On a metric distance between fuzzy sets

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Abstract

Given two fuzzy subsets on support set S in a metric (e.g., Euclidean) space, we consider the problem of defining a distance between them. The inadequacy of an earlier definition is pointed out and a Hausdorff-like distance is defined that is a metric.

Keywords: Fuzzy sets; Distance; Hausdorff metric; Dissimilarity

1. Introduction

Distance is a powerful concept in many disciplines of science. It is desirable that the distance be a metric – in other words, that it be positive definite ($d(x,y) \geqslant 0$, and $d(x,y) = 0$ iff $x = y$), symmetric ($d(x,y) = d(y,x)$ for all x, y), and satisfy the triangle inequality ($d(x,z) \leqslant d(x,y) + d(y,z)$ for all x, y, z).

In the p-dimensional Euclidean space $\mathbb{R}^p$, Minkowski defined a spectrum of metric distances given by

$$d_n(X,Y) = \left\{ \sum_{i=1}^{p} |x_i - y_i|^n \right\}^{1/n}, \qquad (1)$$

where X, Y are two points in $\mathbb{R}^p$ and x_i is the ith coordinate value of X. Three of these distances, namely the city block, chessboard and Euclidean distances, corresponding to $n = 1, \infty$ and 2, respectively, are the most popular in pattern recognition and image processing applications.

The concept of distance has been extended to subsets of a metric space. One popular set distance is the Hausdorff distance (HD), which is a metric. Let W^λ denote the operation of dilating the set W by radius λ (i.e., W^λ is the set of all points within distance λ of W). For any two sets U, V, let

$$L(U,V) = \inf\{\lambda \in \mathbb{R}^+ \mid U^\lambda \supseteq V\}.$$

Then the HD $H(U,V)$ is defined as

$$\max[L(U,V), L(V,U)].$$

The fuzzy set is a generalization of the set concept; a fuzzy subset of a set S is a mapping from S into $[0,1]$, where the value of the mapping for an element of S represents the "degree of membership" or "membership value" of the element in the fuzzy subset. An ordinary ("crisp") subset of S can be regarded as a mapping into $\{0,1\}$, where the mapping takes on value 1 for elements of the subset, and value 0 otherwise.

Two methods of defining the distance between two fuzzy subsets u and v of $\mathbb{R}^p$ have been proposed

(Dubois and Prade, 1980), one of which was later modified (Rosenfeld, 1985).

In one of the methods in (Dubois and Prade, 1980), a distance which is a fuzzy subset of $\mathbb{R}^+$ was defined. For a non-negative real number $r \in \mathbb{R}^+$, the distance is defined as

$$d_{u,v}(r) = \sup_{P,Q,\ d(P,Q)=r} [\min(u(P), v(Q))]. \quad (2)$$

For two non-fuzzy sets u and v, this definition leads to $d_{u,v}(r) = 1$ if there exist $P \in u$, $Q \in v$ such that $d(P,Q) = r$; otherwise $d_{u,v}(r) = 0$. The metric properties of this distance function are discussed in (Dubois and Prade, 1980). $d_{u,v}(r)$ is not a metric in the usual sense of the term.

The definition was slightly modified and renamed in (Rosenfeld, 1985):

$$\Delta_{u,v}(r) \equiv \sup_{P,Q;\ d(P,Q) \leqslant r} [\min(u(P), v(Q))]. \quad (3)$$

Note that $\Delta_{u,v}$ is a monotonically nondecreasing function of r and if $u' \leqslant u$ and $v' \leqslant v$ (i.e., $u'(s) \leqslant u(s)$ for all $s \in S$, and similarly for v' and v), then $\Delta_{u',v'} \leqslant \Delta_{u,v}$. The distance $\Delta_{u,v}$ satisfies some other desirable properties.

The mean distance between two non-empty fuzzy subsets u and v was also defined in (Rosenfeld, 1985) as

$$\bar{d}_{u,v} = \frac{\sum \sum_{P,Q \in S} d(P,Q) \min[u(P), v(Q)]}{\sum \sum_{P,Q \in S} \min[u(P), v(Q)]}. \quad (4)$$

In (Dubois and Prade, 1983) the HD was also generalized to fuzzy subsets. For a fuzzy subset w, for any $\lambda \in \mathbb{R}^+$ let

$$w^{\lambda}(P) = \sup\{w(Q) \mid d(P,Q) \leqslant \lambda\}. \quad (5)$$

Here w^{λ}, the expansion of w by λ, is the result of applying to all points of w a local max operation within a region of radius λ. Now let

$$L(u,v) = \inf\{\lambda \in \mathbb{R}^+ \mid u^{\lambda} \geqslant v\}. \quad (6)$$

The HD $H(u,v)$ is then defined as

$$H(u,v) \equiv \max[L(u,v), L(v,u)]. \quad (7)$$

Note that if $\sup(u) \neq \sup(v)$ then either $L(u,v)$ or $L(v,u)$ does not exist and hence $H(u,v)$ cannot be defined. Thus, all fuzzy subsets in S must have the same supremum for the distance H to exist; this is a serious drawback of this definition.

2. Proposed distance

We propose here a fuzzy generalization of the Hausdorff distance HD that is a metric.

First, let the fuzzy subsets $u_1, u_2, \ldots, u_n$ be defined on a support set S which consists of a finite number of points in a metric space. Also, let each fuzzy subset have a finite number of distinct membership values. Let all the distinct membership values of all subsets, pooled together, be $t_1, \ldots, t_m$. Let the maximum membership value of u_i be $u_{i,\max}$.

To start with, let all the fuzzy subsets have the same maximum $u_{\max}$. In other words, let

$$u_{1,\max} = u_{2,\max} = \cdots = u_{n,\max} \equiv u_{\max}.$$

Consider the t_k level set (or cut set) of u_i, called S_{ik}, which is a non-fuzzy subset of S. Thus, a point p belongs to S_{ik} if and only if $u_i(p) \geqslant t_k$. We can compute the conventional HD between S_{ik} and S_{jk}. Call this HD $h(S_{ik}, S_{jk})$. Now define

$$H(u_i, u_j) = \frac{\sum_{k=1}^{m} t_k\, h(S_{ik}, S_{jk})}{\sum_{k=1}^{m} t_k}. \quad (8)$$

Note that H is a metric. This is so because h is a metric, t_k is a constant multiplier for each k, and the denominator on the right-hand side is a constant independent of u_i and u_j.

Now let us assume that all the fuzzy subsets do not have the same maximum. Then for some fuzzy subsets and for some membership values the level sets are null. There exists no way of defining a conventional HD between one null and another (either non-null or null) subset. Thus, level by level distance computation is not possible, in general.

To take care of this problem let us modify the sets $u_1, u_2, \ldots, u_n$ so that the maximum membership of all the sets is 1 and it occurs at (support) points where the maximum membership occurred previously. Now, Eq. (8) can be applied to the modified u_i's to define the distance.

However, H defined in this way may still not satisfy a desired property of a metric, namely $H(u_i, u_j) = 0$ iff $u_i = u_j$. This is so when the two modified fuzzy sets u_i' and u_j' are equal although originally $u_i \neq u_j$. Such a situation occurs when u_i and u_j are equal at all support points except their maxima.

To counter this problem we can add to H a function

$$e(u_i, u_j) \equiv \varepsilon \frac{\sum_{q \in S} |u_i(q) - u_j(q)|}{\#S}, \tag{9}$$

where ε is a small positive constant and $\#S$ denotes the number of points in S. Note that $e(u_i, u_j)$ is computed on the original fuzzy subsets, not on the modified ones. The modified definition of the HD is then

$$H(u_i, u_j) = \frac{\sum_{k=1}^{m} t_k\, h(S_{ik}, S_{jk})}{\sum_{k=1}^{m} t_k} + \varepsilon \frac{\sum_{q \in S} |u_i(q) - u_j(q)|}{\#S}. \tag{10}$$

where the level sets are computed on the modified fuzzy subsets. (The value of ε should be chosen to reflect the relative importance of the two terms in the given problem domain.) Note that e is a metric and hence H defined by Eq. (10) is a metric. The distance computation algorithm for any two fuzzy subsets u, v of S is as follows:

1. Choose $\varepsilon > 0$. If $u = v$ then $H(u, v) = 0$; Return; Else, continue.
2. Find the maximum $u_{\max}$ of u. Let $S(u_{\max})$ be the (crisp) subset of S where the membership of u is $u_{\max}$. If $u_{\max} \neq 1$ then modify u so that its membership at every point in $S(u_{\max})$ is 1. Let the modified set be called u'. Do the same for v and define v'.
3. Find $H(u', v')$ using Eq. (8); $H(u, v) = H(u', v') + e(u, v)$; return.

In the development above, we assumed that (i) the support S consists of a finite number of points and (ii) there is only a finite number of distinct membership values. In general, the support may be an uncountable set and the fuzzy subsets defined on it can take on membership values in an uncountable subset of $[0, 1]$. To define a distance equivalent to Eq. (10) we assume that the support S is compact, thus has finite hypervolume, and that all the fuzzy subsets (u_i's) defined on S are continuous. Then, Eq. (10) is modified in the following form:

$$H(u_i, u_j) = \int_0^1 \mu\, h(S_{i\mu}, S_{j\mu})\, \mathrm{d}\mu + \varepsilon \frac{\int_S |u_i(s) - u_j(s)|\, \mathrm{d}s}{\int_S \mathrm{d}s}, \tag{11}$$

where $S_{i\mu}$ is the μ level set of the fuzzy subset u_i'.

To see that H defined by Eq. (11) is a metric, note first that by the compactness and continuity conditions we can ensure that

$$\int_S |u_i(s) - u_j(s)|\, \mathrm{d}s = 0 \;\Rightarrow\; u_i = u_j$$

and

$$\int_0^1 \mu\, h(S_{i\mu}, S_{j\mu})\, \mathrm{d}\mu = 0 \;\Rightarrow\; h(S_{i\mu}, S_{j\mu}) = 0 \quad \forall \mu.$$

The triangle inequality can be easily verified for each term on the right-hand side of Eq. (11).

3. Discussion

Some comments about the proposed definition of distance are in order.

1. In defining the distance we modify both fuzzy sets u_i and u_j to set their maximum memberships to 1, or more generally, to some fixed member of the (possibly degenerate) interval $[t_{\max}, 1]$, where $t_{\max} = \max\{u_{i\max} \mid i \in \{1, \ldots, n\}\}$. This seems necessary in order to satisfy the triangle inequality. If, by contrast, we have fuzzy sets u_i, u_j, u_k with $u_{i\max} < u_{j\max} < u_{k\max}$ and we were to compute $H(u_i, u_j)$ by adjusting u_i to u_i' so that the maximum of u_i' is $u_{j\max}$ (and similarly on all pairs of fuzzy sets), we would not satisfy the triangle inequality.

2. The second term in Eq. (10) is introduced to take care of the positive definiteness violation of the first term (in one situation). This term itself is a metric; it denotes a dissimilarity measure, rather than a "geometrical" distance measure, between two fuzzy sets. To illustrate this point, consider two fuzzy sets u_i and u_j with constant membership $u_{ic} \neq u_{jc}$ everywhere on S. To find the distance according to Eq. (10) we set

both of their memberships to 1. Their level sets for any membership value are equal and hence the first term in Eq. (10) is zero, while the second term is nonzero. Note that there is no geometrical distance between u_i and u_j at any level set, but they are dissimilar in their (constant) membership values.

On the other hand, consider two fuzzy sets u_i and u_j on the real line such that the membership of u_i is a constant u_{ic} over $[0, 2]$ and zero elsewhere. The membership of u_j is also a constant u_{jc} over $[3, 5]$ and zero elsewhere. In this situation, one can visualize a geometrical distance between u_i and u_j. Note that the first term in Eq. (10) is nonzero now. Conceptually, there is also a dissimilarity between u_i and u_j (one is nonzero where the other is zero) and the second term contributes positively. However, we are interested here in distance and hence the constant ε is chosen to be small in order to emphasize the distance part.

Dissimilarity between fuzzy sets is also an important issue with practical application potential. We plan to study fuzzy dissimilarity measures in a subsequent paper.

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