PATHWISE STOCHASTIC CALCULUS OF CONTINUOUS SEMIMARTINGALES

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INTRODUCTION

Stochastic integration with respect to Brownian motion was introduced by Ito. Stochastic integration with respect to martingales (and semimartingales) was developed by Kunita-Watanable [24], Fisk [9], Courrege [3] and Meyer [33]. In this thesis, we study the 'pathwise stochastic calculus' restricting ourselves to continuous semimartingales. Here is a brief summary of our results.

In Chapter I, we obtain a 'pathwise formula' for the quadratic variation process $\langle M \rangle$ of a continuous local martingale M. Recall that $\langle M \rangle$ is the natural increasing process in the Doob-Meyer decomposition of M^2. By a pathwise formula for $\langle M \rangle$ we mean a formula describing explicitly a w-path of $\langle M \rangle$ in terms of the corresponding w - path of M. Observe that existence of such a formula already implies that $\langle M \rangle$ depends neither on the underlying probability nor on the underlying filtration. The proof of our formula for $\langle M \rangle$ is simple and does not assume the existence of $\langle M \rangle$, thus providing a simple proof of the existence as well of $\langle M \rangle$. Proceeding as in Kunita-Watanable [24], we also deduce that $\langle M \rangle$ is the only continuous increasing process A such that $M^2 - A$ is a local martingale. We also give an elementary proof of the lesser known fact that almost every path of a continuous local martingale has the property : On any interval, it is either a constant or of unbounded variation. This result also provides a proof of uniqueness of $\langle M \rangle$.

If M is a continuous local martingale such that $d\langle M\rangle(t) \leq dt$, then integral with respect to M can be defined as in the Brownian motion case for all progressively measurable integrands. For a general continuous local martingale M, we show that by applying a 'random time change' we can reduce it to the previous case. Once stochastic integration with respect to a continuous local martingale is done, extensions to cover continuous semimartingale integrators and to allow vector or matrix valued integrands and integrators are immediate. Then using 'random time change' and Doob's maximal inequality, we obtain an estimate on the growth of stochastic integrals. Then, we obtain a 'pathwise formula' for the stochastic integral of a r.c.l.l. process by showing that the 'Riemann sums' calculated for appropriate random partitions do indeed converge to the stochastic integral. Then we shall prove a 'pathwise' version of the well known Ito's formula. The proof highlights the 'pathwise' nature of the stochastic integrals. It should be pointed out that Bichteler has already considered 'pathwise' integration in [1]. He considers right continuous semimartingale integrators. His approach is to look at the stochastic integral as an integral with respect to a vector valued measure as in Metivier-Pellaumail [35] and Kussmaul [25]. Bichteler uses a factorisation theorem of Maurey-Rosenthal and changes the underlying probability to achieve a nice estimate on the growth of the stochastic integral, whereas we use a random time change.

In Chapter II, we consider the stochastic differential equation

$$(1) \qquad Y(t) = \emptyset(t) + \int_0^t b(\cdot, u, Y)\,dX(u)$$

under usual conditions on b, X being a continuous semimartingale. We prove existence and uniqueness of the solution among the class of all continuous processes. In fact after making a time change our proof imitates the proof of the Brownian motion case. A slight modification of the usual successive approximation procedure gives us a 'pathwise formula' to calculate the solution. We also consider stability properties of the solutions. In the last section, we consider the equation

$$Y(t,x) = Y(t) = x + \int_0^t b(\cdot, u, Y(u))\,dX(u)\,,$$

and get a pathwise version Y, which is jointly continuous in t and x. This in turn gives a slight improvement on the well known homeomorphism properties of the solution. The existence and uniqueness of solutions to (1) was proved by Protter [39] (in the case when $\emptyset$ is a semimartingale) and Doleans-Dade [5]. Whereas Protter considered continuous semimartingale integrators Doleans-Dale considered general (r.c.l.l.) semimartingales. However, their techniques are different from ours and do not give a pathwise formula. Independently of us Bichteler [2] gave a different pathwise formula, but as mentioned earlier, his methods are more complex.

In Chapter III, we shall consider multiplicative stochastic integration. Multiplicative integration with respect to Brownian motion was introduced by McKean [28]. Emery [7] considered multiplicative integration with respect to general semimartingales and showed that the multiplicative integral is limit in probability of 'Riemann products'. We follow Emery's approach of using stability properties of stochastic differential equations to obtain results on multiplicative stochastic integral. Working with continuous semimartingales we are able to show that the Riemann products over appropriate random partitions converge almost surely to the multiplicative integral, thus giving a 'pathwise formula' for the multiplicative integral. We obtain 'Peano series' representation of the multiplicative integral; formulae for the determinant and inverse of 'Exponential' of a matrix valued semimartingale; 'integration by parts' formula for the multiplicative integral and a stochastic version of the 'Trotter product' formula. As a consequence of our integration by parts formula, we show that any 'invertible' matrix valued semimartingale can be written in a unique way as a product of a local martingale and a process of bounded variation.

CHAPTER I

STOCHASTIC INTEGRATION

0. Notations :

$(\Omega, \underline{\underline{B}})$ is a fixed measurable space. A filtration $\underline{\underline{F}}$ is a right continuous family $(\underline{\underline{F}}_t)_{t \geq 0}$ of sub-σ fields of $\underline{\underline{B}}$. P will denote a Probability measure on $(\Omega, \underline{\underline{B}})$ and $\underline{\underline{F}}^P$ will denote the filtration obtained by augmenting each $\underline{\underline{F}}_t$ by P-null sets in the P-completion of $\underline{\underline{B}}$. For a $\underline{\underline{F}}$ and P as above, let

$$\underline{\underline{C}}(\underline{\underline{F}}) = \{X : X \text{ is a continuous } \underline{\underline{F}}\text{-adapted process}\}$$

$$\underline{\underline{D}}(\underline{\underline{F}}) = \{X : X \text{ is a right continuous } \underline{\underline{F}}\text{-adapted process having left limits}\}$$

$$\underline{\underline{W}}(\underline{\underline{F}}) = \{X : X \text{ is a } \underline{\underline{F}}\text{-progressively measurable process}\}$$

$$\underline{\underline{T}}(\underline{\underline{F}}) = \{T : T \text{ is a } \underline{\underline{F}}\text{-stop time}\}$$

$$\underline{\underline{A}}(\underline{\underline{F}}) = \{X \in \underline{\underline{C}}(\underline{\underline{F}}) : X(0) = 0 \text{ and for all } w, \text{ the map } t \longrightarrow X(t,w) \text{ is of bounded variation on bounded intervals}\}$$

$$\underline{\underline{A}}^+(\underline{\underline{F}}) = \{A \in \underline{\underline{A}}(\underline{\underline{F}}) : A \text{ is increasing}\}$$

$$\underline{\underline{L}}(\underline{\underline{F}}, P) = \{X \in \underline{\underline{C}}(\underline{\underline{F}}) : X(0) = 0 \text{ and } (X(t), \underline{\underline{F}}_t) \text{ is a } P \text{ local martingale}\}$$

If $X \in \underline{\underline{W}}(\underline{\underline{F}})$ and $T \in \underline{\underline{T}}(\underline{\underline{F}})$, define the stopped processes X^T and X^{T-} by

$$X^T(t,w) = X(t \wedge T(w), w) 1_{\{w' : T(w') > 0\}}(w)$$

and

$$X^{T-}(t,w) = X(t,w)1_{\{w' : T(w') > t\}}(w).$$

Thus X^T is the process X stopped at T except for the fact that $X^T \equiv 0$ on $(T=0)$. And X^{T-} is the process X up to but not including T and 0 after and including time T. Observe that these processes are again in $\underline{\underline{W}}(\underline{\underline{F}})$.

For $A \varepsilon \underline{\underline{A}}(\underline{\underline{F}})$, let $|A| \varepsilon \underline{\underline{A}}(\underline{\underline{F}})$ be its total varia[illegible] process, i.e.

$$|A|(t,w) = \int_0^t |dA(u,w)|$$

For $A_1, A_2 \varepsilon \underline{\underline{A}}(\underline{\underline{F}})$, say that A_2 dominates A_1 $(A_1 << A_2)$ if $A_2 - A_1 \varepsilon \underline{\underline{A}}^+(\underline{\underline{F}})$.

Let $\underline{\underline{E}} = C[0,\infty)$ be the metric space of Real-valued continuous functions on $[0,\infty)$ equipped with the topology of uniform convergence on compacta.

For $\rho \varepsilon \underline{\underline{E}}$ and $0 \leq t \leq \infty$ let

$$|\rho|_t^* = \sup_{0 \leq s \leq t} |\rho(s)|$$

1. <u>Quadratic Variation Process of a Continuous Local Martingale</u>:

It is a well known fact that if $M \varepsilon \underline{\underline{L}}(\underline{\underline{F}}, P)$, then there exists a unique $\langle M \rangle \varepsilon \underline{\underline{A}}^+(\underline{\underline{F}})$ such that $M^2 - \langle M \rangle \varepsilon \underline{\underline{L}}(\underline{\underline{F}}, P)$. This follows from a more general and difficult result of Meyer [29], [30], [31]. A simple proof of the result of Meyer was given by Rao [38]. In this section, we start with yet another simple proof of this result. Moreover, we will give a pathwise formula for $\langle M\rangle$, i.e. $\langle M\rangle(t,w)$ will be defined explicitly in terms of paths of M, thus showing that $\langle M\rangle$ neither depends on the filtration $\underline{\underline{F}}$ nor on the underlying probability measure P. This process $\langle M\rangle$ is called the quadratic variation process of M.

Now let $M \varepsilon \underline{\underline{C}}(\underline{\underline{F}})$ be given such that $M(0) = 0$. For each n, define a process K_n as follows:

$K_n(t,w) = j$ if there exists $\{t_i\}$ such that

$$0 = t_0 < t_1 < \ldots < t_j \leq t,$$

$$|M(t_i) - M(t_{i+1})| = 2^{-n}, \quad 0 \leq i < j$$

and $|M(t_i) - M(u)| < 2^{-n}$ if $u \varepsilon [t_i, t_{i+1}), \ 0 \leq i < j$

or $i = j$ and $u \varepsilon [t_j, t]$.

Let $$X_1(t,w) = \limsup_n \frac{K_n(t,w)}{2^{2n}},$$

$$X_2(t,w) = X_1(t-, w) \quad \text{if} \quad t > 0$$

$$= 0 \quad \text{if} \quad t = 0$$

$$U(w) = \inf\{t \geq 0 : X_2(t,w) \neq X_2(t+, w)\}$$

(As usual infimum of the empty set is ∞).

Thus U is the first point of discontinuity for X_1 and X is the process X_1 truncated at U and so defined at U as to make it continuous.

Then we have

Theorem 1 :

(i) $X \in \underline{\underline{A}}^{+}(\underline{\underline{F}})$

(ii) For all P such that $M \in \underline{\underline{L}}(\underline{\underline{F}}, P)$, $M^2 - X \in \underline{\underline{L}}(\underline{\underline{F}}, P)$.

Proof : Fix a P such that $M \in \underline{\underline{L}}(\underline{\underline{F}}, P)$. For $n \geq 1$ the $\{T_i^n : i \geq 0\}$ inductively by

$$T_0^n = 0$$

$$T_{i+1}^n = \inf \{t \geq T_i^n : |M(t) - M(T_i^n)| \geq 2^{-n}\}.$$

Let $Z_n(t) = \sum_{i=0}^{\infty} (M(t \wedge T_{i+1}^n) - M(t \wedge T_i^n))^2$.

Observe that for each $n \geq 1$, T_i^n goes to infinity as i tends to infinity. (It may be eventually infinity.) Thus for each (t,w), the sum defining $Z_n(t,w)$ is in fact a finite sum. Further,

$$K_n(t,w) = j \quad \text{iff} \quad T_j^n \leq t < T_{j+1}^n .$$

Thus

$$\frac{K_n(t,w)}{2^{2n}} \leq Z_n(t,w) < \frac{K_n(t,w)+1}{2^{2n}},$$

so that

$$X_1(t,w) = \limsup_n Z_n(t,w) .$$

Clearly $Z_n \in \underline{\underline{C}}(\underline{\underline{F}})$ for all n. Thus $X_1, X_2 \in \underline{\underline{W}}(\underline{\underline{F}})$, $U \in \underline{\underline{T}}(\underline{\underline{F}})$ and hence $X \in \underline{\underline{W}}(\underline{\underline{F}})$. Since X is continuous and increasing by definition, we have $X \in \underline{\underline{A}}^+(\underline{\underline{F}})$. Also, if $X_1(\cdot,w)$ is continuous then $X_1(t,w) = X(t,w)$ for all t. Thus complete the proof, suffices to prove

(1) Y_n converges a.s. P (in $\underline{\underline{E}}$) to a $Y \in \underline{\underline{L}}(\underline{\underline{F}}^P, P)$,

where $Y_n(t,w) = M^2(t,w) - Z_n(t,w)$.

Further, suffices to prove (1) assuming M is bounded (say by K). Since, in general we can get stop times $S_m \uparrow \infty$ such that M^{S_m} is bounded, the result will follow from the special case.

Now, writing $M^2(t)$ as

$$M^2(t) = \sum_{i=0}^{\infty} (M^2(t \wedge T^n_{i+1}) - M^2(t \wedge T^n_i))$$

we get

$$Y_n(t) = \sum_{i=0}^{\infty} 2M(t \wedge T^n_i)(M(t \wedge T^n_{i+1}) - M(t \wedge T^n_i))$$

$$= \sum_{i=0}^{\infty} Z_{n,i}(t) \quad \text{(say)}.$$

The fact that M is a bounded martingale implies that for each n, i, $(Z_{n,i}(t), \underline{\underline{F}}_t)$ is a martingale.

Also, for fixed t, n, $\{Z_{n,i}(t) : i \geq 0\}$ is a martingale difference sequence, so that

$$E\left(\sum_{i=r}^{s} Z_{n,i}(t)\right)^2 = \sum_{i=r}^{s} EZ^2_{n,i}(t)$$

$$\leq 4K^2 \sum_{i=r}^{s} E(M(t\wedge T^n_{i+1}) - M(t\wedge T^n_i))^2$$

$$= 4K^2 E(M^2(t\wedge T^n_{s+1}) - M^2(t\wedge T^n_r))$$

$$\longrightarrow 0 \quad \text{as} \quad r,s \longrightarrow \infty .$$

Thus $\sum_{i=0}^{\infty} Z_{n,i}(t)$ converges in L^2, so that for all n, $(Y_n(t), \underline{\underline{F}}_t, P)$ is a martingale.

For each n, let M_n be the process defined by

$$M_n(t) = M(T^n_i) \quad \text{if} \quad T^n_i \leq t < T^n_{i+1} .$$

It is not difficult to verify that for all w, n

$$\left\{T^n_i(w) : i \geq 0\right\} \subseteq \left\{T^{n+1}_j(w) : j \geq 0\right\} .$$

Thus

$$Y_{n-1}(t) = \sum_{j=0}^{\infty} 2M_{n-1}(t\wedge T^n_j)(M(t\wedge T^n_{j+1}) - M(t\wedge T^n_j)).$$

Hence

$$E(Y_n(t) - Y_{n-1}(t))^2 = E\Big[\, 2\sum_{j=0}^{\infty} (M(t\wedge T^n_j) - M_{n-1}(t\wedge T^n_j)).$$
$$(M(t\wedge T^n_{j+1}) - M(t\wedge T^n_j))\,\Big]^2$$

$$\leq 4\sum_{j=0}^{\infty} E(M(t\wedge T^n_j) - M_{n-1}(t\wedge T^n_j))^2 .$$
$$(M(t\wedge T^n_{j+1}) - M(t\wedge T^n_j))^2$$

(Use Fatou's Lemma and the fact that the summands form a martingale difference sequence.)

$$\leq \frac{4}{2^{2(n-1)}} \sum_{j=0}^{\infty} E(M^2(t \wedge T^n_{j+1}) - M^2(t \wedge T^n_j))$$

$$= \frac{16}{2^{2n}} E M^2(t) .$$

Now by Doob's maximal inequality

$$E \sup_{s \leq t} |Y_n(s) - Y_{n-1}(s)|^2 \leq \frac{64}{2^{2n}} E M^2(t) .$$

This, by Borel-Cantelli lemma, implies that $Y_n(.)$ converges a.s. P in $\underline{\underline{E}}$ to some process Y (say). Further, $Y_n(t)$ converges to $Y(t)$ in L^2 for each t. Thus Y is a continuous martingale.

As remarked earlier, this completes the proof.

Remark 1 : If $M \varepsilon \underline{\underline{A}}(\underline{\underline{F}})$, then observe (with the notations of the theorem) that

$$|Z_n(t,w)| \leq \frac{1}{2^n} |M|(t,w)$$

so that $X \equiv 0$. If moreover $M \varepsilon \underline{\underline{L}}(\underline{\underline{F}}, P)$ then by Theorem 1, $M^2 \varepsilon \underline{\underline{L}}(\underline{\underline{F}}, P)$ so that $M \equiv 0$ a.s. P.

Remark 2 : If A is any process in $\underline{\underline{A}}(\underline{\underline{F}})$ such that M and $M^2 - A$ belong to $\underline{\underline{L}}(\underline{\underline{F}}, P)$, then $A - X$ belongs to $\underline{\underline{L}}(\underline{\underline{F}}, P)$ and hence by Remark 1, $A = X$ a.s. P. Thus the process X obtained in Theorem 1 is the process $\langle M \rangle$ mentioned earlier.

Remark 3 : Kunita-Watanabe [24,p 212] proved that if $\{T_i^n : i \geq 0$ is a $\frac{1}{2^n}$ partition for M, $\langle M \rangle$, t and if moreover these partitions form a chain then Z_n defined as above converges a.s. to $\langle M \rangle$. Thus the existence of $\langle M \rangle$ is assumed in their proof.

Remark 4 : If $\{T_i^n\}$ is a sequence of partitions satisfying

(i) $|M(T_i^n) - M(u)| \leq \frac{1}{2^n}$ if $u \in (T_i^n, T_{i+1}^n)$

(ii) $\{T_i^n(w) : i \geq 0\} \subseteq \{T_j^{n+1}(w) : j \geq 0\}$,

then the proof of Theorem 1 shows that Z_n (defined as in Theorem 1) converge to X a.s. P in $\underline{\underline{E}}$.

Now suppose $M_1, M_2 \in \underline{\underline{L}}(\underline{\underline{F}}, P)$. It is possible to choose $\{T_i^n\}$ satisfying (i) and (ii) simultaniously for M_1, M_2 and $M_1 + M_2$. Now using the inequality $(x+y)^2 \leq 2(x^2+y^2)$ and Theorem 1 one can deduce that

$$\langle M_1 + M_2 \rangle << 2(\langle M_1 \rangle + \langle M_2 \rangle).$$

Similarly if $M_1, M_2, \ldots, M_k \in \underline{\underline{L}}(\underline{\underline{F}}, P)$, then using the inequality $(x_1 + \ldots + x_k)^2 \leq k(x_1^2 + \ldots + x_k^2)$, we can deduce

$$\langle \sum_{i=1}^{k} M_i \rangle << k \sum_{i=1}^{k} \langle M_i \rangle .$$

In the proof of uniqueness of $\langle M \rangle$ in Remark 2, we used the formula for $\langle M \rangle$. We now give an independent proof of uniqueness of $\langle M \rangle$. We give an elementary proof of the following result, which implies uniqueness of $\langle M \rangle$, 'almost all paths of a continuous local martingale have the property: on any interval, it is either a constant or of unbounded variation.

For $f \in C[0,\infty)$, let $V_b^a(f)$ be the variation of f on $[a,b]$. It can be easily shown that $V_x^a(f)$ is a left continuous function of x and if $V_b^a(f) < \infty$ then $V_x^a(f)$ and $V_b^x(f)$ are continuous functions of x on $[a,b]$. We shall now prove

<u>Theorem 2</u> : Let $M \in \underline{\underline{L}}(\underline{\underline{F}}, P)$. There exists a P null set N such that for $w \notin N$,

$$V_b^a(M(\cdot,w)) = 0 \quad \text{or} \quad \infty \quad \text{for all} \quad 0 \leq a \leq b \leq \infty .$$

<u>Proof</u> : First observe that it is sufficient to prove the theorem when M is a martingale. We prove the result in 3 steps.

If for every w, $V_\infty^0(M(\cdot,w)) \leq C < \infty$, then $M(\cdot,w) \equiv 0$ a.e. P. To see this observe that for each t,

$$M^2(t,w) = \int_0^t M(u,w)dM(u,w) \quad \text{(Riemann-Stieltjes integral)}$$

$$= \lim_n \sum_{i=1}^{n} M\left(\frac{ti}{n}, w\right) \left(M\left(\frac{t(i+1)}{n}, w\right) - M\left(\frac{ti}{n}, w\right)\right).$$

The assumption that $V_\infty^0(M(\cdot,w)) \leq C$ and the Dominated convergence theorem together imply that the above limit holds also in $L^1(P)$ so that $EM^2(t) = 0$. This implies $M(\cdot,w) \equiv 0$ a.s.P.

For $t \geq 0$, let $A(t,w) = V_t^0(M(\cdot,w))$. Then for a P null set N_0, $w \notin N_0$ implies $A(t,w) = 0$ or ∞. To see this, let $T_n = \inf\{t \geq 0 : A(t,w) \geq n\}$. Then left continuity of $A(t)$

implies that $A^{T_n} \leq n$. Also M^{T_n} is a martingale and $V_\infty^0(M^{T_n}(\cdot,w)) = A(T_n(w),w) \leq n$. Hence $M^{T_n}(\cdot,w) \equiv 0$ a.s. by the previous step. Thus if $T = \lim_n T_n$, then (by continuity of M) $M^T(\cdot,w) \equiv 0$ a.s. P. Now it is easy to check that $A(t,w) = 0$ if $t \leq T(w)$ and $= \infty$ if $t > T(w)$.

Finally, for every rational $r \geq 0$, consider the martingale $(M_{r+t})_{t \geq 0}$ (adapted to the appropriate filtration) and by the previous step get N_r such that $P(N_r) = 0$ and for $w \notin N_r$, $V_b^r(M(\cdot,w)) = 0$ or ∞ for every $b > r$. Let $N = \bigcup_r N_r$. Then for $w \notin N$, $V_b^r(M\cdot,w)) = 0$ or ∞ for all rationals r. Fix $a < b$. Then if $V_b^a(M(\cdot,w)) < \infty$, choose $r_n \in (a,b)$, $r_n \downarrow a$, r_n rationals. $V_b^a(M(\cdot,w)) < \infty$ implies $V_b^{r_n}(M(\cdot,w)) = 0$ for all n and hence $V_b^a(M(\cdot,w)) = 0$. This completes the proof.

<u>Remark 1</u> : Observe that uniqueness of $\langle M\rangle$ follows from Theorem 2.

<u>Remark 2</u> : Concerning a weak form of the above theorem - which is good enough to imply uniqueness of the process $\langle M\rangle$ - see Fisk [10, p 383] and Dellacherie [4, p 111].

We now prove a result connecting M and $\langle M\rangle$. See Kallianpur [16, p 72] for (iii).

<u>Lemma 3</u> : Let $M \in \underline{\underline{L}}(\underline{\underline{F}}, P)$. Then

(i) If M is a martingale and $EM^2(t) < \infty$ for all t, then $M^2 - \langle M\rangle$ is also a martingale.

(ii) If $E\langle M\rangle(t) < \infty$ for all t, then and $M^2 - \langle M\rangle$ are martingales.

(iii) $P(|M|_t^* \geq \alpha) \leq P(\langle M\rangle(t) \geq \beta) + \beta\alpha^{-2}$ for all positive α, β.

<u>Proof</u> : (i) Since $M^2 - \langle M\rangle \in \underline{\underline{L}}(\underline{\underline{F}}, P)$, get $T_n \in \underline{\underline{T}}(\underline{\underline{F}})$, $T_n \uparrow \infty$ such that $(M^2 - \langle M\rangle)^{T_n}$ is a martingale for all n. Thus in particular

$$EM^2(t \wedge T_n) = E\langle M\rangle(t \wedge T_n) \quad \text{for all } n.$$

Doob's maximal inequality implies that $E|M|_t^{*2} \leq 4EM^2(t) < \infty$, and hence $M^2(t \wedge T_n) \longrightarrow M^2(t)$ in L^1 by Dominated convergence theorem. Also

$$\begin{aligned} E\langle M\rangle(t) &= \lim_n E\langle M\rangle(t \wedge T_n) \\ &= \lim_n EM^2(t \wedge T_n) \\ &= EM^2(t) \\ &< \infty. \end{aligned}$$

Thus $M^2(t \wedge T_n) - \langle M\rangle(t \wedge T_n)$ converges in L^1 to $M^2(t) - \langle M\rangle(t)$ and hence $M^2 - \langle M\rangle$ is also a martingale.

(ii) Let $T_n \in \underline{\underline{T}}(\underline{\underline{F}})$, $T_n \uparrow \infty$ be such that M^{T_n}, $(M^2 - \langle M\rangle)^{T_n}$ are martingales. Then

$$EM^2(t \wedge T_n) = E\langle M\rangle(t \wedge T_n) \leq E\langle M\rangle(t) < \infty,$$

and hence $\{M(t \wedge T_n) : n \geq 1\}$ is uniformly integrable. Thus $M(t \wedge T_n) \longrightarrow M(t)$ in L^1 and hence M is a martingale. By Fatou's lemma,

$$EM^2(t) \leq \liminf_n EM^2(t \wedge T_n) \leq E\langle M\rangle(t) < \infty.$$

Now the result follows from (i).

(iii) Let $T_1 = \inf\{s : \langle M\rangle(s) \geq \beta\}$.

and $T = T_1 \wedge t$.

Then

$$P(|M|_t^* \geq \alpha) \leq P(T < t) + P(|M^T|_t^* \geq \alpha)$$

$$\leq P(T < t) + \alpha^{-2}E(M^T(t))^2 \quad \text{(by Doob's maximal inequality and part (ii) above)}$$

$$= P(T < t) + \alpha^{-2}EM^2(T)$$

$$\leq P(\langle M\rangle(t) \geq \beta) + \alpha^{-2}E\langle M\rangle(T)$$

$$\leq P(\langle M\rangle(t) \geq \beta) + \alpha^{-2}\beta .$$

2. Strict Time Change :

Definition : (See Ito-Watanabe [14,p vi]) $\sigma = (\sigma_t)_{t \geq 0}$ ($\subseteq \underline{\underline{T}}(\underline{\underline{F}})$) is called a strict $\underline{\underline{F}}$-time change if $\sigma_0(w) = 0$, $\lim_{t \to \infty} \sigma_t(w) = \infty$ and $t \longrightarrow \sigma_t(w)$ is a strictly increasing continuous function for all $w \in \Omega$.

For a strict $\underline{\underline{F}}$-time change σ and $X \in \underline{\underline{W}}(\underline{\underline{F}})$, let $\sigma\underline{\underline{F}}$ denote the filtration $(\underline{\underline{F}}_{\sigma_t})_{t \geq 0}$ and σX denote the $\sigma\underline{\underline{F}}$ progressively measurable process $\sigma X(t,w) = X(\sigma_t(w),w)$. See Meyer [31,p 67 and p 73]. Let λ, 'the inverse of σ' be defined by $\lambda_t(w) = u$ if $\sigma_u(w) = t$.

The next result gives some properties of a strict time change. See also Ito-Watanabe [14,p vi],Jacod [15,p 311] and Kazamaki [22].

<u>Lemma 4</u> : Let σ be a strict $\underline{\underline{F}}$-time change and λ be its inverse. Then

(a) If $T \varepsilon \underline{\underline{T}}(\underline{\underline{F}})$ then $\lambda_T \varepsilon \underline{\underline{T}}(\sigma\underline{\underline{F}})$ and $\underline{\underline{F}}_T = (\sigma\underline{\underline{F}})_{\lambda_T}$

(b) λ is a strict $\sigma\underline{\underline{F}}$-time change and $\lambda(\sigma\underline{\underline{F}}) = \underline{\underline{F}}$

(c) If $X \varepsilon \underline{\underline{W}}(\underline{\underline{F}})$ and $T \varepsilon \underline{\underline{T}}(\underline{\underline{F}})$ then $\sigma(X^T) = (\sigma X)^{\lambda_T}$ and $\sigma(X^{T-}) = (\sigma X)^{\lambda_T -}$.

(d) The map $X \longrightarrow \sigma X$ is a bijection between $(\underline{\underline{G}}(\underline{\underline{F}}), \underline{\underline{G}}(\sigma\underline{\underline{F}}))$ where $\underline{\underline{G}}$ is any of the class of processes defined in section 0.

(e) Let $A \varepsilon \underline{\underline{A}}(\underline{\underline{F}})$, $f \varepsilon \underline{\underline{W}}(\underline{\underline{F}})$ and X be defined by $X(t) = \int_0^t f(u)dA(u)$ (Riemann-Stieltjes integral). Then

$$(\sigma X)(s) = \int_0^s (\sigma f)(u)d(\sigma A)(u).$$

<u>Proof</u> : (a) Observe that $\bigvee_{t \geq 0} \underline{\underline{F}}_t = \bigvee_{s \geq 0} \underline{\underline{F}}_{\sigma_s}$ ... (i)

Further,

$$A \varepsilon \underline{\underline{F}}_T \Longrightarrow A \cap (T \leq \sigma_t) \varepsilon \underline{\underline{F}}_{\sigma_t} \; \forall \; t \geq 0$$

$$\Longrightarrow A \cap (\lambda_T \leq t) \varepsilon (\sigma\underline{\underline{F}})_t \; \forall \; t \geq 0. \qquad \text{...(ii)}$$

Taking $A = \Omega$, we get $\lambda_T \varepsilon \underline{\underline{T}}(\sigma\underline{\underline{F}})$. Now (i) and (ii) imply

$$\underline{\underline{F}}_T \subseteq (\sigma\underline{\underline{F}})_{\lambda_T}.$$

Also,

$$A \in (\sigma\underline{\underline{F}})_{\lambda_T} \implies A \cap (\lambda_T \leq t) \in (\sigma\underline{\underline{F}})_t \quad \forall\, t \geq 0$$

$$\implies A \cap (T \leq \sigma_t) \in \underline{\underline{F}}_{\sigma_t} \quad \forall\, t \geq 0$$

$$\implies A \cap (T \leq \sigma_t) \cap (\sigma_t < s) \in \underline{\underline{F}}_s \quad \forall\, t \geq 0,\ s \geq 0$$

$$\implies A \cap (T < s) \in \underline{\underline{F}}_s \quad \forall\, s \geq 0 \qquad \ldots\text{(iii)}$$

Now (i), (iii) and right continuity of $\underline{\underline{F}}_t$ imply $(\sigma\underline{\underline{F}})_{\lambda_T} \subseteq \underline{\underline{F}}_T$.

(b) follows from (a)

(c) and (d) are easy to verify and (e) follows from 'change of variable formula' for Riemann Stieltjes integral.

Regarding the following lemma see also Kazamaki [22,p 58]

Lemma 5 : Let $A \in \underline{\underline{A}}(\underline{\underline{F}})$ and $\beta > 0$. Then there exists a strict $\underline{\underline{F}}$ time change σ such that for all w, the map $t \longrightarrow \sigma A(t,w)$ is absolutely continuous with derivatives bounded by β.

Proof : Define $\sigma_t(w) = \inf\{s \geq 0 : (\beta^{-1}|A|(s,w) + s) \geq t\}$. Continuity of A implies that

$$\beta^{-1}(\sigma|A|(t_2,w) - \sigma|A|(t_1,w)) + \sigma_{t_2}(w) - \sigma_{t_1}(w) = t_2 - t_1 .$$

This implies the required result.

3. Definition of Stochastic Integral :

For a filtration $\underline{\underline{F}}$, let $\underline{\underline{U}}(\underline{\underline{F}})$ be the class of all bounded simple processes f of the form $f = \sum_{i=1}^{n} f_i 1_{[T_i, T_{i+1})}$, where $T_i \in \underline{\underline{T}}(\underline{\underline{F}})$ and f_i and are $\underline{\underline{F}}_{T_i}$ measurable random variables.

For $A \in \underline{\underline{A}}(\underline{\underline{F}})$, let

$$\underline{\underline{V}}(\underline{\underline{F}},A,P) = \left\{X \in \underline{\underline{W}}(\underline{\underline{F}}) : \int_0^t X^2(u)d|A|(u) < \infty \text{ for all } t, \text{ a.s. } P\right\}$$

Observe that any process X in $\underline{\underline{D}}(\underline{\underline{F}})$ is already in $\underline{\underline{V}}(\underline{\underline{F}},A,P)$ for all $A \in \underline{\underline{A}}(\underline{\underline{F}})$ and P.

We now define the stochastic integral for simple integrands.

<u>Definition</u> : Let $M \in \underline{\underline{L}}(\underline{\underline{F}}, P)$ and $f = \sum_{i=1}^{n} f_i 1_{[T_i, T_{i+1})} \in \underline{\underline{U}}(\underline{\underline{F}})$.

Then define the stochastic integral $\int fdM$ of f w.r.t. M by

$$\int_0^t fdM = \sum_{i=1}^{n} f_i (M(T_{i+1} \wedge t) - M(T_i \wedge t)) .$$

It is routine to verify that this definition does not depend on the representation of f.

The following proposition is immediate from the definition.

<u>Proposition 6</u> : Let $M \in \underline{\underline{L}}(\underline{\underline{F}}, P)$ and $f \in \underline{\underline{U}}(\underline{\underline{F}})$. Let $X(t) = \int_0^t fdM$ and $Y(t) = \int_0^t f^2 d\langle M\rangle$. Then

(a) $X, X^2 - Y \in \underline{\underline{L}}(\underline{\underline{F}}, P)$

(b) If σ is a strict $\underline{\underline{F}}$-time change, then $\sigma f \in \underline{\underline{U}}(\sigma\underline{\underline{F}})$ and

$$(\sigma X)(t) = \int_0^t (\sigma f)d(\sigma M) .$$

The next step, as usual, is an approximation result.

Lemma 7 : Let $M \in \underline{\underline{L}}(\underline{\underline{F}}, P)$ and $f \in \underline{\underline{V}}(\underline{\underline{F}}, \langle M\rangle, P)$. Then

(i) there exists a sequence $\{f_n\} \subset \underline{\underline{U}}(\underline{\underline{F}}^P)$ such that for all t,

$$\int_0^t |f - f_n|^2 d\langle M\rangle \longrightarrow 0 \text{ in P-probability}$$

(ii) if $\{f_n\}$ is as in (i), then $\int_0^{\cdot} f_n dM$ is an $\underline{\underline{E}}$ valued sequence which is 'cauchy' in P-probability, and hence converges

(iii) if $\{f_n\}$ and $\{\tilde{f}_n\}$ are as in (i) then

$$P - \lim_n \int_0^{\cdot} f_n dM = P - \lim_n \int_0^{\cdot} \tilde{f}_n dM .$$

Proof : By Lemma 5, get a strict $\underline{\underline{F}}$-time change σ such that $\langle \sigma M\rangle(t)$ is absolutely continuous, $\frac{d\langle \sigma M\rangle(t)}{dt} \leq 1$ and $\int_0^{\sigma t} f^2 d\langle M\rangle \leq t$.

Let $g = \sigma f$ and $N = \sigma M$. Then $\langle N\rangle = \sigma\langle M\rangle$, $g \in \underline{\underline{V}}(\sigma\underline{\underline{F}}, \langle N\rangle, P)$ and $E \int_0^t g^2 d\langle N\rangle \leq t < \infty$.

Let $g_k = g.(\frac{|g| \wedge k}{|g|})$. Then $g_k \in \underline{\underline{W}}(\sigma\underline{\underline{F}})$ and is bounded. Get $g_{k,n} \in \underline{\underline{U}}(\sigma\underline{\underline{F}}^P)$ such that

$$E \int_0^t |g_k - g_{k,n}|^2 du \longrightarrow 0 \text{ for all } t, \text{ as } n \longrightarrow \infty .$$

This is the standard approximation result used in defining stochastic integrals for Brownian motion. See Ito [12,p 176] Kallianpur [16,p 60].

Now $\frac{d\langle N\rangle}{dt} \leq 1$ implies

$$E \int_0^t |g_k - g_{k,n}|^2 d\langle N\rangle \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \quad \text{for all} \quad k, t$$

and Dominated convergence theorem gives

$$E \int_0^t |g - g_k|^2 d\langle N\rangle \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \quad \text{for all} \quad t.$$

Get $\{n_k\}$ such that

$$E \int_0^k |g_k - g_{k,n_k}|^2 d\langle N\rangle \leq 2^{-k}.$$

Let $\hat{g}_k = g_{k,n_k}$ and $f_k = \lambda(\hat{g}_k)$, where λ is the inverse of σ. Then

$$E \int_0^t |\hat{g}_k - g|^2 d\langle N\rangle \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \quad \text{for all} \quad t,$$

$$\text{i.e. } E \int_0^{\sigma_t} |f_k - f|^2 d\langle M\rangle \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \quad \text{for all} \quad t.$$

This implies that $\int_0^t |f_k - f|^2 d\langle M\rangle \longrightarrow 0$ in P-probability $\forall\ t$.

For (ii) let $\{f_n\}$ be as in (i) and let $X_n(t) = \int_0^t f_n dM$. Then $\langle X_n - X_m\rangle(t) = \int_0^t |f_n - f_m|^2 d\langle M\rangle$. So, hypothesis implies that $\langle X_n - X_m\rangle(t) \longrightarrow 0$ in P-probability as $n,m \longrightarrow \infty$. This by Lemma 3 implies that X_n is cauchy in P-probability. (iii) follows by the usual interlacing argument.

In view of the above result, we make the following

Definition : Let $M \varepsilon \underline{\underline{L}}(\underline{\underline{F}}, P)$ and $f \varepsilon \underline{\underline{V}}(\underline{\underline{F}}, \langle M \rangle, P)$. Then define

$$\int_0^{\cdot} f dM = P - \lim_n \int_0^{\cdot} f_n dM ,$$

where $\{f_n\}$ approximates f as in Lemma 7.

Usually integration w.r.t. martingales is done only with predictable integrands. See Kunita-Watanabe [24], Meyer [34], Kallianpur [16]. However, when the integrator is a continuous local martingale one can allow progressively measurable integrands as was observed by Ito-Watanabe [14, p vi].

We now list down several properties of the stochastic integral.

Theorem 8 : Let $M \varepsilon \underline{\underline{L}}(\underline{\underline{F}}, P)$.

(a) Let $f \varepsilon \underline{\underline{V}}(\underline{\underline{F}}, \langle M \rangle, P)$, $X = \int_0^{\cdot} f dM$ and $Y = \int_0^{\cdot} f^2 d \langle M \rangle$. Then

(i) $X, X^2 - Y \varepsilon \underline{\underline{L}}(\underline{\underline{F}}^P, P)$

(ii) If σ is a $\underline{\underline{F}}$-time change, then $(\sigma X)(t) = \int_0^t (\sigma f) d(\sigma M)$.

(iii) If $T \varepsilon \underline{\underline{T}}(\underline{\underline{F}})$, then

$$X^T = \int_0^{\cdot} f dM^T = \int_0^{\cdot} f^{T-} dM .$$

(b) Let $f_i \varepsilon \underline{\underline{V}}(\underline{\underline{F}}, \langle M \rangle, P)$ $i = 1,2$ be such that $f_1 = f_2$ on $[T_1, T_2)$, where $T_i \varepsilon \underline{\underline{T}}(\underline{\underline{F}})$, and $X_i(t) = \int_0^t f_i dM$. Then

$$X_1(t \wedge T_2) - X_1(t \wedge T_1) = X_2(t \wedge T_2) - X_2(t \wedge T_1) .$$

(c) Let $f_n \in \underline{\underline{V}}(\underline{\underline{F}}, \langle M\rangle, P)$ be such that $\int_0^{\cdot} |f_n - f|^2 d\langle M\rangle \longrightarrow 0$ in P-probability. Then

$$\int_0^{\cdot} f_n dM \longrightarrow \int_0^{\cdot} f dM \quad (\text{in } \underline{\underline{E}}) \quad \text{in P-probability}.$$

(d) Let $T_k \in \underline{\underline{T}}(\underline{\underline{F}})$ be such that $T_k \uparrow \infty$. For $k \geq 1$, let θ_k be a $\underline{\underline{F}}_{T_k}$ measurable random variable. Define $f = \sum_{k=1}^{\infty} \theta_k 1_{[T_k, T_{k+1})}$. Then $f \in \underline{\underline{V}}(\underline{\underline{F}}, \langle M\rangle, P)$ and

$$\int_0^t f dM = \sum_{k=0}^{\infty} \theta_k (M(T_{k+1} \wedge t) - M(T_k \wedge t)).$$

(e) Let $f \in \underline{\underline{V}}(\underline{\underline{F}}, \langle M\rangle, P)$, $N = \int_0^{\cdot} f dM$.

Then $g \in \underline{\underline{V}}(\underline{\underline{F}}^P, \langle N\rangle, P)$ iff $gf \in \underline{\underline{V}}(\underline{\underline{F}}^P, \langle M\rangle, P)$ and

$$\int_0^{\cdot} g dN = \int_0^{\cdot} gf dM.$$

Proof : (a) Let $\{f_n\}$ be the sequence constructed in Lemma 7(i), σ as in Lemma 7(i), $\hat{g}_n = \sigma f_n$, $g = \sigma f$, $N = \sigma M$, $X_n(t) = \int_0^t f_n dM$, $Y_n(t) = \int_0^t f_n^2 d\langle M\rangle$. Then

$$E \int_0^t |\hat{g}_n - g|^2 d\langle N\rangle \longrightarrow 0$$

$$\left.\begin{array}{ll} \text{so that} & \sigma X_n(t) \longrightarrow \sigma X(t) \quad \text{in } L^2 \\ \text{and} & \sigma Y_n(t) \longrightarrow \sigma Y(t) \quad \text{in } L^1 \end{array}\right\} \quad (1)$$

But σX_n, $\sigma(X_n^2 - Y_n) \in \underline{\underline{L}}(\underline{\underline{F}}, P)$ and $E\langle\sigma X_n\rangle(t) = E\,\sigma Y_n(t) < \infty$. Thus by Lemma 3, σX_n, $\sigma(X_n^2 - Y_n)$ are martingales and hence by (1),

σX, $\sigma(X^2 - Y)$ are martingales. This implies (i). (ii) and (iii) follow from similar properties for $f \in \underline{\underline{U}}(\underline{\underline{F}})$.

For (b), observe that by (iii) above,

$$X_i(t) - X_i(t \wedge T_1) = \int_0^t f_i 1_{[T_1, \infty)} dM$$

and hence

$$X_i(t \wedge T_2) - X_i(t \wedge T_1) = \int_0^t f_i 1_{[T_1, T_2)} dM ,$$

which gives the required result.

For (c), let $X_n(t) = \int_0^t f_n dM$, $X(t) = \int_0^t f dM$. Then

$$\langle X_n - X\rangle(t) = \int_0^t |f_n - f|^2 d\langle M\rangle \quad \text{by (i) above.}$$

Thus $\langle X_n - X\rangle(t) \longrightarrow 0$ in P-probability for all t. This, with Lemma 3, implies that $X_n \longrightarrow X$.

For (d), let $\sum_{k=1}^{n} \theta_k 1_{[T_k, T_{k+1})} = f_n$. Then $\int_0^t |f_n - f|^2 d\langle M\rangle \longrightarrow 0$ in P-probability so that the result follows from (c).

For (e), observe that $\langle N\rangle = \int_0 f^2 d\langle M\rangle$, so that

$$\int_0 g^2 d\langle N\rangle = \int_0 g^2 f^2 d\langle M\rangle ,$$

and hence the first part follows. If g is a simple function, then $\int_0 g dN = \int_0 gf dM$ follows from (iii) above. For a general g, get g_n simple such that $\int_0^t |g_n - g|^2 d\langle N\rangle \longrightarrow 0$ in P-probability for all t. Then $\int_0^t |g_n f - gf|^2 d\langle M\rangle \longrightarrow 0$ in P-probability for all t and hence

$$\int_0^{\cdot} g_n dN \longrightarrow \int_0^{\cdot} g dN$$

and

$$\int_0^{\cdot} g_n f dM \longrightarrow \int_0^{\cdot} g f dM$$

in P-probability by (c). The result follows from these observations.

4. Vector Valued Semimartingale Integrals and the Growth Inequality :

We make the following

Definition : X is called a continuous semimartingale ($X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$) if X can be written as $M + A$, where $M \in \underline{\underline{L}}(\underline{\underline{F}}, P)$ and $A \in \underline{\underline{A}}(\underline{\underline{F}}, P)$.

Observe that in view of remark 1 following Theorem 1, for $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$, the decomposition $X = M + A$ is unique. Let us call it the canonical decomposition of X.

X is called a semimartingale in the literature (See Jacod [15,p 29]) if X can be written as a sum of a local martingale M and a process A of bounded variation on compacta. It is also known that if S is continuous, there exists a decomposition $X = M + A$, where A is predictable. This decomposition is unique if we insist that A be predictable and in such a case both the processes M and A are continuous (See Jacod [15,p 29]). So our definition coincides with the standard definition.

For $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$, let us define $\langle X \rangle$, $|X|$ by

$$\langle X \rangle = \langle M \rangle$$

$$|X| = |A|$$

where $X = M + A$ is the canonical decomposition of X.

Extension of the local martingale integral to semimartingale integral is immediate. Observe that $\underline{\underline{D}}(\underline{\underline{F}}) \subsetneq \underline{\underline{V}}(\underline{\underline{F}}, A, P)$ for all P and all $A \in \underline{\underline{A}}(\underline{\underline{F}})$. From now on we consider integrands from $\underline{\underline{D}}(\underline{\underline{F}})$.

<u>Definition</u> : Let $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ and $X = M + A$ be its canonical decomposition. Let $f \in \underline{\underline{D}}(\underline{\underline{F}})$. Define

$$\int_0^{\cdot} f dX = \int_0^{\cdot} f dM + \int_0^{\cdot} f dA .$$

<u>Remark</u> : The first integral on RHS is stochastic integral defined in Section 3 and the second integral is Riemann Stieltjes integral.

Now, we extend the semimartingale integral to multidimensional case. Let $L(m,k)$ denote the space of $m \times k$ matrices. When $m = k$, $L(m,k)$ is denoted by $L(k)$. To simplify notations, we say that a matrix (or vector) valued process belongs to a class of processes if each of its components belongs to the class. For example if M is a R^d-valued local martingale, we will write

$$M \in \underline{\underline{L}}(\underline{\underline{F}}, P) \quad (\text{or } M \in \underline{\underline{L}}(\underline{\underline{F}}, P) \ (R^d\text{-valued}).)$$

<u>Definition</u> : Let $X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$ (R^k-valued) and $f \varepsilon \underline{\underline{D}}(\underline{\underline{F}})$ ($L(m,k)$ valued). Then $Y = \int_0^{\cdot} f dX$ is the R^m-valued process defined by

$$(Y(t))_i = \sum_{j=1}^{k} \int_0^t f_{ij} dX_j \ , \quad 1 \leq i \leq m .$$

The properties of 'semimartingale' integral follow easily from properties of local martingale integral (Section 3) and the Riemann-Stieltjes integral, the most important being invariance under strict time change (see Theorem 8).

Let $\underline{\underline{E}}_k$, $\underline{\underline{E}}_{m,k}$ denote $C([0,\infty), R^k)$, $C([0,\infty), L(m,k))$ respectively equipped with topology of uniform convergence on compacta. When it is clear from the context, we will write $\underline{\underline{E}}$ for $\underline{\underline{E}}_k$ or $\underline{\underline{E}}_{m,k}$.

For $\rho \varepsilon \underline{\underline{E}}_k$ or $\underline{\underline{E}}_{m,k}$ and $t \geq 0$ let

$$|\rho|_t^* = \sup_{0 \leq s \leq t} |\rho(s)| ,$$

where $|.|$ denotes the norm - root of sum of squares of entries - on R^k or $L(m,k)$.

Now we introduce a subclass of semimartingales, $Q(\beta)$ (also called Ito-processes :see Stroock-Varadhan [42,p 92 and p 113]) and obtain an estimate on the growth of the stochastic integral when the integrator belongs to $Q(\beta)$. This inequality with the invariance of stochastic integral under strict time changes is the key step in all the convergence arguments in the later sections.

For $\beta > 0$, let $Q(\beta) = Q(\beta, \underline{\underline{F}}, P)$ be defined by

$$Q(\beta) = \left\{ X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P) : \langle X\rangle(t, w) \text{ and } |X|(t,w) \text{ are absolutely continuous functions of } t \text{ with derivatives bounded by } \beta \text{ a.e. } P. \right\}$$

Observe that if $X \varepsilon Q(\beta)$ and $X = M + A$ then $\langle M\rangle = \langle X\rangle << \beta t$ so that M is already a martingale.

<u>Lemma 9</u> : (a) Let $X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$ ($L(m,k)$ valued) and $\beta > 0$. Then there exists a strict $\underline{\underline{F}}$-time change σ such that

$$\sigma X \varepsilon Q(\beta) = Q(\beta, \sigma\underline{\underline{F}}, P).$$

(b) Let $X \varepsilon Q(\beta)$ and $h \varepsilon \underline{\underline{D}}(\underline{\underline{F}})$ be bounded by b. Then Y defined by $Y(t) = \int_0^t hdX$ belongs to $Q(b(1+b)\beta)$.

(c) Let $X_i \varepsilon Q(\beta_i)$, $1 \leq i \leq k$. Then X defined by $X = \sum_{i=1}^k X_i$ belongs to $Q(k \sum_{i=1}^k \beta_i)$.

<u>Proof</u> : (a) Let $A = \sum_{i,j} (\langle X_{ij}\rangle + |X_{ij}|)$. Then $A \varepsilon \underline{\underline{A}}(\underline{\underline{F}})$. Choose σ as in Lemma 5, which does the job.

(b) Follows from the relations

$$\langle Y\rangle(t) = \int_0^t h^2 d\langle X\rangle$$

and

$$|Y|(t) = \int_0^t |h| d|X|.$$

(c) As noted earlier, if $M_i \varepsilon \underline{\underline{L}}(\underline{\underline{F}}, P)$, $i = 1, \ldots, k$ then

$$\langle \sum_{i=1}^k M_i\rangle << k \sum_{i=1}^k \langle M_i\rangle$$

so that

$$\langle \sum_{i=1}^{k} X_i \rangle \ll k \sum_{i=1}^{k} \langle X_i \rangle .$$

Also,

$$| \sum_{i=1}^{k} X_i | \ll \sum_{i=1}^{k} |X_i| .$$

The result follows from these observations.

We conclude this section with a growth inequality for stochastic integrals. We have belatedly come to know that a variant of this appears in Metivier and Pellaumail [36], [37].

Theorem 10 : (Growth Inequality)

Let $X \in Q(\beta)$ ($\mathbb{R}^k$ valued) and $h \in \underline{\underline{D}}(\underline{\underline{F}})$ ($L(m,k)$ valued). Then

$$E| \int_0^{\cdot} h dX |_t^{*2} \leq 8k\beta(1+t\beta) \int_0^t E|h|^2(u)du .$$

Proof : Let $X = M + A$ be the canonical decomposition of X. Suffices to show that (i) and (ii) hold.

(i) $$E| \int_0^{\cdot} h dM |_t^{*2} \leq 4k\beta \int_0^t E|h|^2 du$$

and

(ii) $$E| \int_0^{\cdot} h dA |_t^{*2} \leq tk\beta^2 \int_0^t E|h|^2 du .$$

Now,

$$E| \int_0^{\cdot} h dM |_t^{*2} \leq E \sup_{s \leq t} \left(\sum_{i=1}^{m} \left(\sum_{j=1}^{k} \int_0^s h_{ij} dM_j \right)^2 \right)$$

$$\leq k \sum_{i=1}^{m} \sum_{j=1}^{k} E \sup_{s \leq t} \left(\int_0^s h_{ij} dM_j \right)^2$$

Thus by Doob's maximal inequality,

$$E\left|\int_o^{\cdot} h\,dM\right|_t^{*2} \leq 4k \sum_{i=1}^{m} \sum_{j=1}^{k} E\left(\int_o^t h_{ij}\,dM_j\right)^2 .$$

Observe that if $N \in \underline{\underline{L}}(\underline{\underline{F}}, P)$, then $EN^2(t) \leq E\langle N\rangle(t)$ $\forall$ t. In fact if $E\langle N\rangle(t)$ is finite then $EN^2(t) = E\langle N\rangle(t)$ by Lemma 3. Applying this remark to $N = \int_o^{\cdot} f_{ij}\,dM_j$, we get

$$\begin{aligned} E\left|\int_o^{\cdot} h\,dM\right|_t^{*2} &\leq 4k \sum_{i=1}^{m} \sum_{j=1}^{k} E\left(\int_o^t h_{ij}^2\,d\langle M_j\rangle\right) \\ &\leq 4k\beta \sum_{i=1}^{m} \sum_{j=1}^{k} E\left(\int_o^t h_{ij}^2(u)\,du\right) \\ &= 4k\beta \int_o^t E|h|^2(u)\,du . \end{aligned}$$

(ii) follows similarly.

As remarked earlier, this completes the proof of the growth inequalit-.

5. Pathwise Integration Formulae :

Let $h \in \underline{\underline{D}}(\underline{\underline{F}})$ and $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$. Then $Y = \int h\,dX$ as defined in Section 4 depends upon the underlying probability measure in more than one way. Firstly, the canonical decomposition $X = M+A$ depends on the probability measure. Secondly, $\int h\,dM$ is defined as limit in probability of integrals of simple approximations.

A natural question to ask is whether we can choose a process Y such that for all measures P under which X is a semimartingale, Y is a version of $\int hdX$. We answer this question in the affirmative. In fact, we choose the common version pathwise - i.e. $Y(\cdot,w)$ is defined explicitly in terms of $h(\cdot,w)$ and $X(\cdot,w)$. This has also been done by Bichteler [1 ,p 65] by using different techniques.

A similar question can be asked about $\int hd\langle X\rangle$. Again, the answer is in the affirmative and as before, we can choose a pathwise version. In particular, we get $\langle X\rangle$ explicitly in terms of paths of X.

Now we introduce 'random-partitions' which are crucial to get almost surely convergent good approximations to stochastic integrals.

A random partition (of $[0,\infty)$) is a sequence $\{T_i : i \geq 0\}$ of $\underline{\underline{F}}$ - stop times such that $T_0 = 0$ and T_i is increasing to ∞. For a random partition $\{T_i : i \geq 0\}$, define the operators J and H_i, $i \geq 0$ from $\underline{\underline{D}}(\underline{\underline{F}})$ into itself as follows. For a $h \in \underline{\underline{D}}(\underline{\underline{F}})$,

$$(Jh)(t,w) = h(T_i(w),w) \quad \text{if} \quad T_i(w) \leq t < T_{i+1}(w) ,$$

$$(H_i h)(t,w) = h(T_{i+1}(w) \wedge t,w) - h(T_i(w) \wedge t,w) .$$

A random partition $\{T_i : i \geq 0\}$ is said to be an α-partition for the processes $h_1, h_2, \ldots, h_m$ if

$$|Jh_j - h_j| \leq \alpha , \quad j = 1,2,\ldots,m .$$

In the rest of this section (and the subsequent sections) we will be dealing with a sequence $\{T_i^n : i \geq 0\}$, $n \geq 1$ of random partitions. The operators J and H_i for the partition $\{T_i^n : 1 \geq 0\}$ will be denoted by J^n and H_i^n respectively.

Lemma 11 : Let $h_j \in \underline{\underline{D}}(\underline{\underline{F}})$ ($L(m,k)$ valued), $1 \leq j \leq r$, and $\alpha > 0$. Then there exists an α partition $\{T_i\}$ for the processes h_j, $1 \leq j \leq r$. Further, $\{T_i : i \geq 0\}$ can be chosen such that for all w, $\{T_i(w) : i \geq 0\}$ is defined explicity in terms of the paths $\{h_j(\cdot,w) : 1 \leq j \leq r\}$.

Proof : Define $\{T_i : i \geq 0\}$ inductively by

$$T_0 = 0$$

$$T_{i+1} = \inf \{t \geq T_i : |h_j(t) - h_j(T_i)| \geq \alpha \text{ for some } j, 1 \leq j \leq r\}$$

This partition $\{T_i\}$ has the required properties.

Now we are in a position to get a Pathwise integration formula (for $\int hdX$).

Let $h \in \underline{\underline{D}}(\underline{\underline{F}})$ ($L(m,k)$ valued) and $X \in \underline{\underline{C}}(\underline{\underline{F}})$ (R^k valued) be such that $X(0) = 0$. Let $\{T_i^n : i \geq 0\}$ be a (parthwise) $\frac{1}{2^n}$ partition of h (existence is assured by Lemma 11). Let

$$Y_n(t) = \sum_{i=0}^{\infty} h(T_i^n \wedge t)[H_i^n X(t)] .$$

Let $\Omega_0 = \{w : Y_n(\cdot,w) \text{ converges in } \underline{\underline{E}}\}$

and

$$Y(\cdot,w) = \lim_n Y_n(\cdot,w) \quad \text{for } w \in \Omega_0$$
$$= 0 \qquad \text{otherwise}$$

Then we have

<u>Theorem 12</u> : For all P such that $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$, we have

$$\text{(i)} \quad P(\underline{(\bar{\ })}_o) = 1$$

$$\text{and} \quad \text{(ii)} \quad Y(t) = \int_o^t h\,dX$$

<u>Proof</u> : Fix a P such that $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$. Let $Z = \int_o^t h\,dX$. To prove the theorem, suffices to prove that $Y_n \longrightarrow Z$ a.s. P in $\underline{\underline{E}}$. To this end, observe that

$$Y_n(t) = \int_o^t (J^n h)\,dX .$$

Let σ be a strict $\underline{\underline{F}}$-time change such that $\sigma X \in Q(1)$. Fix $t > 0$. Then by the 'growth-inequality' we have

$$E|\sigma Y_n - \sigma Z|_t^{*2} \leq 8k(1+t)E \int_o^t |\sigma(J^n h - h)(u)|^2 du$$

$$\leq 8k(1+t)t2^{-n} .$$

Thus, $\sum_{n=1}^{\infty} \left[E|\sigma Y_n - \sigma Z|_t^{*2} \right]^{1/2} < \infty$. By Minkowski's inequality, this yields

$$\sum_{n=1}^{\infty} |\sigma Y_n - \sigma Z|_t^{*2} < \infty \quad \text{a.s. } P .$$

This implies that $\sigma Y_n \longrightarrow \sigma Z$ a.s. P in $\underline{\underline{E}}$, which is same as $Y_n \longrightarrow Z$ a.s. in $\underline{\underline{E}}$. This completes the proof as pointed out earlier.

Let $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ and $X = M + A$ be its canonical decomposition. Recall that $\langle X \rangle$ is defined by $\langle X \rangle = \langle M \rangle$. For

$X_1, X_2 \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$, define $\langle X_1, X_2\rangle$ by the polarisation formula

$$\langle X_1, X_2\rangle = \frac{1}{4}(\langle X_1 + X_2\rangle - \langle X_1 - X_2\rangle) .$$

In order to get a pathwise integration formula for $\int h d\langle X_1, X_2\rangle$, we need the following lemma which is a special case of 'Ito's formula'.

Lemma 13 : Let $X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$. Then

$$X^2(t) = 2\int_0^t XdX + \langle X\rangle(t) \quad \text{for all } t, \quad \text{a.e. } P.$$

Proof : Since X^2, $\int XdX$ and $\langle X\rangle$ are continuous processes, suffices to prove that for every fixed t,

$$X^2(t) = 2\int_0^t XdX + \langle X\rangle(t) \quad \text{a.e. } P.$$

So, fix a $t > 0$. By a strict time change if necessary, we can assume that $X \varepsilon Q(1)$. If $X = M+A$ is the canonical decomposition of X, then observe that $X \varepsilon Q(1)$ implies $\langle M\rangle(u) \leq u$ and hence by Lemma 3, M and $M^2 - \langle M\rangle$ are martingales. A simple computation shows that for two stop times $S_1 \leq S_2$,

$$E((M(S_2) - M(S_1))^2 - \langle M\rangle(S_2) + \langle M\rangle(S_1) | \underline{\underline{F}}_{S_1}) = 0$$

Now, let $\{T_i^n : i \geq 0\}$ be a $\frac{1}{2^n}$ partition for X, M, A and $\langle M\rangle$ and let $S_i^n = T_i^n \wedge t$. Then

$$X^2(t) = \sum_{i=0}^{\infty}(X^2(S_{i+1}^n) - X^2(S_i^n))$$

$$= \sum_{i=0}^{\infty}(2X(S_i^n)(X(S_{i+1}^n) - X(S_i^n)) + (X(S_{i+1}^n) - X(S_i^n))^2)$$

$$= 2 \int_0^t (J^n X)\,dX + \sum_{i=0}^{\infty} (H_i^n X(t))^2 \qquad \ldots(1)$$

By Theorem 12, $\int_0^t (J^n X)\,dX \longrightarrow \int_0^t X\,dX$ a.e. P $\qquad \ldots(2)$

Let $Y_n = \sum_{i=0}^{\infty} \left[(H_i^n X(t))^2 - (H_i^n M(t))^2 \right]$

and $Z_n = \sum_{i=0}^{\infty} \left[(H_i^n M(t))^2 - H_i^n \langle M \rangle (t) \right]$

$$= \sum_{i=0}^{\infty} (H_i^n M(t))^2 - \langle M \rangle (t)$$

Then $\sum_{i=0}^{\infty} (H_i^n X(t))^2 = Y_n + Z_n + \langle M \rangle (t)$. $\qquad \ldots(3)$

In view of (1), (2) and (3) suffices to prove that $Y_n, Z_n \longrightarrow 0$ a.e. P.

Now, $|Y_n| \leq \sum_{i=0}^{\infty} [H_i^n A(t).(H_i^n X(t) + H_i^n M(t))]$

$$\leq \frac{2}{2^n} \sum_{i=0}^{\infty} |H_i^n A(t)|$$

$$\leq \frac{2}{2^n} |A|(t)$$

Thus $Y_n \longrightarrow 0$ a.e. P .

The observation made at the beginning of the proof implies that the summands in the defination of Z_n form a 'martingale difference' sequence. Thus

$$EZ_n^2 \leq \sum_{i=0}^{\infty} E\left[(H_i^n M(t))^2 - H_i^n \langle M \rangle (t) \right]^2$$

$$\leq 2 \sum_{i=0}^{\infty} E\left[(H_i^n M(t))^4 + (H_i^n \langle M \rangle (t))^2 \right]$$

$$\leq \frac{2}{2^n} \sum_{i=0}^{\infty} E\left[(H_i^n M(t))^2 + H_i^n \langle M \rangle (t) \right]$$

$$= \frac{4}{2^n} E\langle M \rangle (t) . \qquad \ldots(4)$$

Since $E\langle M\rangle(t) < \infty$, (4) implies that $Z_n \longrightarrow 0$ a.e. P (as in Theorem 12) and as remarked earlier, this completes the proof.

Now given $g, X_1, X_2 \in \underline{\underline{C}}(\underline{\underline{F}})$ (Real valued) such that $X_1(0) = X_2(0) = 0$, let $\{T_i^n : i \geq 0\}$ be a $\frac{1}{2^n}$ pathwise partition for g, X_1 and X_2. Existence of such partitions is guaranted by lemma 11. Let

$$Y_n(t) = \sum_{i=0}^{\infty} g(T_i^n \wedge t)(H_i^n X_1(t))(H_i^n X_2(t))$$

and $\Omega_0 = \{w : Y_n(\cdot, w)$ converges in $\underline{\underline{E}}\}$

and $Y(\cdot, w) = \lim_n Y_n(\cdot, w)$ for $w \in \Omega_0$

$= 0$ elsewhere

Then we have

<u>Theorem 14</u> : For any P such that $X_1, X_2 \in \underline{\underline{S}}(\underline{\underline{F}}, P)$, we have $P(\Omega_0) = 1$ and $Y(t,w) = \int_0^t g(u,w)d\langle X_1, X_2\rangle(u,w)$ a.e. P.

<u>Proof</u> : Fix a P such that $X_1, X_2 \in \underline{\underline{S}}(\underline{\underline{F}}, P)$. By polarisation suffices to prove for $X_1 = X_2 = X$. First assume that g is bounded by say K.

By Lemma 12, we have (by considering $X'(t) = X(t)-X(T_i^n \wedge t)$)

$$\left[X(T_{i+1}^n \wedge t) - X(T_i^n \wedge t)\right]^2 = 2 \int_{T_i^n \wedge t}^{T_{i+1}^n \wedge t} \left[X(u)-X(T_i^n \wedge t)\right] dX(u) + H_i^n \langle X \rangle (t).$$

Thus

$$Y_n(t) = 2 \int_0^t (J^n g)(u) \left[X - J^n X\right](u) dX(u) + \int_0^t (J^n g)(u) d\langle X \rangle (u).$$

By choice of $\{T_i^n : i \geq 0\}$, $|J^n g - g| \leq \frac{1}{2^n}$ and $|J^n g(X - J^n X)| \leq \frac{K}{2^n}$.

Proceeding as in Theorem 12, we can easily show that the first integral goes to zero a.e. P in $\underline{\underline{E}}$. Usual properties of Stieltjes integral imply that the second integral converges to $\int_0^t g(u) d\langle X \rangle (u)$ (in $\underline{\underline{E}}$).

Finally if g is unbounded, we can get stop times $S_m \uparrow \infty$ such that $g^{S_n -}$ is bounded for every m. To prove the result it is sufficient to show that for every m,

$Y(t \wedge S_m, w) = \int_0^{t \wedge S_m} g(u,w) d\langle X_1, X_2 \rangle (u,w)$. This follows by considering $g^{S_m -}$ in place of g and applying the earlier calculations.

<u>Remark 1</u> : Theorem 14 shows that the 'pathwise formula' for 'quadratic variation' of a local martingale, given in Section 1, when applied to a semimartingale X gives $\langle X \rangle$: the quadratic variation $\langle M \rangle$ of the local martingale M in its canonical decomposition $X = M+A$.

Remark 2 : Combining the pathwise integration formulae proved earlier, we can get a pathwise integration formula for the symmetric (Strantanovich) stochastic integral (See Ito-Watanabe [14,p x]) defined by

$$\int Y \circ dX = \int Y dX + \frac{1}{2}\langle Y, X\rangle \quad \text{for} \quad Y, X \in \underline{\underline{S}}(\underline{\underline{F}}, P).$$

If $\{T_i^n : i \geq 0\}$ is a $\frac{1}{2^n}$-partition for X and Y then

$$Z_n(t) = \sum_{i=0}^{\infty} \left(\frac{Y(T_i^n \wedge t) + Y(T_{i+1}^n \wedge t)}{2}\right) . (X(T_{i+1}^n \wedge t) - X(T_i^n \wedge t))$$

converges a.s. to $\int Y \circ dX$. (This follows from Theorem 12 and 14 and the identity $\frac{(a+b)}{2}(c-d) = a(c-d) + \frac{1}{2}(b-a)(c-d)$ for real numbers a,b,c,d.).

6. Ito's Formula : a Pathwise Version :

In this section, we obtain 'Ito's formula' for $f(w,t,X(t,w))$. Proceeding as in Kunita-Watanabe [24] and using pathwise integration formulae of the previous section, we shall obtain a pathwise version of Ito's formula.

Let $C^{1,2}([0,\infty) \times R^k)$ be the space of continuous functions $f(t,x)$ on $[0,\infty) \times R^k$ for which the partial derivatives

$f_0 = \frac{\partial f}{\partial t}$, $f_j = \frac{\partial f}{\partial x_j}$ and $f_{jm} = \frac{\partial^2 f}{\partial x_j \partial x_m}$ exist and are continuous.

Let $X \in \underline{\underline{C}}(\underline{\underline{F}})$ (R^k-valued) be such that $X(0) = 0$.

Let $f : \Omega \times [0,\infty) \times R^k \longrightarrow R$ be a function such that

(a) $\forall\, x \in R^k$, $f(\cdot,\cdot,x) \in \underline{C}(\underline{F})$

(b) $\forall\, w \in \Omega$, $f(w,\cdot,\cdot) \in C^{1,2}([0,\infty) \times R^k)$.

Let $Y_j(t,w) = f_j(w,t,X(t,w)) \quad 0 \leq j \leq k$

and $\quad Z_{jm}(t,w) = f_{jm}(w,t,X(t,w)) \quad 1 \leq j \leq k,\ 1 \leq m \leq k$.

Let $\{T_i^n : i \geq 0\}$ be a $\frac{1}{2^n}$ partition for $\{X_j, Y_j : 0 \leq j \leq k\}$ and $\{Z_{j,m} : 1 \leq j \leq k,\ 1 \leq m \leq k\}$, where X_o is the process $X_o(t) \equiv t$.

Let

$$W_n(t,w) = f(w,0,0) + \sum_{j=0}^{k} \sum_{i=0}^{\infty} Y_j(T_i^n \wedge t)(H_i^n X_j(t)) + \frac{1}{2} \sum_{j=1}^{k} \sum_{m=1}^{k} \sum_{i=0}^{\infty} Z_{jn}(T_i^n \wedge t)(H_i^n X_j(t))(H_i^n X_n(t)).$$

Let $\Omega_o = \{w : W_n(\cdot,w)$ converges in $\underline{E}\}$

and

$$W(\cdot,w) = \lim_n W_n(\cdot,w) \quad \text{if } w \in \Omega_o$$
$$= 0 \quad \text{elsewhere.}$$

Let $\Omega_1 = \{w : W(t,w) = f(w,t,X(t,w))$ for all $t\}$.

Then we have

<u>Theorem 15</u> : Let P be such that $X \in \underline{S}(\underline{F}, P)$. Then

(i) $P(\Omega_o) = 1$ and

$$W(t) = f(\cdot,0,0) + \int_o^t \frac{\partial f}{\partial t}(\cdot,u,X(u))du + \sum_{j=1}^{k} \int_o^t \frac{\partial f}{\partial x_j}(\cdot,u,X(u))dX_j(u) + \frac{1}{2} \sum_{j=1}^{k} \sum_{m=1}^{k} \int_o^t \frac{\partial^2 f}{\partial x_j \partial x_m}(\cdot,u,X(u))d\langle X_j, X_m\rangle(u).$$

(ii) $P(\Omega_1) = 1$ i.e.

$$W(t,w) = f(w,t,X(t,w)) \quad \text{for all } t, \quad \text{a.e. } P.$$

Proof : Fix a P such that $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$. Then (i) follows from pathwise integration formulae (Theorems 12 and 14). For (ii) we proceed as follows :

First, fix $s \geq 0$, and let $U_i^n = T_i^n \wedge s$. Observe that

$$f(\cdot,U_{i+1}^n,X(U_{i+1}^n)) - f(\cdot,U_i^n,X(U_i^n)) = f(\cdot,U_{i+1}^n,X(U_{i+1}^n)) - f(\cdot,U_i^n,X(U_{i+1}^n))$$
$$+ f(\cdot,U_i^n,X(U_{i+1}^n)) - f(\cdot,U_i^n,X(U_i^n)).$$

Now by Taylor's formula, there exist $\alpha_i^n(w)$ and $\beta_i^n(w)$ such that $U_i^n \leq \alpha_i^n \leq U_{i+1}^n$ and β_i^n lies on the line segment joining $X(U_i^n)$ and $X(U_{i+1}^n)$ and

$$f(\cdot,U_{i+1}^n,X(U_{i+1}^n)) - f(\cdot,U_i^n,X(U_i^n)) = f_0(\cdot,\alpha_i^n,X(U_{i+1}^n))(U_{i+1}^n - U_i^n)$$
$$+ \sum_{j=1}^{k} f_j(\cdot,U_i^n,X(U_i^n))(X_j(U_{i+1}^n) - X_j(U_i^n))$$
$$+ \frac{1}{2}\sum_{j=1}^{k}\sum_{m=1}^{k} f_{jm}(\cdot,U_i^n,\beta_i^n)(X_j(U_{i+1}^n) - X_j(U_i^n))(X_m(U_{i+1}^n)-X_m(U_i^n))$$

Thus

$$f(w,s,X(s,w)) - f(w,0,0) = \sum_{i=0}^{\infty} (f(w,U_{i+1}^n,X(U_{i+1}^n)) - f(w,U_i^n,X(U_i^n)))$$
$$= W_n(s,w) + \sum_{i=0}^{\infty} R_i^n(s,w) \qquad \ldots(1)$$

where

$$R_i^n(s,w) = \left[f_o(\cdot,\alpha_i^n,X(U_{i+1}^n) - f_o(\cdot,U_i^n,X(U_i^n))\right](U_{i+1}^n - U_i^n)$$
$$+ \frac{1}{2}\sum_{j=1}^{k}\sum_{m=1}^{k}\left[f_{jm}(\cdot,U_i^n,\beta_i^n) - f_{jm}(\cdot,U_i^n,X(U_i^n))\right].$$
$$(H_i^n X_j(t))(H_i^n X_m(t)).$$

For a $g \in C([0,\infty) \times R^k)$, $t > 0$, $\alpha > 0$, $\varepsilon > 0$ let

$$\upsilon(g,t,\alpha,\varepsilon) = \Big\{ \sup \ |g(s_1,x_1) - g(s_2,x_2)| : 0 \leq s_i \leq t, \ |x_i| \leq \alpha, \ i = 1,2 \text{ and } |s_1 - s_2| \leq \varepsilon, \ |x_1 - x_2| \leq \varepsilon \Big\}.$$

Since $[0,t] \times \{x : |x| \leq \alpha\}$ is compact, we have

$$\lim_{\varepsilon \to 0} \upsilon(g,t,\alpha,\varepsilon) = 0 \qquad \ldots(2)$$

for every g,t,α.

Now observe that for $s \leq t$,

$$|R_i^n(s,w)| \leq \upsilon(f_o(w),t,|X|_t^*(w),\tfrac{1}{n})(T_{i+1}^n \wedge s - T_i^n \wedge s)$$
$$+ \frac{1}{2}\sum_{j=1}^{k}\sum_{m=1}^{k}\upsilon(f_{jm}(w),t,|X|_t^*(w),\tfrac{1}{2^n})\,|H_i^n X_j(s)|.|H_i^n X_m(s)|.$$

Hence

$$\left[\sum_{i=0}^{\infty}|R_i^n(\cdot,w)|\right]_t^* \leq \upsilon(f_o(w),t,|X|_t^*(w),\tfrac{1}{2^n}).t$$
$$+ \frac{1}{2}\sum_{j=1}^{k}\sum_{m=1}^{k}\upsilon(f_{jm}(w),t,|X|_t^*(w),\tfrac{1}{2^n})V_{jm}^n(t,w),$$

where $\ldots(3)$

$$V_{jm}^n(t,w) = \sup_{0 \leq s \leq t}\left(\sum_{i=0}^{\infty}|H_i^n X_j(s)|.|H_i^n X_m(s)|\right)$$
$$\leq \left(\sup_{0 \leq s \leq t}\sum_{i=0}^{\infty}(H_i^n X_j(s))^2\right)^{\frac{1}{2}}.\left(\sup_{0 \leq s \leq t}\sum_{i=0}^{\infty}(H_i^n X_m(s))^2\right)^{\frac{1}{2}}$$

Now, by Theorem 14, $\sum_{i=0}^{\infty} (H_i^n X_j(s))^2 \longrightarrow \langle X_j \rangle (s)$ in u.c.c. a.s. P. for all $1 \leq j \leq k$,

so that

$$\lim_n \sup V_{jm}^n(t,w) \leq (\langle X_j \rangle (t,w))^{1/2} (\langle X_m \rangle (t,w))^{1/2} \quad \text{a.s. P}$$

$$< \infty \quad \text{a.s. P.} \qquad \ldots(4)$$

Thus, using (4) (3) and (2), we get

$$\lim_n \left[\sum_{i=0}^{\infty} |R_i^n(\cdot,w)| \right]_t^* = 0 \quad \text{a.s. P.} \qquad \ldots(5)$$

Now, (5) and (1) imply that

$$W_n(\cdot,w) \longrightarrow f(w,\cdot,X(\cdot,w)) - f(w,0,0) \quad \text{a.s. in } \underline{\underline{E}}.$$

This, in view of (i), completes the proof.

CHAPTER II

STOCHASTIC DIFFERENTIAL EQUATIONS

1. Preliminaries :

In this chapter, we consider the stochastic differential equation

$$(2.1) \qquad Y(t) = \emptyset(t) + \int_0^t b(\cdot,u,Y)dX(u)$$

where $\emptyset \in \underline{\underline{C}}(\underline{\underline{F}})$, $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ (R^d , R^k-valued) and b is a function from $\Omega \times [0,\infty) \times \underline{\underline{E}}_d$ into $L(d,k)$ such that (2.2) holds

(2.2)
- (i) If $t \geq 0$ and if $|\rho_1 - \rho_2|_t^* = 0$, then $b(w,t,\rho_1) = b(w,t,\rho_2)$ for all w.
- (ii) For all $Y \in \underline{\underline{C}}(\underline{\underline{F}})$, f defined by $f(t,w) = b(w,t,Y(w))$ belongs to $\underline{\underline{D}}(\underline{\underline{F}})$.

Since condition (2.2) involves all processes in $\underline{\underline{C}}(\underline{\underline{F}})$, here is a simpler condition on b that implies (2.2).

(2.2)'
- (i) $\forall\ \rho \in \underline{\underline{E}}_d$, $b(\cdot,\cdot,\rho) \in \underline{\underline{D}}(\underline{\underline{F}})$
- (ii) $\forall\ t_o \geq 0$, b restricted to $\Omega \times [0,t_o] \times \underline{\underline{E}}_d$ is measurable with respect to $\underline{\underline{F}}_{t_o} \otimes \underline{\underline{B}}_{t_o} \otimes \underline{\underline{N}}_{t_o}$.

(Here $\underline{\underline{B}}_{t_o}$ is the Borel σ field on $[0,t_o]$ and $\underline{\underline{N}}_{t_o}$ is the smallest σ-field on $\underline{\underline{E}}_d$ with respect to which the family of maps $\{\rho \longrightarrow \rho(s) : s \leq t_o\}$ is measurable.)

Observe that if b satisfies (2.2) for the filtration $\underline{\underline{F}}$, then it satisfies (2.2) for the filtration $\underline{\underline{F}}^P$ as well.

For b satisfying (2.2), $T \in \underline{\underline{T}}(\underline{\underline{F}})$ and a strict $\underline{\underline{F}}$-time change σ, define b^{T-} and σb by

$$b^{T-}(w,t,\rho) = b(w,t,\rho)1_{[0,T(w))}(t)$$

and

$$\sigma b(w,t,\rho) = b(w,\sigma_t(w),\lambda\rho)$$

where λ is the inverse of σ and $(\lambda\rho)(t) = \rho(\lambda_t(w))$. Clearly b^{T-} satisfies (2.2). Also, σb satisfies condition (i) of (2.2). To see that σb satisfies (ii) of (2.2), fix $Z \in \underline{\underline{C}}(\sigma\underline{\underline{F}})$ and let $Y = \lambda Z$ and f be defined by $f(t,w) = b(w,t,Y(w))$. Then

$$\begin{aligned}\sigma b(w,t,Z(w)) &= b(w,\sigma_t(w),Y(w)) \\ &= (\sigma f)(t,w)\end{aligned}$$

and hence σb satisfies (ii) of (2.2) (with respect to the filtration $\sigma\underline{\underline{F}}$).

If $Y \in \underline{\underline{C}}(\underline{\underline{F}}^P)$ satisfies (2.1), we say that Y is a solution of (2.1) for $(\emptyset,b,X)$ (or $(\emptyset,b,X,P)$ if we want to stress that the stochastic integral in (2.1) is on $(\underline{\Omega}, \underline{\underline{B}}, P)$).

<u>Lemma 1</u> : (i) If Y is a solution of (2.1) for $(\emptyset,b,X)$ and $T \in \underline{\underline{T}}(\underline{\underline{F}})$, then Y^T is a solution of (2.1) for $(\emptyset^T, b^{T-}, X^T)$.

(ii) For a strict time change σ, Y is a solution of (2.1) for $(\emptyset,b,X)$ if and only if σY is a solution of (2.1) for $(\sigma\emptyset, \sigma b, \sigma X)$.

(iii) Let $Y \in \underline{\underline{C}}(\underline{\underline{F}})$ be such that there exists a sequence $\{T_n\}$ of $\underline{\underline{F}}$ stop times increasing to ∞ with the property that Y^{T_n} is a solution of (2.1) for $(\emptyset^{T_n}, b^{T_n-}, X^{T_n})$ for all $n \geq 1$. Then Y is a solution of (2.1) for $(\emptyset, b, X)$.

Proof : Observe that

$$b^{T-}(w,t,Y(w)) = b(w,t,Y^T(w)).1_{[0,T(w))}(t)$$

and

$$(\sigma b)(w,t,\sigma Y(w)) = \sigma(b(w,t,Y(w))).$$

Now (i), (ii) and (iii) follow from the properties ((a) of Theorem 1.8) of the stochastic integral.

2. Existence and Uniqueness of Solutions :

In this section, we show that under a Lipschitz condition, the existence and uniqueness of solutions of (2.1) can be proved as in the Brownian motion case by using a strict time change.

The existence and uniqueness in the one-dimensional case was proved by Kazamaki [22] using strict time change. In the multidimensional case Protter [39] (assuming that $\emptyset$ is also a semimartingale) and Doleans-Dade [5] proved existence and uniqueness using more complex methods. Whereas Protter considered only continuous semimartingales, Doleans-Dade allowed r.c.l.l. semimartingales.

The form of Lipschitz condition we impose on b is

(2.3) $\left[\begin{array}{l}\text{there exists a locally bounded process } K \text{ such that} \\ |b(w,t,\rho_1) - b(w,t,\rho_2)| \leq K(t,w)|\rho_1 - \rho_2|_t^* \text{ for all } t,w,\rho_1,\rho_2\end{array}\right.$

($K \in \underline{\underline{W}}(\underline{\underline{F}})$ is said to be locally bounded if there exist $T_n \in \underline{\underline{T}}(\underline{\underline{F}})$, $T_n \uparrow \infty$ such that K^{T_n-} is bounded for each n.)

<u>Theorrem 2</u> : Let $\emptyset \in \underline{\underline{C}}(\underline{\underline{F}}), X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ (R^d, R^k-valued) and b satisfy (2.2) and (2.3). Then there exists a solution Y of (2.1) for $(\emptyset, b, X)$. Further, if Y_1 and $Y_2 \in \underline{\underline{C}}(\underline{\underline{F}}^P)$ satisfy (2.1), then $P(w : Y_1(t,w) \neq Y_2(t,w)$ for some $t) = 0$.

<u>Proof</u> : We shall assume that $\emptyset, b$ satisfy (2.4)

There exists a constant C such that

(2.4) $\left[\begin{array}{l}|\emptyset(t,w)| \leq C \\ |b(w,t,0)| \leq C \\ |b(w,t,\rho_1) - b(w,t,\rho_2)| \leq C|\rho_1 - \rho_2|_t^* \text{ for all } t,w,\rho_1,\rho_2.\end{array}\right.$

Here 0 is the function $\equiv 0$.

Otherwise, get $T_n \in \underline{\underline{T}}(\underline{\underline{F}}), T_n \uparrow \infty$ such that $\emptyset^{T_n}, b^{T_n-}(\cdot,\cdot,0)$, K^{T_n-} are bounded and let Y_n be the unique solution of (2.1) for $(\emptyset^{T_n}, b^{T_n-}, X^{T_n})$. By uniqueness and part (i) of Lemma 1, there exists a $Y \in \underline{\underline{C}}(\underline{\underline{F}}^P)$ such that $Y_n = Y^{T_n}$ a.s. for all n. Now by part (iii) of Lemma 1, Y is a solution of (2.1) for $(\emptyset, b, X)$.

If Z is any other solution of (2.1) for $(\emptyset, b, X)$, by Lemma 1 (i), $Z^{T_n} = Y_n = Y^{T_n}$ a.s. for all n and hence $Y \equiv Z$ a.s.

Suppose σ is a strict time change. Then σb satisfies (2.3) with the locally bounded process σK. Thus by Lemma 1 (ii), we can assume that $X \in Q(1)$.

Observe that (2.4) implies

$$|b(w,t,\rho)| \leq C(1+|\rho|_t^*).$$

Now, define Y_n inductively by

$$Y_o(t) = \emptyset(t)$$

$$Y_n(t) = \emptyset(t) + \int_o^t b(\cdot,u,Y_{n-1})dX(u) \ , \ n \geq 1.$$

Then, the 'growth-inequality', the assumptions (2.4) and the fact that $X \in Q(1)$ imply

$$E|Y_1 - Y_o|_t^{*2} \leq 8(1+t)kC^2(1+C)^2t$$

and

$$E|Y_{n+1} - Y_n|_t^{*2} \leq 8(1+t)kC^2 \int_o^t E|Y_n - Y_{n-1}|_u^{*2}du \, .$$

Thus if $\alpha_n(t) = E|Y_{n+1} - Y_n|_t^{*2}$, then for each $t_o > 0$, there exists a constant C_1 such that

$$\left.\begin{aligned} &\alpha_o(t) \leq C_1 \\ \text{and} \quad &\alpha_n(t) \leq C_1 \int_o^t \alpha_{n-1}(u)du \end{aligned}\right\} \quad 0 \leq t \leq t_o \, .$$

By induction, it follows that

$$\alpha_n(t) \leq C_1 \frac{(C_1 t)^n}{n!} \qquad 0 \leq t \leq t_o .$$

As in Theorem 1.12, this estimate implies that Y_n converges a.s. in $\underline{\underline{E}}$ to say Y. Further we have

$$\sum_{n=1}^{\infty} \left[E|Y_n - Y|_t^{*2} \right]^{1/2} < \infty \quad \text{for all } t .$$

Thus by condition (2.4) and the growth-inequality we have

$$\int_o b(\cdot, u, Y_n) dX(u) \longrightarrow \int_o b(\cdot, u, Y) dX(u) \quad \text{a.s. in } \underline{\underline{E}}$$

and hence Y is a solution of (2.1) for $(\emptyset, b, X)$.

Now if Z_1 and Z_2 are two solution of (2.1), then (2.4) and the growth-inequality imply that

$$E|Z_1 - Z_2|_t^{*2} \leq 8(1+t)kC^2 \int_o^t E|Z_1 - Z_2|_u^{*2} du .$$

This and Growmwall's inequality imply that $E|Z_1-Z_2|_t^{*2} = 0 \ \forall\ t$, and hence $P(w : Z_1(t,w) \neq Z_2(t,w)$ for some $t) = 0$ as $Z_1, Z_2 \in \underline{\underline{C}}(\underline{\underline{F}})$. As remarked earlier, this completes the proof.

3. Pathwise Solution :

In the last section, we have seen that the equation (2.1) under the Lipschitz condition (2.3) has a unique solution Y which can be obtained as an almost sure limit of successive iterates Y_n. In view of the 'Pathwise integration formula'

(Theorem 1.12), we can define each Y_n pathwise and thus defining Y to be the uniform limit of Y_n's whereever it exists, we get a 'pathwise solution' which satisfies (2.1) for all P under which X is a semimartingale. This Remark is due to Bichteler [1 ,p 74]. But this definition of Y involves a double limit which is unsatisfactory. In this section we show that by a modification of the successive iteration procedure, we get a 'pathwise solution' as a single limit. Independently, Bichteler [2] has obtained a different formula involving a single limit using more complex methods (when X is r.c.l.l. semimartingale).

Let $\emptyset, X \in \underline{\underline{C}}(\underline{\underline{F}})(R^d, R^k\text{-valued})$ with $X(0) = 0$ and b satisfying (2.2) and (2.3) be given. We shall define processes Y_n for $n \geq 0$ and random partitions $\{T_i^n, i \geq 0\}$ for $n \geq 1$ inductively as follows :

$$Y_0(t,w) = \emptyset(t,w) .$$

Having defind Y_{n-1} let $\{T_i^n : i \geq 0\}$ be a $\frac{1}{2^n}$ (pathwise) random partition for the process f_n defined by

$$f_n(t,w) = b(w,t,Y_{n-1}(w)) ,$$

and then define Y_n by

$$Y_n(t,.) = \emptyset(t,.) + \sum_{i=0}^{\infty} b(.,T_i^n, Y_{n-1})(H_i^n X(t)).$$

Observe that for each (t,w), the sum above is a finite sum.
Let $\Omega_0 = \{w : Y_n(.,w)$ converges in $\underline{\underline{E}}\}$, and

$$Y(\cdot,w) = \lim_n Y_n(\cdot,w) \quad \text{if} \quad w \in \Omega_0$$
$$= 0 \qquad\qquad \text{elsewhere.}$$

Observe that $Y(t,w)$ is defined pathwise i.e. explicitly in terms of $\{ \phi(u,w) ; X(u,w) ; b(w,u,\rho), \; 0 \leq u \leq t \}$.

Theorem 3 : For all P such that $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$, we have

(i) $P(\Omega_0) = 1$

(ii) Y is a solution of (2.1) for (ϕ, b, X, P).

Proof : Fix a P such that $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$. As in Theorem 2, we will assume that b, ϕ satisfy (2.4). Otherwise get $S_m \in \underline{\underline{T}}(\underline{\underline{F}})$, $S_m \uparrow \infty$ such that $\phi^{S_m}, b^{S_m -}$ satisfy (2.4) for each m. The argument that follows will imply that $Y_n^{S_m} \longrightarrow Y^{S_m}$ a.s. in $\underline{\underline{E}}$ and Y^{S_m} is a solution of (2.1) for $(\phi^{S_m}, b^{S_m -}, X^{S_m})$. By Lemma 1 (iii), this will imply Y is a solution of (2.1) for (ϕ, b, X).

We also assume that $X \in Q(1)$. Otherwise let σ be a strict $\underline{\underline{F}}$ time change such that $\sigma X \in Q(1)$. Let $S_i^n = \lambda_{T_i^n}$, where λ is the inverse of σ. Then $\{ S_i^n, i \geq 0, n \geq 1$ and $\sigma Y_n, n \geq 1 \}$ can be obtained by the same formula in terms of $(\sigma\phi, \sigma b, \sigma X)$ by which $\{ T_i^n, i \geq 0, n \geq 1$ and $Y_n, n \geq 1 \}$ was defined and thus the arguments that follow will show that σY is a solution of (2.1) for $(\sigma\phi, \sigma b, \sigma X)$ which implies the required result in view of Lemma 1, (ii).

Now, observe that

$$Y_n(t) = \phi(t) + \int_0^t (J^n f_n)(u)dX(u)$$

and

$$|J^n f_n - J^{n+1} f_{n+1}|^2 \leq 3(|J^n f_n - f_n|^2 + |f_n - f_{n+1}|^2 + |f_{n+1} - J^{n+1} f_{n+1}|^2)$$

$$\leq \frac{6}{2^{2n}} + 3|f_n - f_{n+1}|^2 . \qquad \ldots(1)$$

(See section 1.4 for definition of J^n.)

Now, using the growth inequality, the condition (2.4), the fact that $X \varepsilon Q(1)$ and (1), we get

$$E|Y_{n+1} - Y_n|_t^{*2} \leq 8(1+t)k\left\{t6.2^{-2n} + \int_0^t E|Y_n - Y_{n-1}|_u^{*2} du\right\}$$

and

$$E|Y_1 - Y_0|_t^{*2} \leq 8(1+t)kC^2(1+C^2)t .$$

Thus if $\alpha_n(t) = E|Y_{n+1} - Y_n|_t^{*2}$, then for $t_0 > 0$, there exists a constant C_1 such that for $0 \leq t \leq t_0$,

$$\alpha_0(t) \leq C_1$$

and

$$\alpha_n(t) \leq C_1(2^{-2n} + \int_0^t \alpha_{n-1}(u)du), \quad n \geq 1 .$$

By induction, it follows that

$$\alpha_n(t) \leq 2C_1.2^{-2n} e^{8C_1 t}, \quad 0 \leq t \leq t_0 , n \geq 1 .$$

This as in Theorem 2 implies that Y_n converges a.s. in $\underline{\underline{E}}$. Thus $P(\Omega_0) = 1$ and $Y_n \longrightarrow Y$ a.s. in $\underline{\underline{E}}$. Further, $E|Y_n - Y|_t^{*2} \longrightarrow 0$ for every t. In view of the Lipschitz condition on b, this implies that $E|f_n - f|_t^{*2} \longrightarrow 0$ where $f(t,w) = b(w,t,Y(w))$. Using the fact that $|J^n f_n - f_n| \leq \frac{1}{2^n}$, we get $E|J^n f_n - f|_t^{*2} \longrightarrow 0$. This in turn implies by the growth inequality that $\int_0^{\cdot} (J^n f_n) dX \longrightarrow \int_0 f dX$. This verifies that Y is a solution.

Remark 1 : The proof of Theorem 3 is only a minor modification of the standard successive approximation technique used in Theorem 2. Also, notice that in the proof above, we have not used the existence of the solution.

Remark 2 : Although Y is not $\underline{\underline{F}}$-adapted, Y is $\underline{\underline{F}}^P$ adapted for any P such that $X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$.

4. Convergence of Solutions :

In this section, we consider convergence of solutions as the 'data' $(\emptyset_n, b_n, X_n)$ converges (in an appropriate sense) to $(\emptyset, b, X)$.

For $n \geq 1$, let $Z_n \varepsilon \underline{\underline{D}}(\underline{\underline{F}})$. Let $Z \varepsilon \underline{\underline{D}}(\underline{\underline{F}})$. Following Protter [39] we say that $Z_n \longrightarrow Z$ (LMQM) (read as Locally in Maximal Quadratic mean) if there exist $\{T_i\}$ ($\subseteq \underline{\underline{T}}(\underline{\underline{F}})$; $T_i \uparrow \infty$ a.s. such that $E|Z_n - Z|_{T_i}^{*2}$ goes to zero for all $i \geq 1$. Observe that for a strict time change σ, $Z_n \longrightarrow Z$ (LMQM) if and only if $\sigma Z_n \longrightarrow \sigma Z$ (LMQM). A simple application of Borel-Cantelli lemma gives us the following useful criterion : $Z_n \longrightarrow Z$ (IMQM) if and only if for all $t > 0$, $\varepsilon > 0$, there exists a $T \in \underline{\underline{T}}(\underline{\underline{F}})$ such that $P(T \leq t) < \varepsilon$ and $E|Z_n - Z|_T^{*2} \longrightarrow 0$. From this observation, it easily follows that if for some stop times S_k increasing to ∞, $Z_n^{S_k} \longrightarrow Z^{S_k}$ (IMQM) for each k, then $Z_n \longrightarrow Z$ (IMQM).

Now let b and b_n for $n \geq 1$ satisfy (2.2). Let $\emptyset$ and $\emptyset_n$ for $n \geq 1$ belong to $\underline{\underline{C}}(\underline{\underline{F}})$. Let X and X_n for $n \geq 1$ belong to $\underline{\underline{S}}(\underline{\underline{F}}, P)$. Let Y and Y_n for $n \geq 1$ be given by

$$Y(t) = \emptyset(t) + \int_0^t b(\cdot, u, Y)\, dX$$

and

$$Y_n(t) = \emptyset_n(t) + \int_0^t b_n(\cdot, u, Y_n)\, dX_n .$$

In other words, Y_n solves (2.1) for $(\emptyset_n, b_n, X_n)$ and Y solves (2.1) for $(\emptyset, b, X)$. In the next theorem, we give sufficient conditions for Y_n's to converge to Y.

<u>Theorem 4</u> : Let

(i) $\emptyset_n \longrightarrow \emptyset$ (IMQM)

(ii) $Z_n \longrightarrow Z$ (IMQM), where $Z(t,w) = b(w,t,Y(w))$ and $Z_n(t,w) = b_n(w,t,Y(w))$.

(iii) there exists a time change σ such that $\sigma X \in Q(\beta)$ and $\sigma(X_n - X) \in Q(\beta_n)$, where $\beta_n \longrightarrow 0$.

(iv) b and b_n for $n \geq 1$ satisfy (2.3) with the same process K.

Then $Y_n \longrightarrow Y$ (IMQM).

Proof : By a time change if necessary assume that $X_n - X \in Q(\beta_n)$, $X \in Q(\beta)$. By changing β if necessary, we can assume that $X_n \in Q(\beta)$ for all n, in view of Lemma 1.9. Since we can get $\{S_k\} \subset \underline{\underline{T}}(\underline{\underline{F}})$; $S_k \uparrow \infty$ such that for all $k \geq 1$, $E|\emptyset_n - \emptyset|^{*2}_{S_k} \longrightarrow 0$, $E|Z_n - Z|^{*2}_{S_k} \longrightarrow 0$, Z^{S_k-} and K^{S_k-} are bounded, we can assume that $E|\emptyset_n - \emptyset|^{*2}_{\infty} \longrightarrow 0$, $E|Z_n - Z|^{*2}_{\infty} \longrightarrow 0$, Z is bounded and for some constant K,

$$|b_n(w,u,\rho_1) - b_n(w,u,\rho_2)| \leq K|\rho_1 - \rho_2|^*_u \quad \text{for} \quad n \geq 1$$

and

$$|b(w,u,\rho_1) - b(w,u,\rho_2)| \leq K|\rho_1 - \rho_2|^*_u .$$

Now,

$$Y_n(t) - Y(t) = \emptyset_n(t) - \emptyset(t) + A_n(t) + B_n(t) + C_n(t),$$

where

$$A_n(t) = \int_0^t (b_n(\cdot,u,Y_n) - b_n(\cdot,u,Y))dX_n(u)$$

$$B_n(t) = \int_0^t (b_n(\cdot,u,Y) - b(\cdot,u,Y))dX_n(u)$$

and $$C_n(t) = \int_0^t b(\cdot,u,Y)d(X_n - X)(u).$$

By the growth inequality, for each (fixed) t_o there exists a constant K_1 (depending on β, K etc.) such that for $0 \leq t \leq t_o$,

$$E|A_n|^{*2}_t \leq K_1 \int^t E|Y_n - Y|^{*2}_u du ,$$

$$E|B_n|_t^{*2} \leq K_1 E|Z_n - Z|_t^{*2}$$

and

$$E|C_n|_t^{*2} \leq K_1 \beta_n .$$

Thus for $0 \leq t \leq t_o$,

$$E|Y_n - Y|_t^{*2} \leq K_2\left[E|\emptyset_n - \emptyset|_t^{*2} + E|Z_n - Z|_t^{*2} + \beta_n + \int_\sigma^t E|Y_n - Y|_u^{*2} du\right]$$

$$\leq K_2\left[E|\emptyset_n - \emptyset|_{t_o}^{*2} + E|Z_n - Z|_{t_o}^{*2} + \beta_n + \int_o^t E|Y_n - Y|_u^{*2} du\right].$$

This and Grownwall's inequality imply that

(1) $$E|Y_n - Y|_t^{*2} \leq K_2\left[E|\emptyset_n - \emptyset|_{t_o}^{*2} + E|Z_n - Z|_{t_o}^{*2} + \beta_n\right] e^{K_2 \cdot t}, \quad 0 \leq t \leq t_o .$$

In view of our assumptions, this implies $E|Y_n - Y|_t^{*2} \longrightarrow 0$ for all $t \geq 0$. This completes the proof.

Inequality (1) above shows that if we assume specific rates of convergence (fast enough) in conditions (i),(ii) and (iii) in the above theorem, then Y_n indeed converges to Y almost surely. For instance

<u>Corollary 5</u> : Assume that there exists $\{S_i\}$ ($\subseteq \underline{\underline{T}}(\underline{\underline{F}})$, $S_i \uparrow \infty$ and constants C_i such that

(i) $E|\emptyset_n - \emptyset|_{S_i}^{*2} \leq C_i \frac{1}{2^n}$

(ii) $E|Z_n - Z|_{S_i}^{*2} \leq C_i \frac{1}{2^n}$

(iii) there exists a time change σ such that $X \varepsilon Q(\beta)$ and $X_n \varepsilon Q(\beta \frac{1}{2^n})$

and (iv) b and b_n, $n \geq 1$ satisfy (2.3) with the same process K.

Then $Y_n \longrightarrow Y$ almost surely.

Proof : Under these assumptions, the inequality (1) reduces to

$$E|Y_n - Y|_t^{*2} \leq K_3 \frac{1}{2^n}, \quad 0 \leq t \leq t_o,$$

which implies $Y_n \longrightarrow Y$ a.s.

Remark : The condition $\sigma(X_n - X) \in Q(\beta_n)$, $\beta_n \longrightarrow 0$ is too restrictive. For some convergence results under more general conditions, see Protter [39], [40] and Emery [7], [8].

5. A Homeomorphism Property of Solutions :

In this section we consider the equation

$$\text{(I)} \qquad Y(t) = x + \int_0^t b(\cdot, u, Y(u))dX(u)$$

and show that we can get a pathwise version $Y(t,x)$ which is continuous in (t,x). Combined with a known result this implies that the set of w for which $x \longrightarrow Y(t,x)$ is not a homeomorphism of R^k for some t is P-null for all P under which X is a semimartingale.

We need a L^p growth inequality.

Lemma 5 : (Burkholder's inequality)

For $p \geq 2$, there exists a constant C_p such that for all $M \in L(\underline{F}, P)$ (real valued)

$$E|M|_t^{*p} \leq C_p E[\langle M\rangle(t)]^{p/2} .$$

For a simple proof of this lemma see Stroock-Varadhan [42, p 116]. Using this inequality, we can derive a L^p-growth inequality for stochastic integrals exactly as in the L^2 case.

<u>Lemma 6</u> : (L^p-Growth Inequality)

Let $X \in \underline{\underline{S}}(\underline{\underline{F}}, P) \cap Q(1)$ (R^k valued) and $h \in \underline{\underline{D}}(\underline{\underline{F}})$ ($L(m,k)$ valued) and $p \geq 2$. Then

$$E|\int_0 h\,dX|_t^{*p} \leq C \int_0^t E|h|^p(u)\,du , \quad 0 \leq t \leq t_0$$

where C is a constant depending on p, m, k, t_0 .

We now state a multidimensional analogue of Kolmogorov's theorem on the existence of continuous modification of a stochastic process. A proof of this can be given following the arguments of Corollary 2.1.5 in Stroock-Varadhan [42] using exercise 2.4.1. These details are also worked out in Stroock [41].

<u>Theorem 7</u> : Let B be a separable Banach space and Z be a measurable function from $\Omega \times R^k$ into B, such that for some constants $p, C, \alpha > 0$,

$$E\| Z(x_1) - Z(x_2) \|^p \leq C|x_1 - x_2|^{k+\alpha}$$

for all x_1, x_2 in R^k.

Then

$$P\left\{w : Z(\cdot,w) \begin{array}{l}\text{restricted to any bounded subset of diadic} \\ \text{rationals in } R^k \text{ is uniformly continuous}\end{array}\right\} = 1$$

In particular, Z has a continuous modification.

Now, let 'b' satisfying (2.2) ($L(k)$ valued) be such that

$$b(w,t,\rho_1) = b(w,t,\rho_2) \quad \text{if} \quad \rho_1(t) = \rho_2(t).$$

so that we can write it as $b(w,t,x)$. Assume that b satisfies (2.3) and let $Z(t,x)$ be the 'pathwise' solution of $(\emptyset,b,X)$ for $\emptyset \equiv x$. Fix a P such that $X \in \underline{\underline{S}}(\underline{\underline{F}},P)$. Let T be a stop time such that K^{T-} is bounded. Let σ be a strict $\underline{\underline{F}}$ time change such that $\sigma X \in Q(1)$. Since Z is a solution for (x,b,X), Z^T is a solution for (x,b^{T-},X^T) so that $\sigma(Z^T)$ is a solution of $(x,\sigma(b^{T-}),\sigma(X^T))$. We have for $x_1, x_2 \in R^k$, by the C_r inequal: Lipschitz nature of $\sigma(b^{T-})$, L^p growth inequality and Grownwall's inequality

$$E|\sigma(Z^T(\cdot,x_1) - Z^T(\cdot,x_2))|_t^{*p} \leq C|x_1 - x_2|^p, \quad 0 \leq t \leq t_o$$

for a suitable constant C, so that

$$E|Z(\cdot,x_1) - Z(\cdot,x_2)|^{*p}_{\sigma_t \wedge T} \leq C|x_1 - x_2|^p, \quad 0 \leq t \leq t_o.$$

Thus for a sequence of bounded stop times $S_n \uparrow \infty$, we have

$$E|Z(\cdot,x_1) - Z(\cdot,x_2)|^{*p}_{S_n} \leq C_n|x_1 - x_2|^p,$$

(For instance $S_n = T_n \wedge \sigma_n \wedge n$ where $T_n \uparrow \infty$ is such that K^{T_n-} is bounded for each n). This by Theorem 7 (applied for each fixed n) implies that a.s. P, Z is a uniformly continuous function on diadic rationals in bounded subsets of R^k. Thus if

$Y(\cdot,\cdot,w)$ = extension of $Z(\cdot,\cdot,w)$ by uniform continuity if $Z(t,x,w)$ is a uniformly continuous function from diadic rationals in bounded subsets of R^k into $C[0,\infty)$.

= 0 otherwise.

Then for all P such that $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$,

$$P(w : Y(\cdot,x,w) = Z(\cdot,x,w) \text{ for all } x \in R^k) = 1 .$$

Thus we have

<u>Theorem 8</u> : Let b satisfying (2.2), (2.3) be such that

$$b(t,w,\rho_1) = b(t,w,\rho_2) \text{ if } \rho_1(t) = \rho_2(t)$$

Then there exists a " pathwise solution" $Y(t,x)$ such that

(i) for all w, $(t,x) \longrightarrow Y(t,x,w)$ is continuous

(ii) for all x and for all P such that $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$, $Y(\cdot,x)$ is a solution of (I) for (x,b,X,P).

Kunita [23] (see also Stroock [41]) has given a proof of the fact that if Y is any jointly continuous version of the solution to the SDE (I) then for almost all w, for all t, $x \longrightarrow Y(t,x,w)$ is a homeomorphism (C^{n-1} homeomorphism if $b(t,w,\cdot)$ is C^n) of R^k into itself. The hypotheses on b in Kunita's proof are different but the same proof works for our case as well. Since our solution does not depend upon P, the set (of w) where $Y(\cdot,t,w)$ fails to be a homeomorphism for some is a 'universal' null set. More precisely we have

<u>Theorem 9</u> : Let b, Y be as in Theorem 8. Let

$$\Omega_o = \{ w : x \longrightarrow Y(t,x,w) \text{ is homeomorphism for all } t \} .$$

Then

$$P(\Omega_o) = 1 \text{ for all } P \text{ such that } X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P) .$$

Further, if $n \geq 1$ and if $b(w,t,\cdot)$ is C^n, then

$$P(\Omega_n) = 1 \text{ for all } P \text{ such that } X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$$

where

$$\Omega_n = \{ w : x \longrightarrow Y(t,x,w) \text{ is a } C^{n-1} \text{ homeomorphism for all } t \} .$$

CHAPTER III

MULTIPLICATIVE STOCHASTIC INTEGRATION

1. Definition and Properties of Multiplicative Integral :

In this chapter we define multiplicative stochastic integration and obtain its properties. These include 'integration by parts formula', formulae for the inverse and determinant of 'exponential' of a semimartingale. As an application of 'integration by parts formula', we obtain a 'multiplicative decomposition' of (invertible) matrix valued semimartingales. We start by defining (additive) stochastic integration w.r.t. a matrix valued semimartingale.

Definition : Let $f \in \underline{\underline{D}}(\underline{\underline{F}})$ ($L(m,k)$ valued) and $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ ($L(k,r)$ valued). Then $\int f dX$ is defined to be the $L(m,r)$ valued process whose $(i,j)^{th}$ component is

$$\sum_{s=1}^{k} \int f_{is} dX_{sj} .$$

Remark : $\int dX.g$ and $\int f\, dX.g$ are defined analogously when f, X are as above and $g \in \underline{\underline{D}}(\underline{\underline{F}})$ ($L(r,d)$ valued).

All the properties of $\int f dX$ listed in Section 1.4 continue to hold when X is matrix valued, the important ones being invariance under time change and effect of stopping. We shall state the 'Growth-inequality'. The proof is exactly same as in the case when X is a vector valued semimartingale.

<u>Lemma 1</u> : 'Growth-inequality'.

Let $X \in \underline{\underline{S}}(\underline{\underline{F}}, P) \cap Q(\beta)$ and $h \in \underline{\underline{D}}(\underline{\underline{F}})$ (both $L(k)$ valued). Then

$$E\left|\int_0^{\cdot} h\,dX\right|_t^{*2} \leq 8k^2\beta(1+t\beta)\int_0^t E|h|^2(u,w)\,du.$$

Let $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ and $h \in \underline{\underline{D}}(\underline{\underline{F}})$ (both $L(k)$ valued). Then the SDE

$$Y(t) = I + \int_0^t Y h\,dX$$

has a unique solution. This follows easily from Theorem 2. In fact this result holds even if $h \in \underline{\underline{W}}(\underline{\underline{F}})$ provided $\int h\,dX$ exists. But we shall not deal in this generality.

Following Masani [26], McKean [28], Ibero [11] and Emery [7] we define

<u>Definition</u> : Let $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ and $h \in \underline{\underline{D}}(\underline{\underline{F}})$, (both $L(k)$ valued). The multiplicative stochastic integral of h with respect to X, denoted by $\prod_0^{\cdot}(I + h\,dX)$ is defined to be Y, where Y is the unique solution of the SDE $Y = I + \int_0^{\cdot} Y\,h\,dX$.

<u>Remark</u> : The notation suggests that $\prod_0^{\cdot}(I + h\,dX)$ should be the limit of 'Riemann-Products' in some sense. That this is true when $h \in \underline{\underline{D}}(\underline{\underline{F}})$ is proved in the next section.

We now obtain some properties of the multiplicative integral

<u>Theorem 2</u> : Let $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ and $h \in \underline{\underline{D}}(\underline{\underline{F}})$ (both $L(k)$ valued).

Let $Y = \prod_0 (I + hdX)$.

(i) Define Y_n inductively by

$$Y_0 = I$$

$$Y_{n+1} = I + \int_0 Y_n h dX .$$

Then $Y_n \longrightarrow Y$ a.s. in $\underline{\underline{E}}$.

(ii) $$\prod_0^t (I + hdX) = I + \int_0^t h(u)dX(u) + \int_0^t \int_0^s h(u)dX(u)h(s)dX(s) + \dots$$

(iii) $$\prod_0^t (I + hdX) = \prod_0^s (I + hdX).\prod_s^t (I + hdX), \quad 0 \leq s \leq t$$

(where $\prod_s^t (I + hdX) \equiv \prod_0^t (I + h'dX)$, $h'(u) = h(u)1_{[s,\infty)}(u)$).

(iv) For $T \in \underline{\underline{T}}(\underline{\underline{F}})$, $$Y^T(t) + I.1_{\{T=0\}} = \prod_0^t (I + h^{T-}dX) = \prod_0^t (I + hdX^T) .$$

(v) For a strict $\underline{\underline{F}}$ time change σ,

$$\sigma Y(s) = \prod_0^s (I + (\sigma h)d\sigma X).$$

(vi) Let $h_n \in \underline{\underline{D}}(\underline{\underline{F}})$ be such that $|h_n - h| \leq 2^{-n}$, then

$$\prod_0 (I + h_n dX) \longrightarrow \prod_0 (I + hdX) \text{ a.s. in } \underline{\underline{E}} .$$

Proof : (i) Get a strict time change σ such that $\sigma Z \varepsilon Q(1)$ where $Z = \int hdX$. Fix a stop time T such that Y^T is bounded.

Let $\quad \emptyset_n(t) = E|\sigma(Y_n^T) - \sigma(Y^T)|_t^{*2}$.

Then by Lemma 1,

$$\emptyset_n(t) \leq 8k^2(1+t) \int_0^t \emptyset_{n-1}(u)du$$

and $\emptyset_o(t) \leq C$. (for some constant C).

Thus for fixed $t_o > 0$, for some constant C_1,

$$\emptyset_n(t) \leq \frac{(C_1)^n}{n!} \qquad 0 \leq t \leq t_o \, .$$

Hence $Y_n^T \longrightarrow Y^T$ a.s. in $\underline{\underline{E}}$. Since we can get stop times $T_i \uparrow \infty$ such that Y^{T_i} is bounded, the proof is complete.

(ii) follows from (i) as the first n terms on the right hand side in (ii) add upto Y_{n-1} .

(iii) follows from uniqueness of solution to the SDE

$$Z(t) = H(t) + \int_s^t Z \; hdX \; , \quad t \geq s \, ,$$

for any $H \varepsilon \underline{\underline{C}}(\underline{\underline{F}})$.

(iv) and (v) follow from the corresponding properties of the additive integral.

(vi) follows from Corollary 2.5 .

Remark : (i) is the usual procedure of successive approximation to the solution of SDE given in Section 2.2.(ii) is the 'Peano series' representation of the multiplicative integral. (iii) gives the multiplicative nature of the multiplicative integral.

2. Pathwise Integration Formula :

In this section we obtain a pathwise formula to evaluate $\prod_0 (I + gdX)$ in terms of paths of g and X. This is achieved by proving a.s. convergence of Riemann products to the multiplicative integral.

Let $X \in \underline{\underline{C}}(\underline{\underline{F}})$, $g \in \underline{\underline{D}}(\underline{\underline{F}})$ (both $L(k)$ valued), $X(0) = 0$.

For $n \geq 1$, let $\{T_i^n : i \geq 0\}$ be a $\frac{1}{2^n}$ partition for the processes X and g. For this random partition, define the Riemann sums and products by

$$Y_n(t) = \sum_{i=0}^{\infty} g(T_i^n)(H_i^n X(t))$$

and

$$Z_n(t) = \prod_{i=0}^{\infty} (I + g(T_i^n)(H_i^n X(t)))$$

Let $\Omega_0 = \{w : Y_n(\cdot,w) \text{ and } Z_n(\cdot,w) \text{ converge in } \underline{\underline{E}}\}$

$$\left.\begin{aligned} Y(\cdot,w) &= \lim_n Y_n(\cdot,w) \\ \tilde{Z}(\cdot,w) &= \lim_n Z_n(\cdot,w) \end{aligned}\right\} \quad \text{if } w \in \Omega_0$$

and $Y(\cdot,w) = 0 = \tilde{Z}(\cdot,w)$ otherwise.

Then we have

<u>Theorem 3</u> : Let P be such that $X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$. Then

(i) $P(\underline{(\bar{\ })}_{0}) = 1$

and (ii) $Y = \int_0 gdX$

$$Z = \prod_0 (I + gdX).$$

<u>Proof</u> : Fix a P such that $X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$ and let

$$Y = \int_0 gdX$$

and

$$Z = \prod_0 (I + gdX).$$

We will show that $Y_n \longrightarrow Y$ and $Z_n \longrightarrow Z$ a.s. P. This will complete the proof.

We shall assume that g is bounded by K (say). Since given $g \varepsilon \underline{\underline{D}}(\underline{\underline{F}})$, we can get $S_n \varepsilon \underline{\underline{T}}(\underline{\underline{F}})$, $S_m \uparrow \infty$ such that $g^{S_m^-}$ is bounded, the general case will follow from this. We also assume that $X \varepsilon Q(1)$. Otherwise get σ such that $\sigma X \varepsilon Q(1)$ and let λ be the inverse of σ. Let $S_i^n = \lambda_{T_i^n}$. Then for each $n \geq 1$, σY_n, σZ_n can be obtained from σg, σX and the partition $\{S_i^n : i \geq 0\}$ by the formula defining Y_n, Z_n in terms of g, X, $\{T_i^n : i \geq 0\}$. Thus the arguments that follow will imply that $\sigma Y_n \longrightarrow \sigma Y$, $\sigma Z_n \longrightarrow \sigma Z$ a.s. in $\underline{\underline{E}}$ which is same as $Y_n \longrightarrow Y$ and $Z_n \longrightarrow Z$ a.s. in $\underline{\underline{E}}$.

For the proof of the theorem, we first observe that

(i) $\{T_i^n : i \geq 0\}$ is $\frac{K}{2^n}$ partition for Z_n

(ii) $Z_n(t) = I + \int_0^t Z_n dY_n$

(iii) $Y \varepsilon Q(\beta)$ and $Y_n \varepsilon Q(\beta)$, $Y_n - Y \varepsilon Q(\beta \frac{1}{2^n})$ for $n \geq 1$,

for a suitable constant β.

For (i), observe that $Y_n(t) = \int_0^t (J^n g)dX$ so that

$$Y_n - J^n Y_n = (J^n g).(X - J^n X).$$

Now (i) follows from this as g is bounded by K and $|X - J^n X| \leq \frac{1}{2^n}$.

For (ii) observe that

$$Z_n(t) = \prod_{i=0}^{\infty} (I + Y_n(t \wedge T_{i+1}^n) - Y_n(t \wedge T_i^n))$$

$$= I + \sum_{i=0}^{\infty} Z_n(t \wedge T_i^n).(Y_n(t \wedge T_{i+1}^n) - Y_n(t \wedge T_i^n))$$

$$= I + \int_0^t (J^n Z_n)dY_n .$$

For (iii) observe that

$$Y(t) = \int_0^t g dX$$

and

$$Y_n(t) = \int_0^t J^n g dX$$

so that

$$Y_n(t) - Y(t) = \int_0^t (J^n g - g)dX .$$

Now Lemma 1.8, the fact that $X \in Q(1)$ and the inequalities $|g| \leq K$, $|J^n g - g| \leq \frac{1}{2^n}$ imply (iii).

To complete the proof of the theorem, we show that (i), (ii) and (iii) imply $Y_n \longrightarrow Y$ a.s. P and $Z_n \longrightarrow Z$ a.s. P.

Growth inequality and (iii) (applied to $h = I$, $X = Y_n - Y$) directly gives $Y_n \longrightarrow Y$ a.s. P. We will assume that Z is bounded (say by C) by the obvious stopping argument.

Fix a $t_o > 0$. Now observe that (ii), (iii) and the growth inequality imply

$$E|Z_n|_t^{*2} \leq 2(k + E|\int_o^{\cdot} J^n Z_n dY_n|_t^{*2})$$

$$\leq 2(k + 8k^2(1 + t\beta)\beta \int_o^t E|J^n Z_n|_u^{*2} du)$$

$$\leq 2(k + 8k^2(1 + t\beta)\beta \int_o^t E|Z_n|_u^{*2} du) .$$

Thus by Grownwall's inequality, there exist constants K_1, K_2 (independent of n) such that

$$E|Z_n|_t^{*2} \leq K_1 e^{K_2 t} , \quad 0 \leq t \leq t_o .$$

By (ii)

$$Z_n - J^n Z_n = (J^n Z_n)(Y_n - J^n Y_n) .$$

Thus

$$E|Z_n - J^n Z_n|_t^{*2} \leq \frac{K^2}{2^{2n}} K_1 e^{K_2 t}, \quad 0 \leq t \leq t_o \qquad \ldots(1)$$

Now write

$$Z_n(t) - Z(t) = \int_o^t (J^n Z_n) dY_n - \int_o^t Z dY$$

$$= A_n(t) + B_n(t) + C_n(t),$$

where

$$A_n(t) = \int_o^t (J^n Z_n - Z_n) dY_n$$

$$B_n(t) = \int_o^t (Z_n - Z) dY_n$$

and

$$C_n(t) = \int_o^t Z d(Y_n - Y).$$

Now, growth inequality, estimate (1) and the fact that $Y_n \varepsilon Q(\beta)$ imply (for a suitable constant K_3)

$$E|A_n|_t^{*2} \leq K_3 \frac{1}{2^{2n}}, \quad 0 \leq t \leq t_o. \qquad \ldots(2)$$

Growth-inequality and the fact that $Y_n \varepsilon Q(\beta)$ imply

$$E|B_n|_t^{*2} \leq K_4 \int_o^t E|Z_n - Z|_u^{*2} du, \quad 0 \leq t \leq t_o \qquad \ldots(3)$$

(for a suitable constant K_4).

Finally, boundedness of Z and the fact that $Y_n - Y \varepsilon Q(\beta \frac{1}{2^n})$ together with the growth inequality imply that

$$E|C_n|_t^{*2} \leq K_5 \frac{1}{2^n}, \quad 0 \leq t \leq t_o \qquad \ldots(4)$$

Combining (2), (3) and (4), we get (for suitable K_6, K_7)

$$E|Z_n - Z|_t^{*2} \leq K_7 \cdot \frac{1}{2^n} + K_8 \int_0^t E|Z_n - Z|_u^{*2} du, \quad 0 \leq t \leq t_0$$

and hence by Grownwall's inequality,

$$E|Z_n - Z|_t^{*2} \leq K_7 K_8 \frac{1}{2^n} e^{K_8 t} \qquad 0 \leq t \leq t_0 .$$

This implies $Z_n \longrightarrow Z$ a.s. P.

As remarked earlier, this completes the proof.

3. <u>A More General Product Integral</u> :

Let f be a C^∞ function from R into itself such that $f(0) = 0$. Then symbolically '$f(dt) = f'(0)dt$'. To this symbol statement we attach the following meaning : for a (say) continuou function g, the Riemann sums of the form $\sum_{i=0}^{n-1} g(t_i) f(t_{i+1} - t_i)$ converge to $\int_0^t g(u) f'(0) du$, where $0 = t_0 < t_1 < \ldots < t_n = t$ and the limit is taken as the 'norm' of the partition goes to zero. This can be proved easily by Taylor's formula. The same problem for multiplicative integral can also be considered. See for instance Dollard-Friedman [6 ,p 50].

In the same spirit we write for a Brownian motion $\beta(t)$,

$$f(d\beta(t)) = f'(0)d\beta(t) + \frac{1}{2} f''(0)(d\beta(t))^2$$

$$= f'(0)d\beta(t) + \frac{1}{2} f''(0)dt.$$

Again, this symbolic equation represents the statement : For any continuous $\underline{\underline{F}}$ - adapted process g,

$$\sum_{i=0}^{n-1} g(T_i)f(\beta(T_{i+1}) - \beta(T_i))$$

converges to

$$\int_0^t g(u)f'(0)d\beta(u) + \frac{1}{2}\int_0^t g(u)f''(0)du$$

along a suitable sequence of random partitions $0 = T_0 < T_1 < \ldots < T_n = t$. In this section we prove this (and a similar statement for multiplicative integral) for a continuous semimartingale. Emery [7] considered this problem when $g = 1$ for both the additive as well as multiplicative case. He proved that the Riemann sums and products converge in probability to the respective integrals. We shall allow any process g in $\underline{\underline{D}}(\underline{\underline{F}})$ and obtain almost sure convergence results. This enables us to obtain pathwise integration formulae for $\int g.f(dX)$ and $\prod (I + g.f(dX))$ in terms of paths of g and X.

For the rest of the section we fix a twice continuously differentiable function f from $L(k)$ into itself with $f(0) = 0$ and with the second partial derivatives Lipschitz in a neighbourhood of 0. Here 0 is the matrix with all its entries zero.

For any $X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$ and $D \varepsilon L(k)$, we use the following notations :

$$\left[\nabla f(D), X\right] = \sum_{i,j} \left(\frac{\partial}{\partial x_{ij}} f\right)(D).X_{ij}$$

and

$$\left[\nabla^2 f(D), \langle\langle X \rangle\rangle\right] = \sum_{i,j} \sum_{r,s} \left(\frac{\partial^2}{\partial x_{ij} \partial x_{rs}}\right) f(D).\langle X_{ij}, X_{rs}\rangle .$$

With these notations, Ito's formula takes the simple form :

$$df(X) = \left[\nabla f(X), dX\right] + \frac{1}{2}\left[\nabla^2 f(X), d\langle\langle X \rangle\rangle\right] .$$

Now let $X \varepsilon \underline{\underline{C}}(\underline{\underline{F}})$ and $g \varepsilon \underline{\underline{D}}(\underline{\underline{F}})$ (both L(k) values) be such that $X(0) = 0$. For each $n \geq 1$, let $\{T_i^n : i \geq 0\}$ be a $\frac{1}{2^n}$ random partition for X and g.

Define the Riemann sums and products by

$$Y_n(t) = \sum_{i=0}^{\infty} g(T_i^n).f(X(t \wedge T_{i+1}^n) - X(t \wedge T_i^n))$$

and

$$Z_n(t) = \prod_{i=0}^{\infty} (I + g(T_i^n).f(X(t \wedge T_{i+1}^n) - X(t \wedge T_i^n))) .$$

Again, for each fixed t, w these are finite sums and products and hence $Y_n, Z_n \varepsilon \underline{\underline{C}}(\underline{\underline{F}})$ for all n.

Let $\Omega_0 = \{w : Y_n(\cdot, w)$ and $Z_n(\cdot, w)$ converge in $\underline{\underline{E}}\}$.

Let $\tilde{Y}$ and $\tilde{Z}$ be limits of Y_n and Z_n on Ω_0 and equal to zero on Ω_0^c. Then we have

<u>Theorem 4</u> : For all P such that $X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$, we have $P(\Omega_0) = 1$.

Further, if $R(t) = \left[\nabla f(0), X(t)\right] + \frac{1}{2}\left[\nabla^2 f(0), \langle\langle X \rangle\rangle(t)\right]$

Then

$$Y(t) - \int_0^t gdR$$

and

$$\tilde{Z}(t) = \prod_0^t (I + gdR).$$

Remark : We can write the last two equalities as

$$\int_0^t gf(dX) = \int_0^t gdR$$

and

$$\prod_0^t (I+g.f(dX)) = \prod_0^t (I + gdR)$$

where by definition, the left hand sides are limits of Y_n, Z_n respectively. In view of this theorem, we can write symbolically '$f(dX) = dR$'.

Proof : Fix a P such that $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$. As in Theorem 3, we assume that g is bounded by C_1 and that X belongs to $Q(1)$. Let

$$Y(t) = \int_0^t gdR$$

and

$$Z(t) = \prod_0^t (I + gdR).$$

As in Theorem 3, suffices to prove that $Y_n \longrightarrow Y$ and $Z_n \longrightarrow Z$ a.s. in $\underline{\underline{E}}$. To this end, we will show that

(i) $\left\{T_i^n : i \geq 0\right\}$ is a $\frac{C_1}{2^n}$ partition for Y_n for all $n \geq 1$

(ii) $Z_n(t) = I + \int_0^t (J^n Z_n) dY_n$

and

(iii) $Y \in Q(\beta)$ and $Y_n \in Q(\beta 2^{-n})$ for $n \geq 1$ (for a suitable constant β).

This as in Theorem 3, will imply that $Y_n \longrightarrow Y$ and $Z_n \longrightarrow Z$ a.s. in $\underline{\underline{E}}$ completing the proof of the theorem.

Now,

$$Y_n - J^n Y_n = (J^n g)(X - J^n X) \quad \text{for} \quad n \geq 1$$

and hence $|g| \leq C_1$ and $|X - J^n X| \leq 2^{-n}$ imply (i). (ii) follows exactly as in Theorem 3.

It remains to prove (iii). In the rest of the proof, we will be using Lemma 1.8 time and again. Since $X \in Q(1)$, we conclude that $R \in Q(\alpha)$ (for some α). Now the boundedness of g implies that $Y \in Q(\beta)$ (for some β).

Observe that if $S_1 \leq S_2$ are two stop times and N is a semimartingale, then Ito's formula applied to the semimartingale $N'(t) = N(t) - N(t \wedge S_1)$ and the function f (remembering that $f(0) = 0$) gives

$$f(N(S_2) - N(S_1)) = f(N'(S_2)) - f(0)$$

$$= \int_0^{S_2} \left[\nabla f(N'(u)), dN'(u)\right] + \frac{1}{2} \int_0^{S_2} \left[\nabla^2 f(N'(u)), d\langle\langle N' \rangle\rangle(u)\right]$$

$$= \int_{S_1}^{S_2} \left[\nabla f(N(u) - N(S_1)), dN(u) \right] + \frac{1}{2} \int_{S_1}^{S_2} \left[\nabla^2 f(N(u) - N(S_1)), d\langle\langle N \rangle\rangle(u) \right].$$

Thus

$$f(X(t \wedge T_{i+1}^n) - X(t \wedge T_i^n)) = \int_{t \wedge T_i^n}^{t \wedge T_{i+1}^n} \left[\nabla f(X - J^n X), dX \right]$$

$$+ \frac{1}{2} \int_{t \wedge T_i^n}^{t \wedge T_{i+1}^n} \left[\nabla^2 f(X - J^n X), d\langle\langle X \rangle\rangle \right].$$

Hence, using the definition of Y_n we get

$$Y_n(t) = \int_0^t (J^n g) \left[\nabla f(X - J^n X), dX \right]$$

$$+ \frac{1}{2} \int_0^t (J^n g) \left[\nabla^2 f(X - J^n X), d\langle\langle X \rangle\rangle \right]$$

$$= A_n(t) + B_n(t) \qquad \text{(say)}.$$

Also, let

$$A(t) = \int_0^t g \left[\nabla f(0), dX \right]$$

and

$$B(t) = \frac{1}{2} \int_0^t g \left[\nabla^2 f(0), d\langle\langle X \rangle\rangle \right].$$

Then by the definitions of Y and R, we have

$$Y(t) = A(t) + B(t).$$

Let C_2 be the bound of first and second partial derivatives of f on the unit ball of $L(k)$. Let C_3 be the

Lipschitz constant for the second derivatives on the unit ball. (If the second derivatives are Lipschitz on a ball of radius $\alpha < 1$ around the origin, then the arguments that follow are true for n such that $2^{-n} \leq \alpha$, which suffices for our purpose.)

Write $A_n(t) - A(t) = \eta_1(t) + \eta_2(t)$, where

$$\eta_1(t) = \int_0^t (J^n g - g)\left[\nabla f(X - J^n X), dX\right]$$

and

$$\eta_2(t) = \int_0^t g\left[\nabla f(X - J^n X) - \nabla f(0), dX\right].$$

Now, $|J^n g - g| \leq 2^{-n}$, $X \in Q(1)$ and the fact that the first derivatives of f are bounded on the unit ball imply that $\eta_1 \in Q(C_3 2^{-n})$ for a suitable constant C_3. Also $|g| \leq C_1$, $X \in Q(1)$ and the fact that second derivatives of f are bounded on the unit ball imply that $\eta_2 \in Q(C_4 2^{-n})$ (for some C_4). Combining these two observations, we get $A_n - A \in Q(C_5 2^{-n})$ (for some C_5).

Similarly, using that the second derivatives of f are Lipschitz on the unit ball, we get $B_n - B \in Q(C_6 2^{-n})$ and hence $Y_n - Y \in Q(C_7 2^{-n})$ (for suitable constants C_6, C_7). This completes the proof.

We remark that $\int gf(dX)$ and $\prod (I + gf(dX))$ can be defined when g is $L(k,m)$, X is $L(r,s)$ valued and f is a suitable function from $L(r,s)$ into $L(m,k)$. Theorem 6 holds good in this case as well with the same proof. The same is true of $\int f(dX).g$ and $\prod (I + f(dX).g)$.

4. A Trotter Type Formula :

In this section we apply results of the last section to specific function f. To consider a simple but useful situation assume that f is 'analytic' and is zero at the origin, that is for some $\alpha > 0$ and real numbers b_n, the following expansion is valid for $D \in L(k)$ with $|D| \leq \alpha$

$$f(D) = \sum_{n=1}^{\infty} b_n D^n .$$

For $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ ($L(k)$ valued) with $X = M + A$, let $\langle X, X\rangle = \langle M, M\rangle$ be the unique $L(k)$ valued process in $\underline{\underline{A}}(\underline{\underline{F}})$ such that $M^2 - \langle M, M\rangle \in \underline{\underline{L}}(\underline{\underline{F}})$. In other words,

$$\begin{aligned} \langle X,X\rangle_{ij} &= \langle M,M\rangle_{ij} \\ &= \sum_{m=1}^{k} \langle M_{im}, M_{mj}\rangle \\ &= \sum_{m=1}^{k} \langle X_{im}, X_{mj}\rangle . \end{aligned}$$

Then following the same notations as in the last section we have

$$\left[\nabla f(0), X\right] = b_1 X$$

and

$$\left[\nabla^2 f(0), \langle\langle X\rangle\rangle\right] = 2b_2 \langle X, X\rangle .$$

Let $R = b_1X + b_2\langle X,X\rangle$. Then by Theorem 4, we have

$$\int_0^t f(dX) = R(t)$$

and

$$\prod_0^t (I + f(dX)) = \prod_0^t (I + dR) .$$

Recall that by definition, expressions of the form $\int f(dX)$ or $\prod (I + f(dX)$ represent limits of 'simple' approximations as in Section 3.

<u>Theorem 5</u> : Let $X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$ ($L(k)$ valued). Then

(i) $$\prod_0^t e(dX) = \prod_0^t (I + dX + \frac{1}{2}\, d\langle X,X\rangle)$$

(ii) $$\prod_0^t (I + dX) = \prod_0^t e(dX - \frac{1}{2}\, d\langle X,X\rangle) .$$

(Here $e(D) = \sum_{n=0}^{\infty} \frac{D^n}{n!}$) .

<u>Proof</u> : Let $f(D) = \sum_{n=1}^{\infty} \frac{D^n}{n!}$. Then f satisfies the conditions mentioned at the beginning of this section and $e = I + f$. Thus (i) follows from the considerations preceding the theorem.

For (ii), let $X_1 = X - \frac{1}{2}\langle X,X\rangle$. Then as $X_1 - X \varepsilon \underline{\underline{A}}(\underline{\underline{F}})$, $\langle X_1,X_1\rangle = \langle X,X\rangle$ and hence $X = X_1 + \frac{1}{2}\langle X_1,X_1\rangle$. Now (ii) follows from (i). This completes the proof of the theorem.

For $X \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$ ($L(m,k)$ valued) and $Y \varepsilon \underline{\underline{S}}(\underline{\underline{F}}, P)$ ($L(k,r)$ valued), define $\langle X,Y\rangle$ ($L(m,r)$ valued) by

$$\langle X,Y\rangle_{is} = \sum_{j=1}^{k} \langle X_{ij}, Y_{js}\rangle$$

Now, we obtain 'stochastic version' of a Trotter type product formula for (deterministic) multiplicative integral. (See Masani [27]).

<u>Theorem 6</u> : Let $X, Y \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ ($L(k)$ valued). Then

$$\dot{\prod_{0}} e(dX).e(dY) = \dot{\prod_{0}} e(dX + dY + \tfrac{1}{2}d\langle X,Y\rangle - \tfrac{1}{2}d\langle Y,X\rangle).$$

<u>Proof</u> : Define Z by $Z = (X,Y)$. Define

$$f : L(k,2k) \cong L(k)(\underline{X})L(k) \longrightarrow L(k)$$

by

$$f(A,B) = e(A).e(B) .$$

Then observe that

$$f(A,B) = I + A + B + \tfrac{1}{2}A^2 + \tfrac{1}{2}B^2 + AB + \text{ higher order terms.}$$

Thus

$$\left[\nabla f(0,0), Z\right] = X + Y$$

and

$$\left[\nabla^2 f(0,0), \langle\langle Z\rangle\rangle\right] = \langle X,X\rangle + \langle Y,Y\rangle + 2\langle X,Y\rangle$$

Thus by Theorem 4,.

$$\dot{\prod_{0}} e(dX).e(dY) = \dot{\prod_{0}} f(dZ)$$

$$= \dot{\prod_{0}} (I + dR) \qquad \ldots(1)$$

where $R = X + Y + \tfrac{1}{2}\langle X,X\rangle + \tfrac{1}{2}\langle Y,Y\rangle + \langle X,Y\rangle$.

But, by Theorem 5,

$$\prod_{0}^{\cdot} (I + dR) = \prod_{0}^{\cdot} e(dR_1) \qquad \ldots(2)$$

where $R_1 = R - \frac{1}{2}\langle R,R\rangle$

$$= X + Y + \langle X,Y\rangle + \frac{1}{2}\langle X,X\rangle + \frac{1}{2}\langle Y,Y\rangle$$

$$- \frac{1}{2}(\langle X,X\rangle + \langle Y,Y\rangle + \langle X,Y\rangle + \langle Y,X\rangle)$$

$$= X + Y + \frac{1}{2}\langle X,Y\rangle - \frac{1}{2}\langle Y,X\rangle \qquad \ldots(3)$$

Now, (1), (2) and (3) imply the assertion.

Remark 1 : The result can also be stated in one of the following equivalent forms :

(i) $\prod_{0}^{\cdot} e(dX + dY) = \prod_{0}^{\cdot} e(dX).e(dY_1)$

where $Y_1 = Y - \frac{1}{2}\langle X,Y\rangle + \frac{1}{2}\langle Y,X\rangle$

(ii) $\prod_{0}^{\cdot} e(dX + dY) = \prod_{0}^{\cdot} e(dX_1).e(dY)$

where $X_1 = X - \frac{1}{2}\langle X,Y\rangle + \frac{1}{2}\langle Y,X\rangle$.

To see that (i) is equivalent to the theorem all we need to observe is that if $Y - Y_1 \in \underline{\underline{A}}(\underline{\underline{F}})$ $(Y, Y_1 \in \underline{\underline{S}}(\underline{\underline{F}}, P))$ then $\langle X,Y\rangle = \langle X,Y_1\rangle$ for all $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$. That (ii) is equivalent to the theorem also follows similarly.

Remark 2 : If $X, Y \in \underline{\underline{A}}(\underline{\underline{F}}, P)$, then $\langle X,Y\rangle = \langle Y,X\rangle = 0$ and hence we get the Trotter type product formula for deterministic Multiplicative integral.

Remark 3 : Let β be a Brownian motion (real valued) and let $f,g \in \underline{\underline{D}}(\underline{\underline{F}})$ ($L(k)$ valued) and let $X = \int f d\beta$, $Y = \int g d\beta$. Then $\langle X,Y\rangle = \int fg\,dt$ and $\langle Y,X\rangle = \int gf\,dt$ so that

$$\langle X,Y\rangle - \langle Y,X\rangle = \int (fg - gf)dt$$

$$= \int [f,g]dt . \qquad ([A,B] = AB - BA).$$

Thus Trotter product formula takes the form :

$$\prod_0 e(fd\beta).e(gd\beta) = \prod_0 e((f+g)d\beta + \tfrac{1}{2}[f,g]dt).$$

5. Integration by Parts Formula and Multiplicative Decomposition of Semimartingales :

In this section, we obtain a 'stochastic' analogue of the 'Integration by Parts' formula for multiplicative integral. (See Masani [26], [27]). Using this, we obtain a multiplicative decomposition of matrix valued semimartingales. Before proving the integration by parts formula, we get formulae for the determinant and inverse of exponential of a continuous semimartingale.

Definition : Let $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ ($L(k)$ valued). Then the exponential of X, (denoted by $EXP(X)$ is defined by

$$EXP(X)(t) = \prod_0^t (I + dX) .$$

Remark : If X is a real valued semimartingale, then by Ito's formula, it follows that $EXP(X)(t) = \exp(X(t) - \frac{1}{2}\langle X,X\rangle(t))$. (Here exp is the usual exponential function on $\mathbb{R}$.)

Theorem 7 : Let $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$, $Y = \text{trace}\ X$, $Z = \sum_{i,j} \langle X_{ij}, X_{ji} \rangle$.

Then

$$\text{(i)} \quad \det(\text{EXP}(X)(t)) = \exp(Y(t) - \tfrac{1}{2}Z(t)).$$

$$\text{(ii)} \quad \det(\prod_0^t e(dX)) = \exp(Y(t)).$$

Proof : (ii) follows from the definition of $\prod_0^t e(dX)$ and the relation

$$\det(\prod_{i=1}^n A_i) = \exp(\text{trace} \sum_{i=1}^n A_i),$$

for $A_i \in L(k)$. (i) Follows from (ii) and Theorem 5.

Part (i) of the previous theorem shows that for $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$, EXP(X) is 'invertible'. We now obtain a formula for the inverse of EXP(X). Call a L(k) valued process Z 'invertible' if $Z^{-1}(t,w)$ exists for all t for almost all w.

Theorem 8 : (a) Let $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ and $Y = \text{EXP}(X)$. Then Y is an invertible process such that $Y - I \in \underline{\underline{S}}(\underline{\underline{F}}, P)$. Further

$$(\text{EXP}(X)^{-1})' = \text{EXP}(-X' + \langle X', X' \rangle).$$

(b) If Y is an invertible process such that $Y - I \in \underline{\underline{S}}(\underline{\underline{F}}, P)$, then there exists a unique $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ such that $Y = \text{EXP}(X)$.

Proof : Let $X_1 = X - \frac{1}{2}\langle X, X \rangle$. Then by Theorem 5

$$Y = \prod_0^{\cdot} e(dX_1). \qquad \ldots(1)$$

Now observe that if Z_n, W_n approximate $\prod_0 e(dX_1)$ and $\prod_0 e(d(-X_1'))$ as in Theorem 4, then $Z_n . W_n' = I$ and $W_n' . Z_n = I$. Thus

$$\prod_0 e(dX_1).(\prod_0 e(d(-X_1')))' = I \qquad \text{...(2a)}$$

and

$$(\prod_0 e(d(-X_1')))' . \prod_0 e(dX_1) = I \qquad \text{...(2b)}$$

Another application of Theorem 5 gives

$$\prod_0^t e(d(-X_1')) = \prod_0^t (I + d(-X_1') + \tfrac{1}{2} d\langle -X_1', -X_1'\rangle)$$

$$= \prod_0^t (I + d(-X') + \tfrac{1}{2} d\langle X,X\rangle' + \tfrac{1}{2} d\langle X', X'\rangle)$$

Now $\langle X,X\rangle' = \langle X', X'\rangle$. Thus

$$\prod_0^t e(d(-X_1')) = \prod_0^t (I + d(-X') + d\langle X', X'\rangle)$$

$$= EXP(-X' + \langle X', X'\rangle) \qquad \text{...(3)}$$

Now (a) follows from (1), (2) and (3)

For (b), let $Z = Y - I$ and $X = \int_0 Y^{-1} dZ$. Note that $Y \in \underline{\underline{C}}(\underline{\underline{F}})$ and Y is invertible implies $Y^{-1} \in \underline{\underline{C}}(\underline{\underline{F}}^P)$. Now

$$\int_0 Y dX = \int_0 Y . Y^{-1} dZ$$

$$= Z$$

$$= Y - I.$$

Thus $Y = I + \int_0^{\cdot} YdX$ and hence $Y = EXP(X)$. To prove the uniqueness part, observe that if $Y = EXP(X_1)$, then

$$\int_0^t Y^{-1}dZ = \int_0^t Y^{-1}.YdX_1$$
$$= \int_0^t dX_1$$
$$= X_1(t) .$$

This implies uniqueness. The proof of the theorem is complete.

In view of Theorem 7, we make the following

<u>Definition</u> : Let Y be an invertible process such that $Y - I \in \underline{\underline{S}}(\underline{\underline{F}}, P)$. Then define logarithm of Y, denoted by $LOG(Y)$, by the equation

$$LOG(Y)(t) = \int_0^t Y^{-1}d(Y-I).$$

<u>Remark</u> : Observe that for $X \in \underline{\underline{S}}(\underline{\underline{F}}, P)$, $LOG(EXP(X)) = X$. If X is invertible and $X - I \in \underline{\underline{S}}(\underline{\underline{F}}, P)$, then we also have $EXP(LOG(X)) = X$. Moreover $X \in \underline{\underline{L}}(\underline{\underline{F}}^P, P)$ if and only if $EXP(X) - I \in \underline{\underline{L}}(\underline{\underline{F}}^P, P)$ and $X \in \underline{\underline{A}}(\underline{\underline{F}}^P)$ if and only if $EXP(X) - I \in \underline{\underline{A}}(\underline{\underline{F}}^P)$.

As a first step towards proving integration by parts formula for multiplicative stochastic integration, observe that Ito's formula implies the following 'integration by parts formula for additive stochastic integral : For $X, Y \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ ($L(k)$ valued) we have

$$X(t).Y(t) = \int_0^t X(u)dY(u) + \int_0^t dX(u).Y(u) + \langle X,Y\rangle(t) .$$

<u>Theorem 9</u> : (Integration by Part Formula)

Let $X_1, X_2 \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ ($L(k)$ valued) and let $Y_i = EXP(X_i)$ for $i = 1,2$. Let $W = \int Y_2 dX_1 Y_2^{-1}$ and $Z = EXP(W)$.

Then

$$EXP(X_1 + X_2 + \langle X_1, X_2\rangle) = Z.Y_2$$

$$= EXP(\int Y_2 dX_1 Y_2^{-1}).EXP(X_2).$$

<u>Proof</u> : $Z.Y_2 = -I + (Z - I).(Y_2 - I) + Z + Y_2$

$$= -I + \int (Z - I) d(Y_2 - I) + \int d(Z - I).(Y_2 - I)$$

$$+ Z + Y_2 + \langle Z - I, Y_2 - I\rangle$$

$$= I + \int Z\, d(Y_2 - I) + \int d(Z - I).Y_2 + \langle Z - I, Y_2 - I\rangle .$$

But by the definition of Z and Y_2, we have

$$Z - I = \int Z\; Y_2 dX_1 . Y_2^{-1}$$

and

$$Y_2 - I = \int Y_2 dX_2 .$$

Thus

$$ZY_2 = I + \int ZY_2 dX_2 + \int ZY_2 dX_1 . Y_2^{-1} Y_2 + \langle Z - I, Y_2 - I\rangle$$

$$= I + \int ZY_2 dX_2 + \int ZY_2 dX_1 + \langle Z - I, Y_2 - I\rangle .$$

Further, observe that for any $W_1, W_2 \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ and $g \in \underline{\underline{D}}(\underline{\underline{F}})$ (g invertible) we have

$$\langle \int dW_1 . g^{-1}, \int g dW_2\rangle = \langle W_1, W_2\rangle$$

so that

$$\langle Z - I, Y_2 - I\rangle = \langle \int ZY_2 dX_1 . Y_2^{-1}, \int Y_2 dX_2\rangle$$

$$= \langle \int ZY_2 dX_1, X_2\rangle$$

$$= \int ZY_2 d\langle X_1, X_2\rangle$$

Thus

$$ZY_2 = I + \int ZY_2 d(X_1 + X_2 + \langle X_1, X_2\rangle) .$$

Hence $\quad ZY_2 = EXP(X_1 + X_2 + \langle X_1 , X_2\rangle) .$

This completes the proof of the theorem.

Remembering that if $X_1 - \check{X}_1 \varepsilon \underline{\underline{A}}(\underline{\underline{F}})$ then $\langle X_1 , X_2\rangle = \langle \check{X}_1 , X_2\rangle$ we can rewrite the integration by parts formula in the following equivalent forms :

(i) $\quad EXP(X_1 + X_2) = EXP(\int Y_2 d\check{X}_1 Y_2^{-1}).EXP(X_2)$

where $\check{X}_1 = X_1 - \langle X_1 , X_2\rangle$

and $Y_2 = EXP(X_2)$.

(ii) $\quad EXP(X_1 + X_2) = EXP(\int \check{Y}_2 dX_1 . \check{Y}_2^{-1}).EXP(\check{X}_2)$

where $\check{X}_2 = X_2 - \langle X_1 , X_2\rangle$

and $\check{Y}_2 = EXP(\check{X}_2)$.

We now proceed to show that any 'invertible' (continuous) semimartingale Y can be uniquely factorized as $Y = N.B$

where N is a 'invertible' local martingale and B is a 'invertible' process of bounded variation. This follows as a simple application of 'integration by parts' formula. For multiplicative decomposition of real valued semimartingales, see Ito-Watanabe [13], Meyer [32] and Jacod [15, p 199].

Theorem 10 : Let Y be a 'invertible' process such that $Y - I \in \underline{\underline{S}}(\underline{\underline{F}}, P)$ ($L(k)$ valued). Then there exists a decomposition $Y = N.B$, where N and B are 'invertible' processes such that $N - I \in \underline{\underline{L}}(\underline{\underline{F}}^P, P)$ and $B - I \in \underline{\underline{A}}(\underline{\underline{F}}^P)$. Further, if $X = M + A$ is the canonical decomposition of $X = LOG(Y)$, then

$$B = EXP(A)$$

and

$$N = EXP(\int BdM.B^{-1}).$$

Hence, the decomposition $Y = N.B$ is unique.

Proof : Let $X = LOG(Y)$. Let $X = M + A$ be its canonical decomposition. Let $B = EXP(A)$, $N = EXP(\int BdM.B^{-1})$. Using integration by parts formula and the fact that $\langle M, A\rangle = 0$, we get

$$\begin{aligned} Y &= EXP(M + A) \\ &= EXP(\int BdM.B^{-1}).EXP(A) \\ &= N.B. \end{aligned}$$

It remains to show that the decomposition is unique. Let X, N, B, M and A be as above.

Let $Y = N_1 . B_1$ be any 'decomposition' of Y such that N_1, B_1 are invertible processes and $N - I \in \underline{\underline{L}}(\underline{\underline{F}}^P, P)$ and $B - I \in \underline{\underline{A}}(F^P)$. Define

$$A_1 = LOG(B_1)$$

$$M_2 = LOG(N_1)$$

and $$M_1 = \int B_1^{-1} dM_2 . B_1 .$$

Then $A_1 \in \underline{\underline{A}}(\underline{\underline{F}}^P)$ and $M_1, M_2 \in \underline{\underline{L}}(\underline{\underline{F}}^P, P)$. Further

$$M_2 = \int B_1 dM_1 . B_1^{-1}$$

so that

$$N_1 = EXP(M_2)$$

$$= EXP(\int B_1 dM_1 . B_1^{-1})$$

Thus

$$Y = N_1 . B_1$$

$$= EXP(\int B_1 dM_1 . B_1^{-1}).EXP(A_1)$$

$$= EXP(M_1 + A_1)$$

by integration by parts formula and the fact that $\langle M_1, A_1 \rangle = 0$.

Hence

$$M_1 + A_1 = LOG(Y)$$

$$= X$$

$$= M + A .$$

Now, by the uniqueness of (additive) decomposition $X = M + A$ (see Theorem 1.2) we get $M_1 = M$ and $A_1 = A$ so that $N_1 = N$ and $B_1 = B$. This completes the proof.

Remark 1 : We can also obtain a 'right decomposition' $Y = BN$ of a invertible semimartingale Y. This can be done either by interchanging the roles of M and A in the proof of Theorem 10 or by obtaining the 'left decomposition' of Y'.

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