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OPTIMAL BLOCK DESIGNS WITH MINIMAL AND NEARLY MINIMAL NUMBER OF UNITS

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Abstract. The study of optimal connected block designs when the number of experimental units is minimal or nearly minimal is of fairly recent origin. In this communication, several new optimality results in different classes of connected block designs are obtained for this situation.

Key words and phrases: A-, D-, E-, MV- and ψ_f-optimality, bipartite graph, spanning tree, control-test comparison.

1. Introduction

Suppose it is desired to investigate the effects of $v\,(> 2)$ treatments using a block design d, having b blocks. It is well known that under a standard fixed effects model, all treatment contrasts are estimable using d (i.e., d is *connected*) *only if*

$$n \geq b + v - 1 = n_0, \quad \text{say,} \tag{1.1}$$

where n is the number of experimental units in d.

In the recent past, several papers have appeared in the literature which deal with the optimality of connected block designs when the number of experimental units is small, typically, n_0 or $(n_0 + 1)$ (see e.g., Mukerjee, Chatterjee and Sen (1986), Krafft (1990), Mukerjee and Sinha (1990), Bapat and Dey (1991), Birkes and Dodge (1991) and Mandal, Shah and Sinha (1991)). With such a small number of experimental units, the usual symmetry arguments for arriving at optimality results no longer work and special techniques are needed to identify optimal designs.

The purpose of this communication is to present additional optimality results in different classes of connected block designs with n_0 or $n_1 = (n_0 + 1)$ experimental units. For $i = 0, 1$, let $\mathcal{D}_i(v, b, k)$ denote the class of all connected block designs with v treatments, b blocks and constant block size k, *satisfying*

$$bk = b + v + i - 1. \tag{1.2}$$

Also, let $\mathcal{D}(v, b, n)$ denote the class of all connected block designs with v treatments, b blocks and n experimental units (block sizes being arbitrary).

In Section 2, we identify D- and E-optimal designs in $\mathcal{D}(v,b,n_1)$. It turns out that the design that is A- and MV-optimal in $\mathcal{D}(v,b,n_1)$ (see Birkes and Dodge (1991)) is also D- and E-optimal in the same class. In Section 3, MV-optimal designs in $\mathcal{D}_1(v,b,k)$ are identified. Bapat and Dey (1991) and Mandal *et al.* (1991) have identified a design that is A-, D- and E-optimal in $\mathcal{D}_0(v,b,k)$. We show in Section 4 that the same design is MV-optimal in $\mathcal{D}_0(v,b,k)$ and is also A- and MV-optimal in $\mathcal{D}_0(v,b,k)$ for inference regarding elementary treatment contrasts between a control treatment and each of a set of test treatments. We further show in Section 4 that in $\mathcal{D}_0(v,b,k)$ there is no ψ_f-optimal design. Finally, in Section 5, we refer to some open problems. For the definitions of various optimality criteria, one may refer to the monograph of Shah and Sinha (1989).

2. Optimality Results in $\mathcal{D}(v,b,n_1)$

2.1. D-optimality

For obtaining results on D-optimality of block designs in $\mathcal{D}(v,b,n_1)$, we make use of a graph-theoretic formulation of the D-criterion (see e.g., Gaffke (1982) and Bapat and Dey (1991)). We briefly summarize this formulation first. For graph-theoretic terminology, one may refer to, e.g., Harary (1988).

Any block design d with v treatments and b blocks can be described by a bipartite multigraph H_d; the treatment labels $1,2,\ldots,v$ and the block labels $\beta_1,\beta_2,\ldots,\beta_b$, say, are the vertices of H_d. A pair of vertices (i,β_j) is joined by n_{dij} parallel edges, where n_{dij} is the number of times the ith treatment appears in the jth block of d. The design d is connected if and only if H_d is connected in the graph-theoretic sense. A (simple) graph T is called a *tree* if it is connected and has no cycles. For a multigraph G, a subgraph of G is called a *spanning tree* if it is a tree and has the same number of vertices as G. For a connected multigraph G, the number of spanning trees in G is called the complexity of G and is denoted by $c(G)$.

It is well known (see Chakrabarti (1963)) that for a connected block design d, all cofactors of C_d (the "C-matrix" of d) are equal and positive and

$$\prod_{i=1}^{v-1} \lambda_{di} = vCo(C_d), \tag{2.1}$$

where $Co(C_d)$ is the cofactor of an element of C_d and $0=\lambda_{do}<\lambda_{d1}\le\lambda_{d2}\le\cdots\le\lambda_{d,v-1}$ are the eigenvalues of C_d. If H_d is the bipartite graph associated with a connected block design d, then it is easy to see that (see e.g., Bapat and Dey (1991))

$$c(H_d) = Co(C_d)\prod_{j=1}^{b} k_{dj}, \tag{2.2}$$

where $k_{d1}, k_{d2}, \ldots, k_{db}$ are the block sizes of d. Hence from (2.1) and (2.2), we have

$$\prod_{i=1}^{v-1} \lambda_{di} = vc(H_d)/\{\prod_{j=1}^{b} k_{dj}\}. \tag{2.3}$$

Thus, a design is D-optimal over a class of competing designs if it maximizes the r.h.s. of (2.3). Recall that a design is D-optimal over a certain class of competing designs if it maximizes $\prod_{i=1}^{v-1} \lambda_{di}$ over the class.

Now, observe that for any design $d \in \mathcal{D}(v, b, n_0), H_d$ itself is a tree and hence has precisely one spanning tree. A design in $\mathcal{D}(v, b, n_1)$ has just one additional experimental unit over a design in $\mathcal{D}(v, b, n_0)$. This means that the graph associated with any design in $\mathcal{D}(v, b, n_1)$ has precisely one extra edge over the number of edges in the graph of a design in $\mathcal{D}(v, b, n_0)$. The consequence of adding one more edge to a tree is that in the new graph, there is precisely one cycle, unless the added edge results in a multiple edge. Note that for $d \in \mathcal{D}(v, b, n_1), H_d$ will have a multiple edge if and only if the design d is non binary. But, if there is a multiple edge in H_d for $d \in \mathcal{D}(v, b, n_1)$, the number of spanning trees is just *two*. If H_d has a cycle, then since H_d is bipartite, the length of the cycle is at least four and equals the number of spanning trees. Further, since H_d is bipartite, the length of the cycle is even and cannot exceed 2 min (b, v). Hence we have

Lemma 2.1. *For any $d \in \mathcal{D}(v, b, n_1), c(H_d) = 2r$, for some $r \in [1, \min(b, v)]$.*

We next have the following

Lemma 2.2. *Let $1 \leq x_1 \leq x_2 \leq \cdots \leq x_t$ be integers satisfying the following conditions:*
(i) *at least u of the x_i's are each greater than or equal to 2,*
(ii) $\sum_{i=1}^{t} x_i = s$.
Then $\prod_{i=1}^{t} x_i$ is minimized when $x_1 = x_2 = \cdots = x_{t-u} = 1, x_{t-u+1} = x_{t-u+2} = \cdots = x_{t-1} = 2$ and $x_t = (s - t - u + 2)$.

Proof. Observe that for two integers $p, q, 2 \leq p \leq q, p+q = c$ (a given constant), $pq \geq (p-1)(q+1)$. Using this fact repeatedly, one gets the result.

Now, let $d \in \mathcal{D}(v, b, n_1)$ be arbitrary. If H_d has a multiple edge, then $r = 1$, and in that case, in view of Lemma 2.2, the maximum of the r.h.s. of (2.3) is $2v/(v+1)$. If H_d does not have a multiple edge, it must have a cycle of length $2r$ for some $r \in [2, \min(b, v)]$. In such a case, at least r of the block sizes $\{k_{dj}\}$ must be at least two. Hence, using Lemma 2.2, the maximum of the r.h.s. of (2.3), for given r, is

$$f(r) = 2vr/\{2^{r-1}(v - r + 2)\}, \quad 2 \leq r \leq \min(b, v). \tag{2.4}$$

Thus, we have

Lemma 2.3. *For any $d \in \mathcal{D}(v, b, n_1)$, if $c(H_d) = 2r$, for some $r \in [1, \min(b, v)]$, then the maximum of the r.h.s. of (2.3), for given r, is $f(r), 1 \le r \le \min(b, v)$, and is attained when the design d is such that*
(i) *$(b - r)$ of the blocks of d are each of size unity,*
(ii) *$(r - 1)$ blocks of d are each of size two and*
(iii) *one block is of size $(v - r + 2)$.*

The next step is to maximize $f(r)$ for variation in r. We prove

Lemma 2.4. *$f(2) = \max_{1 \le r \le v} f(r)$, except when $r = 3 = v$.*

Proof. Observe that $f(2) = 2 > f(1) = 2v/(v + 1)$. Also, $f(2) \ge f(3)$ for all $v \ge 4$. Further,

$$
\begin{aligned}
f(2) - f(r) &= \frac{2^{r-1}(v - r + 2) - vr}{2^{r-2}(v - r + 2)} \\
&\ge \frac{2^{r-1}(v - r + 2) - vr - (2^r - r^2)}{2^{r-2}(v - r + 2)}, \quad \text{for } r \ge 4 \\
&= \frac{(2^{r-1} - r)(v - r)}{2^{r-2}(v - r + 2)} \ge 0.
\end{aligned}
$$

Hence the result is proved.

Consider now the design $d^* \in \mathcal{D}(v, b, n_1)$, given by

$$
d^* \equiv \begin{matrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 2 & 2 & & & & \\ 3 & & & & & \\ \vdots & & & & & \\ v & & & & & \end{matrix} \tag{2.5}
$$

where columns represent blocks and there are $(b - 2)$ blocks each of size one. From (2.3), Lemmas 2.3 and 2.4, the following result is easily obtained.

Theorem 2.1. *The design d^* in (2.5) is D-optimal over $\mathcal{D}(v, b, n_1)$ for all $v \ge 4$ and for $v = 3, b = 2$.*

Remark 2.1. Since the dual of the design in (2.5) also has the same structure, it is also D-optimal for estimating the block contrasts, when $b \ge 4$.

Remark 2.2. Note that any design d with one block containing all the v treatments, one block having any two distinct treatments out of $1, 2, \ldots, v$ and $(b - 2)$ blocks, each of size unity, containing any one of the v treatments is D-optimal over $\mathcal{D}(v, b, n_1)$. Further, for $v = 3, b \ge 3$, the design obtained by taking the union of the blocks of a balanced incomplete block design with parameters $v = 3 = b, r = 2 = k, \lambda = 1$ and $(b-3)$ blocks, each of size unity and containing any one of the three treatments, is universally optimal over $\mathcal{D}(3, b, b+3)$, and hence is also D-optimal.

2.2. E-optimality

Birkes and Dodge (1991), among other things, identified A- and MV-optimal designs in $\mathcal{D}(v, b, n_1)$ for all $v \geq 4$ and for $v = 3, b = 2$; in fact, they show that the design d^* of (2.5) is both A- and MV-optimal in $\mathcal{D}(v, b, n_1)$. We show that d^* is also E-optimal over $\mathcal{D}(v, b, n_1)$ for all $v \geq 4$ and for $v = 3, b = 2$.

It is easy to see that the positive eigenvalues of C_{d^*} are

$$\lambda_{d^*1} = 1 \ \text{ with multiplicity } \ v - 2$$

and

$$\lambda_{d^*2} = 2 \ \text{ with multiplicity } \ 1.$$

Also, it is well known that for any connected block design d,

$$\text{Var}(\hat{t}_i - \hat{t}_j)_d \leq 2\sigma^2/\lambda_{d1} \ \text{ for all } \ i \neq j, \ \ i, j = 1, 2, \ldots, v, \tag{2.6}$$

where $\text{Var}(\hat{t}_i - \hat{t}_j)_d$ represents the variance of the best linear unbiased estimator (BLUE) of the elementary treatment contrast $t_i - t_j$, using d, σ^2 is the variance of an observation and λ_{d1} is the smallest positive eigenvalue of C_d. This implies that

$$\max_{1 \leq i < j \leq v} \text{Var}(\hat{t}_i - \hat{t}_j)_d \leq 2\sigma^2/\lambda_{d1} \Rightarrow \lambda_{d1} \leq \frac{2\sigma^2}{\max_{1 \leq i < j \leq v} \text{Var}(\hat{t}_i - \hat{t}_j)_d}. \tag{2.7}$$

But as, shown by Birkes and Dodge (1991), for any $d \in \mathcal{D}(v, b, n_1)$, with $v \geq 4$ or $v = 3$ and $b = 2$,

$$\max_{1 \leq i < j \leq v} \text{Var}(\hat{t}_i - \hat{t}_j)_d \geq 2\sigma^2. \tag{2.8}$$

Combining (2.7) and (2.8), we have, for any $d \in \mathcal{D}(v, b, n_1)$, with $v \geq 4$ or $v = 3, b = 2$

$$\lambda_{d1} \leq \frac{2\sigma^2}{2\sigma^2} = 1 = \lambda_{d^*1}$$

which leads us to the following

Theorem 2.2. *The design d^*, given by* (2.5) *is E-optimal over $\mathcal{D}(v, b, n_1)$ for all $v \geq 4$ and $v = 3, b = 2$.*

Remark 2.3. For $v = 3, b \geq 3$, the design of Remark 2.1 is universally optimal over $\mathcal{D}(v, b, n_1)$ and hence is also E-optimal.

Remark 2.4. The results of this Section (Theorems 2.1 and 2.2) are better viewed as optimality results on main-effect plans for two-factor experiments, rather than on block designs, as the design that turns out to be optimal has little appeal as a block design. See Mukerjee and Sinha (1990) for a related result.

3. *MV*-Optimal Designs in $\mathcal{D}_1(v,b,k)$

In this Section, MV-optimal designs in $\mathcal{D}_1(v,b,k)$ are identified, where $k \geq 3$. Recall that a design is MV-optimal over a class of competing designs if it minimizes the maximum variance of the BLUE of an elementary treatment contrast.

Let $d \in \mathcal{D}_1(v,b,k)$ be arbitrary. Then, arguing as in Section 2, one can show that for $d \in \mathcal{D}_1(v,b,k)$, the bipartite graph H_d has $2r$ spanning trees for some $r \in [1,b]$. Note that in $\mathcal{D}_1(v,b,k), b < v$. Towards MV-optimality, we first show that for $b \geq 4$,

$$\max_{1\leq i<j\leq v} \text{Var}(\hat{t}_i - \hat{t}_j)_d \geq 4\sigma^2 \quad \forall\ d \in \mathcal{D}_1(v,b,k). \tag{3.1}$$

For proving (3.1), we first prove the following result.

Lemma 3.1. *For any $d \in \mathcal{D}_1(v,b,k)$, the variance of the BLUE of a treatment contrast $t_\alpha - t_\beta$ is at least $4\sigma^2$ if*

$$s \leq 4(t-2), \tag{3.2}$$

where $2t$ is the number of observations involved in an unbiased estimator U of $t_\alpha - t_\beta$ and s is the number of cells common between U and the cycle (or, multiple edge) of H_d, the bipartite graph associated with d.

Proof. For any $d \in \mathcal{D}_1(v,b,k)$, there is a unique error function (apart from a scalar multiple), say Z. (Recall that a linear function of observations with expectation zero is called an error function). Clearly, Z must be a linear function of $2r$ observations belonging to the cycle (or, multiple edge) of H_d and the coefficients in this linear function are ± 1. Let U be a linear unbiased estimator of $t_\alpha - t_\beta$, based on $2t$ observations. If U and Z have s observations in common, the covariance between U and Z, $\text{Cov}(U,Z) = \pm s\sigma^2$, where σ^2, as before, is the variance of an observation. Since Z is the only error function, it is easy to see that the BLUE of $t_\alpha - t_\beta$ is

$$\hat{t}_\alpha - \hat{t}_\beta = U - \{\text{Cov}(U,Z)/\text{Var}(Z)\}Z \tag{3.3}$$

and

$$\text{Var}(\hat{t}_\alpha - \hat{t}_\beta) = (2t - s^2/2r)\sigma^2. \tag{3.4}$$

We can assume, without loss of generality that, $s \le r$, for, if $s > r$, we can replace U by $U' = U - Z$, and for U', the corresponding value of s is $s' = 2r - s < r$.

Since, by our assumption, $s \le r$, the condition (3.2) ensures that $\mathrm{Var}(\hat{t}_\alpha - \hat{t}_\beta) \ge 4\sigma^2$.

In subsequent discussions, it is assumed that U is so chosen that $s \le r$. We next prove

Lemma 3.2. *For any $d \in \mathcal{D}_1(v, b, k), b \ge 4$, one can always identify at least one pair of treatments, α and β, such that* (3.2) *holds.*

Proof. We consider three separate cases, viz., (i) $r = b$, (ii) $r = b - 1$ and (iii) $1 \le r \le b - 2$. Here $2r$ is the number of spanning trees in the graph H_d.

Case (i) $r = b$.

Here we have a cycle of length $2b$ in H_d. Let the cells in the cycle be in standard order $(1, B_1), (2, B_1), (2, B_2), (3, B_2), \ldots, (b, B_b), (1, B_b)$, where B_j $(j = 1, 2, \ldots, b)$ is the label of the vertex corresponding to the jth block in H_d and $1, 2, \ldots, v$ are the treatment labels. Note that the error function Z is a linear combination (with coefficients ± 1) of the observations arising from the above cells. Let the remaining $(v - b)$ treatments, each of which is singly replicated in d, appear in the b blocks as follows: $(b + 1), (b + 2), \ldots, (b + k - 2)$ in block $B_1, b + k - 1, b + k, \ldots, b + 2k - 4$ in B_2, and so on. It is now easy to see that for $\alpha = b + 1$ and $\beta = b + 2k - 3$, $t = 3$ and $s = 4$ and thus (3.2) holds.

Case (ii) $r = b - 1$.

Let the cycle of length $2(b - 1)$ in H_d involve the first $(b - 1)$ blocks. The last block contains precisely one treatment from the union of the $(b - 1)$ blocks of the cycle. Let θ be this treatment. Clearly, θ cannot occur in every block of the cycle. Therefore, suppose B_1 does not contain θ. If α is a treatment from B_1, which is not a part of the cycle and $\beta \ne \theta$ is a treatment from the last block, then $t \ge 3$ and $s \le 2(t - 1)$ and thus (3.2) holds.

Case (iii) $1 \le r \le b - 2$.

Let the blocks of the cycle of length $2r$ be $B_1, B_2, \ldots, B_r$. If $r = 1$, i.e., if there is a multiple edge, let B_1 be the block involved in the multiple edge. Further, let S_1 (respectively, S_2) be the set of treatments in (respectively, not in) $B_1, B_2, \ldots, B_r$. The number of experimental units in $B_1 \cup B_2 \cup \cdots \cup B_r$ is rk. This leaves $(b - r)k$ experimental units. Since $|S_1| = r(k - 1)$, we have $|S_2| = v - r(k - 1) = b(k - 1) - r(k - 1) = (b - r)(k - 1)$. Each of the treatments in S_2 must have at least one replication. Further, $B_{r+1} \cup B_{r+2} \cup \cdots \cup B_b$ must contain at least one treatment from S_1. Thus, at most $(b - r - 1)$ experimental units are available for repeating a treatment from S_2 in these units. Hence, at least $(b - r)(k - 1) - (b - r - 1) = (b - r)(k - 2) + 1$ treatments belonging to

S_2 must have only single replication. Since $k \geq 3, (b-r)(k-2)+1 \geq k$ for $1 \leq r \leq b-2$. Thus, there must exist two blocks, say B_{r+1} and B_{r+2} involving treatments $\alpha, \beta \in S_2, \alpha \in B_{r+1}, \beta \in B_{r+2}$, which have only a single replication each.

We now have two possibilities, viz., (i) there exists a chain (path) from α to β not involving any cell of the cycle, and (ii) every chain from α to β involves cells in the cycle. In the first case, $s = 0, t \geq 2$ and in the second case, $t \geq 3, s \leq 2(t-1)$. Thus, the condition (3.2) is met and the lemma is proved.

Lemmas 3.1 and 3.2 leads us to

Theorem 3.1. *For any* $d \in \mathcal{D}_1(v,b,k), b \geq 4$,

$$\max_{1 \leq i < j \leq v} \mathrm{Var}(\hat{t}_i - \hat{t}_j)_d \geq 4\sigma^2.$$

Now, let $d_1 \in \mathcal{D}_1(v,b,k)$ be the following design with columns as blocks:

$$d_1 \equiv \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 2 & k+1 & 2k & & (b-2)k-b+4 & (b-1)k-b+2 \\ 3 & k+2 & 2k+1 & & (b-2)k-b+5 & (b-1)k-b+3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ k & 2k-1 & 3k-2 & & (b-1)k-b+2 & v \end{bmatrix}. \quad (3.5)$$

Then, we have

Lemma 3.3. *For* $d_1 \in \mathcal{D}_1(v,b,k)$,

$$\mathrm{Var}(\hat{t}_i - \hat{t}_j) \leq 4\sigma^2 \quad \forall\, i, j, 1 \leq i < j \leq v.$$

Proof. Since treatment 1 appears in all the blocks, the variance of the BLUE of any elementary treatment contrast cannot exceed $4\sigma^2$.

Combining Theorem 3.1 and Lemma 3.3, we have

Theorem 3.2. *The design* d_1*, given by* (3.5)*, is MV-optimal over* $\mathcal{D}_1(v,b,k)$ *whenever* $b \geq 4$.

We now discuss the cases $b = 2$ and 3. For $b = 2$, there are only two distinct designs, d_{21} and d_{22}, given below and every other design in $\mathcal{D}_1(v,2,k)$ is isomorphic to one of these two designs:

$$d_{21} \equiv \begin{matrix} (1,2,3,\ldots,k-1,k) \\ (k,k+1,k+2,\ldots,v,1) \end{matrix} \;;\; d_{22} \equiv \begin{matrix} (1,1,2,3,\ldots,k-1) \\ (\theta,k,k+1,k+2,\ldots,v). \end{matrix}$$

Here parentheses include blocks and in d_{22}, θ is any treatment from the first block. Easy computations show that d_{21} is strictly MV-better than d_{22}.

For $b = 3$, there are three distinct designs, d_{31}, d_{32}, d_{33}, given below:

$$d_{31} \equiv \begin{array}{c} (1,2,\ldots,k-1,k) \\ (k,k+1,\ldots,2k-2,2k-1) \\ (2k-1,2k,\ldots,v,1) \end{array} \quad d_{32} \equiv \begin{array}{c} (1,2,\ldots,k-1,k) \\ (k,k+1,\ldots,2k-2,1) \\ (\theta,2k-1,\ldots,3k-2,v) \end{array} ;$$

$$d_{33} \equiv \begin{array}{c} (1,1,2,3,\ldots,k-1) \\ (\theta_1,k,k+1,k+2,\ldots,2k-2) \\ (\theta_2,2k-1,2k,2k+2,\ldots,v). \end{array}$$

Here rows are blocks, θ in d_{32} is a treatment from the union of the first two blocks and in d_{33}, θ_1 is a treatment from the first block and θ_2 is a treatment from the union of the first two blocks. One can easily see that d_{31} is MV-optimal in $\mathcal{D}_1(v,3,k)$.

4. Optimality Results in $\mathcal{D}_0(v,b,k)$

Consider now the class $\mathcal{D}_0(v,b,k)$ and recall that the parameters of any design in $\mathcal{D}_0(v,b,k)$ satisfy $b(k-1) = v-1$. Let d_0 be a design constructed as follows: allocate the $v-1 = b(k-1)$ treatments to the b blocks at the rate of $(k-1)$ treatments per block such that these blocks are mutually disjoint. Add the vth treatment to each of the blocks to get the design d_0. It is known (see Bapat and Dey (1991) and Mandal et al. (1991)) that d_0 is uniquely A- and E-optimal and is also D-optimal (all designs in $\mathcal{D}_0(v,b,k)$ are equivalent as per the D-criterion). We now show that d_0 is (i) MV-optimal, and (ii) A- and MV-optimal for inference regarding control vs test comparisons. We first prove the following result at the suggestion of a referee, which was stated without proof by Mandal et al. (1991).

Lemma 4.1. *The variance of the BLUE of any elementary treatment contrast, using a design $d \in \mathcal{D}_0(v,b,k)$, is an even multiple of σ^2.*

Proof. Since for any design in $\mathcal{D}_0(v,b,k)$, there are no error functions, any linear function of observations is the unique linear unbiased estimator of its expectation. Now, suppose $t_i - t_j$ is an elementary treatment contrast. Then, it is easy to see that the unique linear unbiased estimator of $t_i - t_j$ is necessarily of the form $\sum_{u=1}^{b} \delta_u(y_{ru} - y_{su})$, where $\delta_u = 0$ or 1 and y_{ru}, y_{su} are two observations in the u-th block. The lemma is now obvious.

Note that since $t_i - t_j$ has just one linear unbiased estimator, it is also the 'best' and the term BLUE in the statement of the lemma is used in this sense.

We next prove

Theorem 4.1. *The design d_0 is uniquely MV-optimal (up to isomorphism) in $\mathcal{D}_0(v,b,k)$.*

Proof. It is easy to see that

$$\max_{1\le i<j\le v} \mathrm{Var}(\hat{t}_i - \hat{t}_j)_{d_0} = 4\sigma^2.$$

Let $d\,(\neq d_0)$ be an arbitrary design in $\mathcal{D}_0(v,b,k)$. As shown in Mandal et al. (1991), d has at least one pair of disjoint blocks. Hence

$$\max_{1\le i<j\le v} \mathrm{Var}(\hat{t}_i - \hat{t}_j)_{d} \ge 6\sigma^2 \quad \forall\, d \in \mathcal{D}_0(v,b,k), d \neq d_0.$$

This proves the theorem.

A problem that has received considerable attention in the recent past is that of obtaining A- and MV-optimal designs for inference regarding elementary treatment contrasts between a given treatment (called *control* treatment) and each of a set of other treatments (called *test* treatments). For an excellent review on the developments in this area, see Hedayat, Jacroux and Majumdar (1988). For this problem, we now identify A- and MV-optimal designs in $\mathcal{D}_0(v,b,k)$.

Theorem 4.2. *The design d_0 is A- and MV-optimal in $\mathcal{D}_0(v,b,k)$ for inference regarding elementary contrasts between the control and each of the test treatments when the treatment common to all the blocks of d_0 is treated as control and the other $(v-1)$ treatments are test treatments.*

Proof. By Lemma 4.1, for any $d \in \mathcal{D}_0(v,b,k)$,

$$\mathrm{Var}(\hat{t}_0 - \hat{t}_i)_d \ge 2\sigma^2, \text{ for } \ i = 1,2,\ldots,v-1.$$

Also,

$$\mathrm{Var}(\hat{t}_0 - \hat{t}_i)_{d_0} = 2\sigma^2 \quad \forall\, i = 1,2,\ldots,v-1.$$

Here $(\hat{t}_0 - \hat{t}_i)$ is the BLUE of $t_0 - t_i, t_0$ is the effect of the control treatment and t_i, that of the ith test treatment. The A- and MV-optimality of d_0 for inferring on contrasts $\{t_0 - t_i\}$ now follows.

As mentioned earlier, the design d_0 is known to be A-, D- and E-optimal in $\mathcal{D}_0(v,b,k)$ for inference regarding a complete set of orthonormal treatment contrasts, and, by Theorem 4.1 is also MV-optimal. One may become ambitous and ask whether d_0 is optimal in $\mathcal{D}_0(v,b,k)$ according to a wider class of optimality criteria as well, e.g., one may ask whether d_0 is ψ_f-optimal in $\mathcal{D}_0(v,b,k)$ (For a definition of ψ_f-optimality, we refer the reader to Shah and Sinha (1989, p. 8)). We answer this question in the negative in

Theorem 4.3. *There does not exist a ψ_f-optimal design in $\mathcal{D}_0(v,b,k)$.*

Proof. Recall that a design d^* is ψ_f-optimal for every continuous, nonincreasing convex function f if and only if $\boldsymbol{x}_{d^*}$ is weakly upper majorized by $\boldsymbol{x}_d$ where

$\boldsymbol{x}_{d^*} = (x_{d^*1}, x_{d^*2}, \ldots, x_{d^*,v-1})$ and $\boldsymbol{x}_d = (x_{d1}, x_{d2}, \ldots, x_{d,v-1})$ are the vectors of positive eigenvalues of $C_{d^*}(C_d)$ and d refers to any other competing design in the same class to which d^* belongs (see e.g. Shah and Sinha (1989, pp. 14-15)).

If possible, let $d^* \in \mathcal{D}_0(v,b,k)$ be ψ_f-optimal. Then, this implies that

$$\begin{aligned} &\boldsymbol{x}_{d^*} \prec^w \boldsymbol{x}_d \text{ for any } d\,(\neq d^*) \text{ in } \mathcal{D}_0(v,b,k) \\ \Rightarrow\, &\boldsymbol{x}_{d^*} \prec \boldsymbol{x}_d, \text{ as tr } (C_d) = \text{tr}(C_{d^*})\ \forall\, d \in \mathcal{D}_0(v,b,k). \end{aligned}$$

Here tr$(\cdot)$ stands for the trace of a square matrix, and, $\prec^w$ and $\prec$ stand for weak upper majorization and majorization respectively. Now, it is known (Marshall and Olkin (1979, p. 79)) that for a pair of vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$,

$$\boldsymbol{a} \prec \boldsymbol{b} \Rightarrow \prod_{i=1}^{n} a_i \geq \prod_{i=1}^{n} b_i$$

with equality only if $\boldsymbol{a}$ is a permutation of $\boldsymbol{b}$.

By our assumption, d^* is ψ_f-optimal $\Rightarrow d^*$ is A- and E-optimal as well. But we know from the results of Mandal *et al.* (1991) that there is a unique A-optimal design (up to isomorphism) in $\mathcal{D}_0(v,b,k)$, namely the design d_0 of Theorem 4.2. Hence $d^* = d_0$. This shows that $\boldsymbol{x}_{d^*}$ cannot be a permutation of $\boldsymbol{x}_d$, whatever be $d\,(\neq d^*)$ in $\mathcal{D}_0(v,b,k)$. Further, since all designs in $\mathcal{D}_0(v,b,k)$ are D-equivalent (cf. Bapat and Dey (1991)), we have

$$\prod_{i=1}^{v-1} x_{di} = \prod_{i=1}^{v-1} x_{d^*i} \quad \forall \quad d \in \mathcal{D}_0(v,b,k).$$

This leads to a contradiction and hence d^* cannot be ψ_f-optimal in $\mathcal{D}_0(v,b,k)$.

5. Concluding Remarks

In closing this paper, we look at the current state of solved and unsolved optimality problems in the three design classes discussed in the earlier sections.

For $\mathcal{D}(v,b,n_1)$, Birkes and Dodge (1991) have proved the A- and MV-optimality of the design d^*. In this paper, we have established the D- and E-optimality of the same design.

For the class $\mathcal{D}_0(v,b,k)$, the design d_0 is known to be A-, D- and E-optimal. In this paper, we have shown that d_0 is also MV-optimal. We have also proved the non existence of a ψ_f-optimal design in $\mathcal{D}_0(v,b,k)$.

In the class $\mathcal{D}_1(v,b,k)$, we have obtained an MV-optimal design. In an unpublished manuscript, Balasubramanian and Dey have established the D-optimality of a design in $\mathcal{D}_1(v,b,k)$ which is different from the design that is MV-optimal in that class. The problems of finding A- and E-optimal designs in $\mathcal{D}_1(v,b,k)$ remain open.

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