SIAM J. MATRIX ANAL. APPL.
Vol. 16, No. 3, pp. 964–976, July 1995

PATTERN PROPERTIES AND SPECTRAL INEQUALITIES IN MAX ALGEBRA *

R. B. BAPAT†, DAVID P. STANFORD‡, AND P. VAN DEN DRIESSCHE§

Abstract. The max algebra consists of the set of real numbers, along with negative infinity, equipped with two binary operations, maximization and addition. This algebra is useful in describing certain conventionally nonlinear systems in a linear fashion. Properties of eigenvalues and eigenvectors over the max algebra that depend solely on the pattern of finite and infinite entries in the matrix are studied. Inequalities for the maximal eigenvalue of a matrix over the max algebra, motivated by those for the Perron root of a nonnegative matrix, are proved.

Key words. max algebra, eigenvalue, eigenvector, circuit mean, Frobenius normal form

AMS subject classifications. 15A18, 15A42, 05C38

1. Introduction.

The algebraic system called "max algebra" has been used to describe, in a linear fashion, phenomena that are nonlinear in the conventional algebra. Examples include transportation networks, machine scheduling, and parallel computation. A system in which one component must wait for results from other components (a "discrete event dynamic system") can be modeled in max algebra. See [13, Chap. 1] for a detailed description of such systems. As described there, the question of regularizing a system, that is, of initiating a system in such a way that all components begin cycles at the same time, is answered by solving the eigenproblem in max algebra.

An early exposition of max algebra is the monograph of Cuninghame-Green [13]. Related works are Carré [7, Chap. 3] and Gondran and Minoux [19], that discuss more general "path algebras" and describe Gaussian and related solutions of linear systems over path algebras. Currently, work on max algebra systems is progressing in many directions; see [1], [6], [11], [18], [24]. Over the max algebra, eigenproblems for irreducible matrices were studied in [13] and for reducible matrices in [8] and [18].

The max algebra consists of the set $\boldsymbol{M} = \mathbf{R} \cup \{-\infty\}$, where $\mathbf{R}$ is the set of real numbers, equipped with two binary operations, addition and multiplication, denoted by $\oplus$ and $\otimes$, respectively. The operations are defined as follows:

$$a \oplus b = \max(a, b), \text{ the maximum of } a \text{ and } b$$

and

$$a \otimes b = a + b.$$

* Received by the editors July 9, 1993; accepted (in revised form) by R. Cottle July 5, 1994. The research of these authors was undertaken at the University of Victoria.

† Indian Statistical Institute, New Delhi, 110016, India.

‡ Department of Mathematics, College of William and Mary, Williamsburg, Virginia, 23185.
This research was partially supported by a College of William and Mary Faculty Research Grant. This author presented some of the results of this article at the Second Symposium on Matrix Analysis and Applications held on 22–23 October 1993 at Western Michigan University.

§ Department of Mathematics and Statistics, University of Victoria, Victoria, B.C., V8W 3P4, Canada.
This research was partially supported by Natural Sciences and Engineering Research Council of Canada grant A-8965 and the University of Victoria Committee on Faculty Research and Travel
(pvdd@smart.math.uvic.ca).

Clearly, $-\infty$ and 0 serve as identity elements for the operations $\oplus$ and $\otimes$, respectively. We denote $x_1 \oplus \cdots \oplus x_n$ by

$$\sum_{\oplus i=1}^{n} x_i,$$

or by $\sum_{\oplus} x_i$ when the range of summation of the index i is clear from the context.

We deal with vectors and matrices over the max algebra. Basic operations on matrices are defined in the natural way. Thus, if $A = [a_{ij}], B = [b_{ij}]$ are $m \times n$ matrices over $\boldsymbol{M}$, then $A \oplus B$ is the $m \times n$ matrix with (i,j)-entry $a_{ij} \oplus b_{ij}$. If $k \in \boldsymbol{M}$, then $k \otimes A$ is the matrix $[k \otimes a_{ij}] = [k + a_{ij}]$. If A is $m \times n$ and B is $n \times p$, then $A \otimes B$ is the $m \times p$ matrix with (i,j)-entry.

$$\sum_{\oplus k=1}^{n} a_{ik} \otimes b_{kj} = \max_{k}(a_{ik} + b_{kj}).$$

It is easily verified that matrix multiplication is associative and that it distributes over matrix addition.

The transpose of the matrix A is denoted by A^T. The $n \times n$ matrix with each diagonal entry zero and each off-diagonal entry $-\infty$ is the identity matrix over the max algebra. If we permute the rows (and/or columns) of the identity matrix, then we obtain a permutation matrix over the max algebra. If A, B are $m \times n$ matrices over $\boldsymbol{M}$, then $A \geq B$ means that $a_{ij} \geq b_{ij}$ for all i, j. Similarly, $A > B$ means that $a_{ij} > b_{ij}$ for all i, j. A column or row vector x over $\boldsymbol{M}$ is said to be finite if each component x_i of the vector is finite. A vector is called partly infinite if it has a finite component as well as an infinite component. A matrix or vector with each component $-\infty$ is called infinite and we denote it by $-\infty$ as well; this should not cause any confusion.

The exponential function provides a natural one-to-one map from $\boldsymbol{M}$ onto the nonnegative reals. Under this correspondence, matrices over max algebra correspond to nonnegative matrices over the reals, and much of our work is motivated by the theory of nonnegative matrices. Techniques of proof for max algebra sometimes reflect those for conventional algebra. In particular, the directed graph of a matrix, which provides much information in the study of nonnegative matrices, plays an even more central role in matrices over max algebra; see the definition of $\mu(A)$ below.

Let A be an $n \times n$ matrix over $\boldsymbol{M}$. We associate a directed graph (digraph) $G(A)$ with A as follows. The vertices of $G(A)$ are $1, 2, \ldots, n$. There is an edge from vertex i to vertex j, denoted by (i,j), if a_{ij} is finite and in that case we say that a_{ij} is the weight of the edge (i,j). We use standard terminology from the theory of digraphs. Thus a path of length l in a digraph is a sequence of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_l, i_{l+1})$, also denoted by $i_1 \to i_2 \to \cdots \to i_l \to i_{l+1}$; here the vertices are not necessarily distinct. The weight of a path is the sum of the weights of the edges in the path. The average weight of the path $i_1 \to i_2 \to \cdots \to i_l \to i_{l+1}$ is defined as

$$\frac{a_{i_1 i_2} + a_{i_2 i_3} + \cdots + a_{i_l i_{l+1}}}{l}.$$

A circuit τ of length l is a closed path $i_1 \to i_2 \to \cdots \to i_l \to i_1$, where $i_1, \ldots, i_l$ are distinct. A circuit of length one is a loop. We denote the set of circuits in $G(A)$, or in A, by $\boldsymbol{C}(A)$. If $\tau \in \boldsymbol{C}(A)$ then the average weight of τ is called the mean of the circuit τ, denoted by $M_A(\tau)$. We define the maximal circuit mean of A, denoted by $\mu(A)$, as

$$\mu(A) = \max_{\tau \in \boldsymbol{C}(A)} M_A(\tau)$$

if $\boldsymbol{C}(A) \neq \phi$, and we set $\mu(A) = -\infty$ otherwise. A circuit $\tau \in \boldsymbol{C}(A)$ is called a critical circuit if $M_A(\tau) = \mu(A)$. The set of all critical circuits in A is denoted by $\tilde{\boldsymbol{C}}(A)$. The critical graph of A is a digraph with vertices $1, 2, \ldots, n$, defined as follows. For $i, j \in \{1, 2, \ldots, n\}$, edge (i, j) is in the critical graph of A if and only if it belongs to a critical circuit in $\boldsymbol{C}(A)$.

A digraph is strongly connected if there exists a path from any vertex to any other vertex. We say that the matrix A is irreducible if $G(A)$ is strongly connected. If A is not irreducible then we say that it is reducible. If A is an $n \times n$ matrix over $\boldsymbol{M}$ then clearly A is irreducible if and only if $[e^{a_{ij}}]$ is a nonnegative, irreducible matrix in the usual sense (see, e.g., [4]). We also remark that A is reducible if and only if either A is 1×1 containing $-\infty$ or there exists a permutation matrix Q_1 over the max algebra such that

$$Q_1 \otimes A \otimes Q_1^T = \begin{bmatrix} A_{11} & -\infty \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices of order at least one. For A reducible and not 1×1 containing $-\infty$, there exist $q \geq 2$ and a permutation matrix Q over the max algebra such that

$$Q \otimes A \otimes Q^T = \begin{bmatrix} A_{11} & -\infty & \cdots & -\infty \\ A_{21} & A_{22} & \cdots & -\infty \\ \vdots & \vdots & \ddots & \vdots \\ A_{q1} & A_{q2} & \cdots & A_{qq} \end{bmatrix}, \tag{1.1}$$

where each A_{ii} is either square and irreducible or is 1×1 containing $-\infty$. This is the Frobenius normal form of A.

In §2 we give the basic definitions and state results for eigenvalues and eigenvectors of general square matrices over the max algebra. Proofs of these results can be found in the literature. In applications to discrete event dynamic systems such as machine scheduling or parallel computing, it may be useful to obtain information about eigenvalues and eigenvectors given only partial information concerning the entries of the matrix. In particular, it may be known which components of the system must wait for input from which other components, while the waiting times are unknown. It will then be known where the finite entries of the matrix of interest occur, but their magnitudes will be unknown; that is, only the "pattern" of the matrix will be specified. In §3 we obtain results concerning eigenvalues and eigenvectors that depend only on the pattern of the given matrix. In §4 we present new inequalities concerning the maximal circuit mean of a matrix over the max algebra. Most of these are motivated by known corresponding inequalities for the spectral radius of a nonnegative matrix.

2. Eigenvalues and eigenvectors. Let A be an $n \times n$ matrix over $\boldsymbol{M}$, then $\lambda \in \boldsymbol{M}$ is an eigenvalue of A if there exists a vector $x \neq -\infty$ such that

$$A \otimes x = \lambda \otimes x.$$

In this case, x is an eigenvector of A corresponding to the eigenvalue λ. Furthermore, we call (λ, x) an eigenpair of A. Note that (λ, x) is an eigenpair of A if and only if

$x \neq -\infty$ and $\max_j(a_{ij} + x_j) = \lambda + x_i, i = 1, 2, \ldots, n$. For example, if

$$A = \begin{bmatrix} 3 & -\infty \\ 2 & 4 \end{bmatrix},$$

then

$$A \otimes \begin{bmatrix} -\infty \\ 0 \end{bmatrix} = 4 \otimes \begin{bmatrix} -\infty \\ 0 \end{bmatrix},$$

thus 4 is an eigenvalue of A. It can be checked that 4 is the only eigenvalue of A. Note that A^T has both 3 and 4 as eigenvalues.

If Q is a permutation matrix over the max algebra and $\lambda \in \boldsymbol{M}$ then (λ, x) is an eigenpair of A if and only if $(\lambda, Q \otimes x)$ is an eigenpair of $Q \otimes A \otimes Q^T$. In particular, A and $Q \otimes A \otimes Q^T$ have the same eigenvalues. In view of these observations we often find it convenient to deal with the Frobenius normal form of A in (1.1) instead of the matrix A itself. Note that $G(A)$ and $G(Q \otimes A \otimes Q^T)$ are identical except for labeling of the vertices.

We need the following basic spectral results, which can be found in [1], [8], [10], [12]–[14], [18]. Detailed proofs are also given in [3]. The first result deals with the occurrence of $-\infty$ as an eigenvalue, the other results deal with $\mu(A)$ as an eigenvalue, with A irreducible in the third result.

THEOREM 2.1. *Let A be an $n \times n$ matrix over $\boldsymbol{M}$. Then,*

(i) *$-\infty$ is an eigenvalue of A if and only if A has an infinite column, and*

(ii) *$-\infty$ is the only eigenvalue of A if and only if $\boldsymbol{C}(A) = \phi$.*

THEOREM 2.2. *Let A be an $n \times n$ matrix over $\boldsymbol{M}$. Then $\mu(A)$ is an eigenvalue of A. Moreover, if (λ, x) is an eigenpair with x finite, then $\lambda = \mu(A)$.*

THEOREM 2.3. *Let A be an $n \times n$ irreducible matrix over $\boldsymbol{M}$. Then,*

(i) *$\mu(A)$ is the only eigenvalue of A, and every eigenvector of A is finite,*

(ii) *A has a unique eigenvector (up to scalar multiple over the max algebra) if and only if the critical graph of A is strongly connected.*

Now suppose that A is reducible and is in Frobenius normal form (1.1). For $k = 1, 2, \ldots, q$, let V_k denote the set of indices of rows in A that intersect the diagonal block A_{kk}. The sets V_k are called the classes of A. If V_i and V_j are classes, we say V_j has access to V_i provided either $i = j$ or there is a $u \in V_j$ and a $v \in V_i$ such that there is a path from u to v in $G(A)$. Since each A_{jj} is either irreducible or $[-\infty]$, the relation "has access to" is reflexive and transitive. If $\mu(A_{jj}) > \mu(A_{ii})$ then we say that class V_j dominates class V_i. These definitions are used in the following result to specify the eigenvalues of A, for proofs see [3], [8, Thm. 1], [18, Chap. 4, Coro. 2.2.5].

THEOREM 2.4. *Let A be an $n \times n$ matrix over $\boldsymbol{M}$, which is in Frobenius normal form (1.1), and let $\lambda \in \boldsymbol{M}$. Then λ is an eigenvalue of A if and only if there is an i such that $\mu(A_{ii}) = \lambda$ and no class which dominates V_i has access to V_i.*

3. Pattern properties in max algebra. In this section we investigate spectral properties that depend only on the placement of finite and infinite entries in the matrix, and not on the magnitudes of the finite entries. Such properties are called "pattern properties" of the matrix.

A (square) pattern is an $n \times n$ array $P = [p_{ij}]$ of symbols chosen from $\{*, -\infty\}$. If A is an $n \times n$ matrix over $\boldsymbol{M}$, we write $A \in P$ provided

$$a_{ij} \in \mathbf{R} \text{ if } p_{ij} = *, \qquad a_{ij} = -\infty \text{ if } p_{ij} = -\infty.$$

Following [21], a pattern P is said to allow a particular property if there is a matrix $A \in P$ which has the property. P is said to require the property if every matrix $A \in P$

has the property. We determine which patterns allow, and which patterns require, various spectral properties in the max algebra.

The digraph $G(P)$ of an $n \times n$ pattern P has vertices $\{1, 2, \dots, n\}$, and an edge from i to j if and only if $p_{ij} = *$. We denote the set of circuits in $G(P)$ by $\mathbf{C}(P)$. The concept of reducibility of a square matrix, introduced in §2, extends in an obvious way to patterns. Pattern P is irreducible if and only if $G(P)$ is strongly connected. It follows that P is reducible if and only if P is 1×1 containing $-\infty$, or if by an identical permutation of rows and columns P can be brought to the form

$$\begin{bmatrix} P_{11} & -\infty \\ P_{21} & P_{22} \end{bmatrix},$$

where P_{11} and P_{22} are square with order at least one. We will also deal with the Frobenius normal form of the pattern, defined analogously to that of a matrix; see (1.1).

We first discuss properties of the eigenvalues of a matrix determined by its pattern.

LEMMA 3.1. *Let P be a pattern. The following are equivalent.*

(i) *P requires a finite eigenvalue.*

(ii) *P allows a finite eigenvalue.*

(iii) *$\mathbf{C}(P)$ is not empty.*

Proof. The proof follows easily from Theorems 2.1 and 2.2. □

LEMMA 3.2. *Let P be a pattern. The following are equivalent.*

(i) *P requires $-\infty$ as an eigenvalue.*

(ii) *P allows $-\infty$ as an eigenvalue.*

(iii) *P has an infinite column.*

Proof. The proof follows immediately from Theorem 2.1. □

The following corollary is an immediate consequence of Lemmas 3.1 and 3.2.

COROLLARY 3.3. *Let P be a pattern.*

(i) *P requires that $-\infty$ be the only eigenvalue if and only if P allows the same property, and this occurs if and only if $\mathbf{C}(P)$ is empty.*

(ii) *P requires that all eigenvalues be finite if and only if P allows the same property, and this occurs if and only if P has no infinite column.*

THEOREM 3.4. *Let P be a pattern.*

(i) *P requires a unique and finite eigenvalue if and only if P has no infinite column and the Frobenius normal form of P has exactly one irreducible diagonal block.*

(ii) *P allows a unique and finite eigenvalue if and only if P has no infinite column.*

Proof. (i). We may assume without loss of generality that P is in Frobenius normal form. Suppose P requires a unique and finite eigenvalue. By Lemma 3.2, P has no infinite column. If P had a 1×1 diagonal block $[-\infty]$ in the lower right corner, P would have an infinite column. Hence the lower right diagonal block is irreducible. If P had another irreducible diagonal block, a matrix $A \in P$ could be constructed with the lower right diagonal block having eigenvalue 0 and another irreducible diagonal block having a positive eigenvalue. It follows from Theorem 2.4 that A would have two eigenvalues, one 0 and one positive, violating the fact that P requires a unique eigenvalue.

Now suppose that P has no infinite column and exactly one irreducible block, which then must be P_{qq}, the lower right block. Let $A \in P$. By Theorem 2.1, $-\infty$ is not an eigenvalue of A. By Theorem 2.2, A has an eigenvalue which, by Theorem 2.4, is $\mu(A_{ii})$ for some diagonal block A_{ii} in A. Since A_{qq} is the only irreducible diagonal block in A, $\mu(A_{qq}) > -\infty$ is the only eigenvalue of A.

(ii) If P allows a unique and finite eigenvalue, then P does not require $-\infty$ as an eigenvalue, so by Lemma 3.2 P has no infinite column. Conversely, if P has no infinite column, then the matrix $A \in P$ which has 0 in all the $*$ positions has the unique and finite eigenvalue 0. □

We now turn to pattern properties concerning the eigenvectors of a matrix. We obtain necessary and sufficient conditions on a pattern that it allow (or require) all (or some) eigenvectors to be finite (or partly infinite). Some of the results parallel those concerning partly zero eigenvectors in the conventional algebra presented in [23]. We remark that in the context of a discrete event dynamical system, the existence of a finite eigenvector implies that the system can be regularized. Note that the eigenpairs of the matrix (pattern) with each entry $-\infty$ are of the form $(-\infty, x)$ with $x \neq -\infty$. We exclude that pattern from consideration in the following.

THEOREM 3.5. *Let P be a pattern with at least one $*$. Then P requires that all eigenvectors be partly infinite if and only if P has an infinite row.*

Proof. First suppose P has no infinite row. Let $A \in P$ be obtained by replacing all $*$'s with 0's. Then the vector of all 0's is a finite eigenvector of A corresponding to the eigenvalue 0. Hence P does not require that all eigenvectors be partly infinite.

Now suppose that row i of P is infinite, but that $A \in P$ has a finite eigenvector x corresponding to eigenvalue λ. Then entry i of $A \otimes x$ is $-\infty$, so $\lambda \otimes x_i$ is $-\infty$. Since x_i is finite, $\lambda = -\infty$. Now if a_{jk} is finite, then entry j of $A \otimes x$ is finite, whereas entry j of $\lambda \otimes x = -\infty$. Hence $A = -\infty$, so $P = -\infty$, a contradiction. Therefore if P has an infinite row, then P requires that all eigenvectors be partly infinite. □

COROLLARY 3.6. *Let P be a pattern with at least one $*$. Then P allows a finite eigenvector if and only if P has no infinite row.*

THEOREM 3.7. *Let P be a pattern with at least one $*$. The following are equivalent.*

(i) *P is irreducible.*

(ii) *P requires that all eigenvectors be finite.*

(iii) *P allows all eigenvectors to be finite.*

Proof. (i) $\Rightarrow$ (ii). If $A \in P$ then A is irreducible, so by Theorem 2.3, all eigenvectors of A are finite. Therefore (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i). Suppose that P is reducible, so that without loss of generality we may assume $P = \begin{bmatrix} P_{11} & -\infty \\ P_{21} & P_{22} \end{bmatrix}$. Let $A = \begin{bmatrix} A_{11} & -\infty \\ A_{21} & A_{22} \end{bmatrix} \in P$ be partitioned as P is. Let $x_{(2)}$ be an eigenvector of A_{22} corresponding to $\mu(A_{22})$, and let $x = \begin{bmatrix} -\infty \\ x_{(2)} \end{bmatrix}$. Then $A \otimes x = \mu(A_{22}) \otimes x$, so x is an eigenvector of A which is partly infinite. Hence P does not allow all eigenvectors to be finite. Therefore (iii) $\Rightarrow$ (i). □

COROLLARY 3.8. *Let P be a pattern with at least one $*$. The following are equivalent.*

(i) *P is reducible.*

(ii) *P allows a partly infinite eigenvector.*

(iii) *P requires a partly infinite eigenvector.*

Proof. The equivalence of (i) through (iii) in Theorem 3.7 implies the corollary. □

THEOREM 3.9. *Let P be a pattern with at least one $*$. Then P requires a finite eigenvector if and only if P has no infinite row and the Frobenius normal form of P has exactly one irreducible diagonal block.*

Proof. We may assume without loss of generality that P is in Frobenius normal form. Suppose P requires a finite eigenvector. By Theorem 3.5, P has no infinite row. Therefore the upper left diagonal block P_{11} in P is irreducible. Suppose there is a $k > 1$ such that P_{kk} is irreducible. We will construct a matrix $A \in P$ with all eigenvectors partly infinite, contradicting the hypothesis on P. To do this, let

$$U_1 = [P_{21}^T \quad P_{31}^T \quad \cdots \quad P_{q1}^T]^T \quad \text{and} \quad U_2 = \begin{bmatrix} P_{22} & -\infty & -\infty & \cdots & -\infty \\ P_{32} & P_{33} & -\infty & \cdots & -\infty \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{q2} & P_{q3} & \cdots & \cdots & P_{qq} \end{bmatrix}.$$

Then $P = \begin{bmatrix} P_{11} & -\infty \\ U_1 & U_2 \end{bmatrix}$, and P_{kk} is one of the diagonal blocks in U_2. Since P_{kk} is irreducible, P_{kk} has a circuit. Select a circuit in P_{kk} and set all its $*$ entries equal to 1. Set the other $*$ entries in U_2 to 0 to create a matrix $A_2 \in U_2$. Set all $*$ entries in P_{11} and U_1 to 0 to complete $A = \begin{bmatrix} A_{11} & -\infty \\ A_1 & A_2 \end{bmatrix} \in P$ with $\mu(A_{11}) = 0$ and $\mu(A) = \mu(A_2) = 1$. Suppose A has a finite eigenvector $x = \begin{bmatrix} x_{(1)} \\ x_{(2)} \end{bmatrix}$ partitioned to conform to the partition of A above. Since x is finite, the corresponding eigenvalue must be $\mu(A)$ by Theorem 2.2. But then $A_{11} \otimes x_{(1)} = 1 \otimes x_{(1)}$, which is impossible because the only eigenvalue of A_{11} is 0. Hence A cannot have a finite eigenvector and the desired contradiction is reached.

Now suppose P has no infinite row and has exactly one irreducible diagonal block, which must then be P_{11}. If $P = P_{11}$, that is if P is irreducible, then P requires all eigenvectors finite and we are through. Otherwise, let $q \geq 2$ be the number of diagonal blocks in P. Let $A \in P$ be partitioned as P is. We construct a finite eigenvector of A inductively as follows. Since A_{11} is irreducible, A_{11} has a finite eigenvector $x_{(1)}$ corresponding to its eigenvalue $\mu(A_{11})$. Let $x_2 = A_{21} \otimes x_{(1)} - \mu(A_{11})$, and for $2 \leq i < q$, let $x_{i+1} = [A_{i+1,1} A_{i+1,2} \ldots A_{i+1,i}] \otimes [x_{(1)}^T x_2 \ldots x_i]^T - \mu(A_{11})$, a finite member of $\boldsymbol{M}$. It then follows that $x = [x_{(1)}^T x_2 \ldots x_q]^T$ is a finite eigenvector of A corresponding to $\mu(A_{11})$, so P requires a finite eigenvector. □

COROLLARY 3.10. *Let P be a pattern with at least one $*$. Then P allows all eigenvectors to be partly infinite if and only if P has an infinite row or the Frobenius normal form of P has two irreducible diagonal blocks.*

Proof. Upon observing that a pattern with no infinite row must have a Frobenius normal form with the upper left diagonal block irreducible, the corollary follows immediately from Theorem 3.9. □

THEOREM 3.11. *Let P be a pattern. Then P allows a unique and finite eigenvector if and only if P is irreducible.*

Proof. Assume P is irreducible. Let $a_{ij} = 0$ whenever $p_{ij} = *$. Then A has a unique and finite eigenvector by Theorem 2.3. Assume P is reducible, then P requires a partly infinite eigenvector by Corollary 3.8. Thus P does not allow a unique and finite eigenvector. □

THEOREM 3.12. *Let P be a pattern. Then P requires a unique and finite eigenvector if and only if P is irreducible and the directed graph $G = G(P)$ does not contain two vertex-disjoint circuits.*

Proof. Assume P is irreducible and G does not have two vertex-disjoint circuits. Let $A \in P$. Then by Theorem 2.3, A has a unique eigenvalue which is $\mu(A)$, each eigenvector of A is finite, and A has a unique eigenvector if and only if the critical graph $\boldsymbol{C}$ of A is strongly connected. Now $\boldsymbol{C}$ is a subgraph of G and is a union of circuits. Since G does not have two vertex-disjoint circuits, $\boldsymbol{C}$ does not have two vertex-disjoint circuits. If i and j are vertices in $\boldsymbol{C}$, then i lies on a circuit $\boldsymbol{C}_i$ and j lies on a circuit $\boldsymbol{C}_j$. If $\boldsymbol{C}_i = \boldsymbol{C}_j$ there are paths from i to j and from j to i in

$\boldsymbol{C}$. Otherwise $\boldsymbol{C}_i$ and $\boldsymbol{C}_j$ share a vertex, and again there are paths from i to j and from j to i in $\boldsymbol{C}$. Hence $\boldsymbol{C}$ is strongly connected and the eigenvector of A is unique up to scalar multiples in the max algebra. Therefore P requires a unique and finite eigenvector.

Now assume P requires a unique and finite eigenvector. If P were reducible, then by Corollary 3.8 P would allow a partly infinite eigenvector. Hence P is irreducible. Suppose G has two vertex-disjoint circuits. Then we may select two vertex disjoint circuits in G and construct a matrix $A \in P$ which has 1 in the positions belonging to either of the two circuits and 0 and $-\infty$ elsewhere. Then the circuit means are 1 on each of the two circuits and less than 1 on each other circuit, so the critical graph of A is the union of the two disjoint circuits and is not strongly connected. Hence the eigenvector of A is not unique, contradicting the assumption on P. Hence G does not have two vertex-disjoint circuits. □

4. Inequalities. Many of the results in this section are motivated by known inequalities for the spectral radius (or the Perron root) $\rho(B)$ of a nonnegative matrix B. Thus, Lemma 4.1 and Corollary 4.2 are analogs of well-known bounds for the Perron root; see, for example, [4, p. 28] and [25, p. 31]. Theorem 4.3 is the max algebra version of a result due to Birkhoff and Varga [5]. The parallels between inequalities for $\mu(A)$, where A is a matrix over $\boldsymbol{M}$ and $\rho(B)$, where B is a nonnegative matrix, are quite striking and remain to be fully explored. Theorem 4.9 is yet another result in this direction. Let A be an $n \times n$ matrix over $\boldsymbol{M}$ and let B be the Hadamard exponential of A, i.e., $b_{ij} = e^{a_{ij}}$ for all i, j. Then $e^{\mu(A)}$ is the maximal circuit geometric mean of the nonnegative matrix B. We remark that the maximal circuit geometric mean of a nonnegative matrix has been considered in the literature; see, e.g., [15], [17], [22].

The following lemma is stated and proved in [18, Chap. 4, Lemmas 1.3.8, 1.3.9].

LEMMA 4.1. *Let A be an $n \times n$ matrix over $\boldsymbol{M}$ and $\eta \in \boldsymbol{M}$. Then $\mu(A) \geq \eta$, if and only if there exists a vector $z \neq -\infty$ such that $A \otimes z \geq \eta \otimes z$. Furthermore, if A is irreducible, then $\mu(A) \leq \eta$, if and only if there exists a vector $z \neq -\infty$ such that $A \otimes z \leq \eta \otimes z$.*

COROLLARY 4.2. *Let A be an $n \times n$ matrix over $\boldsymbol{M}$. Then*

$$\min_i \max_j a_{ij} \leq \mu(A) \leq \max_{i,j} a_{ij}.$$

Proof. Let $\alpha = \min_i \max_j a_{ij}$ and let 0 denote the vector with each component zero. Then $A \otimes 0 \geq \alpha \otimes 0$. It follows from Lemma 4.1 that $\mu(A) \geq \alpha$. It is easy to see that for any $\sigma \in C(A), M_A(\sigma) \leq \max_{i,j} a_{ij}$, and hence $\mu(A) \leq \max_{i,j} a_{ij}$, giving the second inequality. □

Let A be an $n \times n$ matrix over $\boldsymbol{M}$. By Theorem 2.2, $\mu(A)$ is an eigenvalue of A and there is a vector $x \neq -\infty$ such that $A \otimes x = \mu(A) \otimes x$. We refer to x as a right eigenvector of A corresponding to $\mu(A)$. Since $\mu(A) = \mu(A^T)$, there is a vector $y \neq -\infty$ as a left eigenvector of A corresponding to $\mu(A)$. We note that (by Theorem 2.3) if A is irreducible, then x and y are finite and $\mu(A)$ is the only eigenvalue of A.

THEOREM 4.3. *Let A be an $n \times n$ irreducible matrix over $\boldsymbol{M}$. Then the following assertions hold.*

(i) $\mu(A) = \max_{x > -\infty} \min_{y > -\infty} (y^T \otimes A \otimes x - y^T \otimes x)$.

(ii) $\mu(A) = \min_{y > -\infty} \max_{x > -\infty} (y^T \otimes A \otimes x - y^T \otimes x)$.

Proof. For any finite vectors x, y, we have

$$\begin{aligned} y^T \otimes A \otimes x &= \max_{i,j}(a_{ij} + y_i + x_j) \\ &= \max_{i,j}(a_{ij} + x_j - x_i + y_i + x_i) \\ &\geq \min_i \max_j (a_{ij} + x_j - x_i) + y^T \otimes x. \end{aligned}$$

Therefore,

$$y^T \otimes A \otimes x - y^T \otimes x \geq \min_i \max_j (a_{ij} + x_j - x_i). \tag{4.1}$$

Suppose

$$\min_i \max_j (a_{ij} + x_j - x_i) = \max_j (a_{kj} + x_j - x_k).$$

Let z be the vector with $z_k = -x_k$, with the remaining components chosen finite and so that $z^T \otimes x = 0$ and satisfying

$$\max_j (a_{ij} + z_i + x_j) \leq \max_j (a_{kj} + z_k + x_j), \qquad i = 1, 2, \ldots, n.$$

When we set $y = z$, equality holds in (4.1) and hence we have shown that for any finite x,

$$\min_{y > -\infty} (y^T \otimes A \otimes x - y^T \otimes x)$$

exists. Thus by (4.1)

$$\min_{y > -\infty} (y^T \otimes A \otimes x - y^T \otimes x) = \min_i \max_j (a_{ij} + x_j - x_i).$$

Let $S = [a_{ij} + x_j - x_i]$. Then $\mu(A) = \mu(S)$ and by Corollary 4.2

$$\min_i \max_j (a_{ij} + x_j - x_i) \leq \mu(S).$$

Therefore, we conclude that

$$\mu(A) \geq \sup_{x > -\infty} \min_{y > -\infty} (y^T \otimes A \otimes x - y^T \otimes x). \tag{4.2}$$

When we set x to be a right eigenvector of A, we see that for any finite $y, y^T \otimes A \otimes x - y^T \otimes x = \mu(A)$. Thus, (i) follows from (4.2). The proof of (ii) is similar. □

We next give an easy inequality, and then characterize the case of equality.

LEMMA 4.4. *Let X, Y be $n \times n$ matrices over $\boldsymbol{M}$ such that $X \geq Y$. Then $\mu(X) \geq \mu(Y)$.*

Proof. The result is obvious if $\boldsymbol{C}(Y) = \phi$, since in that case, $\mu(Y) = -\infty$. So suppose $\boldsymbol{C}(Y) \neq \phi$. For any $\sigma \in \tilde{\boldsymbol{C}}(Y)$,

$$\mu(Y) = M_Y(\sigma) \leq M_X(\sigma) \leq \mu(X)$$

and the proof is complete. □

Observe that Lemma 4.4 shows that if Z is a principal submatrix of X, then $\mu(X) \geq \mu(Z)$.

To discuss the case of equality in Lemma 4.4, we now introduce some notation. Suppose σ is the circuit $(i_1 \quad i_2 \quad \cdots \quad i_k)$; in this notation we assume i_1 to be the least integer among $i_1, i_2, \ldots, i_k$ and this convention makes the representation of the circuit uniquely determined. If X is an $n \times n$ matrix and if $\sigma = (i_1 \quad i_2 \quad \cdots \quad i_k) \in \boldsymbol{C}(X)$, then we define $X(\sigma)$ as the vector

$$[x_{i_1 i_2} \quad x_{i_2 i_3} \quad \cdots \quad x_{i_k i_1}]^T.$$

LEMMA 4.5. *Let X, Y be $n \times n$ matrices over $\boldsymbol{M}$ such that $X \geq Y$ and suppose $\mu(Y)$ is finite. Then the following conditions are equivalent.*

(i) $\mu(X) = \mu(Y)$.

(ii) *There exists $\sigma \in \tilde{\boldsymbol{C}}(X) \cap \tilde{\boldsymbol{C}}(Y)$ such that $M_X(\sigma) = M_Y(\sigma)$.*

(iii) *There exists $\sigma \in \tilde{\boldsymbol{C}}(X)$ such that $M_X(\sigma) = M_Y(\sigma)$.*

(iv) *$\tilde{\boldsymbol{C}}(Y) \subset \tilde{\boldsymbol{C}}(X)$ and for all $\sigma \in \tilde{\boldsymbol{C}}(Y)$, $X(\sigma) = Y(\sigma)$.*

(v) *$\tilde{\boldsymbol{C}}(Y) \subset \tilde{\boldsymbol{C}}(X)$ and there exists $\sigma \in \tilde{\boldsymbol{C}}(Y)$ such that $X(\sigma) = Y(\sigma)$.*

Proof. First observe that since $\mu(Y)$ is finite, and $X \geq Y$, $\mu(X)$ is finite and $C(Y), C(X)$ are nonempty.

(i) $\Rightarrow$ (ii). Let $\sigma \in \tilde{\boldsymbol{C}}(Y)$. Then

$$\mu(Y) = M_Y(\sigma) \leq M_X(\sigma) \leq \mu(X) \tag{4.3}$$

and since $\mu(X) = \mu(Y)$, equality holds throughout in (4.3). It follows that $\sigma \in \tilde{\boldsymbol{C}}(X) \cap \tilde{\boldsymbol{C}}(Y)$ and $M_X(\sigma) = M_Y(\sigma)$.

(iii) $\Rightarrow$ (i). Let $\sigma \in \tilde{\boldsymbol{C}}(X)$ such that $M_X(\sigma) = M_Y(\sigma)$. Then $\mu(X) = M_X(\sigma) = M_Y(\sigma) \leq \mu(Y) \leq \mu(X)$, and hence $\mu(X) = \mu(Y)$.

(i) $\Rightarrow$ (iv). Let $\sigma \in \tilde{\boldsymbol{C}}(Y)$. As in the proof of (i) $\Rightarrow$ (ii), equality holds throughout in (4.3). It follows that $\sigma \in \tilde{\boldsymbol{C}}(X)$ and $M_X(\sigma) = M_Y(\sigma)$. Since $X \geq Y$, we have $X(\sigma) \geq Y(\sigma)$. If $X(\sigma) \neq Y(\sigma)$, then it will follow, after taking the sum of the entries in $X(\sigma), Y(\sigma)$, that $M_X(\sigma) > M_Y(\sigma)$, which is a contradiction. Thus $X(\sigma) = Y(\sigma)$.

It is easy to see that (ii) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (v), and (v) $\Rightarrow$ (i). That completes the proof. □

THEOREM 4.6. *Let $X_1, \ldots, X_m$ be $n \times n$ matrices and let $X = \sum_{\oplus} X_i$. Then*

$$\mu(X) \geq \sum\nolimits_{\oplus} \mu(X_i). \tag{4.4}$$

Furthermore, equality holds in (4.4) *if and only if one of the following conditions is satisfied.*

(i) $\mu(X) = -\infty$.

(ii) *$\mu(X)$ is finite and there exists $\sigma \in \tilde{\boldsymbol{C}}(X)$ and $k \in \{1, 2, \ldots, m\}$ such that $X_k(\sigma) \geq X_i(\sigma), i = 1, 2, \ldots, m$.*

Proof. If $\mu(X) = -\infty$, then $\mu(X_i) = -\infty, i = 1, 2, \ldots, m$ and both sides in (4.4) are $-\infty$. So we assume that $\mu(X)$ is finite. Since $X \geq X_i, i = 1, 2, \ldots, m$, by Lemma 4.4, we have $\mu(X) \geq \mu(X_i), i = 1, 2, \ldots, m$ and hence (4.4) holds.

If equality holds in (4.4) then there exists $k \in \{1, 2, \ldots, m\}$ such that $\mu(X) = \mu(X_k)$. Thus $\mu(X_k)$ is finite. By Lemma 4.5 (see (i) $\Rightarrow$ (v)), there exists $\sigma \in \tilde{\boldsymbol{C}}(X)$ such that $X(\sigma) = X_k(\sigma)$. It follows that $X_k(\sigma) \geq X_i(\sigma), i = 1, 2, \ldots, m$.

Conversely, if (ii) holds, then $X(\sigma) = X_k(\sigma)$. Thus $\mu(X_k)$ is finite. Set $Y = X_k$ and use implication (iii) $\Rightarrow$ (i) of Lemma 4.5 to conclude that equality holds in (4.4). □

A square matrix D is a diagonal matrix over the max algebra if $d_{ij} = -\infty$ for all $i \neq j$. A well-known result due to Cohen [9] (see also [20, p. 364]) asserts that the Perron root of a nonnegative matrix B is a convex function of the diagonal entries of B. In this context the next result is somewhat surprising since it says that $\mu(A)$, considered as a function of the diagonal entries of A, is linear over the max algebra.

THEOREM 4.7. *Let X be an $n \times n$ matrix over $\boldsymbol{M}$ and let $D_1, \ldots, D_m$ be $n \times n$ diagonal matrices over the max algebra. Then*

$$\mu\left(X \oplus \sum\nolimits_{\oplus} D_j\right) = \sum\nolimits_{\oplus} \mu(X \oplus D_j). \tag{4.5}$$

Proof. Let $X_j = X \oplus D_j, j = 1, 2, \ldots, m$. Then $X \oplus \sum_{\oplus} D_j = \sum_{\oplus} X_j$. If $\mu(X \oplus \sum_{\oplus} D_j) = -\infty$, then (4.5) is true by Theorem 4.6. So let $\mu(X \oplus \sum_{\oplus} D_j)$ be finite. If there exists $\sigma \in \tilde{\boldsymbol{C}}(X \oplus \sum_{\oplus} D_j)$ of length more than one, then $\sigma \in \tilde{\boldsymbol{C}}(X \oplus D_j)$ $j = 1, 2, \ldots, m$ and (4.5) is proved, in view of (ii) $\Rightarrow$ (i) of Lemma 4.5. So suppose that every circuit in $\tilde{\boldsymbol{C}}(X \oplus \sum_{\oplus} D_j)$ is of length one, and let σ be one such. Clearly, there exists $k \in \{1, 2, \ldots, m\}$ such that $D_k(\sigma) \geq D_j(\sigma)$, and hence $X_k(\sigma) \geq X_i(\sigma), i = 1, 2, \ldots, m$. Thus (ii), Theorem 4.6 is satisfied, and (4.5) holds. $\square$

Let $C \neq -\infty$ be an $n \times n$ matrix over $\boldsymbol{M}$ and

$$\Omega(C) = \left\{(i, j) : c_{ij} = \max_{k,l} c_{kl}\right\}.$$

Construct a $(0, 1)$ matrix $\hat{C} = [\hat{c}_{ij}]$ by setting $\hat{c}_{ij} = 1$ if $(i, j) \in \Omega(C)$ and $\hat{c}_{ij} = 0$ otherwise. Let $\gamma = \sum_{s=1}^{n} \sum_{t=1}^{n} \hat{c}_{st}$, and for $i, j = 1, 2, \ldots, n$, let

$$\alpha_i(C) = \frac{1}{\gamma} \sum_{t=1}^{n} \hat{c}_{it} \quad \text{and} \quad \beta_j(C) = \frac{1}{\gamma} \sum_{s=1}^{n} \hat{c}_{sj}.$$

With this notation, we have the following result, which is the max algebra analog of [2, Thm. 3].

LEMMA 4.8. *Let A be an $n \times n$ matrix over $\boldsymbol{M}$, with $A \neq -\infty$, let u, v, w, z be vectors over $\boldsymbol{M}$ with w and z finite, and let $C = [a_{ij} \otimes z_i \otimes w_j]$. Then*

$$v^T \otimes A \otimes u - z^T \otimes A \otimes w \geq \sum_{i=1}^{n} \alpha_i(C)(v_i - z_i) + \sum_{j=1}^{n} \beta_j(C)(u_j - w_j).$$

Proof. For any i, j, we have

$$a_{ij} \otimes v_i \otimes u_j - a_{ij} \otimes z_i \otimes w_j = v_i - z_i + u_j - w_j. \tag{4.6}$$

If $(i, j) \in \Omega(C)$, then $a_{ij} \otimes z_i \otimes w_j = z^T \otimes A \otimes w$. Apply (4.6) to each $(i, j) \in \Omega(C)$ and add the resulting equations to get

$$\sum_{(i,j) \in \Omega(C)} a_{ij} \otimes v_i \otimes u_j - \gamma(z^T \otimes A \otimes w) = \sum_{(i,j) \in \Omega(C)} (v_i - z_i) + \sum_{(i,j) \in \Omega(C)} (u_j - w_j). \tag{4.7}$$

Now

$$\sum_{(i,j) \in \Omega(C)} (v_i - z_i) = \sum_{i=1}^{n} \left\{(v_i - z_i) \sum_{j=1}^{n} \hat{c}_{ij}\right\} = \gamma \sum_{i=1}^{n} \alpha_i(C)(v_i - z_i),$$

and similarly

$$\sum_{(i,j)\in\Omega(C)} (u_j - w_j) = \gamma \sum_{j=1}^{n} \beta_j(C)(u_j - w_j).$$

Since $\sum_{(i,j)\in\Omega(C)} a_{ij} \otimes v_i \otimes u_j \leq \gamma(v^T \otimes A \otimes u)$ the result follows from (4.7) after a trivial simplification. $\square$

Let B be an $n \times n$ nonnegative, irreducible matrix. Then it is known, see [16], that

$$f^T B g \geq \rho(B) f^T g,$$

where f and g are right and left Perron eigenvectors of B, respectively. We now obtain a max algebra analog of this result. In the special case of an irreducible matrix A with $G(A)$ having a unique critical circuit, a proof based on Lemma 4.8 is contained in [3]. For the more general result, we give an alternative proof that was suggested by an anonymous referee.

THEOREM 4.9. *Let A be an $n \times n$ matrix over $\boldsymbol{M}$ and let x and y be right and left eigenvectors, respectively, of A corresponding to the eigenvalue $\mu(A)$. Let u, v be n-vectors over $\boldsymbol{M}$ such that $u_i \otimes v_i = x_i \otimes y_i$ for $i = 1, 2, \ldots, n$. Then $v^T \otimes A \otimes u \geq \mu(A) \otimes y^T \otimes x$. In particular, $x^T \otimes A \otimes y \geq \mu(A) \otimes y^T \otimes x$.*

Proof. The result is trivial if $\mu(A) = -\infty$. Assume then that $\mu(A)$ is finite, and so there is a critical circuit in $G(A)$. Let $F = \{i\colon x_i \text{ is finite}\}$ and let H be the digraph with vertex set F and edge set $E = \{(i,j)\colon i,j \in F \text{ and } a_{ij} + x_j - x_i = \mu(A)\}$. Thus, from the right eigenequation,

$$\max_j (a_{ij} + x_j) = \mu(A) + x_i,$$

every vertex in H has outdegree at least one in H. Furthermore, for each $i \in F$, there is a path from i to a circuit τ_i in H, which must be a critical circuit in $G(A)$. The left eigenequation gives

$$a_{ij} + y_i \leq \mu(A) + y_j,$$

for each $(i,j) \in E$. Hence $x_i + y_i \leq x_j + y_j$ for each $(i,j) \in E$. Thus if τ is a circuit of length $|\tau|$ in H, there is a number k_τ such that $x_i + y_i = k_\tau$ for all vertices i lying along the circuit τ. Also if $i \in F$, then $x_i + y_i \leq k_{\tau_i} \leq \max_{\tau\in\Gamma} k_\tau$, where Γ denotes the set of all circuits in H, thus $\Gamma \subseteq \tilde{\boldsymbol{C}}(A)$. We have

$$\begin{aligned} v^T \otimes A \otimes u &= \max_{i,j}(v_i + a_{ij} + u_j) = \max_{i,j}(x_i + y_i - u_i + a_{ij} + u_j) \\ &\geq \max_{\tau\in\Gamma}\{\max_{(i,j)\in\tau}(x_i + y_i + a_{ij} - u_i + u_j)\} \\ &\geq \max_{\tau\in\Gamma}\left\{\frac{1}{|\tau|}\sum_{(i,j)\in\tau}(x_i + y_i + a_{ij} - u_i + u_j)\right\} \\ &= \max_{\tau\in\Gamma}\left\{\frac{1}{|\tau|}\sum_{(i,j)\in\tau}(x_i + y_i + a_{ij})\right\} = \mu(A) + \max_{\tau\in\Gamma} k_\tau \\ &= \mu(A) + \max_{i\in F}(x_i + y_i) = \mu(A) \otimes y^T \otimes x. \end{aligned}$$

The second inequality in the theorem follows by setting $v = x, u = y$. $\square$

Acknowledgments. We thank Dr. Stéphane Gaubert of INRIA, France, for sending us a copy of his thesis [18]. We also thank anonymous referees for pointing out reference [8], and for their constructive comments that led to improvements, especially in Theorem 4.9.

REFERENCES

[1] F. L. BACCELLI, G. COHEN, G. J. OLSDER, AND J.-P. QUADRAT, *Synchronization and Linearity: An Algebra for Discrete Event Systems*, Wiley, Chichester, 1992.

[2] R. B. BAPAT, *Applications of an inequality in information theory to matrices*, Linear Algebra Appl., 78 (1986), pp. 107–117.

[3] R. B. BAPAT, D. P. STANFORD, AND P. VAN DEN DRIESSCHE, *The eigenproblem in max algebra*, DMS-631-IR, University of Victoria, British Columbia, 1993.

[4] A. BERMAN AND R. J. PLEMMONS, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.

[5] G. BIRKHOFF AND R. S. VARGA, *Reactor criticality and nonnegative matrices*, SIAM J., 6 (1958), pp. 354–377.

[6] J. G. BRAKER AND G. J. OLSDER, *The power algorithm in max algebra*, Linear Algebra Appl., 182 (1993), pp. 67–89.

[7] B. CARRÉ, *Graphs and Networks*, Clarendon Press, Oxford, 1979.

[8] W. CHEN, X. QI, AND S. DENG, *The eigen-problem and period analysis of the discrete-event system*, Systems Sci. Math. Sci., 3 (1990), pp. 243–260.

[9] J. E. COHEN, *Convexity of the dominant eigenvalue of an essentially nonnegative matrix*, Proc. Amer. Math. Soc., 81 (1981), pp. 657–658.

[10] G. COHEN, D. DUBOIS, J.-P. QUADRAT, AND M. VIOT, *Analyse du comportment périodique des systèmes de production par la théorie des dioïdes*, INRIA Rep. 191, Le Chesnay, France, 1983.

[11] ———, *A linear-system-theoretic view of discrete-event processes and its use for performance evaluation in manufacturing*, IEEE Trans. Automat. Control, 30 (1985), pp. 210–220.

[12] G. COHEN, P. MOLLER, J.-P. QUADRAT, AND M. VIOT, *Algebraic tools for the performance evaluation of discrete event systems*, Proc. IEEE, 77 (1989), pp. 39–58.

[13] R. A. CUNNINGHAME-GREEN, *Minimax Algebra*, Lecture Notes in Economics and Math. Systems, 166, Springer-Verlag, 1979.

[14] P. I. DUDNIKOV AND S. N. SAMBORSKII, *Endomorphisms of finitely generated free semimodules*, in Idempotent Analysis, V. P. Maslov and S. N. Samborskii, eds., Advances Soviet Math., 13 (1992), pp. 65–85.

[15] G. M. ENGEL AND H. SCHNEIDER, *Diagonal similarity and equivalence for matrices over groups with* 0, Czechoslovak Math. J., 25 (1975), pp. 389–403.

[16] M. FIEDLER, C. R. JOHNSON, T. MARKHAM, AND M. NEUMANN, *A trace inequality for M-matrices and the symmetrizability of a real matrix by a positive diagonal matrix*, Linear Algebra Appl., 71 (1985), pp. 81–94.

[17] S. FRIEDLAND, *Limit eigenvalues of nonnegative matrices*, Linear Algebra Appl., 74 (1986), pp. 173–178.

[18] S. GAUBERT, *Théorie des Systèmes Linéaires dans des Dioïdes*, Thèse, L'École des Mines de Paris, Paris, 1992.

[19] M. GONDRAN AND M. MINOUX, *Graphs and Algorithms*, John Wiley & Sons Ltd., 1984.

[20] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[21] C. R. JOHNSON, *Combinatorial matrix analysis: an overview*, Linear Algebra Appl., 107 (1988), pp. 3–15.

[22] S. KARLIN AND F. OST, *Some monotonicity properties of Schur powers of matrices and related inequalities*, Linear Algebra Appl., 68 (1985), pp. 47–65.

[23] J. S. MAYBEE, D. D. OLESKY, AND P. VAN DEN DRIESSCHE, *Partly zero eigenvectors*, Linear Multilinear Algebra, 28 (1990), pp. 83–92.

[24] G. J. OLSDER AND C. ROOS, *Cramer and Cayley-Hamilton in max algebra*, Linear Algebra Appl., 101 (1988), pp. 87–108.

[25] R. S. VARGA, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1962.