.4.1.</u>

$$M = \begin{bmatrix} 2 & -6 \\ 5 & -3 \end{bmatrix}$$

The complementary cones are situated as shown in the following diagram.



From the above diagram it is clear that $(M,0)$ has unique solution and there is a $d > 0 \in R^2$ such that (M,d) has unique solution -

$d \doteq e_2$ is one such. Thus $M \in L(e_2)$. But the cones incident on the (n-1) face $Pos(I_{.1})$ are not properly situated.

1.4.4. The principal pivoting method : The principal pivoting method was first proposed by R.W. Cottle [6] and later extended by Cottle and Dantzig [7, pp.119] to solve (M, q). The steps of the method are as follows. Consider the equation

$$w = Mz + q$$

The initial solution is taken as $z = 0$ and $w = q$. This solves (M, q) if $q \geqq 0$. Otherwise it is a complementary basic solution which is not feasible (i.e. which is not nonnegative).

Step 1 : Terminate if $q \geqq 0$. We have a solution to (M, q). Otherwise go to step 2.

Step 2 : Assume with out loss of generality that $q_1 < 0$. We increase z_1 until it is blocked by a positive basic variable decreasing to zero or by the negative w_1 increasing to zero.

Step 3 : We make the blocking variable nonbasic by pivoting its complement into the basic set. The major cycle is terminated if w_1 drops out of the basic set of variables. Otherwise we return to step 2.

Step 4 : At the end of a major cycle we obtain a new system of equations

$$\bar{w} = \bar{M}\bar{z} + \bar{q}$$

We go to step 1.

Theorem 1.4.6 : (Cottle and Dantzig) The above procedure computes a solution to (M,q) for any arbitrary $q \in R^n$ in a finite number of steps, if $M \in P$.

1.4.5. Saigal Chandrasekaran algorithm : Romesh Saigal [33, p.180] and Chandrasekaran [1, pp.267] have given an algorithm to solve (M, q) when $M \in Z$, which is based on the principal pivoting method described above. The steps of the algorithm are as follows.

We consider the tableau $\{-M, q\}$ and use the symbols $\{-M, q\}$ at every step so that $-M$, q stand for the given matrix $-M$ and given $q \in R^n$ as well as for their principal transforms (i.e. principal pivot transformations).

Step 1 : Let $J = \{i \mid q_i < 0\}$.

If $J = \emptyset$ go to step 5. Otherwise go to step 2.

Step 2 : Let $j \in J$. If $m_{jj} > 0$ go to step 3. Otherwise go to step 4.

Step 3 : Using $-m_{jj} < 0$ as a pivotal element obtain a principal transformation of the tableau. Go to step 1.

Step 4 : Terminate. There is no solution to (1.1.1).

Step 5 : Terminate with the current tableau. The column q of the tableau gives the values of the basic variables in the solution.

The above algorithm is applicable when $M \in Z$. It has been shown that this algorithm computes a solution to (M, q) or detects infeasibility in atmost n iterations if $M \in Z$.

We also note the following theorem proved by various authors.

<u>Theorem 1.4.7</u> : Let either $M \in Z$ or $M \in \bar{L}(d)$. Then (M, q) has a solution whenever (1.1.1) has solution. Thus D(M) is a convex cone with nonempty interior, as $R^n_+ \subsetneq D(M)$ always.

<u>Remark 1.4.4</u> : We say that M is a Q matrix if (M,q) has a solution for each $q \in R^n$ and a K^* matrix if (M,q) has a solution whenever (1.1.1) has a solution. The problem of getting a complete characterisation of Q and K^* matrices is not yet solved. However we see that $\bar{L}^*(d) \subsetneq Q$ for all $d > 0 \in R^n$ and $\bar{L}(d)$ and $Z \subsetneq K^*$.

1.5. Constant parity property :

The problem of determining the number of solutions to (M,q) when $q \in D_2(M)$ or $q \in D_1(M)$ is interesting from the point of view of its relevance to mathematical programming and engineering applications. Also, it was noted in the begining that the classes of matrices such as L(d) or L, for which L(M,q,d) was found applicable, possessed the property of having an odd number of solutions to (M,q) for all $q \in D_2(M)$. (This is not true of $\bar{L}(d)$ or Z). This prompted the study of such properties. See [33, p.176]. In this section we summarise the important results on the number of solutions to (M,q) when $q \in D_2(M)$ or $D_1(M)$. These results are relevant to our study of such properties in Chapter 3.

Theorem 1.5.1. (K.G. Murty) : The number of solutions to (M,q) is finite for all $q \in R^n$ if and only if all the principal minors of M are nonzero. (equivalently if and only if all the complementary cones of (I, -M) have nonempty interior). See [31, p.73].

Theorem 1.5.2. (K.G. Murty) : Suppose $Y \subseteq R^n$ has nonempty interior. Then there is a $q \in Y$ such that q is nondegenerate with respect to M. See [31, p.75].

A number of authors have contributed to the following result.

Theorem 1.5.3 : (M,q) has a unique solution for each $q \in R^n$ if and only if $M \in P$. In terms of complementary cones this means that the complementary cones of (I, -M) form a partition of R^n if and only if $M \in P$. See [7, 18, 31, 38].

<u>Definition 1.5.1</u> : For any nonnegative integer r we say that its parity is odd if it is an odd number, even if it is 0 or an even number. Let $Y \subseteq R^n$. M is said to have constant parity property over the set Y if for all $q \in Y$, (M,q) has a finite number of solutions and this number has the same parity.

The following theorem has been observed by Romesh Saigal in [35, p.43].

<u>Theorem 1.5.4</u> (Romesh Saigal) : Let C(M) be the class of all complementary cones of (I, -M) and their (n-1) faces. For a (n-1) face $F \in C(M)$ let H(F) denote the (n-1) dimensional space which contains F. A necessary and sufficient condition for M to have constant parity property over $D_1(M)$ is that if F be a (n-1) face in C(M) then any q in H(F) which is not in any (n-2) or less faces of the cones in C(M) is contained in an even number of degenerate (n-1) cones lying in H(F).

A number of results observed by many authors follow from this theorem. We however note that the conditions of the theorem are difficult to verify. A sufficient condition for constant parity property of M over $D_1(M)$ is given by the following theorem.

<u>Theorem 1.5.5</u> (S.R. Mohan) : Suppose (M,0) has unique solution. Then for all $q \in D_1(M)$, (M,q) has the same parity of number of solutions. See [29, p.21].

Theorem 1.5.6 (Karamardian) : Let (M, 0) have unique solution and let for some $q^o \in D_1(M)$ there be an odd number of solutions to (M,q^o). Then M has odd parity over $D_1(M)$. See [35, p.40].

Theorem 1.5.7 (K.G. Murty) : Let all the principal minors of M be nonzero. Then M has constant parity property over $D_1(M)$. See [31, p.85] and also see [35, p.44].

Theorem 1.5.8 (R. Saigal) : Let E(M) be regular. When M is not a P matrix and all principal minors of M are nonzero, (M,q) has exactly 2 solutions for all $q \in D_2(M)$. See [36, p.53].

Theorem 1.5.9 (Romesh Saigal) : Suppose there exists a vector $z \geq 0 \in R^n$ such that $z^T M < 0$. Then (M,q) has an even number of solutions for all $q \in D_1(M)$. See [35, p.45].

Theorem 1.5.10 (B.C. Eaves) : Let $M \in P_o$. (M,q) has unique solution for all $q \in D_2(M)$. See [13, p.626].

Theorem 1.5.11 (B.C. Eaves) : Let $M \in L$. (M,q) has an odd number of solutions for all $q \in D_2(M)$. See [13, p.620].

Remark 1.5.1 : From theorem 1.5.11 it follows that the linear complementarity problems arising from linear programming, convex quadratic programming problems and bimatrix game problems all have an odd number of solutions, when they have atleast one solution and all the basic feasible solutions are nondegenerate.

Remark 1.5.2 : There are also Q-matrices which have even parity of number of solutions for all $q \in D_1(M)$. In [31, p.107] K.G. Murty gives an example of such a matrix.

We also note the following theorem which is a consequence of theorems 1.3.8 and 1.5.8.

Theorem 1.5.12 : Let $M \in \bar{Z}$, and let all the principal minors of M be nonzero. If $M \notin K$ then there are exactly two solutions to (M,q) for each $q \in D_2(M)$.

1.6: Solution rays :

In this section we introduce the concept of a ray of solutions and present Cottle's result on the existence of a ray of solutions.

In one of his recent papers R.W. Cottle studies a question posed to him by Maier in a private communication [4, p.60]. The question posed by Maier is as follows:

Let $q \in R^n$, $p \in R^n$ and $\alpha \geqq 0 \in R$. Let M be a given square matrix of order n. Consider the problem $(M, q+\alpha p)$. Suppose M is symmetric and positive semi-definite and $(M, q+\alpha p)$ has solutions $(\bar{w}(\alpha), \bar{z}(\alpha))$ for each α in the interval $[0,\bar{\alpha}]$ but no solution when $\alpha > \bar{\alpha}$. Does there then exist a $\bar{v} \geqq 0 \in R^n$ such that for some $\bar{w}(\lambda) \geqq 0$, $(\bar{w}(\lambda), \bar{z}(\bar{\alpha}) + \lambda\bar{v})$ solves $(M, q+\bar{\alpha}p)$ for each $\lambda \geqq 0$? Can the symmetry assumption be dropped? This question was raised by Maier in the context of structural mechanics and according to him mechanical considerations indicate that the answer is "yes" atleast in the symmetric case.

In his paper Cottle gives affirmative answers to the above questions. In addition he shows that the assumption of positive semi-definiteness can be weakened to the assumption that $M \in C P^+$. When $M \in C P^+$, $D(M)$ is a convex polyhedral cone (Theorem 1.4.7) with nonempty interior. Therefore Maier's question is about $q + \bar{\alpha} p$ which is in the boundary of $D(M)$. Cottle proves the following theorem.

<u>Theorem 1.6.1</u> : Let $M \in C P^+$ and consider (M, q). There exist $\bar{w}(\lambda) \geqq 0$, $\bar{z} \geqq 0$, $\bar{v} \geqq 0$, all $\in R^n$ such that $(\bar{w}(\lambda), \bar{z} + \lambda \bar{v})$ solves (M, q) for each $\lambda \geqq 0 \in R$ if and only if q is in the boundary of $D(M)$. Moreover in the above we may take $(\bar{w}(0), \bar{z})$ as any solution to (M, q).

We note that existence of rays of solutions is a special case of the existence of infinitely many solutions to (M, q). A definite result on the existence of infinite number of solutions to (M, q) is given by theorem 1.5.1. due to K.G. Murty. To the best of our knowledge the only results about the existence of infinitely many solutions and ray of solutions are the ones due to K.G. Murty and

R.W. Cottle quoted above. In chapter 4 we study conditions under which infinitely many solutions and solution rays exist for (M, q) when $M \in Z$.

1.7. A summary of results obtained in this dissertation :

Our results on the linear complementarity problem (M,q) with $M \in Z$ are presented in three chapters. Chapter 2 deals with the computational aspects of (M,q) when $M \in Z$. Chapter 3 considers the problems of constant parity property and of determining the number of solutions to (M,q) for those q for which finitely many solutions to (M,q) exist. Chapter 4 presents results on the existence of infinitely many solutions and solution rays. We give a chapterwise summary of results here.

Chapter 2 :

i) We consider the problem

minimise z_o

subject to $w - Mz - dz_o = q$

$$(w, z, z_o) \geqq 0$$

We show that when the ordinary simplex method is applied to the above problem when (1.1.1) has a solution, the sequence of basic feasible solutions generated by the simplex method and the sequence of near complementary solutions generated by L(M,q,d) for any $d > 0 \in R^n$ are the same. When (1.1.1) has a solution the simplex method finds it and so does L(M,q,d) in atmost n iterations.

ii) Based on the above result we describe a new algorithm to solve (M,q) when $M \in Z$. This we call S(M,q,d).

iii) When (1.1.1) does not have solution S(M,q,d) never terminates later than L(M,q,d), but in some problems may terminate earlier. S(M,q,d) always terminates in atmost n iterations whereas in some problems L(M,q,d) may require more tham n iterations to terminate.

iv) When (1.1.1) does not have solution there are problems for which S(M,q,d) terminates earlier than Chandrasekaran - Saigal pivot algorithm presented in 1.4.5. But there are also problems where Chandrasekaran - Saigal algorithm terminates earlier than S(M,q,d).

v) The solutions generated by S(M,q,d) have isotonicity property. Also S(M,q,d) obtains the least element of $X(q) = \{ z \mid Mz \geqq -q, z \geqq 0 \}$; Whatever $d > 0 \in R^n$ be considered, when (1.1.1) has solution, S(M,q,d) and Chandrasekaran Saigal algorithm require ~~the~~ same number of iterations to find the solution.

vi) Let $(M, q)_a$ be the problem

Find (w, z) such that

$$w - Mz = q$$
$$w \geqq 0, \quad z \geqq 0, \quad z \leqq a$$
$$w^T z = 0.$$

where $a > 0 \in R^n$ is given. Let $R(a,M) = \{ q \mid (M,q)_a$ has a solution $\}$. We show that if $M \in Z$, $R(a,M)$ is convex for each $a \in R^n$ and a modification of the algorithm $S(M,q,d)$ solves $(M,q)_a$.

Chapter 3 : Let $M \in Z$.

(i) A necessary and sufficient condition for $(M,0)$ to have a unique solution (only the trivial solution) is that there is no $x \geq 0 \in R^n$ such that $Mx = 0$. (This is not true in general). Thus a sufficient condition for M to have constant parity property over $D_1(M)$ is that there is no $x \geq 0 \in R^n$ such that $Mx = 0$.

(ii) If for some $q < 0 \in R^n$, (M,q) has a solution then $M \in K$.

(iii) If M is nonsingular and if for some $q > 0 \in R^n$, (M,q) has a unique solution then $M \in K$.

(iv) Let $M \in K_0 - K$. (i.e. all the principal minors of M are nonnegative and there is atleast one principal minor which is 0). Then there exists $x \geq 0 \in R^n$ such that $Mx = 0$.

(v) We then consider the case when there is nontrivial solution to (M,0). For Z matrices we are able to give some sufficient conditions on M so that constant parity property holds. These conditions are in terms of the representation of M in the partitioned form and are easily verifiable. We also obtain a necessary condition on M for it to have constant parity property over $D_1(M)$.

(vi) Let $M \in \bar{Z}$, and let N(q) be the number of solutions to (M, q). If $N(q) < \infty$ then $N(q) \leqq 2$. This is a generalisation of the result of theorem 1.5.12.

Chapter 4 :

(i) First we obtain some results on the representationof M in the partitioned form when $M \in K_0 - K$. Using these forms we introduce the class of matrices $\bar{K}_0$.

(ii) We prove some properties of $\bar{K}_0$ matrices such as if $M \in \bar{K}_0$ then $M^T \in \bar{K}_0$, if $M \in \bar{K}_0$ then for any $L \subsetneq N$, $M_L \in \bar{K}_0$.

(iii) We show that if $M \in \bar{K}_0$ then a ray of solutions exists for (M,q) at some solution $(\bar{w}, \bar{z})$ to (M,q) if and only if q is in the boundary of D(M).

(iv) If $M \in K_0$ and at some solution $(\bar{w}, \bar{z})$ to (M,q) a ray of solutions existsthen q is in the boundary of D(M).

(v) In terms of the representation of M in the partitioned forms obtained by us we prove a necessary condition and a sufficient condition on $M \in Z$ so that no q in the interior of D(M) possesses a solution ray.

(vi) For a symmetric Z-matrix M, (M,q) has a ray of solution only if q is in the boundary of D(M). If $M \in K_0$ and symmetric then at each solution $(\bar{w}, \bar{z})$ to (M, q), the problem has a solution ray, for all those q which are in the boundary of D(M).

(vii) If $M \in K_0$, (M,q) has infinitely many solutions if and only if q belongs to a complementary cone of (I, -M) whose interior is empty (The "if part" is not true in general).

(viii) If $M \in K_0$, (M,q) has infinitely many solutions if and only if q is in the boundary of D(M). Thus the complementary cones with empty interior constitute the boundary of D(M).

Most of the results appearing in Chapter 2 are already published in [28]. Those of Chapter 3 are published in [29]. Some of the results appearing in Chapter 4 are due to appear in [30].

2. Computational methods for Z - matrices

2.1. Introduction :

In this Chapter we discuss the various algorithms available for solving (M,q) when $M \in Z$. The first to propose an algorithm for this class of matrices was R. Chandrasekaran [1]. His algorithm was later modified by Romesh Saigal who called it a greedy algorithm. In section 1.4.5. we presented a modification of their algorithms. In [34] Romesh Saigal showed that $L(M,q,e_n)$ is applicable to (M,q). This was presented in theorem 1.4.5. The main interest in this chapter is in obtaining a modification of the algorithm L(M,q,d) so that the modified algorithm becomes more efficient than L(M,q,d) for any $d > 0$. In order to do this we show that solving (M, q) when $M \in Z$ is equivalent to solving the following problem using the simplex method.

$$\text{Minimise} \quad z_o$$

$$\text{subject to} \quad w - Mz - dz_o = q$$

$$(w, z, z_o) \geqq 0$$

This result is proved in section 2.2.

In section 2.3 we compare our method with L(M,q,d) and the principal pivoting method of Saigal and Chandrasekaran.

In section 2.4 we summarise the recent results obtained by A. Tamir [39], I. Kaneko [19] and R.W. Cottle [3] on the least element and isotonicity properties of solutions to (M,q) when $M \in Z$. We show how our results are related to these.

Most of the results appearing in this chapter are published in [28].

2.2. The simplex method for (M, q) when $M \in Z$:

We can attempt to solve the system of equations (1.1.1) and (1.1.2) by applying the simplex method to the problem

$$\begin{aligned} &\text{Minimise} \quad z_o \\ &\text{Subject to} \quad w - Mz - dz_o = q \\ &\qquad\qquad (w, z, z_o) \geqq 0 \qquad \ldots \quad \ldots (2.2.1) \end{aligned}$$

where $d > 0 \in R^n$.

The initial basic feasible solution can be taken as in Lemke's algorithm L(M,q,d). The difference between these two methods essentially lies in the choice of variable to enter the basis at each iteration. The criterion for the choice of variable to be removed from the basis is the same. In general therefore the simplex method applied to (2.2.1) does not necessarily obtain a near complementary basic feasible solution at each iteration and therefore may not solve (1.1.1) and (1.1.2).

Given (M, q) and $d > 0 \in R^n$ we shall assume without loss of generality that $\frac{q_n}{d_n} = \min \left[\frac{q_j}{d_j}\right]$. With this assumption $L(M,q,d)$ when applied to (M, q) generates before terminating, a sequence of near complementary basic feasible solutions and a corresponding sequence of near complementary basis matrices the last columns of which are $-d$, if $q_n < 0$. In [34, p.203] Romesh Saigal observes the following lemma.

Lemma 2.2.1 : Let $M \in Z$. Consider $L(M,q,d)$ for (M,q). Let B be a near complementary basis matrix generated by the algorithm. Then B can be written in the form

$$C = \begin{bmatrix} I & D_1 & -\bar{d}^1 \\ 0 & D_2 & -\bar{d}^2 \\ 0 & g & -\bar{d}_n \end{bmatrix}$$

where C is obtained from B, if necessary by a principal rearrangement of rows and columns of B and $\bar{d}$ is obtained from d by the corresponding permutation of its coordinates. The first set of columns corresponds to the w_i variables in the basis, the second set corresponds to columns of $-M$ in the basis and the last column is $-\bar{d}$. Any one of the first two sets of columns may be empty.

Notation 2.2.1 : Let $\bar{M}$ be the matrix obtained by applying to M the same principal rearrangement as was applied to B in obtaining C. We note that the last column of $-\bar{M}$ is $-\bar{M}_{.n}$ and $-\bar{M}_{.n}$ and $I_{.n}$ are the pair of nonbasic complementary columns. We continue to use the symbol z_n for the variable corresponding to $-\bar{M}_{.n}$. We also note that g is a row vector of order $1 \times n-k-1$ where k is the order of I in C. The coordinates of g are $(-\bar{m}_{n\,k+1}, \ldots, -\bar{m}_{n\,n-1})$ where $\bar{m}_{ij}$ denotes the ij^{th} element of $\bar{M}$. We note the following trivial lemma.

<u>Lemma 2.2.2</u> : Let $M \in Z$ and let C be as in lemma 2.2.1. and $\bar{M}$ as in notation 2.2.1. If either (a) the second set of columns in C is empty so that $C = \begin{bmatrix} I & -d^1 \\ 0 & -d_n \end{bmatrix}$ or (b) $-D_2$ is a P-matrix then

i) $h \leq 0$ where h is the last row of C^{-1}

ii) $h (-\bar{M}_{.j}) \leqq 0$ for all $j \neq n$.

Proof :- Let k be the number of w_i variables in the basis (i.e. k is the order of I in C).

If case (a) holds then note that

$$C^{-1} = \begin{bmatrix} I & -\frac{d^1}{d_n} \\ 0 & -\frac{1}{d_n} \end{bmatrix}$$

so that the last row $h \leq 0$.

If case (b) holds then $k < n-1$ and $C^{-1}C = I$ gives us the equations

$$h\, I_{\cdot j} = 0, \quad j = 1,2,\ldots, k \quad \text{which} \Longrightarrow h_j = 0, \quad j = 1,2,\ldots, k$$

$$\sum_{i=k+1} h_i\,(-\bar{d}_i) = 1$$

and $\quad h^*(-D_2) = h_n g \quad$ where $h^* = (h_{k+1} \ldots\ldots h_{n-1})$

Also note that $g \geqq 0$ since $M \in Z$ and $-\bar{m}_{nn}$ is not a coordinate of g.

Since $-D_2$ is a P-matrix from theorem 1.3.2 it follows that $(-D_2)^{-1} \geq 0$ and therefore all the coordinates of h^* have the same sign as h_n. From here it follows that $h \leq 0$.

Since $h \leq 0$, $h(I_{\cdot j}) \leqq 0$ for all $j = 1,2,\ldots\ldots n$. Now let $j \neq n$ be such that $-\bar{M}_{\cdot j}$ is not a column of C. It follows that $I_{\cdot j}$ is a column of C, $1 \leqq j \leqq k$ and that $h_j = 0$. Therefore, we have

$$h(-\bar{M}_{\cdot j}) = \sum_{i \neq j} h_i\,(-\bar{m}_{ij}) \leqq 0 \quad \text{as} \quad -\bar{m}_{ij} \geqq 0 \quad \text{for } i \neq j.$$

Also it is clear that if $j \neq n$ is such that $-\bar{M}_{\cdot j}$ is a column of C, then $h(-\bar{M}_{\cdot j}) = 0$. This completes the proof of (ii) in both of the cases (a) and (b).

<u>Theorem 2.2.1</u> : Let $M \in Z$. The simplex method applied to solve problem (2.2.1) solves (1.1.1) and (1.1.2) or shows that no solution exists to

(1.1.1). Moreover the sequence of basic feasible solutions generated by the simplex method applied to problem (2.2.1) is the same as the sequence of near complementary basic feasible solutions generated by L(M,q,d) applied to solve (M,q), if (1.1.1) has a solution.

Proof : If (1.1.1) has no solution then the simplex method will find an optimal solution to (2.2.1) with $z_0 > 0$ and this will show that (1.1.1) has no solution.

Let (1.1.1) have solution. Note that the simplex method and L(M,q,d) starts with the same initial near complementary basic feasible solutions. (i.e. the solutions that correspond to near complementary basis matrices). Also initially

$$C_1 = \begin{bmatrix} I & -d^1 \\ 0 & -d_n \end{bmatrix}; \quad \text{so that } k\text{, the order of } I \text{ in } C_1 \text{ is}$$

(n-1) and the second set of columns is empty. From lemma 2.2.2 it follows that $h \leq 0$ and $h(-\bar{M}_{.j}) \leqq 0$ for $j \neq n$ where h is the last row of C_1^{-1}.

We note that the simplex multipliers ('$z_j - {}_j$' in the standard notation) corresponding to the variables w_j's are $h(I_{.j})$'s and the variables z_j's are $h(-\bar{M}_{.j})$'s which are all negative except possibly for $h(-\bar{M}_{.n})$. Since (1.1.1) has solution the simplex method applied to (2.2.1) can not terminate with C_1. Therefore $h(-\bar{M}_{\cdot n}) > 0$ and both the simplex method and Lemke's algorithm $L(M,q,d)$ select the same column $-\bar{M}_{\cdot n}$ to be included in the basis at the next iteration, each according to its own criterion.

Let $D_2(2)$ be the D_2 of C_2, the second near complementary or complementary basis matrix (depending on which variable is eliminated from the basis in obtaining C_2) which is the same for both $L(M,q,d)$ and the simplex method. As we have observed in the previous paragraph $D_2(2)$ is of order 1×1 and is $-\bar{m}_{nn} < 0$. Therefore $-D_2(2)$ is a P-matrix of order 1×1.

Induction hypothesis : Let the first s near complementary basis matrices generated by $L(M,q,d)$ and the simplex method applied to problem (2.2.1) be the same. Let $D_2(r)$ be the D_2 of C_r and let $-D_2(r)$ be a P-matrix for $r \leq s$.

We shall show that the $(s+1)^{th}$ near complementary (or complementary) basis matrices generated by the simplex method and $L(M,q,d)$ are the same and that $-D_2(s+1)$ is a P-matrix.

If in the basic feasible solution corresponding to C_s, $z_o = 0$ then both $L(M,q,d)$ and the simplex method obtain a degenerate solution to (M,q) and terminate with C_s. In this case therefore the theorem holds. Let us therefore assume that in the solution corresponding to C_s, $z_o > 0$. Since (1.1.1) has solution this means that the simplex method does not terminate with C_s. Since $-D_2(s)$ is a P-matrix lemma 2.2.2 applies. In the notation of 2.2.1 we shall have,

$$h\,(I_{.j}) \leqq 0, \quad j = 1,2,\ldots\ldots n, \quad h(-\bar{M}_{.j}) \leqq 0 \quad \text{if} \quad j \neq n.$$

Therefore we must have $h(-\bar{M}_{.n}) > 0$.

Thus the column to enter the basis under the simplex criterion is $-\bar{M}_{.n}$ . ... (2.2.2)

$(I_{.n}, -\bar{M}_{.n})$ are the pair of nonbasic complementary columns in notation 2.2.1. $L(M,q,d)$ must also select $-\bar{M}_{.n}$ to be included in the basis in the next iteration. For, if otherwise $-\bar{M}_{.n}$ must have been eliminated from C_{s-1} which is incompatible with (2.2.2) in view of the induction hypothesis, as a variable eliminated in an iteration can not reenter the basis in the immediate next iteration under the simplex criterion.

Thus C_{s+1} generated by $L(M,q,d)$ and the simplex method are the same.

To complete the proof we must show that $-D_2(s+1)$ is a P-matrix. We proceed as follows:

Let $y = C_s^{-1}\,(-\bar{M}_{.n})$. We have already seen that $y_n > 0$. We note that $y_{k+1}, \ldots\ldots, y_n$ satisfy the equations

$D_2(s)\, f = a + \bar{d}^{\,2}\, y_n$ where $f = \begin{bmatrix} y_{k+1} \\ \vdots \\ y_{n-1} \end{bmatrix}$. and

$a^T = (-\bar{m}_{k+1\,n}, \ldots\ldots, -\bar{m}_{n-1\,n})$.

Since $-D_2(s)$ is a P-matrix by induction hypothesis and since $a \geq 0$, $y_n > 0$, $\bar{d}^{\,2} > 0$ by theorem 1.3.2, we have $f \leqq 0$. Thus $C_s^{-1}(-\bar{M}_{.n})_i \leqq 0$ for $k+1 \leqq i \leqq n-1$.

This shows that in obtaining C_{s+1} simplex method eliminates from C_s a column $I_{.t}$, $t \leqq k$ and not any column $-\bar{M}_{.j}$ in C_s. Thus $-D_2(s+1)$ is equal to

$$\begin{bmatrix} -D_2(s) & -a \\ -g & \bar{m}_{nn} \end{bmatrix}$$

We also note that

$$-D_2(s+1)\begin{pmatrix} -f \\ 1 \end{pmatrix} = \begin{bmatrix} \bar{d}^{\,2} \\ \bar{d}_n \end{bmatrix} y_n$$

where $\begin{pmatrix} -f \\ 1 \end{pmatrix} \geq 0$, $y_n > 0$, $\begin{bmatrix} \bar{d}^{\,2} \\ \bar{d}_n \end{bmatrix} > 0$.

Therefore from theorem 1.3.2 it follows that $-D_2(s+1)$ is a P-matrix.

The following corollary is an immediate consequence.

Corollary 2.2.1 : Let $M \in Z$ and (1.1.1) have solution. Let B_1, B_2, ..., B_s be the sequence of near complementary basic feasible solutions generated by L(M,q,d); for any $d > 0$. Then (a) in the representation C_s of B_s as in lemma 2.2.1 $-D_2(s)$ is a P-matrix; (b) $(C_s^{-1}(-\bar{M}_{.n}))_i \leqq 0$, $k+1 \leqq i \leqq n-1$ where k is the order of I in C_s and $-\bar{M}_{.n}$ is, as in notation 2.2.1, the column selected to be included in the basis in the next iteration. Also L(M,q,d) solves (M, q).

Corollary 2.2.2 : When the simplex method is applied to problem (2.2.1) a nonbasic w_j never becomes a basic variable and a basic z_j never becomes nonbasic.

Proof : The proof of theorem 2.2.1 shows that when the simplex method is applied to problem (2.2.1) the simplex multipliers corresponding to w_j's , j = 1,2, ... n are always nonpositive. Therefore a nonbasic w_j never becomes a basic variable. Corollary 2.2.1 shows that the variables z_j's are nondecreasing from iteration to iteration. Therefore a basic z_j never becomes nonbasic.

Corollary 2.2.3 : The simplex method applied to problem (2.2.1) finds a solution to (M,q) or shows that (1.1.1) does not have solution in atmost n iterations. If (1.1.1) has a solution L(M,q,d) determines a solution to (M,q) in atmost n iterations.

Proof : The simplex method applied to problem (2.2.1) replaces one after another of the variables w_j's by z_j's. Also a nonbasic w_j never enters the basis again. Therefore the algorithm must terminate in atmost n iterations. Since when (1.1.1) has a solution L(M,q,d) is equivalent to simplex method, L(M,q,d) also finds a solution to (M,q) in atmost n iterations.

Remark 2.2.1 : When (1.1.1) does not have solutions L(M,q,d) and the simplex method applied to (2.2.1) are not equivalent. We shall illustrate this with an example in the next section.

Corollary 2.2.4 : When the simplex method applied to problem (2.2.1) terminates with $z_0 > 0$ the row corresponding to z_0 in the terminal tableau contains no positive element except in the column corresponding to z_0.

Proof : We observe that the row corresponding to z_0 in the tableau contains the simplex multipliers.

Remark 2.2.2 : In a recent paper C.B. Garcia [17] observes that when $M \in L$ and when L(M,q,d) applied to (M,q) terminates in a secondary ray then the row corresponding to z_0 in the terminal tableau contains no positive element. Corollary 2.2.4 is comparable to this result. However, we shall show in the next section that corollary 2.2.4 does not hold for L(M,q,d) applied to (M,q) when $M \in Z$.

2.3. Comparisions among the algorithms to solve (M,q) when $M \in Z$:

Based on the results of the previous section we present the following algorithm for (M,q) when $M \in Z$. We call this algorithm $S(M,q,d)$.

The algorithm $S(M,q,d)$: Let $d > 0 \in R^n$ be given.

Step 1 : Given (M,q) choose minimum nonnegative z_o so that $z_o d + q \geqq 0$. If $z_o = 0$ terminate with the solution $w = q$, $z = 0$. This solves (M,q). Otherwise go to step 2.

Step 2 : Suppose $(q + z_o d)_\alpha = 0$ for some $1 \leqq \alpha \leqq n$. Choose the initial basic near complementary solution as $w = q + z_o d$, $z = 0$ and $z_o > 0$ as in step 1. The initial near complementary basis matrix is the matrix B whose j^{th} column is $I_{.j}$ for $j \neq \alpha$ and whose α^{th} column is $-d$. The columns of the initial tableau are $B^{-1}(I_{.j})$, $B^{-1}(-M_{.j})$, $B^{-1}(-d)$ and $B^{-1}(q)$. We note that B^{-1} initially is of the form

$$\begin{bmatrix} I & \dfrac{-d^1}{d_n} \\ 0 & -\dfrac{1}{d_n} \end{bmatrix}.$$

Step 3 : At any iteration we obtain a nonbasic pair of complementary variables. (In the first iteration (w_α, z_α)). Let it be

(w_β, z_β). If in the α^{th} row the column $B^{-1}(-M_{.\beta})$ contains a nonpositive element then terminate. There is no feasible solution to (1.1.1). Otherwise go to step 4.

Step 4 : Choose z_β as the variable to be introduced into the basic set. Select the variable to be eliminated from the basic set by the usual minimum ratio criterion. Carry out the usual pivotal transformation of the simplex method. Go to step 5.

Step 5 : If in step 4 z_o is the variable selected to be eliminated from the basic set, terminate. The current solution does not contain z_o and solves (M,q). Otherwise go to step 3.

Remark 2.3.1 : We note that since the problem (2.2.1) always has an optimal solution, S(M,q,d) does not encounter the 'unboundedness' case of the simplex method at step 4. We also note that it is not necessary to maintain columns corresponding to w_j variables in the tableau. Also, since a basic z_j never becomes nonbasic and a nonbasic w_j never enters the basic set it is not necessary to introduce any nondegeneracy resolving mechanism. This observation holds true also for L(M,q,d).

As we have seen in the previous section if (1.1.1) has a solution then S(M,q,d) and L(M,q,d) generate the same sequence of basic feasible solutions. However if (1.1.1) does not have solution the two methods are not equivalent. The following examples illustrate this.

Example 2.3.1 : Consider (M,q) where

$$M = \begin{bmatrix} 5 & -2 & -3 & -1 \\ -3 & 6 & -1 & -2 \\ -2 & -4 & 3 & -2 \\ -1 & -3 & -4 & 1 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 \\ -2 \\ -3 \\ -4 \end{bmatrix}$$

$$\text{Take } d = e_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

At the end of the second iteration both S(M,q,d) and L(M,q,d) obtain the following tableau.

w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_o	sol	Basic variables
1	0	-2/3	1/3	-20/3	-5/3	11/3	0	0	13/3	w_1
0	1	-1	0	1	-10	4	0	0	1	w_2
0	0	1/3	-1/3	1/3	1/3	-7/3	1	0	1/3	z_4
0	0	-1/3	-2/3	- 4/3	-10/3	-25/7	0	1	11/3	z_o

At this stage both $S(M,q,e_4)$ and $L(M,q,e_4)$ determine the variable to be included in the basic set as z_3. However $S(M,q,e_4)$

terminates at this stage, since the element in the z_3 column and the z_o row in the tableau is $-25/7$, negative. $L(M,q,e_4)$ at this stage does not terminate in secondary ray. In fact $L(M,q,e_4)$ takes two more iterations before terminating in a secondary ray.

We also note that all the elements in the z_o row of the current tableau except for the 1 in the z_o- column are negative.

Theorem 2.3.1 : If (1.1.1) does not have solutions $S(M,q,d)$ detects this infeasibility never later than $L(M,q,d)$ does.

Proof : The termination requirement for $S(M,q,d)$ is only that $(B^{-1}(-\bar{M}_{.n}))_n \leqq 0$ where $\bar{M}_{.n}$ is as given by notation 2.2.1. On the otherhand the termination rule for $L(M,q,d)$ requires that $(B^{-1}(-\bar{M}_{.n}))_i \leqq 0$ for $1 \leqq i \leqq n$. This concludes the proof.

Remark 2.3.2 : When $L(M,q,d)$ terminates in a secondary ray for the problem in example 2.3.1, the terminal tableau has positive elements in the row corresponding to z_o. This is because the terminal solution is not an optimal solution for problem (2.2.1), with M and q as in example 2.3.1.

Example 2.3.2 : Consider (M, q) where

$$M = \begin{bmatrix} 1 & -4 & -2 & -3 \\ -4 & 1 & -4 & -2 \\ -1 & -1 & 5 & -2 \\ -1 & -1 & -3 & 6 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 \\ 1 \\ -3 \\ -4 \end{bmatrix}$$

Take $d = e_4$

For this problem $L(M,q,e_4)$ takes 6 iterations (more than $n = 4$) to terminate in a secondary ray. $S(M,q,e_4)$ detects that (1.1.1) does not have solution in 3 iterations. Thus we see that $L(M,q,d)$ may require more than n iterations to terminate.

We shall show in the next section that whatever may be the given $d > 0 \in R^n$, $S(M,q,d)$ or $L(M,q,d)$ require the same number of iterations to solve (M,q) if (1.1.1) has solutions. This however is not true if (1.1.1) does not have solutions. The following example illustrates this.

Example 2.3.3 : Consider (M,q) where

$$M = \begin{bmatrix} 1 & -2 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & -1 & -3 & 1 \\ -1 & -1 & 1 & -4 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} -1 \\ -1 \\ -2 \\ -3 \end{bmatrix}$$

$S(M,q,e_4)$ takes 4 iterations before terminating with $z_0 > 0$. However if we take $d = \begin{bmatrix} 1 \\ 2 \\ 10 \\ 10 \end{bmatrix}$, $S(M,q,d)$ requires only 2 iterations before concluding that (1.1.1) does not have solution. Note that even if (1.1.1) has solutions the sequences of basic near complementary solutions generated by $S(M,q,d)$ for different $d \in R^n$ may not be the same.

We shall now compare our algorithm with the Chandrasekaran - Saigal algorithm presented in section 1.4.5. We shall make the comparision only when (1.1.1) does not have solutions. In the next section it will be shown that when (1.1.1) has solution $S(M,q,d)$ and the Chandrasekaran - Saigal algorithm require the same number of iterations, although the sequence of near complementary basis matrices generated may be different.

We shall slightly modify the algorithm presented in 1.4.5 as follows.

(i) In step 2 consider all $j \in J$ and if for any $j \in J$, $m_{jj} \leqq 0$ go to step 4.

(ii) In step 3 choose the pivotal element as m_{kk} if $k \in J$ and $q_k = \min_{j \in J} q_j$.

Modification (i) above helps faster termination in case (1.1.1) has no solution. Modification (ii) removes the arbitrariness in the selection of pivotal element in step 3.

Example 2.3.4 : Consider (M,q) where

$$M = \begin{bmatrix} 3 & -1 & -8 & 0 & 0 & 0 \\ -2 & 2 & -6 & 0 & 0 & 0 \\ -1 & -2 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -1 & -3 \\ 0 & 0 & 0 & -1 & 4 & -2 \\ -4 & 0 & 0 & -1 & -1 & 3 \end{bmatrix} ; \quad q = \begin{bmatrix} -5 \\ -2 \\ -1 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

$S(M,q,e_6)$ terminates in the 4^{th} iteration concluding that (1.1.1) has no solution. However the Chandrasekaran - Saigal algorithm with the modifications (i) and (ii) above terminates only after 6 pivotal iterations. The choice of pivotal element under $S(M,q,e_6)$ differs from that of Chandrasekaran - Saigal method from the second iteration onwards.

However we can also construct examples to show that $S(M,q,e_n)$ does worse than Chandrasekaran-Saigal method in detecting infeasibility. Thus the two methods are not comparable.

Example 2.3.5 : Consider (M,q) where

$$M = \begin{bmatrix} 5 & -1 & -3 & 0 & 0 & 0 \\ -1 & 4 & -2 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 & -8 \\ 0 & 0 & 0 & -2 & 2 & -6 \\ -4 & 0 & 0 & -4 & -2 & 8 \end{bmatrix} ; \quad q = \begin{bmatrix} -5 \\ -2 \\ -1 \\ -1 \\ -2 \\ -1 \end{bmatrix}$$

$S(M,q,e_6)$ takes 6 iterations to detect that (1.1.1) has no solution. The Chandrasekaran - Saigal algorithm takes only 3 iterations to reach the same conclusion.

The isotonicity and least element properties of solutions generated by $S(M,q,d)$:

Consider the parametric linear complementarity problem $(M, q + \alpha p)$ where α is a nonnegative real number. Suppose $(M, q + \alpha p)$ has unique solution for $0 \leq \alpha < \infty$ and let $(\bar{w}(\alpha), \bar{z}(\alpha))$ be the solution. We say that the solutions possess isotonicity property if $\bar{z}_i(\alpha)$ is a nondecreasing function of α for each $i = 1,2, \ldots\ldots n$. In the context of structural mechanics Maier [25] has posed the problem of determining conditions on M, p, and $q \geq 0 \in R^n$ so that isotone solutions exist for $(M, q + \alpha p)$, $0 \leq \alpha < \infty$. In [3] Cottle considers the above problem for positive semi-definite matrices M and proves that for each $q \geq 0 \in R^n$, $p \in R^n$ $(M, q + \alpha p)$ has isotone solutions if and only if $M \in K$. He also provides a monotonicity checking algorithm for positive semidefinite matrices. Further he observes that if $M \in K^*$ to determine $\max_{\alpha \geq 0} \{\alpha \mid (M, q + \alpha p) \text{ has solutions}\}$ one need only solve the following linear programming problem.

$$\text{Maximise} \quad \alpha$$

$$\text{Subject to} \quad \alpha p + Mz \geq -q$$

$$\alpha \geq 0, \quad z \geq 0.$$

The isotonicity property of solutions to $(M, q + \alpha p)$ when $M \in Z$ has been considered by Kaneko [19]. Because of the possibility of nonuniqueness or nonexistence of solutions to (M, q) when $M \in Z$ the definition of isotonicity is modified as follows. $(M, q + \alpha p)$ for some $q \geqq 0 \in R^n$ and $p \in R^n$ is said to have isotone solutions if for $0 \leqq \alpha_1 < \alpha_2$, $(M, q + \alpha_1 p)$ and $(M, q + \alpha_2 p)$ have solutions imply that they have solutions $(\bar{w}(\alpha_1), \bar{z}(\alpha_1))$ and $(\bar{w}(\alpha_2), \bar{z}(\alpha_2))$ respectively such that $\bar{z}(\alpha_1) \leqq \bar{z}(\alpha_2)$. I. Kaneko [19, p.15] shows that if $M \in Z$, $(M, q + \alpha p)$ has isotonicity property for each $q \geqq 0 \in R^n$ and $p \in R^n$. These results have further been entended by Kaneko [20] and have been used to characterise the classes Z and K.

In relation to these we observe the following.

<u>Remark 2.4.1</u> : Let $q > 0 \in R^n$, $p \in R^n$ be given. We note that when $S(M, p, q)$ is applied to solve (M, p) the algorithm generates solutions to $(M, z_o q + p)$ for $z_o^* \leqq z_o < \infty$, where z_o^* is the minimal value of z_o obtained by the algorithm for problem (2.2.1). Suppose $(\bar{w}(z_o), \bar{z}(z_o))$ is a solution to $(M, z_o q + p)$ for $z_o^* \leqq z_o < \infty$. Note that $(\frac{1}{z_o} \bar{w}(z_o), \frac{1}{z_o} \bar{z}(z_o))$ solves $(M, q + \alpha p)$ where $\alpha = \frac{1}{z_o}$, so that for $0 \leqq \alpha < \frac{1}{z_o^*}$ we can get solutions to $(M, q + \alpha p)$. We call these solutions the solutions generated by $S(M, p, q)$ to $(M, q + \alpha p)$ for $0 \leqq \alpha < \frac{1}{z_o^*}$. (If $z_o^* = 0$, we get a solution to $(M, q + \alpha p)$ for $0 \leqq \alpha < \infty$).

Theorem 2.4.1 : Let $q > 0 \in R^n$, $p \in R^n$ be given. The solutions generated by $S(M, p, q)$ to $(M, q + \alpha p)$ for $0 \leqq \alpha \leqq \frac{1}{z_o^*}$ (or $0 \leqq \alpha < \infty$, if $z_o^* = 0$) where z_o^* is the value of z_o at the termination of the algorithm are isotone. If $z_o^* \neq 0$, $(M, q + \alpha p)$ does not have solutions for $\alpha > \frac{1}{z_o^*}$.

Proof :- Let at the m^{th} iteration $S(M, p, q)$ generate the solution $(\bar{w}(m), \bar{z}(m))$ to $(M, z_o(m)\, q + p)$. Without loss of generality let us assume that $\bar{w}_j(m)$, $1 \leqq j \leqq k$, $\bar{z}_j(m)$, $k+1 \leqq j \leqq n-1$ are the basic variables. Let z_n be selected to be included in the basis at the next iteration and let w_i, $1 \leqq i \leqq k$ be the variable to be excluded from the basis. Let $y_{.n}$ denote the column in the $S(M,p,q)$ tableau of the m^{th} iteration corresponding to the variable z_n. We note that

$$
\begin{aligned}
w_r^*(\alpha) &= \alpha(\bar{w}_r(m) - \theta y_{rn}), && 1 \leqq r \leqq k \\
&= 0, && k+1 \leqq r \leqq n \\
z_r^*(\alpha) &= \alpha(\bar{z}_r(m) - \theta y_{rn}), && k+1 \leqq r \leqq n-1 \\
z_n^*(\alpha) &= \alpha\theta && \\
z_r^*(\alpha) &= 0, && 1 \leqq r \leqq k
\end{aligned}
$$

where $\theta = (z_o(m) - \frac{1}{\alpha}) \cdot \frac{1}{y_{nn}}$ and $\frac{1}{z_o(m)} \leqq \alpha < \frac{1}{z_o(m+1)}$ solves $(M, q + \alpha p)$ for $\frac{1}{z_o(m)} \leqq \alpha < \frac{1}{z_o(m+1)}$.

The isotonicity of $z^*(\alpha)$ now follows from corollary 2.2.1. as $y_{rn} \leqq 0$, $k+1 \leqq r \leqq n-1$. (If there is degeneracy $z_0(m) = z_0(m+1)$). If $z_0(m+1) = 0$ then we have isotone solutions for $\frac{1}{z_0(m)} \leqq \alpha < \infty$. This completes the proof of the theorem.

Let $X(q) = \{z \mid Mz \geqq -q, \quad z \geqq 0\}$; for a given $q \in R^n$. We say that $z^* \in X(q)$ is a least element of $X(q)$ if $z_i^* \leqq z_i$ $i = 1,2, \ldots n$ for any $z^T = (z_1, \ldots, z_n) \in X(q)$. We note that if $X(q)$ possesses a least element then it is unique. In [10, p.246] Cottle and Veinott showed that the least element of $X(q)$ is a solution to (M, q) if and only if $M \in K$. A. Tamir [39, p.28] extended this result by showing that M is a Z matrix if and only if when $X(q) \neq \emptyset$, $X(q)$ has a least element which is a solution to (M,q).

In relation to the above we observe the following.

<u>Theorem 2.4.2</u> : Suppose $X(q) \neq \emptyset$. Then for any $d > 0$, $S(M,q,d)$ computes the least element of $X(q)$ in atmost n iterations.

Proof : $S(M,q,d)$ terminates with a solution to (M,q) since $X(q) \neq \emptyset$. We shall show that this solution also solves the problem

$$\text{minimise} \quad \Sigma\, c_i z_i$$

$$\text{subject to} \quad w - Mz = q, \quad w \geqq 0, \quad z \geqq 0$$

for any $(c_1 \ldots\ldots c_n) \geq 0$.

Let us assume without loss of generality that at the terminal tableau of $S(M,q,d)$, w_j, $1 \leqq j \leqq k$, z_j, $k+1 \leqq j \leqq n$ are the basic variables. Then the basis matrix corresponding to the terminal tableau is

$$B = \begin{bmatrix} I & D_1 \\ 0 & D_2 \end{bmatrix}$$

where I is of order k. Let $f = (f_1, \ldots, f_n)$ be the cost coefficients of the basic variables. Note that $f_i = 0$ if $1 \leqq i \leqq k$ and $f_i = c_i$ for $k+1 \leqq i \leqq n$.

We note that by theorem 2.2.1 $-D_2$ is a P-matrix and $-D_2^{-1} \geq 0$.

For any nonbasic variable z_j, $1 \leqq j \leqq k$, its simplex multiplier is

$$f(B^{-1}(-M_{.j})) - c_j = (0, \bar{f} D_2^{-1})(-M_{.j}) - c_j$$

where $\bar{f} = (c_{k+1}, \ldots\ldots, c_n)$.

We note that since $-D_2^{-1} \geq 0$ and $\bar{f} \geqq 0$, we get $\bar{f} D_2^{-1} \leqq 0$.

$$\therefore\ f B^{-1}(-M_{.j}) - c_j = \sum_{i=k+1}^{n} (\bar{f} D_2^{-1})_i (-m_{ij}) - c_j \leqq 0; \text{ as}$$

$(-m_{ij}) \geqq 0$ for $i \geqq k+1$ and $(\bar{f} D_2^{-1})_i \leqq 0$.

Similarly we can show that the simplex multipliers of the nonbasic w_j's, $j \geqq k+1$ are nonpositive. This proves that the terminal solution of $S(M,q,d)$ minimises $\sum c_j z_j$ for any $(c_1, \ldots, c_n) \geqq 0$, and therefore is the least element of $X(q)$.

Corollary 2.4.1 : When (1.1.1) has solution $S(M,q,d)$ takes the same number of iterations to terminate for any $d > 0 \in R^n$.

Proof :- Note that when (1.1.1) has solution $S(M,q,d)$ computes the least element of $X(q)$ for any $d > 0$. Now in view of corollary 2.2.2. it is clear that the number of iterations required by $S(M,q,d)$ to terminate is the same for all $d > 0$.

Remark 2.4.2 : Consider $M = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$. Take $q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $p = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$; $q^1 = q + p = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $q + 2p = q^2 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

Now (M, q^1) has two solutions $(\bar{w}^1, \bar{z}^1)$ and (w^{*1}, z^{*1}) where $\bar{w}^1 = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$; $\bar{z}^1 = \begin{bmatrix} .5 \\ 0 \end{bmatrix}$; $w^{*1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $z^{*1} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Similarly (M, q^2) has two solutions $(\bar{w}^2, \bar{z}^2)$ and (w^{*2}, z^{*2}) where $\bar{w}^2 = \begin{bmatrix} 0 \\ .5 \end{bmatrix}$ $\bar{z}^2 = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$, $w^{*2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $z^{*2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

We note that the pair $(\bar{w}^1, \bar{z}^1), (\bar{w}^2, \bar{z}^2)$ satisfy the isotonicity property whereas the pair $(w^{*1}, z^{*1}), (w^{*2}, z^{*2})$ does not. Theorem 1.4.1. implies that $S(M,p,q)$ does not generate (w^{*1}, z^{*1}) and (w^{*2}, z^{*2}). We also note that z^{*1}, z^{*2} are not the least elements of $X(q^1)$ and $X(q^2)$ respectively so that $S(M,q^1,d)$ or $S(M,q^2,d)$ for any $d > 0 \in R^2$ do not terminate with these solutions.

When $X(q) \neq \emptyset$, the Chandrasekaran - Saigal algorithm also computes the least element. See [19, p. 15]. Thus it follows that when $X(q) \neq \emptyset$, the number of iterations required by $S(M,q,d)$ and the Chandrasekaran - Saigal algorithm are the same.

In the context of structural mechanics the following problem has been considered by O. De Donato and Maier [12].

Find (w, z) such that

$$w - Mz = q$$

$$w \geqq 0, \quad z \geqq 0, \quad z \leqq a \qquad \dots \quad \dots \quad \dots \quad \dots \quad (2.4.1)$$

$$w^T z = 0 \qquad \dots \quad \dots \quad \dots \quad \dots \quad (2.4.2)$$

where $a > 0 \in R^n$ is a given vector. This problem is denoted by $(M,q)_a$.

Assuming that M is a P-matrix Cottle [5] showed that the set

$$R(a, M) = \{ q \mid (M, q)_a \text{ has solutions} \}.$$

is convex for each $a > 0$, if and only if $M \in K$. In what follows we shall extend this result to Z-matrices. First we show that a result analogous to that of Chandrasekaran for (M,q) when $M \in Z$ namely, theorem 1.4.7, holds for $(M, q)_a$ also.

Theorem 2.4.3 : Let $M \in Z$. If (2.4.1) has solution, then $(M,q)_a$ has solution.

Proof : Since $M \in Z$ the set $X(q)$ has a unique least element which is also a solution to (M, q), when $X(q) \neq \emptyset$.

Let z^0 be the least element. If $z^0 \not\leqq a$ then (2.4.1) has no solution. If $z^0 \leqq a$ then (w^0, z^0) is a solution to $(M,q)_a$ where $w^0 = Mz^0 + q$. If $X(q) = \emptyset$ then (2.4.1) has no solution. These observations conclude the proof of the theorem.

Theorem 2.4.4 : Let $M \in Z$. $R(a, M)$ is convex for each $a > 0 \in R^n$.

Proof : This follows from theorem 2.4.3.

Remark 2.4.3 : We note that the algorithm $S(M,q,d)$ with the following modification solves $(M, q)_a$ in atmost n iterations.

At each iteration we verify if $z_j \leqq a_j$ for each $j = 1,2,\ldots n$. If for some j, $z_j > a_j$, $S(M,q,d)$ is terminated. There is no solution to (2.4.1). Otherwise $S(M,q,d)$ solves $(M, q)_a$ in atmost n iterations.

We note that S(M, q, d) is applicable to the class of resource allocations problems considered by A. Tamir in [40, p.320].

We conclude this chapter by noting that for K-matrices with special structures special algorithms to solve (M,q) have been developed by R.S. Sacher [32], R.W. Cottle and R.S. Sacher [8] and R.W. Cottle, G.H. Golub and R.S. Sacher [9]. Also, O.L Mangasarian shows in [26] that (M, q) can be solved as a linear programming problem for some classes of matrices M which include the Z-matrices.

3. Number of solutions and constant parity property

3.1. Introduction :

Let $M \in Z$. In this chapter we consider the problem of determining how many solutions does (M,q) have when there are only finitely many solutions and what conditions need M satisfy so that M has constant parity property over the sets $D_1(M)$ or $D_2(M)$.

Theorems 1.5.9, 1.5.10 and 1.5.12 are some known results on this problem. Some more results are obtained in an unpublished report by Romesh Saigal [37]. The results we prove in section 2 extend these known results. In this section we first prove a necessary and sufficient condition for M to be a K-matrix. We next consider the constant parity property problem and discuss only the case where $(M, 0)$ has unique solution. For Z-matrices uniqueness of solution to $(M, 0)$ leads to a simple condition on M. We also consider a conjucture by Romesh Saigal stated in [33, p.182] and show that this conjucture is not correct.

In section 3 we consider the case when $(M, 0)$ has nontrivial solutions. We prove some theorems on the representation of M in the partitioned form and using these forms we obtain some sufficient conditions and a necessary condition on M for constant parity property to hold over $D_2(M)$. We extend theorem 1.5.12. by showing that when $M \in \bar{Z}$ if $N(q)$ denotes the number of solutions to (M,q) then $N(q) < \infty$ implies that $N(q) \leqq 2$. Many results appearing in this chapter are already published. See [29].

3.2. The case where (M, 0) has unique solution.

Theorem 3.2.1 : Let $M \in Z$. If (1.1.1) has a solution for some $q^o < 0 \in R^n$ then M is a K-matrix and (M,q) has unique solution for each $q \in R^n$.

Proof :- Since (1.1.1) has a solution from Farkas' lemma [27, p.34] it follows that the system

$$u \leq 0, \quad -M^T u \leqq 0, \quad u^T q^o > 0$$

has no solution $u \in R^n$.

Since $q^o < 0$ any $u \leq 0$ will satisfy $u^T q > 0$. Therefore $\nexists$ $u \leq 0$ such that $-M^T u \leqq 0$.

$\therefore$ $\nexists$ $u \geq 0 \in R^n$ such that $-M^T u \geqq 0$.

$\therefore$ $-M^T \notin S_o$.

From theorem 1.3.1 it follows that $M \in S$ and from theorem 1.3.2 we can conclude that $M \in K$.

From theorem 1.5.3 we see that (M,q) has unique solution for each $q \in R^n$. This completes the proof.

<u>Remark 3.2.1</u> : Thus $Z \cap Q \subseteq K$. We note that theorem 3.2.1 is stronger than the necessary and sufficient condition for a matrix to be a P-matrix proved by A. Tamir [41].

In [33, p.182] Romesh Saigal makes the following conjucture.

Let $M \in \tilde{Z}$. For $q \in D_2(M)$, (M,q) has constant parity of number of solutions; either odd or even.

<u>Theorem 3.2.2</u> : The above conjucture is not correct.

Proof :- Consider the examples (M,q^1) and (M, q^2) where

$$M = \begin{bmatrix} 2 & -2 & -2 & -3 \\ -1 & 1 & -4 & -1 \\ 0 & 0 & 6 & -8 \\ 0 & 0 & -9 & 2 \end{bmatrix}; \quad q^1 = \begin{bmatrix} 7 \\ 8 \\ 2 \\ 7 \end{bmatrix}; \quad q^2 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 7 \end{bmatrix}$$

We note that (M, q^1) has exactly two solutions, namely $(w^1,z^1)^T = (7,8,2,7,0,0,0,0)$ and $(w^2,z^2)^T = (2,3,0,0,0,0,1,1)$ whereas (M,q^2) has the unique solution $(w,z)^T = (1,3,2,7,0,0,0,0)$. Thus constant parity does not hold. In [37] Romesh Saigal gives a similar example.

<u>Remark 3.2.2</u> : The above conjucture is true under the additional assumption that $(M,0)$ has unique solution. This follows from theorem 1.5.5. Note that in the above example $(M, 0)$ has a nontrivial solution namely $w = 0$ and $z^T = (1,1,0,0)$.

<u>Theorem 3.2.3</u> : Suppose $M \in Z$. If there is no $x \geq 0 \in R^n$ such that $M x = 0$ then $(M,0)$ has unique solution.

Proof :- Suppose $(M, 0)$ has a nontrivial solution (w^*, z^*). Without loss of generality let us assume that $z_i^* > 0$ for $i = 1,2,\ldots k$ and $z_i^* = 0$ for $i = k+1,\ldots n$.

The equation $w^* - Mz^* = 0$ gives us

$$-\sum_{j=1}^{k} m_{ij} z_j^* = 0 , \quad i = 1,2, \ldots k \quad \ldots(3.2.1)$$

$$w_i^* - \sum_{j=1}^{k} m_{ij} z_j^* = 0 , \quad i = k+1, \ldots n \quad \ldots(3.2.2)$$

But for $i \geqq k+1$, $-m_{ij} \geqq 0$ for all $j = 1,2, \ldots k$.

Therefore (3.2.2) implies that $w_i^* = 0$, $i = k+1,\ldots n$.

Thus $\sum_{j=1}^{k} M_{.j} z_j^* = 0$

Therefore $\exists$ $x^T = (z_1^* \ldots z_k^*, 0 \ldots 0) \in R^n$ such that $x \geqq 0$ and $Mx = 0$.

This completes the proof.

<u>Corollary 3.2.1</u> : Let $M \in Z$. $(M,0)$ has unique solution if either

(a) M is non-singular, or

(b) there exists $y \in R^n$ such that $M^T y > 0$.

Proof :- Case (a) is trivial. Case (b) follows from Gordan's theorem of the alternative. See [27, p.34].

Example 3.2.1 : If M is singular it does not follow that (M,0) has a nontrivial solution. The following example shows this.

$$\text{Let } M = \begin{bmatrix} 1 & -2 & -4 \\ -1 & -3 & -6 \\ -2 & 0 & 0 \end{bmatrix}$$

Clearly M is singular as column 3 is a multiple of column 2. But (M,0) has unique solution since if we take $y^T = (0,1,0)$, $y^T M > 0$, and by Gordan's theorem of the alternative there does not exist $x \geq 0 \in R^n$ such that $Mx = 0$.

Remark 3.2.3 : We note that the (b) part of corollary 3.2.1 and theorem 1.5.5 strengthen and extend theorem 1.5.9 for Z-matrices. We also note that case (a) implies case (b).

Theorem 3.2.4 : Let $M \in Z$. Suppose that there exists $y \in R^n$ such that $M^T y > 0$ and that for some $q^0 \in R^n$, $q^0 > 0$, (M, q^0) has a unique solution. Then M is a K-matrix. (Note that y need not be nonnegative).

Proof :- From corollary 3.2.1 it follows that (M, 0) has unique solution. Therefore $M \in L^*(q^0)$ and from theorem 1.4.3 it follows that M is a Q matrix. The conclusion of the theorem now follows from theorem 3.2.1.

<u>Theorem 3.2.5</u> : Let $M \in Z$. Suppose (i) M is singular and (ii) there exists $y \in R^n$ such that $M^T y > 0$. Then for all $q \in D_1(M)$, $N(q)$ is even where $N(q)$ is the number of solutions to (M,q).

Proof :- From case (b) of corollary 3.2.1, $(M,0)$ has unique solution. Suppose for some $q^o \in D_1(M)$, (M,q^o) has an odd number of solutions. Then from theorem 1.5.6 it follows that for all $q \in D_1(M)$, (M,q) has an odd number of solutions. Since the set

$$Y = \left\{ q \mid q_i < 0, \; i = 1,2, \ldots . n \right\}$$ has non-empty interior

from theorem 1.5.2 it follows that there exists $q < 0 \in D_1(M)$ for which (M,q) has an odd number of solutions. Therefore from theorem 3.2.1 we conclude that M is a K-matrix. However this contradicts the hypothesis that M is singular. The conclusion of the theorem follows.

<u>Theorem 3.2.6</u> : Let $M \in K_o$. For all $q \in D_2(M)$, (M,q) has unique solution.

Proof :- This follows from theorem 1.3.3 and 1.5.10.

<u>Theorem 3.2.7</u> : Let $M \in Z$. If there is a $x > 0 \in R^n$ such that $Mx \geqq 0$ then for all $q \in D_2'(M)$, (M,q) has unique solution.

Proof :- This follows from theorems 1.3.4 and 3.2.6.

Theorem 3.2.8 : Let $M \in K_0 - K$. Then there is a $x \geq 0 \in R^n$ such that $Mx = 0$.

Proof :- Note that from theorems 1.3.2 and 1.3.3 the principal minors of M are nonnegative and there is atleast one principal minor which is zero.

Suppose there does not exist $x \geq 0 \in R^n$ such that $Mx = 0$. Theorem 3.2.3 implies that (M,0) has unique solution. By theorem 1.5.5 we therefore conclude that for all $q \in D_1(M)$, (M,q) has the same parity of number of solutions. Noting that $D_2(M) \subseteq D_1(M)$, we see from theorem 3.2.6 that this parity is odd. Thus we conclude that for all $q \in D_1(M)$, (M,q) has an odd number of solutions. (infact a unique solution). Since the set

$$Y = \{q \mid q_i < 0 \;,\; i = 1,2, \ldots.. n\}$$

has nonempty interior there is a $q^0 \in Y$ such that $q^0 \in D_2(M)$, and (M,q^0) has solution.

Theorem 3.2.1. now implies that M is a K-matrix. This contradicts our hypothesis. The proof is complete.

Remark 3.2.4 : We note that the above theorem is a partial converse of theorem 1.3.4 and generalises a part of theorem 1.3.5. This theorem will be useful for us in chapter 4 to obtain a convenient from of representing $M \in K_o - K$ and to obtain a few useful results about such matrices.

<u>Theorem 3.2.9</u> : Let $M \in Z$. The constant parity property of number of solutions to (M,q) holds

(i) for all $q \in D_1(M)$ ~~if~~ there exists no $x \in R^n$, $x \geq 0$ such that $Mx = 0$ or equivalently if there exists $y \in R^n$ such that $M^T y > 0$. The parity is odd if all the principal minors of M are positive. It is even if otherwise,

(ii) for all $q \in D_2(M)$ if all the principal minors are non-negative but atleast one is zero; i.e. if $M \in K_o - K$. (K-matrices have been covered under case (i)). In this case there is unique solution for all $q \in D_2(M)$ and therefore M has odd parity over $D_2(M)$. Also there exists $x \geq 0 \in R^n$ such that $Mx = 0$.

Proof :- This theorem follows from the earlier theorems.

(i) follows from theorems 1.5.5, 3.2.3 and 3.2.1 and corollary 3.2.1.

(ii) follows from theorems 3.2.6 and 3.2.8. In fact this theorem only summarises the results of earlier theorems.

Example 3.2.2 : The following example shows that uniqueness of solution to (M,0) when $M \in Z$ is not a necessary condition for constant parity to hold over $D_2(M)$.

$$\text{Let} \quad M = \begin{bmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -1 & -2 \\ 0 & 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -6 & -4 \end{bmatrix}$$

Note that (M,0) has the nontrivial solution (w,z) where $w^T = (0,0,0,0)$, $z^T = (1,1,0,0)$.

Let $J_1 = \{1,2\}$; $J_2 = \{3,4,5\}$. We note that the principal submatrix M_{J_1} is a K_0-matrix and M_{J_2} has atleast one negative principal minor. We also observe that with $y^T = (0,0,-1)$ we have $y^T M_{J_2} = M_{J_2}^T y > 0$. Thus $(M_{J_2}, 0)$ has a unique solution and from theorem 3.2.9 it follows that for all $q \in D_1(M_{J_2})$, (M_{J_2}, q) has an even number of solutions. Similarly for all $q \in D_1(M_{J_1})$, it follows from theorem 3.2.6, that (M_{J_1}, q) has either unique solution or no solution.

We note further that (w,z) is a solution to (M,q) for $q \in R^5$ if and only if (w_{J_1}, z_{J_1}) is a solution to (M_{J_1}, q_{J_1}) and (w_{J_2}, z_{J_2})

is a solution to (M_{J_2}, q_{J_2}). Thus $q \in D_1(M) \Longleftrightarrow q_{J_1} \in D_1(M_{J_1})$ and $q_{J_2} \in D_1(M_{J_2})$. From these observations it is easy to see that (M,q) has an even number of solutions for all q in $D_1(M)$. Thus M has constant parity property over the set $D_1(\bar{M})$.

Remark 3.2.5 : Consider 2×2 $\bar{Z}$-matrices M. Suppose M is not in K_0. Since $m_{11} \geqq 0$ and $m_{22} \geqq 0$, it follows that the determinant of M is negative. That is, we have

$$m_{11}\, m_{22} - m_{12}\, m_{21} < 0,$$ which implies that

$m_{12} < 0$, $m_{21} < 0$. Since M is nonsingular $(M, 0)$ has unique solution. Thus for 2×2 $\bar{Z}$ matrices either

(i) $\nexists\ x \geq 0 \in R^2$ such that $Mx = 0$ or

(ii) $M \in K_0$

holds. We see from theorems 3.2.6 and 1.5.5 that in either case Saigal's conjucture holds. Therefore for 2 x 2 case Saigal's conjucture always holds.

3.3. Constant parity property when $(M,0)$ has nontrivial solutions :

We first state a result on the representation of M in the partitioned form proved by Saigal in [37, p.7].

Lemma 3.3.1 (R. Saigal) : Let $M \in Z$. Suppose all the principal submatrices of M are in $Z \cap S_o$ then $M \in K_o$.

Lemma 3.3.2 (R. Saigal) : Let $M \in Z$. Then exactly one of the following holds.

(i) $-M^T \in S$, (ii) $M \in K_o$ and (iii) There exists a partition $N = J_1 \cup J_2$ and a representation for M if necessary with a principal rearrangement so that

$$M = \begin{bmatrix} M_{J_1} & M_{J_1 J_2} \\ 0 & M_{J_2} \end{bmatrix}$$

where $M_{J_1} \in K_o$, $-M^T_{J_2} \in S$, $J_1 \neq \emptyset$, $J_2 \neq \emptyset$.

We next observe the following result.

Lemma 3.3.3 : Let $M \in Z$ and case (iii) of lemma 3.3.2 hold with $M_{J_1} \in K$. Then there exists $y \in R^n$ such that $M^T y > 0$.

Proof : We have $M = \begin{bmatrix} M_{J_1} & M_{J_1 J_2} \\ 0 & M_{J_2} \end{bmatrix}$ with $M_{J_1} \in K$ and $-M^T_{J_2} \in S$.

Since $M_{J_1} \in K$ and since the principal minors of $M^T_{J_1}$ are the same as those of M_{J_1}, it follows from theorem 1.3.2 that $M^T_{J_1} \in K$ and that there exists $x \geq 0 \in R^{|J_1|}$ such that $M^T_{J_1} x > 0$. Also, since $-M^T_{J_2} \in S$, by definition there exists $y \geq 0 \in R^{|J_2|}$ such that $M^T_{J_2}(-y) > 0$.

Choose $\lambda > 0$ such that

$\lambda (M_{J_1 J_2})^T x + M^T_{J_2}(-y) > 0$. It is easy to see that such a $\lambda > 0$ exists.

Now $\binom{\lambda x}{y} \in R^n$ is the required vector and the conclusion of the lemma follows.

Remark 3.3.1 : In [37, p.8] Saigal proves the result that if in lemma 3.3.2 either (a) case (i) holds or (b) case (ii) holds with $M \in K$ or (c) case (iii) holds with $M_{J_1} \in K$, then M has constant parity property over the set $D_1(M)$. We shall show how this result is related to our results in section 3.2. First we note that if $-M^T \in S$ then by definition there exists $y \geq 0 \in R^n$ such that $-M^T y > 0$ or $M^T(-y) > 0$. If $M \in K$ from theorem 1.3.2 $M^T \in K$ and there exists $y \geq 0 \in R^n$ such that $M^T y > 0$. If case (c) holds then lemma 3.3.3. shows that there exists $y \in R^n$ such that $M^T y > 0$. Thus in all cases of (a), (b) and (c) there exists $y \in R^n$ such that

$M^T y > 0$. Therefore Saigal's conditions (a), (b) or (c) implies, in view of corollary 3.2.1, that (M,0) has unique solution. Now to completely establish the equivalence between our results in section 3.2 and the conditions (a), [illegible] or (c) of Saigal it is enough to show that if either $M \in K_o - K$ or if case (iii) of lemma 3.3.2. holds with $M_{J_1} \in K_o - K$, then (M, 0) has nontrivial solutions. For $M \in K_o - K$ this follows immediately from theorem 3.2.8. If case (iii) of lemma 3.3.2. holds with $M_{J_1} \in K_o - K$, by theorem 3.2.8 there exists $x \geq 0 \in R^{|J_1|}$ such that $M_{J_1} x = 0$. Now consider $y = \binom{x}{0} \in R^n$. It is easy to see that $My = 0$. Thus the equivalence between our results in section 3.2 and Saigal's conditions (a), (b) or (c) for constant parity property of M over the set $D_1(M)$ is established.

We shall now consider the case (iii) of lemma 3.3.2. with $M_{J_1} \in K_o - K$. To study the constant parity property of such matrices we first require a few results on the representation of M in the partitioned form when $M \in (Z - K) \cap S_o$.

<u>Lemma 3.3.4</u> : Let $M \in K_o - K$. Then there exists a $J_1 \subseteq N$ such that (i) $\det(M_{J_1}) = 0$ (ii) no proper principal minor of M when defined (i.e. when J_1 is not a singleton set) is zero.

Proof : Trivial.

<u>Lemma 3.3.5</u> : Let $M \in K_o - K$. Then there is a partition $N = J_1 \cup J_2 \ldots\ldots \cup J_r$ and a representation of M if necessary with principal rearrangement, as

$$M = \begin{bmatrix} M_J & M_{JJ_r} \\ M_{J_rJ} & M_{J_r} \end{bmatrix}$$

where $r \geqq 2$. If $J_r \neq \emptyset$, M_{J_r} is defined and $M_{J_r} \in K$. $J \neq \emptyset$ and has the partition

$J = J_1 \cup J_2 \ldots\ldots \cup J_{r-1}$, each J_i satisfying

(i) $\det (M_{J_i}) = 0$, $1 \leqq i \leqq r-1$

and (ii) no proper principal minor of M_{J_i} is 0.

Proof :- Since $M \in K_o - K$ from lemma 3.3.4 we see that there is a $J_1 \neq \emptyset$, $J_1 \subseteq N$ such that

(i) $\det (M_{J_1}) = 0$ (ii) no proper principal minor of M_{J_1} is 0.

If $J_1 = N$, we have the above partition for M with $r = 2$, $J = J_1 = N$, $J_2 = \emptyset$ so that M_{J_2} is not defined.

If $N - J_1 \neq \emptyset$ then consider M_{N-J_1}. This is a K_o-matrix. If all the principal minors of M_{N-J_1} are positive then the above partition is obtained with $J = J_1$, $J_2 = N - J_1$ and M_{J_2} a K-matrix.

If $M_{N-J_1} \in K_o - K$ we apply lemma 3.3.4 to M_{N-J_1} to obtain J_2. Proceeding thus in a finite number of steps we shall obtain the above partition ~~for~~ M.

Lemma 3.3.6 : Let $M \in (Z - K_o) \cap S_o$. Then case (iii) of lemma 3.3.2 holds. The converse is also true.

Proof :- Note that $M \in S_o \Longrightarrow -M^T \notin S$ by theorem 1.3.1, so that case (i) of lemma 3.3.2 can not occur. Case (ii) is excluded because $M \in Z - K_o$. Case (iii) must therefore hold.

Suppose case (iii) of lemma 3.3.2 holds with $M_{J_1} \in K$. Then, since there exists $x \geq 0 \in R^{|J_1|}$ such that $M_{J_1} x > 0$, taking $y = \binom{x'}{0} \in R^n$ we have $My \geq 0$. Therefore $M \in S_o$. If case (iii) holds with $M_{J_1} \in K_o - K$ then in remark 3.3.1 it was shown that there exists $y \geq 0 \in R^n$ such that $My = 0$. Therefore $M \in S_o$. From here the converse follows.

Remark 3.3.2 : From the proof of the above lemma we also see that the conditions (i) $M \in (Z - K_o) \cap S_o$ (ii) there exists $x \geq 0 \in R^n$ such that $Mx = 0$ hold if and only if in lemma 3.3.2 case (iii) occurs with $M_{J_1} \in K_o - K$.

Lemma 3.3.7 : Let (a) $M \in (Z - K_o) \cap S_o$ and (b) There exist $x \geq 0 \in R^n$ such that $Mx = 0$. Then there is a partition

$N = J_1 \cup J_2 \ldots\ldots \; J_r \cup J_{r+1}$, where $r \geqq 2$ and a representation for M as

$$M = \begin{bmatrix} M_J & M_{JJ_r} & M_{JJ_{r+1}} \\ M_{J_rJ} & M_{J_r} & M_{J_rJ_{r+1}} \\ 0 & 0 & M_{J_{r+1}} \end{bmatrix}$$

such that (i) $J = \bigcup_{i=1}^{r-1} J_i \neq \emptyset$, J_r may be empty, $J_{r+1} \neq \emptyset$.

(ii) $\exists \; x \geq 0 \in R^{|J_{r+1}|}$ such that $x^T M_{J_{r+1}} < 0$.

and (iii) J_i's, $1 \leqq i \leqq r-1$ satisfy the conditions on J_i's of lemma 3.3.5.

Proof :- Using lemma 3.3.6 we conclude that case (iii) of lemma 3.3.2 holds and there is a partition

$$N = L \cup (N-L), \quad L \neq \emptyset, \quad N-L \neq \emptyset$$

and M has the representation

$$M = \begin{bmatrix} M_L & M_{L\,N-L} \\ 0 & M_{N-L} \end{bmatrix}$$

where

(i) M_L is a K_0 matrix. (ii) $-M^T_{N-L} \in S$, or there exists $x \geq 0 \in R^{|N-L|}$ such that $x^T M_{N-L} < 0$.

Also because of condition (b) and remark 3.3.2 we have $M_L \in K_o - K$. Now applying lemma 3.3.5 to M_L we get a partition $L = J_1 \cup J_2 \cup \ldots$ $\ldots \cup J_r$ which satisfy the conditions of lemma 3.3.5. We take $N - L = J_{r+1}$. This gives the desired partition and concludes the proof of the lemma.

<u>Theorem 3.3.1</u> : Let (a) $M \in (Z - K_o) \cap S_o$ and let (b) there exist $x \geq 0 \in R^n$ such that $Mx = 0$. Consider the partition $N = J_1 \cup J_2 \cup \ldots \ldots J_r \cup J_{r+1}$ as given by lemma 3.3.7. M has constant parity property over the set $D_1(M)$ if (i) $M_{JJ_{r+1}} = 0$ and (ii) when $J_r \neq \emptyset$ either $M_{JJ_r} = 0$ or $M_{J_r J_{r+1}} = 0$.

Proof :- The case $J_r = \emptyset$ is easy. We shall consider only the case $J_r \neq \emptyset$.

Case (1) : $M_{JJ_r} = 0$; $M_{JJ_{r+1}} = 0$.

Let $L = J_r \cup J_{r+1}$ and consider

$$M_L = \begin{bmatrix} M_{J_r} & M_{J_r J_{r+1}} \\ 0 & M_{J_{r+1}} \end{bmatrix}$$

Here M_{J_r} is a K-matrix and $-M^T_{J_{r+1}} \in S$. Therefore case (c) in remark 3.3.1 holds and thus there is even parity of number of solutions for all $q \in D_1(M_L)$ by theorem 3.2.9, as there is atleast one principal minor which is negative $M_{J_{r+1}}$ not being a K_0-matrix.

By the hypothesis of the case we note that

$$M = \begin{bmatrix} M_J & 0 \\ M_{LJ} & M_L \end{bmatrix}$$

Consider any $q = \begin{bmatrix} q_J \\ q_L \end{bmatrix} \in D_1(M)$. Note that if (w, z) is a solution to (M, q), because $M_{JL} = 0$, (w_J, z_J) is a solution to (M_J, q_J). Therefore it follows that $q_J \in D_1(M_J)$.

Since $M_J \in K_0 - K$, by theorem 3.2.6 it follows that either (M_J, q_J) has unique solution or no solution.

Suppose (M_J, q_J) has a solution. Let the unique solution be $(\bar{w}_J, \bar{z}_J)$ and let

$$\bar{q}_L = \begin{bmatrix} q_{J_r} + M_{J_r J}\, \bar{z}_J \\ q_{J_{r+1}} \end{bmatrix}$$

and note that (w, z) is a solution to (M, q) if and only if (w_L, z_L) is a solution to (M_L, q_L). It therefore follows that $q \in D_1(M)$ $\Longleftrightarrow \bar{q}_L \in D_1(M_L)$, $q_J \in D_1(M_J)$.

Therefore either there are an even number of solutions to $(M_L, \bar{q}_L)$ or there is no solution. If there are an even number of solutions to $(M_L, \bar{q}_L)$, $(\bar{w}_J, \bar{z}_J)$ with each of these solutions gives a solution to (M, q).

From these observations we conclude that (M, q) has either no solution or has an even number of solutions. M has even parity over $D_1(M)$. This concludes the proof for this case.

Case (2) : $M_{JJ_{r+1}} = 0$; $M_{J_r J_{r+1}} = 0$. We let $L = J \cup J_r$ and note that M is of the form

$$M = \begin{bmatrix} M_L & 0 \\ 0 & M_{J_{r+1}} \end{bmatrix}$$

where M_L is a K_0-matrix and $-M^T_{J_{r+1}} \in S$.

We note that for any $q \in R^n$, (w, z) is a solution to (M, q) if and only if (w_L, z_L) is a solution to (M_L, q_L) and $(w_{J_{r+1}}, z_{J_{r+1}})$ is a solution to $(M_{J_{r+1}}, q_{J_{r+1}})$. Therefore it follows that

$$q \in D_1(M) \Longleftrightarrow q_L \in D_1(M_L) \text{ and } q_{J_{r+1}} \in D_1(M_{J_{r+1}}).$$

We note that $(M_{J_{r+1}}, q_{J_{r+1}})$ has an even number of solutions or has no solution for all $q_{J_{r+1}} \in D_1(M_{J_{r+1}})$. Also from theorem 3.2.6

we see that (M_L, q_L) has either a unique solution or no solution for any $q_L \in D_1(M_L)$. From these observations it follows that (M,q) either has an even number of solutions or no solution for all $q \in D_1(M)$. The conclusion of the theorem follows.

The case $J_r = \emptyset$ may be treated similarly.

Lemma 3.3.8 : Suppose $M \in K_o$ such that

(i) $\det(M) = 0$ (ii) no proper principal minor of M is zero. Then M is irreducible, rank of M is (n-1) and there exists $x > 0 \in R^n$ such that $Mx = 0$.

Proof :- Suppose M is reducible and consider the representation

$$M = \begin{bmatrix} M_J & 0 \\ M_{J\ N-J} & M_{N-J} \end{bmatrix}$$

Since $\det(M) = \det(M_J).\ \det(M_{N-J}) = 0$ it follows that either $\det(M_J) = 0$ or $\det(M_{N-J}) = 0$. This contradicts the hypothesis of the lemma. The other conclusions of the lemma now follow from theorem 1.3.5.

Lemma 3.3.9 : Let $M \in (Z - K_o) \cap S_o$ and let there exist $x \geq 0 \in R^n$ such that $Mx = 0$. Consider the partition $N = J_1 \cup \ldots\ J_{r-1} \cup J_r \cup J_{r+1}$ given by lemma 3.3.7. Let $J = J_1 \cup \ldots\ldots \cup J_{r-1}$. There exists $x \in R^{|J|}$, $x > 0$ such that $x^T M_J \leq 0$.

Proof : Note that each $M^T_{J_i}$ satisfies the conditions of lemma 3.3.8 for $1 \leq i \leq r-1$. By lemma 3.3.9 therefore there exists $x_{J_i} > 0 \in R^{|J_i|}$ such that $x^T_{J_i} \cdot M_{J_i} = 0$.

Define $x^* \in R^{|J|}$ by taking $x^*_{J_i} = x_{J_i}$, $1 \leq i \leq r-1$.

Since M_J is a Z-matrix it is easy to see that

$$x^{*T}_J \; M_J \leq 0$$

This concludes the proof of the lemma.

Lemma 3.3.10 : Let $M \in (Z-K_o) \cap S_o$ and let there exist $x \geq 0 \in R^n$ such that $Mx = 0$. Consider the partition of N and the representation of M as in lemma 3.3.7 and let $J = \bigcup_{i=1}^{r-1} J_i$. Given any $p \geq 0 \in R^{|J|}$ there exists $\bar{q} > 0 \in R^{|J|}$ and a real number $\lambda_o > 0$ such that $\bar{q} \in D_2(M_J)$ and (M_J, q) has no solution for all $q \leq \bar{q} - \lambda_o p$.

Proof : Consider any $q^* > 0 \in R^{|J|}$. Look at $q^* - \lambda p$.
Let $x > 0 \in R^{|J|}$ be given by lemma 3.3.9 satisfying $x^T M_J \leq 0$.
Choose $\lambda_o > 0$ so that $x^T (q^* - \lambda_o p) < 0$. Such a λ_o exists since $x^T p > 0$.

The set $Y = \{q \mid q \leq q^*, \; q \geq 0\}$ has non-empty interior. Therefore from theorem 1.5.2 there is a $\bar{q} \in Y$ such that $\bar{q} \in D_2(M_J)$.

Note that $\bar{q} - \lambda_o p \leq q^* - \lambda_o p$.

Therefore for all $q \leq \bar{q} - \lambda_o p$, the following inequalities are satisfied.

$$x^T M_J \leq 0, \quad x^T q < 0.$$

Therefore using Farkas' lemma we conclude that the system

$w - M_J z = q, \quad w \geq 0, \quad z \geq 0$ does not have solution for any $q \leq \bar{q} - \lambda_o p$.

This concludes the proof of the lemma.

Theorem 3.3.2 : Let $M \in (Z - K_o) \cap S_o$ and let there exist $x \geq 0 \in R^n$ such that $Mx = 0$. Consider the partition of N and representation of M as in lemma 3.3.7. Let $(M_{J_r J_{r+1}})_{i.}$ denote the i^{th} row of M in $M_{J_r J_{r+1}}$ for $i \in J_r$ and let $(M_{JJ_r})_{.i}$ denote the i^{th} column of M in M_{JJ_r} for $i \in J_r$. A necessary condition for M to have constant parity property over $D_2(M)$ is that if $J_r \neq \emptyset$,

$$(M_{JJ_{r+1}})_{i.} < 0 \Longrightarrow (M_{JJ_r})_{.i} = 0.$$

Proof : Suppose $J_r \neq \emptyset$ and for some $i \in J_r, (M_{J_r J_{r+1}})_{i.} < 0$ and $0 \neq (M_{JJ_r})_{.i} \leq 0$.

Considering the set $R_+^{|J_{r+1}|} = \{q \mid q \in R^{|J_{r+1}|}, \ q > 0\}$ which has non empty interior and using theorem 1.5.2 we obtain a $q^*_{J_{r+1}} > 0 \in R^{|J_{r+1}|}$ such that $q^*_{J_{r+1}} \in D_2(M_{J_{r+1}})$.

Since $-M^T_{J_{r+1}} \in S$, by theorem 3.2.9 and remark 3.3.1 there are an even number of solutions to $(M_{J_{r+1}}, q^*_{J_{r+1}})$. Since $q^*_{J_{r+1}} > 0$ one such solution is $\bar{w}_{J_{r+1}} = q^*_{J_{r+1}}$, $\bar{z}_{J_{r+1}} = 0$. Therefore there are an odd number s of solutions to $(M_{J_{r+1}}, q^*_{J_{r+1}})$ such that $z^{(m)}_{J_{r+1}} \neq 0$, m = 1,2, ... s $(w^{(m)}_{J_{r+1}}, z^{(m)}_{J_{r+1}})$ being the solutions.

Let $Y = \left\{q_{J_r} \mid q_{J_r} > 0, (q_{J_r} + M_{J_r J_{r+1}} \cdot z^{(m)}_{J_{r+1}})_i < 0, m = 1,2,\dots s\right\}$.

Since $(M_{J_r J_{r+1}})_{i.} < 0$, Y is nonempty and has a nonempty interior.

Therefore there exists $q^*_{J_r} \in Y$ such that $q^*_{J_r} + M_{J_r J_{r+1}} z^{(m)}_{J_{r+1}}$ are in $D_1(M_{J_r})$ for m = 1,2, ... s. Since M_{J_r} is a K-matrix there is a unique solution to each of $(M_{J_r}, q_{J_r} + M_{J_r J_{r+1}} z^{(m)}_{J_{r+1}})$, m = 1,2, ... s.

Let $L = J_r \cup J_{r+1}$. Consider

$$M_L = \begin{bmatrix} M_{J_r} & M_{J_r J_{r+1}} \\ 0 & M_{J_{r+1}} \end{bmatrix} \quad \text{and take} \quad q^*_L = \begin{bmatrix} q^*_{J_r} \\ q^*_{J_{r+1}} \end{bmatrix}.$$

From our observations above we note that $q_L^* \in D_1(M_L)$ and (M_L, q_L^*) has (s+1) solutions, one of which is $\bar{w}_L = q_L^*$; $z_L^* = 0$ and in the remaining s solutions $z_{J_{r+1}} = z_{J_{r+1}}^{(m)}$, m = 1,2, ... , s.

We also note that since $(q_{J_r}^* + M_{J_r J_{r+1}} z_{J_{r+1}}^{(m)})_i < 0$, in each of these s solutions $z_i > 0$. Let $(w_L^{(m)}, z_L^{(m)})$ denote these s solutions in each of which $z_i^{(m)} > 0$.

Let $z_i^* = \min_{1 \leqq m \leqq s} z_i^{(m)}$ and $\beta > \max_{\substack{1 \leqq m \leqq s \\ j \in J}} (M_{JL} z_L^{(m)})_j$.

Using lemma 3.3.10 we can obtain a $q_J^* > 0 \in R^{|J|}$ such that $q_J^* \in D_2(M_J)$ and (M_J, q_J) has no solution for all

$$q_J \leqq q_J^* - \lambda_o (-M_{JJ_r})_{.i} \; z_i^* .$$

From here it follows that $(M_J, q_J^* - \lambda_o z_i^{(m)} (-M_{JJ_r})_{.i})$ has no solution for $1 \leqq m \leqq s$. Thus if we take $q^{*T} = (q_J^{*T}, \lambda_o q_L^*) > 0 \in R^n$ there exists exactly one solution to (M, q^*), namely $\bar{w} = q^*$; $\bar{z} = 0$. We also note that $q^* \in D_2(M)$.

On the other hand we can choose $\bar{q}_J \in D_2(M_J)$, $\bar{q}_J > 0 \in R^{|J|}$ so that $\bar{q}_J - \beta e_{|J|} > 0$. If we now take $\bar{q}^T = (\bar{q}_J^T , q_L^{*T})$, then $\bar{q} \in D_2(M)$ and $(M, \bar{q})$ has (s+1) solutions, s of which correspond to the nondegenerate solutions to

$$(M_{J}, \bar{q}_{J} - (-M_{J\,L})\ z_{L}^{(m)}), \quad 1 \leq m \leq s .$$

Thus constant parity of number of solutions does not hold even over $D_2(M)$. This completes the proof.

Example 3.3.1 : The following example shows that $M_{JJ_r} = 0$; $M_{JJ_{r+1}} = 0$; $M_{J_r J_{r+1}} = 0$ are not necessary conditions for constant parity property over $D_1(M)$ to hold

$$M = \begin{bmatrix} 1 & -1 & -2 & -1 & 0 & 0 \\ -2 & 2 & -1 & -2 & 0 & 0 \\ 0 & 0 & 4 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -2 & 0 \end{bmatrix}$$

In this example $r = 2$, $J_1 = J = \{1,2\}$, $J_2 = \{3\}$; $J_3 = \{4,5,6\}$.

We note that if any complementary set of column vectors contains $-M_{.4}$, then in the matrix of these columns the fourth row is 0. Therefore such complementary sets of column vectors which contain $-M_{.4}$ generate complementary cones with empty interior. Because of this $D_2(M)$ is a subset of the union of the 2^{n-1} complementary cones in which $I_{.4}$ appears as a column. Thus if $q \in D_2(M)$ then $q_4 > 0$. In fact it is easy to see that if $q \in D_2(M)$ then $q_5 > 0$ and $q_6 > 0$.

Let $\bar{M}$ be the matrix obtained by replacing $+M_{.4}$ by $-I_{.4}$ i.e. the fourth column of $-\bar{M}$ is $I_{.4}$. Now the complementary cones of $(I, -\bar{M})$ are the same as the complementary cones of $(I, -M)$ in which $I_{.4}$ appears as a generating column, but each such cone appearing twice. Thus $D_2(M) = D_2(\bar{M})$. But there are twice as many solutions to $(\bar{M}, q)$ for each $q \in D_2(\bar{M})$ as there are solutions to (M,q).

We shall show that for each $q \in D_2(\bar{M})$, $(\bar{M},q)$ has four solutions.

Let $L_1 = \{1,2,3\}$; $L_2 = \{4\}$ and $L_3 = \{5,6\}$.

We note that

$$\bar{M} = \begin{bmatrix} M_{L_1} & 0 & 0 \\ 0 & M_{L_2} & 0 \\ 0 & 0 & M_{L_3} \end{bmatrix}$$ where M_{L_2} is of order 1×1, $M_{L_2} = -1$, $M_{L_1} \in K_0$ and M_{L_3} is a 2×2 $\bar{Z}$- matrix whose determinant is negative. If $q \in D_2(\bar{M})$, as noticed earlier $q_{L_3} > 0$, $q_{L_2} > 0$. Now in view of remark 3.2.5 if $q_{L_3} \in D_2(\bar{M}_{L_3})$, there are exactly two solutions to (M_{L_3}, q_{L_3}). It is also easy to see that

there are two solutions to (M_{L_2}, q_{L_2}) for each $q_{L_2} \in D(\bar{M}_{L_2})$. Thus there are four solutions to each $q_{L_2 \cup L_3} \in D_2(\bar{M}_{L_2 \cup L_3})$, noting that $q_{L_2 \cup L_3} \in D_2(\bar{M}_{L_2 \cup L_3})$ if and only if $q_{L_2} \in D_2(\bar{M}_{L_2})$ and $q_{L_3} \in D_2(\bar{M}_{L_3})$. Also it is easy to see that $q \in D_2(\bar{M})$ if and only if $q_{L_1} \in D_2(\bar{M}_{L_1})$ and $q_{L_2 \cup L_3} \in D_2(\bar{M}_{L_2 \cup L_3})$. Noting that M_{L_1} is a K_o-matrix, we conclude that for each $q \in D_2(\bar{M})$ there are exactly four solutions to $(\bar{M}, q)$. From here it follows that there are exactly two solutions to (M,q) for each $q \in D_2(M)$.

Thus we see that M has constant parity property over $D_1(M)$. (If $q \in D_1(M) - D_2(M)$, (M,q) has no solution and by definition even parity holds).

We shall now consider $\bar{Z}$-matrices and extend the result of theorem 1.5.12.

<u>Theorem 3.3.3</u> : Let $M \in \bar{Z}$ and let $-M^T \in S$. Then (M,q) has a constant number of solutions for all q in $D_2(M)$. This constant is 2.

Proof : Since $-M^T \in S$ and $M \notin K$ if all the principal minors of M are nonzero then the conclusion of the theorem is just the result of theorem 1.5.12.

Suppose now some principal minors are zero. Since $-M^T \in S$ by remark 3.3.1 and theorem 3.2.9 it follows that M has even parity over $D_2(M)$. Thus there exist an even number of solutions (≥ 2) for all $q \in D_2(M)$.

Let us suppose that for some $\bar{q} \in D_2(M)$, $(M,\bar{q})$ has atleast four solutions. We make the following observations.

(i) For any $J \subsetneqq N$ consider the equation

$$\det (M_J + \theta I) = 0 .$$

This is a polynomial of degree n in θ and has only a finite number of real solutions. Therefore there exists $\theta(J) > 0$ such that for $0 < \theta < \theta(J)$, $\det (M_J + \theta I) \neq 0$. Thus, choosing $\theta_0 = \min_{\substack{J \subseteq N \\ J \neq \emptyset}} \theta(J)$, we conclude that for $0 < \theta < \theta_0$, all the principal minors of $(M + \theta I)$ are nonzero.

(ii) Let

$$B_i = \begin{bmatrix} M_{J_i} & 0 \\ M_{J_i\, N-J_i} & I \end{bmatrix}; \quad i = 1,2,3,4 \text{ be the four}$$

complementary basis matrices corresponding to four solutions y^i, $i = 1,2,\ldots,4$ of the even number of solutions to (M,q). Note that these complementary basis matrices are distinct and $J_i = \emptyset$ is permitted for one i in which case the corresponding B_i is I. Since all the solutions are nondegenerate we have

$$B_i\, y^i = q , \quad y^i > 0 .$$

Now we can find $\beta_i < \theta_o$ such that with

$$C_i = \begin{bmatrix} M_{J_i} + \theta I & 0 \\ M_{J_i\, N-J_i} & I \end{bmatrix}$$

we have $x_\theta^i > 0$ such that $C_i x_\theta^i = q$, for all $\theta < \beta_i$, $i = 1, \ldots 4$.

Thus if we choose $\theta_1 = \min_{1 \leq i \leq 4} \beta_i$ then for all $0 < \theta < \theta_1$, $\bar{q}$ is in the interior of the four complementary cones generated by C_i's of $(I, -M - \theta I)$

From observations (i) and (ii) above it follows that there exists $\theta > 0$ such that $\bar{q}$ is in the interior of atleast four of the complementary cones of $(I, -M - \theta I)$ and the principal minors of $M + \theta I$ are nonzero. Now consider the set

$$Y = \{ q \mid \ || q - \bar{q} || < \alpha \}$$ where $|| q - \bar{q} ||$ is the usual norm in R^n. We can choose $\alpha > 0$ so that Y is wholly contained in all of the four complementary cones of $(I, -M - \theta I)$ which contain $\bar{q}$. Since Y has nonempty interior from theorem 1.5.2 there exists $q^* \in Y$ which is also in $D_2(M + \theta I)$. Moreover (M, q^*) has atleast four solutions. This contradicts theorem 1.5.12.

The conclusion of the theorem follows.

Theorem 3.3.4 : Let $M \in \bar{Z}$ and let $q \in D_1(M)$. Then $N(q) \leqq 2$, where $N(q)$ is the number of solutions to (M,q).

Proof : We note that according to lemma 3.3.2 there are only three cases to be considered.

(i) $-M^T \in S$ (ii) $M \in K_o$ and (iii) There is a partition of N as $N = J_1 \cup J_2$ and a representation for M as

$$M = \begin{bmatrix} M_{J_1} & M_{J_1 J_2} \\ 0 & M_{J_2} \end{bmatrix}$$

with $M_{J_1} \in K_o$ and $-M^T_{J_2} \in S$.

If case (i) holds then from theorem 3.3.3 for all $q \in D_2(M)$, $N(q) = 2$. If case (ii) holds then from theorem 3.2.6 for all $q \in D_2(M)$, $N(q) = 1$. We need to consider only case (iii)

Let $q \in D_2(M)$. Note that this implies $q_{J_2} \in D_2(M_{J_2})$. From theorem 3.3.3 it follows that (M_{J_2}, q_{J_2}) has exactly two solutions. Let the two solutions be $(\bar{w}_{J_2}, \bar{z}_{J_2})$ and $(w^*_{J_2}, z^*_{J_2})$. These solutions lead to the problems $(M_{J_1}, q_{J_1} + M_{J_1 J_2} \bar{z}_{J_2})$ and $(M_{J_1}, q_{J_1} + M_{J_1 J_2} z^*_{J_2})$. Note that $q \in D_2(M) \Longrightarrow$ (i) $q_{J_1} + M_{J_1 J_2} \bar{z}_{J_2} \in D_1(M_{J_1})$; $q_{J_1} + M_{J_1 J_2} \bar{z}_{J_2} \in D_1(M_{J_1})$ and (ii) at least one of $q_{J_1} + M_{J_1 J_2} \bar{z}_{J_2}$

and $q_{J_1} + M_{J_1J_2} \; z^*_{J_2}$ is in $D_2\,(M_{J_1})$. Since $M_{J_1} \in K_o$ it follows that each problem can have atmost one solution (theorem 3.2.6). Thus it follows that (M,q) has atmost two solutions.

Since, if $q \in D_1(M) - D_2(M)$, $N(q) = 0$, the conclusion of the theorem follows.

Corollary 3.3.1 : Let $M \in \bar{Z}$. If M has constant parity property over $D_2(M)$ then (M,q) has a constant number of solutions for all $q \in D_2(M)$.

Proof : Trivial.

4. Infinitely many solutions and solution rays

4.1. Introduction :

The definition of solution ray was given in section 1.6. In the same section we also posed the problem of determining the conditions on M and q so that (M,q) has a ray of solutions and pointed out how this problem arose. We also noted that existence of a ray of solutions to (M,q) implies the existence of infinitely many solutions. In this chapter for a subclass of the class Z we characterise the set of q for which (M,q) possesses a solution ray and also the set of q for which there are infinitely many solutions to (M,q).

In section 4.2 we prove some more results on the representation of M in the partitioned form when $M \in K_o - K$. Using these results we introduce the class $\bar{K}_o \subseteq K_o$. We prove some properties of the class $\bar{K}_o$. The main result proved in section 4.3 is that for $M \in \bar{K}_o$ a weaker version of the result of Cottle for CP^+ matrices stated in theorem 1.6.1. is true. More precisely the result we prove is as follows: Let $M \in \bar{K}_o$. At some solution $(\bar{w}(0), \bar{z}(o))$ to (M,q) there exists $v \geq 0 \in R^n$ which generates a solution ray if and only if q is in the boundary of D(M). We also prove that if $M \in K_o$ then the set of q for which (M,q) has a ray of solutions at some solution

$(\bar{w}(0), \bar{z}(0))$ is contained in the boundary of D(M). Further if $M \in (Z-K_0) \cap S_0$ we obtain a necessary and sufficient condition on M so that for no q in the interior of D(M), (M,q) possesses a ray of solutions. In section 4.4 we show that for $M \in K_0$, (M,q) has infinitely many solutions if and only if q is in the boundary of D(M). We also give examples to illustrate the situation in cases of the other subclasses of Z.

We conclude this section after stating two preliminary lemmas which will be useful in the latter sections.

Lemma 4.1.1 : Suppose $M \in Z$ and suppose $(\bar{u}, \bar{v}) \geq 0$ is a nonzero solution to (M,0). Then $\bar{u} = 0$ and $M\bar{v} = 0$.

Proof :- Similar to the proof of theorem 3.2.3.

Lemma 4.1.2 (Cottle) : Let $(\bar{w}, \bar{z})$ be a solution to (M, q*). A vector $\bar{v} \geq 0 \in R^n$ generates a solution ray for (M, q*) at $\bar{z}$ if and only if

(i) there is $\bar{u} \geq 0 \in R^n$ such that $(\bar{u}, \bar{v})$ is a nonzero solution to (M, 0).

(ii) $\bar{v}^T \bar{w} = 0$

(iii) $\bar{z}^T M \bar{v} = 0$

4.2. The class $\bar{K}_o$:

Let $M \in K_o - K$. We consider the following partition of N and the corresponding representation of M as in lemma 3.3.5.

$$M = \begin{bmatrix} M_{J_1} & M_{J_1 J_2} & \cdots\cdots & M_{J_1 J_r} \\ M_{J_2 J_1} & M_{J_2} & \cdots\cdots & M_{J_2 J_r} \\ \vdots & & & \\ M_{J_{r-1} J_1} & M_{J_{r-1} J_2} & \cdots\cdots & M_{J_{r-1} J_r} \\ M_{J_r J_1} & M_{J_r J_2} & \cdots\cdots & M_{J_r} \end{bmatrix}$$

where $r \geqq 2$, $\bigcup_{i=1}^{r} J_i = N$, J_r may be empty and

(i) $\det (M_{J_k}) = 0, \quad 1 \leqq k \leqq r-1$

(ii) No proper principal minor of M_{J_k} is 0, $1 \leqq k \leqq r-1$.

(iii) If $J_r \neq \emptyset$, M_{J_r} is defined and is a K-matrix.

<u>Lemma 4.2.1</u> : Let $M \in K_o - K$ and let M be singular with no proper principal minor of M as zero. Then there does not exist $x \in R^n$, $x \geqq 0$ such that $Mx \geq 0$.

Proof :- We note that $M^T \in K_o$ with $\det (M^T) = 0$ and no proper principal minor of M^T as zero. From lemma 3.3.8 it follows that there exists $y > 0$, $y \in R^n$ such that $M^T y = 0$. Now the conclusion of the lemma follows from Tucker's theorem of the alternative [27, p.34].

<u>Theorem 4.2.1</u> : Let $M \in K_o - K$ and consider the representation of M as in lemma 3.3.5. Let $V \subseteq \{1,2, \ldots\ldots r-1\}$. Then

(i) There exists $m \in V$ such that $M_{J_k J_m} = 0$ for all $k \in V$ and $k \neq m$.

(ii) If $J_r \neq \emptyset$ and if $i \in J_r$, the i^{th} row $(M_{J_r J_k})_{i.} \neq 0$ implies that the i^{th} column $(M_{J_k J_r})_{.i} = 0$ for $1 \leq k \leq r-1$.

Proof :- Let $L = \bigcup_{k \in V} J_k$. Look at M_L. $M_L \in K_o - K$. Therefore by theorem 3.2.8 there exists $y \in R^{|L|}$, $y \geq 0$ such that $M_L y = 0$. Thus we have for this y

$$\sum_{s \in V} M_{J_k J_s} \; y_{J_s} = 0 \quad \text{for each} \quad k \in V$$

Or

$$\sum_{\substack{s \in V \\ k \neq s}} M_{J_k J_s} \, y_{J_s} + M_{J_k} \, y_{J_k} = 0 \quad \text{for each} \quad k \in V \quad \ldots (4.2.1)$$

Since for $k \neq s$ $M_{J_k J_s} \leq 0$, and since $M_{J_k} \in K_o - K$ with no proper principal minor as zero, it follows from lemma 4.2.1 that equation (4.2.1) can hold for any $k \in V$ only if

$$M_{J_k} y_{J_k} = 0 \quad \text{and} \quad \sum_{\substack{s \in V \\ k \neq s}} M_{J_k J_s} y_{J_s} = 0, \quad k \in V \quad \ldots (4.2.2)$$

Since $y \neq 0$ it follows from lemma 3.3.8 that equations (4.2.2) can hold only if for atleast one $m \in V$, $y_{J_m} > 0$.

Now consider $M_{J_k J_m} \; y_{J_m}$ for each $k \in V$.

Noting that $y_{J_s} \geqq 0$ and $M_{J_k J_s} \leqq 0$ for $k \neq s$, equations (4.2.2) can hold only if $M_{J_k J_m} \; y_{J_m} = 0$ for $k \in V$.

Therefore $M_{J_k J_m} = 0$. This completes the proof of (i).

To prove (ii) let $1 \leqq k \leqq r-1$ and let the i^{th} row $(M_{J_r J_k})_{i.}$ be denoted by X and the i^{th} column $(M_{J_k J_r})_{.i}$ by Y. Look at

$$\begin{bmatrix} M_{J_k} & Y \\ X & m_{ii} \end{bmatrix}$$

This is a $K_o - K$ matrix and proceeding as in the proof of (i) we can show that either $X = 0$ or $Y = 0$.

<u>Corollary 4.2.1</u> : Let $M \in K_o - K$ and consider the representation of M given by lemma 3.3.5. Let $J = \bigcup_{k=1}^{r-1} J_k$. By a principal rearrangement of rows and columns if necessary, M_J can be written as an upper diagonal block matrix with zeros in the blocks of matrices below the diagonal blocks.

Proof : We take $V = \{1, 2, \ldots\ldots r-1\}$ and apply result (i) of theorem 4.2.1. This gives us a m, $1 \leqq m \leqq r-1$ such that $M_{J_k J_m} = 0$ for $1 \leqq k \leqq r-1$, $k \neq m$. These blocks will form the first column blocks. We now omit m from V and apply result (i) of theorem 4.2.1, to get the blocks of matrices that will form the second column. Thus in a finite number of steps we obtain the desired representation for M_J. This concludes the proof.

<u>Lemma 4.2.2</u> : Let $M \in K_o - K$ and consider the representation of M given by lemma 3.3.5. Let

$$T = \{ i \mid 1 \leqq i \leqq r-1, \text{ there exists } x \geq 0 \in R^n \text{ such that } Mx = 0 \text{ and } x_{J_i} > 0 \}.$$

Then

(i) T is nonempty (ii) Let $J_T = \bigcup_{i \in T} J_i$. M_{J_T} has only 0 blocks in the off diagonal positions.

Proof :- Since $M \in K_o - K$ by theorem 3.2.8 there exists $x \geq 0 \in R^n$ such that $Mx = 0$.

Considering the equations

$$M_{J_k} x_{J_k} + \sum_{k \neq m}^{r-1} M_{J_k J_m} x_{J_m} = 0 , \quad 1 \leqq k \leqq r-1$$

we see from lemma 4.2.1 that if these equations were to hold

$$\sum_{\substack{m=1 \\ k \neq m}}^{r-1} M_{J_k J_m} \; x_{J_m} = 0 \quad \text{for each} \quad 1 \leqq k \leqq r-1$$

or equivalently, using the facts $x_{J_m} \geqq 0$, and $M_{J_k J_m} \leqq 0$, we get

$$M_{J_k J_m} \; x_{J_m} = 0 \quad \text{for} \quad k \neq m.$$

Therefore it follows that $M_{J_k} \; x_{J_k} = 0$.

Now lemma 3.3.8 implies that either $x_{J_k} > 0$ or $x_{J_k} = 0$.

Since $x \neq 0$, we must have atleast one i such that $x_{J_i} > 0$.

Thus T is nonempty.

Suppose $i \in T$. Then there exists $x^{(i)} \geq 0 \in R^n$ such that $x^{(i)}_{J_i} > 0$ and $Mx^{(i)} = 0$.

As before we can show that this implies that $M_{J_k J_i} \; x^{(i)}_{J_i} = 0$ for $1 \leqq k \leqq r-1$, $k \neq i$, and which in turn implies

$$M_{J_k J_i} = 0 \quad \text{for} \quad 1 \leqq k \leqq r-1, \quad k \neq i,$$

Moreover by corollary 4.2.1, $M_{J_i J_k} = 0$ for $i \neq k$.

Thus M_T has only 0 blocks in the off diagonal positions.

<u>Corollary 4.2.2</u> : Let $M \in K_o - K$. There exists a partition of N, $N = J_1 \cup \ldots\ldots \cup J_r$, $r \geqq 2$ and if necessary by a principal rearrangement of rows and columns of M, M can be represented in the partitioned form given by lemma 3.3.5, with $T = \{1,2, \ldots.s\}$, $1 \leqq s \leqq r-1$ where T is as defined in lemma 4.2.2. Also in such a representation M_{J_T} has nonzero blocks of matrices only in the diagonal positions, M_J, where $J = \bigcup_{i=1}^{r-1} J_i$, is an upper diagonal block matrix with 0's in the blocks below the diagonal blocks, and if for $x \geq 0 \in R^n$, $Mx = 0$ then $x_{J_k} > 0$ for some $1 \leqq k \leqq s$, and $x_{J_i} = 0$ if $s+1 \leqq i \leqq r-1$.

Proof :- This follows from lemma 3.3.5, corollary 4.2.1 and lemma 4.2.2.

<u>Definition 4.2.1</u> : Let $M \in K_o$. We say that $M \in \bar{K}_o$ if either

(i) $M \in K$ or (ii) In the representation of M as in corollary 4.2.2, $T = \{1,2, \ldots. r-1\}$; i.e. Given any $1 \leqq k \leqq r-1$, there exists $x^{(k)} \in R^n$, $x^{(k)} \geq 0$ such that $x^{(k)}_{J_k} > 0$ and $Mx = 0$.

<u>Lemma 4.2.3</u> : Let $M \in \bar{K}_o - K$ and suppose in the representation of M as in corollary 4.2.2, $J = \bigcup_{i=1}^{r-1} J_i$. M_J is a block diagonal matrix with '0' blocks at the off diagonal positions.

Proof : This follows directly from the definition 4.2.1 and corollary 4.2.2.

<u>Lemma 4.2.4</u> : Let $M \in K_o - K$ and suppose that in the representation of M as in corollary 4.2.2, $J_r = \emptyset$. $M \in \bar{K}_o$ if and only if all the off diagonal blocks of M_J, where $J = \bigcup_{i=1}^{r-1} J_i$, are 0's.

Proof :- Suppose $M \in \bar{K}_o$. From lemma 4.2.3 it follows that the off diagonal blocks of M_J are 0's.

Now suppose that the off diagonal blocks of M_J are 0's. For each $1 \leqq k \leqq r-1$, by lemma 3.3.8, there exists $x_{J_k} \in R^{|J_k|}$, $x_{J_k} > 0$ such that $M_{J_k} x_{J_k} = 0$.

We note that $y \in R^n$ defined by $y_{J_k} = x_{J_k}$; $y_{J_i} = 0$; if $i \neq k$, satisfies $My = 0$, $y \geq 0$, $y_{J_k} > 0$. This completes the proof of the lemma.

<u>Lemma 4.2.5</u> : Let $M \in K_o - K$ and consider the representation of M as in corollary 4.2.2. Suppose $J_r \neq \emptyset$ and let $J = \bigcup_{i=1}^{r-1} J_i$. If M_J has '0' blocks at the off diagonal positions and if either $M_{J_r J} = 0$ or $M_{J J_r} = 0$ then $M \in \bar{K}_o$.

Proof : As in the proof of lemma 4.2.4, for each $1 \leqq k \leqq r-1$, there exists $x_J^{(k)} \geq 0 \in R^{|J|}$, $x_{J_k}^{(k)} > 0$ such that

$$M_J \, x_J^{(k)} = 0$$

For this $x_J^{(k)}$ consider the equations

$$M_J x_J^{(k)} + M_{JJ_r} y_{J_r} = 0 \quad \dots \quad \dots \quad \dots \quad (4.2.3)$$

$$M_{J_r J} x_J^{(k)} + M_{J_r} y_{J_r} = 0 \quad \dots \quad \dots \quad \dots \quad (4.2.4)$$

If $M_{J_r J} x_J^{(k)} = 0$ take $y_{J_r} = 0$

If $M_{J_r J} x_J^{(k)} \neq 0$ it follows that $M_{JJ_r} = 0$ ... (4.2.5)

Since M_{J_r} is a K matrix there exists a unique $y_{J_r} \geq 0$ such that equation (4.2.4) is satisfied, as $M_{J_r J} \leqq 0$, $x_J^{(k)} \geqq 0$ and, by theorem 1.3.2, $M_{J_r}^{-1} \geq 0$.

From (4.2.5) it follows that for such a y_{J_r} equation (4.2.3) is also satisfied.

Define $y^{(k)}$ by taking $y_J^{(k)} = x_J^{(k)}$ and $y_{J_r}^{(k)} = y_{J_r}$

Then $My = 0$, $y_{J_k} > 0$, $y \geq 0$. Therefore $k \in T$ and the conclusion of the lemma follows.

<u>Lemma 4.2.6</u> : Let $M \in K_o - K$ and consider the representation of M as in corollary 4.2.2. $M \in \bar{K}_o$ if and only if for each $1 \leqq k \leqq r-1$, there exists $\bar{y}^{(k)} \geq 0 \in R^n$ such that

(i) $\bar{y}_{J_k}^{(k)} > 0$ (ii) $\bar{y}_{J_m}^{(k)} = 0$ if $m \neq k$, $1 \leqq m \leqq r-1$, and

(iii) $M \bar{y}^{(k)} = 0$.

Proof :- Definition 4.2.1. of $\bar{K}_0$ matrices ensures for any $1 \leq k \leq r-1$ the existence of a $y^* \geq 0 \in R^n$ satisfying $y^*_{J_k} > 0$ and $M y^* = 0$. We must show that there is a $\bar{y} \geq 0 \in R^n$ which in addition to (i) and (iii) satisfies (ii) also. We proceed as follows:

First we note that for any $\beta > 0$ the following equalities hold.

$$\beta M_{J_r J} \; y^*_J + \beta M_{J_r} \; y^*_{J_r} = 0 \quad \ldots \quad \ldots \quad (4.2.6)$$

$$\beta M_{JJ_r} \; y^*_{J_r} + \beta M_J \; y^*_J = 0 \quad \ldots \quad \ldots \quad (4.2.7)$$

Also using lemma 4.2.1. as in the proof of lemma 4.2.2., we get

$$\beta M_{JJ_r} \; y^*_{J_r} = 0 \quad \ldots \quad \ldots \quad (4.2.8)$$

Now, as in the proof of lemma 4.2.4 noting that $M_J \in \bar{K}_0$, we can get a $y_J \in R^{|J|}$ such that $y_{J_k} > 0$, $y_{J_m} = 0$ if $k \neq m$, $1 \leq m \leq r-1$, and $M_J \, y_J = 0$.

With this y_J consider the equation

$$M_{J_r J} \; y_J + M_{J_r} \; y_{J_r} = 0 \quad \ldots \quad \ldots \quad (4.2.9)$$

Suppose $M_{J_r J} \, y_J = 0$; Choose $y_{J_r} = 0$. This satisfies the above equation and $\bar{y} = \begin{pmatrix} y_J \\ 0 \end{pmatrix} \in R^n$ is the required vector.

Suppose $M_{J_rJ}\, y_J \neq 0$. Since $M_{J_rJ}\, y_J = M_{J_rJ_k}\, y_{J_k}$ and $y_{J_k} > 0$, we conclude that $M_{J_rJ_k} \leq 0$. (i.e. $M_{J_rJ_k} \neq 0$).

Since $y^*_{J_k} > 0$ it follows from equation (4.2.6) that

$$M_{J_rJ}\, y^*_J = \sum_{i=1}^{r-1} M_{J_rJ_i}\, y^*_{J_i} \leq 0.$$

Therefore there exists $\beta_o > 0$ such that

$$\beta_o\, M_{J_rJ}\, y^*_J \geqq M_{J_rJ}\, y_J .$$

Multiplying by $M_{J_r}^{-1} \geq 0$ both sides and using (4.2.7) and (4.2.9) we get, $-\beta_o\, y^*_{J_r} \geqq -y_{J_r} = M_{J_r}^{-1}\, (M_{J_rJ}\, y_J)$.

Now using (4.2.8) with $\beta = \beta_o$ and noting that $M_{JJ_r} \leqq 0$ we see that

$$M_{JJ_r}\, y_{J_r} = 0$$

It is now easy to verify that $\bar{y} = \begin{bmatrix} y_J \\ y_{J_r} \end{bmatrix}$ satisfies (i), (ii) and (iii). This concludes the proof.

<u>Lemma 4.2.7</u> : Let $M \in K_o - K$. Consider the representation of M as in corollary 4.2.2. $M \in \bar{K}_o$ if and only if there exists $y \geq 0 \in R^n$ such that

(i) $My = 0$ (ii) $y_J > 0$.

Proof : Follows immediately from the definition 4.2.1.

<u>Theorem 4.2.2</u> : Let $M \in \bar{K}_0 - K$ and consider the representation of M given by corollary 4.2.2. If $L \subseteq N$ such that M_L is singular then there is a $1 \leqq k \leqq r-1$ such that $J_k \subseteq L$.

Proof : Let $L_k = L \cap J_k$, $1 \leqq k \leqq r$

Case (i) $L_r = L \cap J_r = \emptyset$.

Suppose L_k is a proper subset of J_k for each $1 \leqq k \leqq r-1$.

The principal submatrix M_L has '0' blocks at the off diagonal positions in the partition $L = \bigcup_{k=1}^{r-1} L_k$, the result of lemma 4.2.3. Since each det (M_{L_k}) is a proper principal minor of M_{J_k} it follows that det $(M_{L_k}) > 0$ for all $1 \leqq k \leqq r-1$. Therefore det $(M_L) = \prod_{k=1}^{r-1}$ det $(M_{L_k}) > 0$, which contradicts the hypothesis that M_L is singular. This concludes the proof in this case.

Case (ii) $L_r \neq \emptyset$. Once again suppose that $J_k \neq L_k$ for all $1 \leqq k \leqq r-1$.

Let $\bar{L} = \bigcup_{k=1}^{r-1} L_k$, $L_{r+1} = J_r - L_r$. Let $L_{r+1} \neq \emptyset$.

Look at M_L in the partitioned form

$$M_L = \begin{bmatrix} M_{\bar{L}} & M_{\bar{L} L_r} \\ M_{L_r \bar{L}} & M_{L_r} \end{bmatrix}$$

We note that as shown in case (i) above $M_{\bar{L}}$ is a K-matrix.

Since $M \in \bar{K}_o - K$, using lemma 4.2.7, we get $y^* \geq 0 \in R^n$ such that

(i) $My^* = 0$ (ii) $y^*_J > 0$ and using lemma 4.2.1, we also have (iii) $M_{JJ_r} \; y^*_{J_r} = 0$.

Therefore we have the following equalities for any $\beta > 0 \in R$.

$$\beta(-M_{L_rJ}\, y^*_J - M_{L_rL_{r+1}}\, y^*_{L_{r+1}}) = \beta M_{L_r}\, y^*_{L_r} \quad \ldots \quad (4.2.9)$$

$$\beta(M_{\bar{L}L_r}\, y^*_{L_r} + M_{\bar{L}L_{r+1}}\, y^*_{L_{r+1}}) = 0 \quad \ldots \quad \ldots \quad (4.2.10)$$

Also, since $M_{\bar{L}L_r}$ and $M_{\bar{L}L_{r+1}}$ are nonpositive matrices, equation (4.2.10) implies, in fact, that

$$\beta M_{\bar{L}L_r}\, y^*_{L_r} = \beta M_{\bar{L}L_{r+1}}\, y^*_{L_{r+1}} = 0 \quad \ldots \quad (4.2.11)$$

Suppose now $\begin{bmatrix} y_{\bar{L}} \\ y_{L_r} \end{bmatrix} \geq 0 \in R^{|L|}$ satisfies the equations

$$M_{\bar{L}}\, y_{\bar{L}} + M_{\bar{L}L_r}\, y_{L_r} = 0 \quad \ldots \quad \ldots \quad (4.2.12)$$

and

$$M_{L_r\bar{L}}\, y_{\bar{L}} + M_{L_r}\, y_{L_r} = 0 \quad \ldots \quad \ldots \quad (4.2.13)$$

Choose $\beta_o > 0$ such that

$$\beta_o \left(-M_{L_r J}\, y_J^* - M_{L_r L_{r+1}}\, y^*_{L_{r+1}} \right) \leqq -M_{L_r \bar{L}}\, y_{\bar{L}} \,.$$

Such a $\beta_o > 0$ exists because $y_J^* > 0$, $-M_{L_r J}$, $-M_{L_r \bar{L}}$ are nonnegative matrices and if $-M_{L_r J}\, y_J^* = 0$ it follows that $-M_{L_r \bar{L}}\, y_{\bar{L}}$ is also zero, as $\bar{L} \subseteq J$.

Multiplying both sides of the above inequality by $M_{L_r}^{-1} \geq 0$, we get

$$\beta_o\, y^*_{L_r} \leqq y_{L_r} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4.2.14)$$

(4. 2.14) and (4.2.11) imply, because of the nonpositivity of $M_{\bar{L}L_r}$, that

$$M_{\bar{L}\, L_r}\, y_{L_r} = 0$$

Therefore $y_{\bar{L}} = 0$, $y_{L_r} = 0$. Thus the only solution to (4.2.12) and (4.2.13) is $y_{\bar{L}} = 0$, $y_{L_r} = 0$. This contradicts theorem 3.2.8 and concludes the proof. (The case $L_{r+1} = \emptyset$ is similar).

<u>Theorem 4.2.3</u> : Suppose $M \in \bar{K}_o$ and $L \subsetneq N$. Then $M_L \in \bar{K}_o$.

Proof : If $M_L \in K$ then clearly $M_L \in \bar{K}_o$.

Suppose $M_L \in K_o - K$.

By theorem 3.2.8 M_L is singular and by theorem 4.2.2. there is a $1 \leqq k \leqq r-1$ such that $J_k \subseteq L$. Without loss of generality assume that $J_i \subseteq L$ for $i = 1,2, \ldots s$ and let $L_i = J_i \cap L$ for $1 \leqq i \leqq r$. M_L has the representation

$$M_L = \begin{bmatrix} M_{J_1} & & & & & M_{J_1 L_r} \\ & \ddots & & & & \\ & & M_{J_s} & & & M_{J_s L_r} \\ & & & M_{L_{s+1}} & & \vdots \\ & & & & \ddots & \vdots \\ M_{L_r J_1} & \cdots & M_{L_r J_s} & M_{L_r L_{s+1}} & \cdots & M_{L_r} \end{bmatrix}$$

Let $\bar{L} = \bigcup_{i=s+1}^{r} L_i$. Since $J_i \not\subseteq \bar{L}$, for any $1 \leqq i \leqq r-1$, from theorem 4.2.2 it follows that $M_{\bar{L}}$ is nonsingular and hence $M_{\bar{L}}$ is a K-matrix. Thus the above representation of M_L is also the representation given by lemma 3.3.5 and corollary 4.2.2. with $r = s+1$. It is therefore enough to show that given any $1 \leqq k \leqq s$ there exists $y^* \geq 0 \in R^{|L|}$ such that $y^*_{J_k} > 0$ and $M_L y^* = 0$. This will conclude the proof of the theorem.

In case $L_r = \emptyset$, since $M_{\bar{L} J_k} = 0$ for $1 \leqq k \leqq r-1$, the existence of such a y^* is immediate from lemma 4.2.5.

Consider the case $L_r \neq \emptyset$. Let $1 \leqq k \leqq s$ and let $L_{r+1} = J_r - L_r$. We shall consider only the case $L_{r+1} \neq \emptyset$, the proof for the case $L_{r+1} = \emptyset$ being similar and easier.

Since $M \in \bar{K}_o - K$ applying lemma 4.2.6 we get a $y \geq 0 \in R^n$ such that

(i) $My = 0$, (ii) $y_{J_k} > 0$ and (iii) $y_{J_m} = 0$, if $1 \leqq m \leqq r-1$, $m \neq k$.

We therefore obtain the following equality

$$-M_{L_r J_k} \; y_{J_k} - M_{L_r \, L_{r+1}} \; y_{L_{r+1}} = M_{L_r} \, y_{L_r}$$

Take $y^*_{J_k} = y_{J_k}$. Since $-M_{L_r J_k} \, y_{J_k} - M_{L_r L_{r+1}} \, y_{L_{r+1}} \geqq -M_{L_r J_k} \, y^*_{J_k}$, multiplying both sides by $M^{-1}_{L_r} \geq 0$ we get

$$y_{L_r} \geqq M^{-1}_{L_r} \left(-M_{L_r J_k} \; y^*_{J_k}\right) = y^*_{L_r}$$

Noting that $M_{J_k L_r} \, y_{L_r} = 0$, $1 \leqq k \leqq s$, $M_{L_k L_r} \, y_{L_r} = 0$, $s+1 \leqq k \leqq r-1$, we conclude that

$$M_{J_k L_r} \; y^*_{L_r} = 0, \quad 1 \leqq k \leqq s \quad \text{and}$$

$$M_{L_k L_r} \; y^*_{L_r} = 0, \quad s+1 \leqq k \leqq r-1$$

Define $\bar{y} \in R^{|L|}$ by taking $\bar{y}_{J_k} = y^*_{J_k}$; $\bar{y}_{L_r} = y^*_{L_r}$; $\bar{y}_{J_m} = 0$ for $1 \leqq m \leqq s$, $m \neq k$ and $\bar{y}_{L_i} = 0$, $s+1 \leqq i \leqq r-1$. This $\bar{y}$ is the required vector and this concludes the proof.

<u>Theorem 4.2.4</u> : Let $M \in \bar{K}_o$. Then $M^T \in \bar{K}_o$.

Proof : If $M \in K$ obviously $M^T \in K$ and therefore $M^T \in \bar{K}_o$.

So let us assume that $M \in \bar{K}_o - K$.

By lemma 4.2.7. there exists $\bar{y} \geq 0 \in R^n$ such that $\bar{y}_J > 0$ and $M\bar{y} = 0$, where $J = \bigcup_{k=1}^{r-1} J_k$ in the representation of M as in corollary 4.2.2.

Let $I_1 = \{ i \in J_r \mid \bar{y}_i = 0 \}$; $I_2 = J_r - I_1$. We note that if $I_1 = \emptyset$ and $I_2 = J_r$ then $M_{JJ_r} = 0$. On the otherhand if $I_1 = J_r$ and $I_2 = \emptyset$ then $M_{J_rJ} = 0$. We shall consider only the more general case $I_1 \neq \emptyset$, $I_2 \neq \emptyset$, the proof for the other cases being similar.

We note that since the equations

$$M_{I_1J}\ \bar{y}_J + M_{I_1}\ \bar{y}_{I_1} + M_{I_1I_2}\ \bar{y}_{I_2} = 0, \quad \bar{y}_{I_1} = 0, \quad \bar{y}_{I_2} > 0$$

hold, we must have $M_{I_1J} = 0$ and $M_{I_1I_2} = 0$.

Also, from the equations $M_{JI_2}\ \bar{y}_{I_2} = 0$, $M_{JI_2} \leqq 0$ it follows that $M_{JI_2} = 0$.

Consider M^T. Since the determinants of any principal submatrix $(M_L{}^T)$ of M^T and M_L of M are the same for any $L \subseteq N$, $J_1, J_2, \ldots, J_r$

of the representation of M and of M^T as in corollary 4.2.2. can be assumed to be the same.

We also note that $M^T_{J_k J_m} = (M_{J_m J_k})^T$.

Consider $M^T_J = (M_J)^T$. By lemma 4.2.4. $M_J \in \bar{K}_o - K$ and by lemma 4.2.7. there exists $y^*_J \in R^{|J|}$, $y^*_J > 0$ such that $M^T_J y^*_J = 0$.

With this y^*_J consider the equations

$$M^T_{I_1 J} \; y^*_J + M^T_{I_1} \; x_{I_1} + M_{I_1 I_2} \; x_{I_2} = 0 \qquad \ldots \qquad (4.2.15)$$

$$M^T_{I_2 J} \; y^*_J + M^T_{I_2 I_1} \; x_{I_1} + M^T_{I_2} \; x_{I_2} = 0 \qquad \ldots \qquad (4.2.16)$$

We note that in the above $M^T_{I_2 J} = (M_{J I_2})^T = 0$ and

$$M^T_{I_2 I_1} = (M_{I_1 I_2})^T = 0.$$

Therefore (4.2.16) reduces to

$$M^T_{I_2} \; x_{I_2} = 0.$$

Since M_{I_2} is a K-matrix the only solution to (4.2.16) is therefore $x_{I_2} = 0$.

Now equation (4.2.15) reduces to

$$M^T_{I_1 J} \; y^*_J + M^T_{I_1} \; x_{I_1} = 0 \qquad \ldots \qquad (4.2.17)$$

Since $M^T_{I_1J} \leqq 0$ and M_{I_1} is a K-matrix there is a solution $x^*_{I_1} \geqq 0$ to (4.2.17).

Let $x^*_{J_r} = \begin{pmatrix} x^*_{I_1} \\ 0 \end{pmatrix}$. It is easy to see that

$$M^T_{JJ_r}\, x^*_{J_r} = (M_{J_rJ})^T\, x^*_{J_r} = (M_{I_1J})^T\, x^*_{I_1} + (M_{I_2J})^T\, x^*_{I_2} = 0,$$

where $x^*_{I_2} = 0$.

Therefore $y^* = \begin{pmatrix} y^*_J \\ x^*_{J_r} \end{pmatrix}$ satisfies $M^T y^* = 0$, $y^*_J > 0$.

The conclusion of the theorem now follows from lemma 4.2.7.

<u>Theorem 4.2.5</u> : Let $M \in K_o$ be symmetric. Then $M \in \bar{K}_o$.

Proof : If $M \in K$ then by definition $M \in \bar{K}_o$. So let $M \in K_o - K$. and consider the representation as in corollary 4.2.1. By symmetry it follows that M_J is block diagonal having 0's in the off diagonal blocks. Similarly result (ii) of theorem 4.2.1 and symmetry imply that $M_{JJ_r} = M_{J_rJ} = 0$.

The conclusion of the theorem now follows from lemma 4.2.5.

4.3. Existence of solution rays :

In this section we characterise the set of q for which (M,q) possesses a ray of solutions when M belongs to some subclasses of Z.

Theorem 4.3.1 : Let $M \in K_o$. If there exists a ray of solutions to (M, q^*) at some solution $(\bar{w}, \bar{z})$ to (M, q^*) then q^* is in the boundary of $D(M)$.

Proof : Let $(\bar{w}, \bar{z})$ be a solution to (M, q^*) and let there be a solution ray for (M, q^*) at $\bar{z}$.

From lemma 4.1.2. it follows that there exists $(\bar{u}, \bar{v}) \geq 0$, $\bar{v} \neq 0$ such that $(\bar{u}, \bar{v})$ is a nonzero solution to $(M, 0)$ with $\bar{v}^T \bar{w} = 0$ and $\bar{z}^T M\bar{v} = 0$. From lemma 4.1.1. it follows that $\bar{u} = 0$ and $M\bar{v} = 0$.

Let $L_1 = \{i \mid \bar{v}_i > 0\}$; $L_2 = \{i \mid \bar{w}_i = 0\}$; $L_3 = \{i \mid \bar{z}_i > 0\}$.

Note that $L_1 \subseteq L_2$ and $L_3 \subseteq L_2$.

Also, $M\bar{v} = 0 \Longrightarrow M_{L_1} \bar{v}_{L_1} = 0$.

Now $M^T_{L_2}$ is a K_o-matrix with $M^T_{L_1}$ as a principal submatrix whose determinant is zero. Therefore $M^T_{L_2} \in K_o - K$ and by theorem 3.2.8 there exists $x \geq 0 \in R^{|L_2|}$ such that $x^T M_{L_2} = 0$.

Define x^* by taking $x^*_{L_2} = x$; $x^*_{N-L_2} = 0$.

It is easy to verify that (i) $x^{*T} M \leq 0$ (ii) $x^{*T} M \bar{z} = 0$ (iii) $x^{*T} \bar{w} = 0$ and (iv) $x^{*T} q^* = 0$.

Since $x^* \geq 0$ there is a $p \in R^n$ such that ${x^*}^T p < 0$.

Therefore we have

$${x^*}^T \geq 0, \quad {x^*}^T M \leqq 0 \quad \text{and} \quad {x^*}^T (q + \theta p) < 0 \text{ for all } \theta > 0.$$

Therefore by Farkas' lemma $q^* + \theta p \notin D(M)$ for any $\theta > 0$. Since $D(M)$ is a convex cone whose interior is nonempty, this implies that q^* is in the boundary of $D(M)$. This completes the proof.

Lemma 4.3.1 : Let $M \in (Z - K_o) \cap S_o$ and suppose that there exists $z \geq 0 \in R^n$ such that $Mx = 0$. Then there exists a partition $N = J_1 \cup \ldots\ldots \cup J_{r-1} \cup J_r \cup J_{r+1}$ and a representation of M in the partitioned form such that

(i) $r \geqq 2$, $J_i \neq \emptyset$, $1 \leqq i \leqq r-1$, $J_{r+1} \neq \emptyset$.

(ii) $\det (M_{J_i}) = 0$, but all proper principal minors of M_{J_i} are positive.

(iii) J_r may be empty. If $J_r \neq \emptyset$, M_{J_r} is a K-matrix.

(iv) $-M^T_{J_{r+1}} \in S$.

(v) There exists a s, $1 \leqq s \leqq r-1$, such that for any k, $1 \leqq k \leqq s$, there is a $x^{(k)} \geq 0 \in R^n$ with $x_{J_k} > 0$ and $Mx = 0$. (s is the largest such)

(vi) $M_{J_{r+1} J_k} = 0$, $1 \leqq k \leqq r$; Also $M_{J_k J_m} = 0$ if $k \neq m$, $1 \leqq k \leqq s$ and $1 \leqq m \leqq s$.

Proof : Under the hypothesis of the lemma, lemma 3.3.7 applies and we get a partition $N = J_1 \cup \ldots\ldots \cup J_{r+1}$, $r \geq 2$ such that M has the representation

$$M = \begin{bmatrix} M_J & M_{JJ_r} & M_{JJ_{r+1}} \\ M_{J_rJ} & M_{J_r} & M_{J_rJ_{r+1}} \\ 0 & 0 & M_{J_{r+1}} \end{bmatrix}$$

The J_i's satisfy conditions (i) - (iv). Also $M_{J_{r+1}J} = M_{J_{r+1}J_r} = 0$. Now consider M_J. $M_J \in K_o - K$. Therefore corollary 4.2.2. applies and M_J has a representation, if necessary with a principal rearrangement of rows and columns, as given by corollary 4.2.2. This rearrangement of M_J does not upset the form of M as obtained above. We can now renumber the J_i's, $1 \leq i \leq r-1$, obtaining a representation for M which satisfies all of (i) - (vi) of the lemma. This concludes the proof of the lemma.

Theorem 4.3.2 : Let $M \in (Z - K_o) \cap S_o$ and let there be a $x \geq 0 \in R^n$ such that $Mx = 0$. Consider the representation of M as in lemma 4.3.1. Let $-M_{J_{r+1}} \in S$ and let

$$R(M) = \{ q \mid q \in D(M), \text{ there is a ray of solutions for } (M,q) \text{ at some solution } (\bar{w}, \bar{z}) \text{ to } (M, q) \}.$$

Then,

$$R^{o}(M) = R(M) \cap \text{Interior of } D(M) = \phi \text{ only if}$$

$$M_{J_k J_{r+1}} = 0, \text{ for } 1 \leqq k \leqq s, \text{ and } M_{J_t J_{r+1}} \neq 0 \Longrightarrow M_{J_i J_t} = 0$$

$$\text{for all } 1 \leqq i \leqq s \text{ and for any } s+1 \leqq t \leqq r.$$

Proof : Case (A): Suppose there exists k, $1 \leqq k \leqq s$, such that

$$M_{J_k J_{r+1}} \neq 0.$$

Consider the system of inequalities

$$\begin{bmatrix} M^T_{J_k} \\ (M_{J_k J_{r+1}})^T \\ I \end{bmatrix} u \geq 0, \quad u \in R^{|J_k|} \quad \ldots \quad \ldots \quad (4.3.1)$$

We note that $M^T_{J_k} u \geqq 0$, $u \geqq 0 \Longrightarrow$ either $u = 0$, or in view of lemma 4.2.1 and lemma 3.3.8, $M^T_{J_k} u = 0$ and $u > 0$.

Since $u = 0$ does not satisfy the system of inequalities (4.3.1) we need consider only $M^T_{J_k} u = 0$ and $u > 0$.

However since $M_{J_k J_{r+1}} \leq 0$, such a $u > 0$ does not give

$$(M_{J_k J_{r+1}})^T u \geqq 0.$$

Therefore there is no solution to the system (4.3.1).

Thus by Steimke's theorem of the alternative [27, p.34] there exist $\begin{bmatrix} y_{J_k} \\ y_{J_{r+1}} \end{bmatrix} \in R^{|J_k \cup J_{r+1}|}$, $\bar{y} \in R^{|J_k|}$ such that

$$(M_{J_k} \quad M_{J_{r+1}} \quad I) \begin{bmatrix} y_{J_k} \\ y_{J_{r+1}} \\ \bar{y} \end{bmatrix} = 0, \quad y_{J_k} > 0, \quad y_{J_{r+1}} > 0 \text{ and } \bar{y} > 0.$$

From here we obtain the following inequality.

$$- M_{J_k} y_{J_k} - M_{J_k J_{r+1}} y_{J_{r+1}} > 0 \qquad \ldots \qquad (4.3.2)$$

Further, since $-M_{J_{r+1}} \in S$, there exists $x_{J_{r+1}} \in R^{|J_{r+1}|}$ such that

$$x_{J_{r+1}} > 0 \text{ and } -M_{J_{r+1}} x_{J_{r+1}} > 0 \qquad \ldots \qquad (4.3.3)$$

From (4.3.2) and (4.3.3) it follows that there is a real number $\lambda > 0$ such that

$$- M_{J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}}) > 0 \text{ and}$$

$$- M_{J_k} y_{J_k} - M_{J_k J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}}) > 0.$$

Let $J = \bigcup_{i=1}^{r} J_i$. Since $M_J \in K_o - K$ and since $1 \leqq k \leqq s$,

using corollary 4.2.2. and proceeding as in the proof of lemma 4.2.6, we obtain a $\bar{v}_J \in R^{|J|}$ such that

(i) $\bar{v}_{J_k} > 0$ (ii) $\bar{v}_{J_m} = 0$, $m \neq k$, $1 \leqq m \leqq r-1$ and

(iii) $M_J \bar{v}_J = 0$.

Define $\bar{v} \in R^n$ by taking $\bar{v} = \begin{bmatrix} \bar{v}_J \\ 0 \end{bmatrix}$. Note that $\bar{v} \geq 0$ and $M\bar{v} = 0$.

We consider two cases

Case (i) $J_r = \emptyset$

Define $q^* \in R^n$ by taking

$$q^*_{J_{r+1}} = -M_{J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}})$$

$$q^*_{J_k} = -M_{J_k} y_{J_k} - M_{J_k J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}})$$

$$\text{and} \quad q^*_{J_i} = e_{|J_i|} - M_{J_i J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}}),$$

$$\text{for} \quad 1 \leqq i \leqq r-1, \quad i \neq k.$$

Note that $q^* > 0$ and (M, q^*) has the solution

$$w^*_{J_i} = e_{|J_i|} - M_{J_i J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}}); \quad 1 \leqq i \leqq r-1, \quad i \neq k.$$

$$z^*_{J_i} = 0 \; ; \quad 1 \leqq i \leqq r-1, \quad i \neq k.$$

$$w^*_{J_k} = 0 \; ; \quad z^*_{J_k} = y_{J_k}.$$

$$w^*_{J_{r+1}} = 0 \; ; \quad z^*_{J_{r+1}} = y_{J_{r+1}} + \lambda x_{J_{r+1}}.$$

It is easy to verify now that $\bar{v}$ generates a ray of solutions to (M, q^*) at the solution (w^*, z^*).

Case (ii) : $J_r \neq \emptyset$.

Let $I_1 = \{ i \mid i \in J_r ,\ \bar{v}_i > 0 \}$; $I_2 = J_r - I_1$.

If $I_1 = \emptyset$ and $I_2 = J_r$ we proceed as in case (i) above and define $q^*_{J_r} = e_{|J_r|} - M_{J_r J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}})$; we note that $q^* > 0$. If we define $w^*_{J_r} = e_{|J_r|} - M_{J_r J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}})$, $z^*_{J_r} = 0$ and the other components of w^* and z^* as in case (i) above, it is easy to see that (w^*, z^*) solves (M, q^*) and $\bar{v}$ generates a ray of solutions at (w^*, z^*).

In what follows the possibility that $I_2 = \emptyset$ is permitted, in which case the equations and inequalities we consider need only be slightly changed.

We note that $\bar{v}_{I_1} > 0$, $\bar{v}_{I_2} = 0$ and that

$$M_{I_1 J_k} \bar{v}_{J_k} + M_{I_1} \bar{v}_{I_1} = 0. \text{ Also } (\bar{v}_{J_k}, \bar{v}_{I_1}) > 0 \quad \ldots (4.3.4)$$

Further,

$$M_{I_2 J_k} \bar{v}_{J_k} + M_{I_2 I_1} \bar{v}_{I_1} + M_{I_2} \bar{v}_{I_2} = 0$$

which implies $M_{I_2 J_k} = 0$; $M_{I_2 I_1} = 0$ ... ... (4.3.5)

We also have $M_{J_m I_1} = 0$, for $1 \leq m \leq r-1$ ... (4.3.6)

equation (4.4.4) and Steimke's theorem of the alternative imply that $u^T(-M_{I_1 J_k}, -M_{I_1}) \leq 0$ has no solution.

Also, $u^T(-M_{I_1}) = 0 \Longrightarrow u = 0$, since M_{I_1} is a K-matrix.

Thus there is no solution to

$$u^T(-M_{I_1 J_k}, -M_{I_1}, I) \leq 0$$

Therefore by Steimke's theorem of the alternative, there exists

$$\begin{bmatrix} \bar{y}_{J_k} \\ y_{I_1} \end{bmatrix} \geq 0 \in R^{|J_k \cup I_1|} \text{ such that}$$

$$-M_{I_1 J_k} \bar{y}_{J_k} - M_{I_1} y_{I_1} > 0$$

Choose β, a positive real number, so that

$$-M_{J_k}(y_{J_k} + \beta \bar{y}_{J_k}) - M_{J_k J_{r+1}}(y_{J_{r+1}} + \lambda x_{J_{r+1}}) > 0.$$

Also, we have

$$-M_{I_1 J_k}(y_{J_k} + \beta \bar{y}_{J_k}) - M_{I_1}(\beta y_{I_1}) > 0.$$

Define q^* by taking

$$q^*_{J_i} = e_{|J_i|} - M_{J_i J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}}), \text{ for } 1 \leqq i \leqq r-1, \quad i \neq k.$$

$$q^*_{J_k} = - M_{J_k} (y_{J_k} + \beta \bar{y}_{J_k}) - M_{J_k J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}})$$

$$q^*_{I_1} = - M_{I_1 J_k} (y_{J_k} + \beta \bar{y}_{J_k}) - M_{I_1} (\beta y_{I_1}) - M_{I_1 J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}})$$

$$q^*_{I_2} = e_{|I_2|} - M_{I_2 J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}})$$

and $q^*_{J_{r+1}} = - M_{J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}})$.

Using (4.3.5) and (4.3.6) it is easy to verify that, as in case (i), we can obtain a solution (w^*, z^*) to (M, q^*) at which $\bar{y}$ generates a ray of solutions.

We also note that since $q^* > 0$ it is in the interior of $D(M)$. This completes the proof for case (A).

Case (B) : Suppose now there is a t, $s+1 \leqq t \leqq r$ and a k, $1 \leqq k \leqq s$ such that

$$M_{J_t J_{r+1}} \neq 0 \quad \text{and} \quad M_{J_k J_t} \neq 0.$$

We proceed as in case (A) and consider the system of inequalities

$$\begin{bmatrix} M^T_{J_t} \\ (M_{J_t J_{k+1}})^T \\ I \end{bmatrix} u \geq 0, \qquad u \in R^{|J_t|}$$

Using Steimke's theorem of the alternative and the arguments of (4.3.3) we obtain $y^*_{J_t} > 0 \in R^{|J_t|}$, $y_{J_{r+1}} > 0 \in R^{|J_{r+1}|}$, $x_{J_{r+1}} > 0 \in R^{|J_{r+1}|}$ and $\lambda > 0 \in R$ such that

$$- M_{J_t} y^*_{J_t} - M_{J_t J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}}) > 0 \quad \ldots \quad (4.3.7)$$

Similarly considering the system of inequalities

$$\begin{bmatrix} M^T_{J_k} \\ (M_{J_k J_t})^T \\ I \end{bmatrix} u \geq 0, \qquad u \in R^{|J_k|}$$

and proceeding as in case (A) using Steimke's theorem of the alternative, we obtain $y_{J_k} > 0$, $\bar{y}_{J_t} > 0$ so that

$$- M_{J_k} y_{J_k} - M_{J_k J_t} \bar{y}_{J_t} > 0 \quad \ldots \quad \ldots \quad (4.3.8)$$

Using (4.3.7) and (4.3.8) we get a $\alpha > 0 \in R$ and $y_{J_t} = y^*_{J_t} + \alpha \bar{y}_{J_t}$ such that

$$- M_{J_t} y_{J_t} - M_{J_t J_{r+1}} (y_{J_{r+1}} + \lambda x_{J_{r+1}}) > 0$$

$$- M_{J_k} y_{J_k} - M_{J_k J_t} y_{J_t} > 0.$$

Now we proceed just as in case (A) above, with a few necessary changes to construct a $q^* > 0$ and a solution (w^*, z^*) to (M, q^*) at which $\bar{v} = \begin{pmatrix} \bar{v}_J \\ 0 \end{pmatrix}$, where $\bar{v}_J$ is obtained from corollary 4.2.2. satisfying $\bar{v}_{J_k} > 0$, $\bar{v}_{J_m} = 0$, $1 \leqq m \leqq r-1$, $m \neq k$, generates a ray of solutions.

This completes the proof of the theorem.

<u>Remark 4.3.1</u> : Let $M \in (Z - K_o) \cap S_o$ and let there exist a $x \geq 0 \in R^n$ such that $Mx = 0$. Consider the representation of M given by lemma 4.3.1. In what follows we shall assume that there is a t, $s \leqq t \leqq r$ such that $M_{J_i J_{r+1}} = 0$ for $s+1 \leqq i \leqq t$ and $M_{J_i J_{r+1}} \neq 0$ for $t+1 \leqq i \leqq r$. If $t = s$ then for all $s+1 \leqq i \leqq r$, $M_{J_i J_{r+1}} \neq 0$; If $t = r$ then for all $s+1 \leqq i \leqq r$, $M_{J_i J_{r+1}} = 0$. This assumption can be made without loss of generality, because a principal rearrangement involving the blocks $M_{J_i J_k}$, $s+1 \leqq i \leqq r$, $s+1 \leqq k \leqq r$, will satisfy the above assumption. However in the rearranged representation M_{J_r} may not be a K-matrix.

<u>Theorem 4.3.3</u> : Let $M \in (Z-K_o) \cap S_o$ and let there be a $x \geq 0 \in R^n$ such that $Mx = 0$. Consider the representation of M given by lemma 4.3.1. and remark 4.3.1. Let $R^o(M)$ be defined as in theorem 4.3.2.

$R^o(M) = \emptyset$, if $M_{J_k J_{r+1}} = 0$ for all $1 \leqq k \leqq s$ and $M_{J_k J_i} = 0$ for all $t+1 \leqq i \leqq r$; $1 \leqq k \leqq t$.

Proof : Suppose $M_{J_k J_{r+1}} = 0$ for all $1 \leqq k \leqq s$ and $M_{J_k J_i} = 0$ for all $t+1 \leqq i \leqq r$, $1 \leqq k \leqq t$.

Let $q^* \in D(M)$ and let $\bar{v} \geq 0 \in R^n$ generate a ray of solutions to (M, q^*) at some solution (w^*, z^*).

From lemma 4.1.1. and 4.1.2., we have,

(i) $M\bar{v} = 0$ (ii) $\bar{v}^T w^* = 0$ and (iii) $z^{*T} M \bar{v} = 0$.

Now, since $M\bar{v} = 0$ and $M_{J_i J_{r+1}} = 0$ for all $1 \leqq i \leqq r$, it follows that

$$M_{J_{r+1}} \bar{v}_{J_{r+1}} = 0.$$

Since $-M^T_{J_{r+1}} \in S$, this implies that $\bar{v}_{J_{r+1}} = 0$.

Let $J = \bigcup_{i=1}^{t} J_i$. We note that $M\bar{v} = 0 \Longrightarrow M_J \bar{v}_J = 0$.

Define $\bar{q}_J \in R^{|J|}$ by taking

$$\bar{q}_{J_k} = q^*_{J_k} + \sum_{i=t+1}^{r+1} M_{J_k J_i} z^*_{J_i} \quad \text{for } 1 \leqq k \leqq t.$$

Since by our hypothesis $M_{J_k J_i} = 0$ for $1 \leqq k \leqq t$, $t+1 \leqq i \leqq r$, we have,

$$\bar{q}_J = q^*_J$$

We note that $\bar{q}_J = w_J^* - M_J z_J^*$ and that $\bar{v}_J^T w_J^* = 0$; $z_J^{*T} M_J \bar{v}_J = 0$. Thus $\bar{q}_J \in D(M_J)$ and by lemma 4.1.2 ; $\bar{v}_J$ generates a ray of solutions to $(M_J, \bar{q}_J)$ at the solution (w_J^*, z_J^*).

From theorem 4.3.1. it follows that $\bar{q}_J$ is in the boundary of $D(M_J)$.

The set $A = \left\{ x \mid x \geq 0, \sum_{i=1}^{|J|} x_i = 1, \quad x^T M_J \leq 0 \right\}$

is nonempty because $M_J^T \in K_o - K$, and by theorem 3.2.8, there is a $x \geq 0 \in R^{|J|}$ such that $M_J^T x = 0$.

Thus A is a nonempty convex compact set and therefore there is a $\bar{x} \in A$ such that

$$d_o = \bar{x}^T \bar{q}_J = \min_{x \in A} x^T \bar{q}_J .$$

Now, since $\bar{q}_J \in D(M_J)$, using Farkas' lemma, we have

$\bar{q}_J^T x \geq 0$ for all $x \in A$. Hence $d_o \geq 0$. In fact we must have $d_o = 0$, for if $d_o > 0$, for any $0 \neq p_J \in R^{|J|}$, there is a $\theta(p_J) > 0$ such that

$$x^T (\bar{q}_J + \theta p_J) \geq 0 , \quad \text{for all } x \in A, \ \theta < \theta(p_J).$$

This implies, by Farkas' lemma, that $\bar{q}_J + \theta\, p_J \in D(M_J)$, which, because of the convexity of $D(M_J)$, contradicts our earlier conclusion that $\bar{q}_J$ is in the boundary of $D(M_J)$.

Thus $d_o = 0$.

Since $\bar{x}^T \bar{q}_J = 0$, it follows that $\bar{x}^T w_J^* = 0$ and $\bar{x}^T M_J z_J^* = 0$.

Choose $\bar{p}_J \in R^{|J|}$ such that $\bar{x}^T \bar{p}_J < 0$.

Define $\bar{y}$ by taking $\bar{y}_J = \bar{x}$; $\bar{y}_{J_{r+1}} = 0$.

We note that $\bar{y}^T M \leqq 0$; Also since $\bar{q}_J = q_J^*$, $\bar{y}^T q^* = 0$.

We define $p \in R^n$ by taking $p = \begin{bmatrix} \bar{p}_J \\ e_{|N-J|} \end{bmatrix}$. We have

$$\bar{y}^T (q^* + \theta p) < 0 \quad \text{for all} \quad \theta > 0$$

Therefore, using Farkas' lemma and the convexity of $D(M)$, we conclude that q^* is in the boundary of $D(M)$.

This concludes the proof of the theorem.

<u>Remark 4.3.2</u> : We note that we did not assume $-M_{J_{r+1}} \in S$ in the above theorem. This was required in theorem 4.3.2.

Remark 4.3.3 : Suppose $M \in (Z - K_o) \cap S_o$ and there is no $x \geq 0 \in R^n$ such that $Mx = 0$. Then, in view of remark 3.3.2, in lemma 3.3.2

case (iii) occurs with $M_{J_1} \in K$. M has the representation

$$M = \begin{bmatrix} M_{J_1} & M_{J_1 J_2} \\ 0 & M_{J_2} \end{bmatrix}$$

where $M_{J_1} \in K$ and $-M_{J_2}^T \in S$. In this case $(M,0)$ has a unique solution and therefore, by lemma 4.1.2., $R(M) = \emptyset$.

<u>Corollary 4.3.1</u> : Consider the representation of M as in lemma 4.3.1, for some $M \in (Z - K_o) \cap S_o$ with the assumption that there exists a $x \geq 0 \in R^n$ such that $Mx = 0$. Let $J = \bigcup_{i=1}^{r} J_i$. $R^o(M) = \emptyset$ if for $1 \leq i \leq r$, $M_{J_i J_{r+1}} = 0$.

Proof : Follows from theorem 4.3.3.

<u>Corollary 4.3.2</u> : Let $M \in (Z - K_o) \cap S_o$ and let there exist a $x \geq 0 \in R^n$ such that $Mx = 0$. Consider the representation of M as in lemma 4.3.1 and let $J = \bigcup_{i=1}^{r} J_i$. If M_J is a $\bar{K}_o$- matrix with $-M_{J_{r+1}} \in S$, then a necessary and sufficient condition for $R^o(M) = \emptyset$ is that $M_{J_i J_{r+1}} = 0$ for $1 \leq i \leq r-1$ and

$$M_{J_r J_{r+1}} \neq 0 \Longrightarrow M_{J_i J_r} = 0 \quad \text{for } 1 \leq i \leq r-1.$$

Proof : This follows from the definition 4.2.1. of $\bar{K}_o$-matrices and theorems 4.3.2 and 4.3.3.

<u>Corollary 4.3.3</u> : Let $M \in Z$ be symmetric. Then $R^o(M) = \emptyset$.

Proof : According to lemma 3.3.2. and lemma 3.3.6. exactly one of the following cases holds (i) $-M^T \in S$ (ii) $M \in K_o$ and (iii) $M \in (Z - K_o) \cap S_o$.

If case (i) holds $(M,0)$ has a unique solution and by lemma 4.1.2. $R(M) = \emptyset$; $R^o(M) = \emptyset$.

If case (ii) holds theorem 4.3.1. shows that $R^o(M) = \emptyset$.

Suppose case (iii) holds and there is no nonzero solution to $(M, 0)$. Then obviously $R^o(M) = \emptyset$. If $(M,0)$ has a nonzero solution, note that in the representation of M as in lemma 4.3.1, M_J is a $\bar{K}_o$-matrix, where $J = \bigcup_{i=1}^{r} J_i$, according to theorem 4.2.5. Also because $M_{J_{r+1}J} = 0$, by symmetry $M_{JJ_{r+1}} = 0$ and corollary 4.3.2. applies. The conclusion follows.

<u>Example 4.3.1</u> :

The following example shows that in theorem 4.3.2. it is necessary to assume $-M_{J_{r+1}} \in S$.

$$M = \begin{bmatrix} 2 & -2 & -1 & 0 \\ -3 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$

M has the representation given by lemma 4.3.1. with $r = 2$, $J_r = \emptyset$, $J_1 = \{1, 2\}$; $J_3 = \{3, 4\}$, $s = 1 = r-1$.

We note that $-M^T_{J_3} \in S$ but $-M_{J_3} \notin S$. Also $M_{J_1 J_3} \neq 0$.

We observe that if $q \in D(M)$ then $q_3 \geqq 0$. Also, if $q \in$ interior $D(M)$ then $q_3 > 0$. If $q \in D(M)$, then in any solution (w, z) to (M, q), $w_3 > 0$ if $q_3 > 0$ and therefore $z_3 = 0$.

Now let $q \in D(M)$ and let $\bar{\bar{v}} \geq 0 \in R^4$ generate a ray of solutions to (M, q) at the solution (w^*, z^*). If $q_3 = 0$ then clearly q is in the boundary of D(M). If $q_3 > 0$ then our observations imply that the method of proof of theorem 4.3.3. can be applied. This implies that q is in the boundary of D(M). Thus $R^o(M) = \emptyset$.

<u>Example 4.3.2</u> : This example shows that in theorem 4.3.3. the assumption that $M_{J_k J_i} = 0$ for all $t+1 \leqq i \leqq r$, and for all $1 \leqq k \leqq t$ is necessary.

$$M = \begin{bmatrix} 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 & -6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 2 \end{bmatrix}$$

M has the representation given by lemma 4.3.1. and remark 4.3.1. with $r = 4$; $J_r = \emptyset$, $J_1 = \{1,2\}$; $J_2 = \{3,4\}$; $J_3 \{5,6\}$ and $J_5 = \{7,8\}$. We also have $s = 1 < r-1 = 3$, and $t = 2$.

We note that $M_{J_i J_3} = 0$ for $i = 1$, but not zero for $i = 2$.

Let $q^{*T} = (2,1,1,1,3,2,4,1) > 0 \in R^8$.

It is easy to verify that (w^*, z^*) solves (M, q^*) if we take $w^* = 0$ and $z^* = e_8$. Also if we take

$v^T = (1, 1, 0, 0, 0, 0, 0, 0)$ then $(w^*, z^* + \lambda v)$ solves (M, q^*) for all $\lambda \geqq 0 \in R$. Thus $R^o(M) \neq \emptyset$.

<u>Theorem 4.3.4</u> : Let $M \in \bar{K}_o$ and let q^* be in the boundary of $D(M)$. Then there exist $(\bar{w}, \bar{z}) \geq 0$ and $\bar{v} \geq 0 \in R^n$ such that $(\bar{w}, \bar{z} + \lambda \bar{v})$ solves (M, q^*) for each real number $\lambda \geqq 0$.

Proof : Since $q^* \in D(M)$ there exists (w^*, z^*) which solves (M, q^*). Also, since q^* is in the boundary, there exists $0 \neq p \in R^n$ such that $q^* + \theta p \notin D(M)$ for any $\theta > 0$.

Using Farkas' lemma we have, for any $\theta > 0$, a $v_\theta \geq 0 \in R^n$ such that

$$v_\theta^T M \leqq 0 , \quad v_\theta^T (q^* + \theta p) < 0 \quad \ldots . \quad (4.3.9)$$

Consider the set $A = \left\{ v \mid v \in R^n, \sum_{i=1}^{n} v_i = 1, v \geq 0, v^T M \leqq 0 \right\}$.

By (4.3.9). A is nonempty and it is also convex and compact. Therefore there exists a $v^* \in A$ such that

$$v^{*T} q^* = \min_{v \in A} v^T q^*$$

Since $q^* \in D(M)$, by Farkas' lemma, $v^T q^* \geqq 0$ for all $v \in A$. From (4.3.9) it now follows that $v^{*T} q^* = 0$.

Since $q^* = w^* - M z^*$, it follows that $v^{*T} w^* = 0$, $v^{*T} M z^* = 0$.

Let $L = \{ i \mid v_i^* > 0 \} \subseteqq \{ i \mid w_i^* = 0 \}$.

$$v^{*T} M \leqq 0 \Longrightarrow v_L^{*T} M_L \leqq 0 \qquad \ldots \qquad (4.3.10)$$

By theorem 4.2.3, $M_L \in \bar{K}_o$. Moreover, if $M_L \in K$ then (4.3.10) can not hold. Therefore $M_L \in \bar{K}_o - K$, and by theorem 3.2.8. M_L is singular. Hence appealing to theorem 4.2.2, we conclude that in the representation of M as in corollary 4.2.2, there is a k, $1 \leqq k \leqq r-1$, such that $J_k \subseteqq L$.

Case (i) : In the representation of M given by corollary 4.2.2 either $J_r = \emptyset$ or $M_{J_r J_k} = 0$.

Using lemma 4.2.6, we get a $y \geq 0 \in R^n$ such that $y_{J_k} > 0$, $y_{N-J_k} = 0$ and $My = 0$.

Take $\bar{v} = y$. This $\bar{v}$ generates a solution ray at (w^*, z^*).

Case (ii) : $J_r \neq \emptyset$, $M_{J_r J_k} \neq 0$.

From lemma 4.2.6, we get a $y \geq 0 \in R^n$ such that $y_{J_k} > 0$, $y_{J_m} = 0$, if $1 \leqq m \leqq r-1$, $m \neq k$, $y_{J_r} \neq 0$ and $My = 0$ ..(4.3.11)

Let $I_1 = \{ i \in J_r \mid y_i > 0 \}$; $I_2 = J_r - I_1$

In what follows we assume that $I_2 \neq \emptyset$. The modification in the steps for the case $I_2 = \emptyset$ will be obvious.

(4.3.11) $\Longrightarrow$

$$M_{I_1 J_k} y_{J_k} + M_{I_1} y_{I_1} = 0, \; y_{J_k} > 0, \; y_{I_1} > 0 \quad ..(4.3.12)$$

and

$$M_{I_2 J_k} y_{J_k} + M_{I_2 I_1} y_{I_1} = 0, \text{ which implies that}$$

$$M_{I_2 J_k} = 0, \quad M_{I_2 I_1} = 0 \quad \ldots \quad \ldots \quad (4.3.13)$$

(4.3.12) and Steimke's theorem of the alternative imply that

$$u^T (-M_{I_1 J_k} , \; -M_{I_1}) \leq 0 \text{ has no solution.}$$

Also, $u^T (-M_{I_1 J_k} , -M_{I_1}) = 0 \Longrightarrow u = 0$ as M_{I_1} is a K-matrix.

Therefore, applying Farkas' lemma, we get $x_{J_k} \geqq 0 \in R^{|J_k|}$ and $x_{I_1} \geqq 0 \in R^{|I_1|}$ such that

$$w^*_{I_1} = - M_{I_1 J_k} x_{J_k} - M_{I_1} x_{I_1}$$

Or

$$- M_{I_1}^{-1} (w_{I_1}^* + M_{I_1 J_k} x_{J_k}) \geqq 0.$$

We can choose $\beta > 0$, a real number, such that

$\beta M_{I_1 J_k} y_{J_k} \leqq M_{I_1 J_k} x_{J_k}$ and, since $-M_{I_1}^{-1} \leq 0$, we have

$$-M_1^{-1} (w_{I_1}^* + \beta M_{I_1 J_k} y_{J_k}) \geqq -M_1^{-1} (w_{I_1}^* + M_{I_1 J_k} x_{J_k}) \geqq 0.$$

Therefore there exists $z_{I_1} \geq 0 \in R^{|I_1|}$ such that

$$w_{I_1}^* = -\beta M_{I_1 J_k} y_{J_k} - M_{I_1} z_{I_1} \quad \ldots \quad \ldots (4.3.14)$$

Define $(\bar{w}, \bar{z})$ by taking

$$\bar{z}_{J_m} = z_{J_m}^*, \text{ if } m \neq k, \quad 1 \leqq m \leqq r-1, \quad \bar{z}_{I_1} = z_{I_1}^* + z_{I_1},$$

$$\bar{z}_{I_2} = z_{I_2}^*, \quad \bar{z}_{J_k} = z_{J_k}^* + \beta y_{J_k},$$

$$\bar{w}_{N-I_1} = w_{N-I_1}^*, \quad \bar{w}_{I_1} = 0.$$

Using (4.3.11), (4.3.13) and (4.3.14) it is easy to verify that $(\bar{w}, \bar{z})$ solves (M, q^*) and $\bar{v} = y$ generates a ray of solutions at $\bar{z}$.

This completes the proof of the theorem.

Corollary 4.3.4 : Let $M \in K_o$ be symmetric. If q^* is in the boundary of $D(M)$, at every solution $(\bar{w}, \bar{z})$ to (M, q^*) there exists a $\bar{v} \geq 0 \in R^n$ which generates a solution ray.

Proof : From theorem 4.2.5. $M \in \bar{K}_o$. Start with any solution $(\bar{w}, \bar{z})$ to (M, q^*) and proceed as in the proof of theorem 4.3.4. Note that case (ii) of the proof does not arise because of symmetry. Therefore it follows that there is a $v \geq 0 \in R^n$ which generates a ray of solutions to (M, q^*) at $(\bar{w}, \bar{z})$.

This completes the proof.

Corollary 4.3.5 : Let $M \in \bar{K}_o$, and let $q \in D(M)$. There exist $\bar{w} \geqq 0 \in R^n$, $\bar{z} \geqq 0 \in R^n$, $\bar{v} \geq 0 \in R^n$ such that $(\bar{w}, \bar{z} + \lambda\bar{v})$ solves (M, q) for each real number $\lambda \geqq 0$ if and only if q is in the boundary of $D(M)$.

Proof : This follows immediately from theorem 4.3.1 and theorem 4.3.4.

4.4. Existence of infinitely many solutions to (M, q) when $M \in Z$:

In this section we present some results relating the existence of infinitely many solutions for (M, q) to the boundary of $D(M)$, when $M \in Z$.

Lemma 4.4.1 : Let $q \in D(M)$ and let (M, q) have an infinite number of solutions. Then q is contained in a complementary cone of $(I, -M)$, whose interior in R^n is empty.

Proof : If (M, q) has infinitely many solutions, since there are only finitely many complementary cones, there is a set $L \subseteq N$ such that

$$q = \begin{bmatrix} -M_L & 0 \\ -M_{N-L\,L} & I \end{bmatrix} \begin{bmatrix} z_L^1 \\ w_{N-L}^1 \end{bmatrix} = \begin{bmatrix} -M_L & 0 \\ -M_{N-L\,L} & I \end{bmatrix} \begin{bmatrix} z_L^2 \\ w_{N-L}^2 \end{bmatrix}$$

where (z_L^1, w_{N-L}^1), (z_L^2, w_{N-L}^2) solve (M, q), $(z_L^1, w_{N-L}^1) \neq (z_L^2, w_{N-L}^2)$ and $L \neq \emptyset$. ($L = N$ is permitted in which case the complementary matrix considered is $-M$).

It follows immediately that the matrix

$$\begin{bmatrix} -M_L & 0 \\ -M_{N-L\,L} & I \end{bmatrix}$$

is singular and therefore the interior of the complementary cone generated by this complementary set of column vectors is empty.

Remark 4.4.1 : In the above proof if we take $v_L = z_L^1 - z_L^2$ and $u_{N-L} = w_{N-L}^1 - w_{N-L}^2$ we see that $(\bar{u}, \bar{v})$, where $\bar{u} = \begin{pmatrix} 0 \\ u_{N-L} \end{pmatrix} \in R^n$ and $\bar{v} = \begin{pmatrix} v_L \\ 0 \end{pmatrix} \in R^n$, is a nonzero solution to the system

$$w - Mz = 0, \quad w_i z_i = 0, \quad 1 \leq i \leq n \qquad \ldots (4.4.1)$$

For $0 < \theta < 1$ consider $(\bar{w}, \bar{z})$ defined by

$$\bar{w}_{N-L} = \theta\, w^1_{N-L} + (1-\theta)\, w^2_{N-L} \; ; \; \bar{w}_L = 0 .$$

$$\bar{z}_L = \theta\, z^1_L + (1-\theta)\, z^2_L \; ; \; \bar{z}_{N-L} = 0 .$$

It is easy to see that $\bar{u}_i \neq 0 \Longrightarrow \bar{w}_i > 0$ and $\bar{v}_i \neq 0 \Longrightarrow \bar{z}_i > 0$. Also therefore, $\bar{u}_i \neq 0 \Longrightarrow \bar{z}_i = 0$ and $\bar{v}_i \neq 0 \Longrightarrow \bar{w}_i = 0$.

Thus if we choose,

$$\lambda_0 = \min \left[\min_{1 \leqq i \leqq n} \left\{ \frac{-\bar{w}_i}{\bar{u}_i} , \; \bar{u}_i < 0 \right\}, \; \min_{1 \leqq i \leqq n} \left\{ \frac{-\bar{z}_i}{\bar{v}_i} , \; \bar{v}_i < 0 \right\} \right]$$

then $\lambda_0 > 0$ and $(\bar{w} + \lambda \bar{u}, \; \bar{z} + \lambda \bar{v})$ solves (M, q) for all $0 \leqq \lambda < \lambda_0$.

Thus it follows from lemma 4.4.1 that, in general, if (M, q) has infinitely many solutions, then an infinite number of them can be written as $(\bar{w} + \lambda\bar{u}, \; \bar{z} + \lambda\bar{v})$, where $0 \leqq \lambda < \lambda_0$, $\lambda_0 > 0$ and $(\bar{u}, \bar{v})$ is a nonzero solution to (4.4.1) which satisfies $\bar{v}_i \neq 0 \Longrightarrow \bar{w}_i = 0$, $\bar{u}_i = 0$, $\bar{u}_i \neq 0 \Longrightarrow \bar{z}_i = 0$, and $\bar{v}_i = 0$. If in addition, $\bar{u} \geqq 0$ and $\bar{v} \geqq 0$, then $(\bar{u}, \bar{v})$ generates a ray of solutions to (M, q) at $(\bar{w}, \bar{z})$. (i.e. λ_0 can be taken as ∞).

We also note that in lemma 4.4.1. M is any square matrix. The converse of lemma 4.4.1. is not in general true. However it is true for K_o-matrices. We state it as our next lemma.

Lemma 4.4.2 : Let $M \in K_o$ and let $q \in D(M)$ be contained in a complementary cone whose interior is empty. Then (M, q) has infinitely many solutions.

Proof : Let Pos (B) be a complementary cone of $(I, -M)$ whose interior is empty and which contains q. For some $L \subseteq N$, $L \neq \emptyset$ we have,

$$B = \begin{bmatrix} -M_L & 0 \\ -M_{N-L\ L} & I \end{bmatrix}$$

Also, there exists $\begin{pmatrix} v_L \\ u_{N-L} \end{pmatrix} \geq 0 \in R^n$ such that

$$B \begin{bmatrix} v_L \\ u_{N-L} \end{bmatrix} = q \qquad \ldots \quad \ldots \quad \ldots \ (4.4.2)$$

Now, we assume without loss of generality, that $u_{N-L} > 0$. Because, if for some $j \in N-L$, $u_j = 0$ then we can replace $I_{.j}$ from B by $-M_{.j}$, redefine L by including j in it and after a principal rearrangement obtain the above form of B with j in L and $v_j = 0$. These steps can be repeated until we obtain a B, v_L, u_{N-L}

satisfying (4.4.2) with $u_{N-L} > 0$. (If $N-L = \emptyset$, u_{N-L} is not defined, $B = -M$).

Since B is singular it follows that $M_L \in K_o - K$. Appealing to theorem 3.2.8 we get $x \geq 0 \in R^{|L|}$ such that

$$M_L x = 0$$

For this x, $-M_{N-L\ L}\ x \geqq 0$ so that if we take $y = M_{N-L\ L}\ x \leqq 0$ then $B\binom{x}{y} = 0$.

Let $\lambda_o = \min_{j \in N-L} \left\{ \frac{-u_j}{y_j}, y_j < 0 \right\}$. (If $y_j = 0$ for all $j \in N-L$, set $\lambda_o = \infty$). We note that $\lambda_o > 0$.

Define $\bar{w} \in R^n$, $\bar{z} \in R^n$, $\bar{u} \in R^n$, and $\bar{v} \in R^n$ by taking $\bar{w}_L = 0$, $\bar{w}_{N-L} = u_{N-L}$; $\bar{z}_L = v_L$, $\bar{z}_{N-L} = 0$; $\bar{u}_L = 0$, $\bar{u}_{N-L} = y$ and $\bar{v}_L = x$ and $\bar{v}_{N-L} = 0$.

We note that $(\bar{w} + \lambda\bar{u}, \bar{z} + \lambda\bar{v})$ solves (M, q) for all $0 \leqq \lambda < \lambda_o$. Thus there are infinitely many solutions to (M, q). This concludes the proof.

Example 4.4.1 : The following example shows that lemma 4.4.2 is not in general true.

Let $M = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix}$; $q = \begin{bmatrix} 3 \\ -3 \\ -4 \\ -5 \end{bmatrix}$.

It is clear that if (M, q) has solutions then in any solution $z_1 > 0$, $z_2 > 0$.

Consider $B = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ -3 & -1 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{bmatrix}$

We note that if we take $y = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ then $q = By$. Therefore $q \in \text{Pos}(B)$. Also det (B) = 0, and rank (B) = 3. Further the space $\{x \mid Bx = 0\}$ is one dimensional and contains all scalar multiples of $\bar{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ -4 \end{bmatrix}$.

If $q = By^1$, $y^1 \neq y$ is another solution then $y - y^1$ is a scalar multiple of $\bar{x}$, so that y^1 can not be nonnegative. This shows that the only nonnegative solution to $Bz = q$ is $z = y$.

Any complementary cone which contains q must contain $-M_{.1}$ and $-M_{.2}$ as generating columns. But any such complementary cone of (I, -M) is just Pos (B). This shows that (M, q) has four solutions. We also note that the interior of Pos (B) is empty.

<u>Theorem 4.4.1</u> : Let $M \in K_o$. (M, q) has infinitely many solutions if and only if q is contained in some complementary cone of (I, -M) whose interior is empty.

Proof : Immediately follows from lemma 4.4.1. and lemma 4.4.2.

The following theorem relates the existence of infinite number of solutions to the boundary of D(M) when $M \in K_o$.

<u>Theorem 4.4.2</u> : Let $M \in K_o$. (M, q) has infinitely many solutions if and only if q is in the boundary of D(M).

Proof : Suppose (M, q) has infinitely many solutions. As in the proof of lemma 4.4.1 and remark 4.4.1 there is a $(\bar{w}, \bar{z})$ which solves (M, q) and a $(\bar{u}, \bar{v}) \neq 0$ such that (i) $\bar{u} - M\bar{v} = 0$, (ii) $\bar{u}_i \neq 0 \Longrightarrow \bar{z}_i = 0$, $\bar{v}_i = 0$, (iii) $\bar{v}_i \neq 0 \Longrightarrow \bar{w}_i = 0$, $\bar{u}_i = 0$.

Let $L_1 = \{ i \mid \bar{v}_i \neq 0 \}$; $L_2 = \{ i \mid \bar{w}_i = 0 \}$; $L_3 = \{ i \mid \bar{z}_i > 0 \}$.
We note that $L_3 \subseteq L_2$; $L_1 \subseteq L_2$.

M_{L_2} is a K_o-matrix with M_{L_1} as a principal submatrix and M_{L_1} is singular, since $M_{L_1} \bar{v}_{L_1} = 0$.

Therefore $M^T_{L_2} \in K_o - K$ and using theorem 3.2.8. we get a $x_{L_2} \geq 0 \in R^{|L_2|}$ such that $x^T_{L_2} M_{L_2} = 0$. Now proceeding from this point as in the proof of theorem 4.3.1. We can show that q is in the boundary of D(M).

Now suppose that q is in the boundary of D(M). Let $(\bar{w}, \bar{z})$ be a solution to (M, q). Proceeding as in the proof of theorem 4.3.4. we can show that there is a $v^* \geq 0 \in R^n$ such that

$$v^{*T} M \leqq 0, \quad v^{*T} q = 0, \quad v^{*T} \bar{w} = 0, \quad v^{*T} M \bar{z} = 0.$$

Let $L_1 = \{ i \mid v^*_i > 0 \}$; $L_2 = \{ i \mid \bar{w}_i = 0 \}$; $L_3 = \{ i \mid \bar{z}_i > 0 \}$.

We note that $L_1 \subseteq L_2$, $L_3 \subseteq L_2$.

$$v^{*T} M \leqq 0 \Longrightarrow v^{*T}_{L_2} M_{L_2} \leqq 0 \Longrightarrow M_{L_2} \in K_o - K .$$

Therefore from theorem 3.2.8 we can get a $\bar{v}_{L_2} \geq 0 \in R^{|L_2|}$ such that $M_{L_2} \bar{v}_{L_2} = 0$.

Thus there exists a $0 \geqq \bar{u}_{N-L_2} \in R^{|N-L_2|}$ such that

$$\begin{bmatrix} I_{N-L_2} & -M_{N-L_2\, L_2} \\ 0 & -M_{L_2} \end{bmatrix} \begin{bmatrix} \bar{u}_{N-L_2} \\ \bar{v}_{L_2} \end{bmatrix} = 0 .$$

Define $\bar{u} \in R^n$ and $\bar{v} \in R^n$ by taking $\bar{u} = \begin{bmatrix} 0 \\ \bar{u}_{N-L_2} \end{bmatrix}$, $\bar{v} = \begin{bmatrix} \bar{v}_{L_2} \\ 0 \end{bmatrix}$.

We note that $u_i < 0 \Longrightarrow i \in N-L_2 \Longrightarrow \bar{w}_i > 0$

Take

$$\bar{\lambda} = \begin{cases} \infty \text{ if } \bar{u}_i = 0 \quad \text{for } i = 1,2, \ldots, n. \\ \min\limits_{1 \leq i \leq n} \left\{ \frac{-\bar{w}_i}{u_i} \mid u_i < 0 \right\}, \text{ otherwise.} \end{cases}$$

It follows that $\bar{\lambda} > 0$ and $(\bar{w} + \lambda\bar{u}, \bar{z} + \lambda\bar{v})$ solves (M, q), for $0 \leq \lambda < \bar{\lambda}$. Thus there are an infinite number of solutions to (M, q).

Theorem 4.4.3 : Let $M \in K_o$. The boundary of $D(M)$ is equal to the union of all complementary cones of $(I, -M)$ each of whose interior is empty.

Proof : This follows from theorem 4.4.1. and theorem 4.4.2. we note that if $M \in K$ then the boundary of $D(M)$ is empty since $D(M) = R^n$ and all complementary cones have nonempty interior.

Theorem 4.4.4 : Let $M \in K_o$. If q is in the boundary of $D(M)$, (M, q) has infinitely many solutions. If q is in the interior of $D(M)$ then there is a unique solution to (M, q).

Proof : The first part of the assertion follows from theorem 4.4.2. To prove the second part, we notice that $D_2(M) \subseteq$ interior $D(M)$.

To complete the proof therefore it is enough to show that if $q \in$ interior $D(M) - D_2(M)$ then (M, q) has a unique solution.

Consider a q in the interior of $D(M)$ but not in $D_2(M)$. Such a q belongs to only the cones which have nonempty interior. If such a q were to have two distinct solutions then it must belong to two different cones with nonempty interior but must not be in the common boundary (if they have a common boundary) between them. However in this case it is possible to find a q^1 in the θ neighbourhood of q which is in $D_2(M)$ and belongs to the interior of two different complementary cones with nonempty interior. This contradicts theorem 3.2.6.

The conclusion of the theorem follows.

Example 4.4.2 : The following example shows that even for strictly copositive plus matrices, for some q in the interior of $D(M)$, (M, q) can have infinitely many solutions.

$$M = \begin{bmatrix} 2 & 2 & 1 & 2 \\ 3 & 3 & 2 & 3 \\ -2 & 1 & 5 & -2 \\ 1 & -2 & 1 & 2 \end{bmatrix} \qquad q = \begin{bmatrix} -4 \\ -6 \\ 4 \\ 4 \end{bmatrix}$$

We note that for $x \geq 0 \in R^n$

$$x^T M x = x_1^2 + 3x_2^2 + 3x_3^2 + 2x_4^2 + (x_1 - x_3)^2 + (x_3 - x_4)^2 + 5x_1x_2 + x_1x_3 + 3x_1x_4 + 3x_2x_3 + x_2x_4 + x_3x_4 > 0 .$$

Therefore $M \in SCP$.

A solution to (M, q) is $(\bar{w}, \bar{z})$ where $\bar{w} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}$ and $\bar{z} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Setting $u = \begin{bmatrix} 0 \\ 0 \\ -3 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ we note that $(\bar{w} + \lambda u, \bar{z} + \lambda v)$ solves (M, q), for all $0 \leqq \lambda \leqq 1$. Thus (M, q) has infinitely many solutions. Also we note that since $M \in SCP$, $D(M) = R^n$ and therefore q is in the interior of $D(M)$.

<u>Example 4.4.3</u> : This example shows that for $M \in K_0 - \bar{K}_0$ a solution ray need not exist for all q in the boundary of $D(M)$.

$$M = \begin{bmatrix} 2 & -1 & -3 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 2 \end{bmatrix} \quad ; \quad q = \begin{bmatrix} 5 \\ 6 \\ -1 \\ 2 \end{bmatrix}$$

This is a K_0-matrix and is in the form given by corollary 4.2.2., with $r = 3$, $J_r = \emptyset$, $J_1 = \{1,2\}$, $J_2 = \{3,4\}$.

We note that in any solution to (M, q), $z_3 > 0$. Also, in view of lemma 4.2.1 applied to M_{J_1}, we note that in any solution to (M, q), $w_1 > 0$, $w_2 > 0$. Similarly, since $M_{J_1 J_2} \neq 0$, in any solution to $Mx = 0$, $x \geq 0$, $x_{J_1} > 0$. These facts imply that there is no solution to (M, q) at which a solution ray exists.

However, taking $\bar{w} = \begin{bmatrix} 2 \\ 6 \\ 0 \\ 0 \end{bmatrix}$; $\bar{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$; $\bar{u} = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and

$\bar{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ we see that $(\bar{w} + \lambda\bar{u}, \bar{z} + \lambda\bar{v})$ solves (M, q) for all $0 \leqq \lambda \leqq 2/3$. Thus (M, q) has infinitely many solutions.

<u>Example 4.4.4</u> : We noted that when $M \in CP^{+}$, then Cottle's theorem (theorem 1.6.1) asserted the existence of a ray of solutions to (M, q) for any q in the boundary of D(M) at every solution to (M, q). The following example shows that if $M \notin \bar{K}_0$ and if q is in the boundary of D(M) there may be solutions to (M, q) at which a solution ray does not exist.

$$M = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 0 \\ -3 & -4 & 7 \end{bmatrix} \quad ; \quad q = \begin{bmatrix} -1 \\ 2 \\ 11 \end{bmatrix}$$

M has the representation given by corollary 4.2.2. with $r = 2$, $J_r = \{3\}$, $J_1 = \{1,2\}$; $\det(M_{J_1}) = 0$; $M_{J_2} \in K$. $\bar{z} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$;

$\bar{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a solution to (M, q).

Since $Mx = 0; \; x \geq 0 \Longrightarrow x_3 > 0$, at this solution there does not exist a ray of solutions to (M, q). However $z^* = \begin{bmatrix} 3 \\ 2 \\ 6/7 \end{bmatrix}$ and $w^* = 0$ is another solution to (M, q) and at this solution $\bar{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ generates a ray of solutions to (M, q).

Example 4.4.5 : Suppose $-M^T \in S$; $M \in Z$. The following example shows that the set of $q \in D(M)$ for which (M, q) has infinitely many solutions need not be contained in the boundary of $D(M)$.

$$M = \begin{bmatrix} 2 & -6 & -3 & -1 \\ -5 & 3 & -1 & -1 \\ -4 & -2 & 2 & -8 \\ -1 & 0 & -4 & 3 \end{bmatrix} ; \quad q = \begin{bmatrix} 8 \\ 4 \\ 12 \\ 2 \end{bmatrix}$$

Note that q is in the interior of $D(M)$. Also $w = 0$, $z = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is a solution to (M, q). Let $u = 0$; $v = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$. Note that $(w + \lambda u, \; z + \lambda v)$ solves (M, q) for all $0 \leqq \lambda \leqq 1$. Thus (M, q) has infinitely many solutions. We note also that since $-M^T \in S$, $(M, 0)$ has a unique solution and therefore, by lemma 4.1.2., (M, q) does not possess solution rays for any $q \in D(M)$. But the boundary of $D(M)$ is nonempty.

Example 4.4.6 : The following example shows that when $-M^T \in S$ the set of all $q \in D(M)$ for which infinitely many solutions to (M, q) exist need not contain the boundary of $D(M)$.

$$M = \begin{bmatrix} 2 & -3 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ -1 & -2 & 2 & -8 \\ -2 & -3 & -4 & 3 \end{bmatrix} ; \quad q = \begin{bmatrix} 3 \\ 2 \\ -2 \\ 4 \end{bmatrix}$$

It is clear that in view of lemma 4.2.1., in any solution to (M, q), $w_1 > 0$, $w_2 > 0$. Also in any solution $z_3 > 0$. This means that we need consider only $Pos(B_1)$ and $Pos(B_2)$ where

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 8 \\ 0 & 0 & 4 & -3 \end{bmatrix} ; \quad B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix}$$

Either of this leads to the unique solution $(\bar{w}, \bar{z})$ where

$$\bar{w} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} ; \quad \bar{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$. Now let us consider $$p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$.

We note once more that if (w, z) is any solution to $q + \theta p$ for some $\theta > 0$ then $w_1 > 0$, $w_2 > 0$ and $z_3 > 0$. This means that we need consider only $Pos(B_1)$ and $Pos(B_2)$. It is now easy to verify that

$$q + \theta p \notin Pos(B_1), \; Pos(B_2) \quad \text{for any} \quad \theta > 0.$$

Thus q is in the boundary of $D(M)$.

Example 4.4.7 : The following example shows that for $M \in (Z-K_o) \cap S_o$, the set of q for which (M, q) has infinitely many solutions need not be contained in the boundary of $D(M)$ even if M satisfies the conditions imposed in theorem 4.3.2.

$$M = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -4 & -2 \end{bmatrix} \quad ; \quad q = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

M is in the form given by lemma 4.3.1., with $r = 2$, $J_r = \emptyset$, $J_1 = \{1,2\}$ and $J_3 = \{3, 4\}$; $-M^T_{J_3} \in S$ and $M_{J_1} \in K_o - K$ with no proper principal minor of M_{J_1} as zero. We also note that q is in the interior of D(M).

$$\bar{w} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$ solves (M, q); Also if we take $$\bar{v} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

then $(\bar{w}, \bar{z} + \lambda\bar{v})$ solves (M, q) for all $0 \leqq \lambda \leqq 1$. Thus (M, q) has infinitely many solutions.

<u>Example 4.4.8</u> : In this example we consider a $M \in (Z - K_o) \cap S_o$ which satisfies the conditions stated in theorem 4.3.2. We show that there is a q in the boundary of D(M) and (M, q) has a unique solution.

$$M = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & -6 & 4 \end{bmatrix} \quad ; \quad q = \begin{bmatrix} 3 \\ 5 \\ -2 \\ 6 \end{bmatrix}$$

M is given in the form obtained by lemma 4.3.1. with $r = 2$, $J_r = \emptyset$, $-M_{J_3}^T \in S$, $-M_{J_3} \in S$ and $\det(M_{J_1}) = 0$ with no proper principal minor of M_{J_1} as 0 where $J_1 = \{1,2\}$, and $J_3 = \{3,4\}$. We also note that the conditions stated in theorem 4.3.2. hold.

Using lemma 4.2.1., we can conclude that in any solution (w, z) to (M, q); $w_1 > 0$, $w_2 > 0$, $z_3 > 0$. We can then show, proceeding as in example 4.4.6. that (M, q) has a unique solution and that q is in the boundary of $D(M)$.

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Appendix - 1

Consider the linear fractional programming problem mentioned in section 1.1. Let

$$C = \{ x \mid Ax \leqq b, \quad x \geqq 0 \in R^n \}$$

We shall assume that $d^T x + \beta \neq 0$ for all $x \in C$, whence it follows that either $d^T x + \beta > 0$ for all $x \in C$ or $d^T x + \beta < 0$ for all $x \in C$. We shall assume without loss of generality that $d^T x + \beta > 0$ for all $x \in C$.

This problem is extensively discussed in the literature. A linear programming formulation with the addition of one more variable is available for this problem. See [2]. We shall show that it can be directly cast as a linear complementarity problem.

Under our assumption about $d^T x + \beta$, the fractional function $(c^T x + \alpha)/(d^T x + \beta)$ is both pseudo convex and pseudo concave [27, p.149]. Therefore the Kuhn - Tucker conditions for a point $x^o \in C$ to be optimal are necessary and sufficient. [27, p.152 and 156]. Using these conditions we see that $x \geq 0 \in R^n$ is a solution to the linear fractional programming problem if and only if there exist $y \geqq 0 \in R^m$, $u \geqq 0 \in R^m$, $v \geqq 0 \in R^n$ such that

$$\frac{(d^T x + \beta)c - (c^T x + \alpha)d}{(d^T x + \beta)^2} = A^T u - v$$

$$
\begin{aligned}
Ax + y &= b \\
x^T v &= 0 \\
u^T y &= 0
\end{aligned}
$$

Since $d^T x + \beta > 0$ it is equivalent to finding a solution to the set of equations

$$
\begin{bmatrix} D & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} v \\ y \end{bmatrix} = \begin{bmatrix} \alpha d - \beta c \\ b \end{bmatrix}
$$

$$
x^T v + u^T y = 0
$$

where D is a $n \times n$ matrix whose ij^{th} element is $c_i d_j - d_i c_j$.

This is a linear complementarity problem (M, q) with

$$
M = \begin{bmatrix} -D & A^T \\ -A & 0 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} \alpha d - \beta c \\ b \end{bmatrix} .
$$

We note that the diagonal elements of M are 0's and M is antisymmetric (i.e. $M = -M^T$). Such a M is positive semi-definic and therefore $M \in CP^+$. L (M, q, d) is applicable to this problem with any $d > 0 \in R^n$.

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