ON SOME CHARACTERIZATIONS OF THE NORMAL DISTRIBUTION

By D. BASU and R. G. LAHA
Indian Statistical Institute, Calcutta

1. Introduction

If $x_1, x_2, \ldots, x_n$ are n independent normal variables with equal variances then, as is well known, $\bar{x} = \frac{1}{n} \Sigma x_i$ is independent of the set of variables $y_i = x_i - \bar{x}$, $i = 1, 2, \ldots, n-1$, and so it follows that $\bar{x}$ is independent of any function $g(x_1, x_2, \ldots, x_n)$ such that

$$g(x_1+\lambda, \ldots, x_n+\lambda) = g(x_1, x_2, \ldots, x_n) \text{ for all } \lambda. \quad \ldots \quad (A)$$

For, putting $\lambda = -\bar{x}$, we have

$$g(x_1, x_2, \ldots, x_n) = g(x_1-\bar{x}, x_2-\bar{x}, \ldots, x_n-\bar{x}) = g(y_1, y_2, \ldots, y_{n-1}, -\Sigma y_i).$$

That the direct converse of the above proposition is not true is seen from the following example.

Let x_1 and x_2 be independent and identically distributed continuous chance variables with density function $f(x)dx$ and let

$$g(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 < x_2, \\ 1, & \text{if } x_1 \geq x_2. \end{cases}$$

If $u = \frac{1}{2}(x_1+x_2)$ and $v = \frac{1}{2}(x_1 - x_2)$ then $f(u, v)\, du dv = 2f(u+v)f(u-v)\, du dv$ and hence the conditional frequency function for u given that $v < 0$ is

$$f(u \mid v<0)du = \frac{du \int_{-\infty}^{0} 2f(u+v)f(u-v)dv}{\int_{-\infty}^{\infty} du \int_{-\infty}^{0} 2f(u+v)f(u-v)dv}$$

$$= \frac{du \int_{0}^{\infty} 2f(u+v)f(u-v)dv}{\int_{-\infty}^{\infty} du \int_{0}^{\infty} 2f(u+v)f(u-v)dv}$$

$$= f(u \mid v \geq 0)du$$

which proves that $\bar{x}$ is independent of $g(x_1, x_2)$.

However, the converse of (A) is true if the function $g(x_1, x_2, \ldots, x_n)$ be of some particular types. Bernstein (1941) proved under certain assumptions that if x_1 and x_2 are independent chance variables such that x_1+x_2 is independent of x_1-x_2 then both x_1 and x_2 are normal with equal variances. Bernstein's Theorem has been considerably extended by Frechet (1951), Darmois (1951), and Basu (1951). It has been proved that if $x_1, x_2, \ldots, x_n$ are independent chance variables such that $a_1x_1+a_2x_2+\ldots+a_nx_n$ is independent of $b_1x_1+b_2x_2+\ldots+b_nx_n$ then x_i is normal if $a_ib_i \neq 0$ $(i=1, 2, \ldots, n)$.

Geary (1936) proved that if $x_1, x_2, \ldots, x_n$ are independent observations from the same population then the sample mean can be independent of the sample variance if and only if the parent population is normal. Geary proved his result under the implicit assumption that the parent population has moments of all orders. Lukacs (1942) proved Geary's Theorem under the assumption that moments up to the second order exist. It is believed that the result is true without any such assumptions. Below we give an extension of Geary's Theorem.

2. Extension of Geary's theorem

Let $x_1, x_2, \ldots, x_n$ be independent samples from a population that has finite moments up to the rth order and let K_r be the K-statistic of the rth order, i.e. K_r is the unbiassed estimator as given by Fisher (1928), of the rth population cumulant k_r.

Theorem : *If Σx_i is independent of any $K_r (r \geqslant 2)$ then the parent population is normal.**

Let μ_i $(i = 1, 2, \ldots, r)$ be the ith population moment about the origin. Then k_r is the co-efficient of t^r in the power series expansion of

$$\log\left(1+\mu_1 t+\frac{1}{2!}\mu_2 t^2+\ldots+\frac{1}{r!}\mu_r t^r\right)$$

and so
$$k_r = G(\mu_1, \mu_2, \ldots, \mu_r) \qquad \ldots \quad (2.1)$$
where G is some polynomial in the μ's.

Now suppose we want to estimate a term like $\mu_1^{\alpha_1}\mu_2^{\alpha_2}\ldots\mu_r^{\alpha_r}$ where the α's are non-negative integers. If we want to use only polynomials in the x's then our estimator must necessarily consist of terms like $x_1^{i_1} x_2^{i_2}\ldots x_n^{i_n}$ where of the n indices $i_1, i_2, \ldots, i_n$, α_j must be equal to j $(j = 1, 2, \ldots, r)$ and the rest must be zeros.

Hence from considerations of symmetry (Basu, 1952) the best unbiassed polynomial estimator of $\mu_1^{\alpha_1}\mu_2^{\alpha_2}\ldots\mu_r^{\alpha_r}$ is

$$t_{\alpha_1\alpha_2\ldots\alpha_r} = \frac{\alpha_1!\ldots\ldots\alpha_r!(n-\alpha_1\ldots-\alpha_r)!}{n!}\Sigma x_1^{i_1}\ldots x_n^{i_n} \qquad \ldots \quad (2.2)$$

where the summation is taken over all the $n!/\{\alpha_1!\ldots\alpha_r!(n-\alpha_1\ldots-\alpha_r)!\}$ possible permutations of the indices.

* When this paper was in press, Laha (1953) extended Geary's Theorem in another direction by proving the following:—

"Let $x_1, x_2, \ldots, x_n$ be independently and identically distributed chance variables with a finite variance σ^2. If now the conditional expectation of any unbiased quadratic estimate of $c\sigma^2$ $(c \neq 0)$ for the fixed sum $x_1+x_2+\ldots+x_n$ does not involve the latter, then the distribution of each x_i is normal."

It is thus clear that in general we can set up unbiassed polynomial estimators for $\mu_1^{\alpha_1}\ldots\mu_r^{\alpha_r}$ only if $\alpha_1+\alpha_2+\ldots+\alpha_r \leqslant n$ and so k_r admits of an unbiassed estimator K_r only if $r \leqslant n$.

Now $$k_r = G(\mu_1, \mu_2, \ldots, \mu_r)$$
$$= \Sigma C_{\alpha_1\alpha_2\cdots\alpha_r}\mu_1^{\alpha_1}\ldots,\mu_r^{\alpha_r} \quad \ldots \quad (2.3)$$
and hence $$K_r = \Sigma C_{\alpha_1\alpha_2\cdots\alpha_r}\, t_{\alpha_1\alpha_2\cdots\alpha_r} \quad \ldots \quad (2.4)$$
where the summation is taken over all α's such that $\alpha_1+2\alpha_2+\ldots+r\alpha_r = r$.

Let $\Phi(t)$ and $\psi(t)$ be the characteristic functions of Σx_i and K_r respectively. If Σx_i be independent of $K_r(2 \leqslant r \leqslant n)$ then

$$\Phi(t, u) = E\{e^{t\Sigma x_i+uK_r}\} = \Phi(t)\psi(u) \quad \ldots \quad (2.5)$$

(we are writing t and u for it and iu respectively). Differentiating both sides with respect to u (which is permissible because $E(K_r)$ exists) and then putting $u=0$ we have

$$E(K_r e^{t\Sigma x_i}) = \{E(K_r)\}\Phi(t)$$
$$= k_r\Phi(t). \quad \ldots \quad (2.6)$$

Now $$E\,(x_1^{i_1}\ldots x_n^{i_n}\,e^{t\Sigma x_i}) = \Big(E(x_1^{i_1}\,e^{tx_1})\Big)\Big(E(x_2^{i_2}\,e^{tx_2})\Big)\ldots(E(x_n^{i_n}\,e^{tx_n}))$$
$$= \varphi_{i_1}(t)\;\ldots\;\varphi_{i_n}(t)$$
$$= \{\varphi_1(t)\}^{\alpha_1}\ldots\{\varphi_r(t)\}^{\alpha_r}\{\varphi(t)\}^{n-\alpha_1-\cdots-\alpha_r} \quad \ldots \quad (2.7)$$

where $\varphi(t)$ is the c.f. of the parent population and

$$\varphi_r(t) = \frac{d^r}{dt^r}\varphi(t), \qquad \varphi_0(t) = \varphi(t).$$

Hence from (2.2), (2.4), (2.6) and (2.7) we have

$$E(K_r\,e^{t\Sigma x_i}) = \Sigma C_{\alpha_1\ldots\alpha_r}\,\varphi_1^{\alpha_1}\cdots\varphi_r^{\alpha_r}\,\varphi^{n-\alpha_1-\cdots-\alpha_r}$$

(where the summation, as usual, extends over all the α's for which $\alpha_1+2\alpha_2+\ldots+r\alpha_r = r$)

$$= k_r\Phi(t).$$

Now since $\Phi(t) = \{\varphi(t)\}^n$ we at once have from (2.3) that

$$G\left(\frac{\varphi_1}{\varphi}, \frac{\varphi_2}{\varphi}, \ldots, \frac{\varphi_r}{\varphi}\right) = k_r. \quad \ldots \quad (2.8)$$

We now observe that the l.h.s. of (2.8) is $\dfrac{d^r}{dt^r}\log\varphi(t)$, for

$$\log\varphi(t+h) = \log\{\varphi(t)+h\varphi_1(t)+\frac{h^2}{2!}\varphi_2(t)+\ldots\}$$

$$= \log\varphi(t)+\log\{1+h\frac{\varphi_1}{\varphi}+\frac{h^2}{2!}\,\frac{\varphi_2}{\varphi}+\ldots\}$$

King, A. J. and Jessen, R. J. (1945) : The master sample of agriculture. *J. Amer. Stat. Ass.*, **40**, 38–56.

Kitagawa, T. (1950a) : Successive process of statistical inferences (1). *Mem. Fract. Sci. Kyushu Univ.*, Ser. A, **5**, 139–180.

——— (1950b) : Estimation-formulae used in the sampling survey of the Fishing catches in Fukuoka Prefecture : Appendix to Sampling survey of fishery catch in Fukuoka Prefecture. Statistics and Research Division, Agricultural Improvement, Bureau, Ministry of Agriculture and Forestry, 1–39.

——— (1951a) : Successive process of statistical inferences (2). *Mem. Frac. Sci. Kyushu Univ.*, Ser. A, **6**, 55–95.

——— (1951b) : Random integration. *Bull. Math. Stat.*, **4**.

——— (1952a) : Successive process of statistical inferences (3). *Mem. Frac. Sci. Kyushu Univ.*, Ser. A, **7**, 131–135.

——— (1952b) : Successive process of statistical inferences (4). *Bull. Math. Stat.*, **5**, 35–50.

——— (1953a) : Successive process of statistical inferences (5). *Mem. Frac. Sci. Kyusyu Univ.*, Ser. A, **7**, 95–120.

——— (1953b) : Successive process of statistical inferences (6). *Mem. Fract. Sci. Kyusyu Univ.*, Ser. A, **8**, 1–29.

——— (1953c) : Some stochastic considerations concerning empirical functions. *Bull. Math. Stat.*, **5**, 19–33.

Kitagawa, T. and Fujita, T. (1951) : Sampling surveys of living costs of labourers in Mieko coal mine (in Japanese). Toyo Keizai Shimpo, Tokyo.

Mahalanobis, P. C. (1940) : A sample survey of the acreage under jute in Bengal. *Sankhyā*, **4**, 511–530.

——— (1944) : On large-scale sample surveys. *Phil. Trans. Roy. Soc.*, B, **231**, 329–451.

——— (1946) : Recent experiments in statistical sampling in the Indian Statistical Institute. *J. Roy. Stat. Soc.*, **109**, 325–370.

——— (1952) : Some aspects of the design of sample surveys. *Sankhyā*, **12**, 1–7.

Neyman, J. (1925) : Contributions to the theory of small sample drawn from a finite population. *Biometrika*, **27**, 472–478.

——— (1934) : On the two different aspects of the representative method : the method of stratified sampling and the method of purposive selection. *J. Roy. Stat. Soc.*, **97**, 558–606.

——— (1938) : Contribution to the theory of sampling human populations. *J. Amer. Stat. Ass.*, **33**, 101–116.

Paull, A. E. (1950) : On a preliminary test for pooling mean squares in the analysis of variance. *Ann. Math. Stat.*, **21**, 539–556.

Robbins, H. (1952) : Some aspects of the sequential designs of experiments. *Bull. Amer. Math. Soc.*, **58**, 527–535.

Robbins, H. and Monro, S. (1951) : A stochastic approximation method. *Ann. Math. Stat.*, **22** 400–407.

Stephan, F. F. (1945) : The expected value and variance of the reciprocal and other negative powers of a positive Bernoullian variate. *Ann. Math. Stat.*, **16**, 50–61.

Sukhatme, P. V. (1935) : Contribution to the theory of the representative method. *J. Roy. Stat. Soc., Supplement*, **2**, 253–268.

Wald, A. (1950) : *Statistical Decision Functions*, John Wiley and Sons, New York.

Yates, F. (1939) : *Sampling Methods for Censuses and Surveys*, Charles Griffin & Co., London.

Paper received : August, 1953.