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A general composition theorem and its applications to certain partial orderings of distributions

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Abstract

A composition theorem for functions obeying certain positive ordering is proved. The novelty of the present version is that unlike earlier results which assume both components of the composition to be distributions or survival functions, one of the components is allowed to be negative and unbounded. The theorem is applied to yield very simple proof of characterizations for failure rate orderings of distributions given recently by Capéraà (1988). We also use this composition theorem to give a characterization of two distributions with ordered mean residual life functions.

Key words: Likelihood ratio ordering; TP_2 ordering; Failure rate ordering; Mean residual life ordering

1. Introduction and summary

Stochastic ordering of distributions has been an important tool in the theory of reliability and statistical inference in general. One of the earlier definitions of stochastic ordering was given by Lehmann (1955): distribution function F_2 is said to be stochastically larger than F_1 if $F_2(z) \leqslant F_1(z)$ for every z, or equivalently, if $\bar{F}_2(z) \geqslant \bar{F}_1(z)$ for every z, where $\bar{F}_1 = 1 - F_1$ and $\bar{F}_2 = 1 - F_2$ are corresponding *survival functions.* If X_1 and X_2 are random variables with distribution functions F_1 and F_2 respectively, then one of the basic properties of this ordering is that for every nondecreasing function g,

$$E[g(X_2)] \geqslant E[g(X_1)].$$

In some cases a pair of distributions may satisfy a stronger condition called *likelihood ratio* ordering. Suppose distributions F_1 and F_2 possess densities f_1 and f_2 respectively. Then the condition required for

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likelihood ratio ordering is given by

$$\frac{f_2(z)}{f_1(z)} \text{ is nondecreasing in } z. \tag{1.1}$$

Condition (1.1) is related to TP_2 functions. A nonnegative measurable function $h(x, y)$ is said to be TP_2 if

$$\begin{vmatrix} h(x_1, y_1) & h(x_1, y_2) \\ h(x_2, y_1) & h(x_2, y_2) \end{vmatrix} \geqslant 0 \quad \text{for every } x_1 \leqslant x_2 \text{ and } y_1 \leqslant y_2. \tag{1.2}$$

Let $h(i, z)$ denote a probability density function $f_i(z)$ for $i = 1, 2$. Then the monotone likelihood ratio condition is clearly seen to be equivalent to the condition h is TP_2.

Keilson and Sumita (1982) studied orderings which lie between likelihood ratio and stochastic ordering. These are defined by the condition

$$\frac{\bar{F}_2(z)}{\bar{F}_1(z)} \text{ is monotone increasing in } z \tag{1.3}$$

and

$$\frac{F_2(z)}{F_1(z)} \text{ is monotone increasing in } z. \tag{1.4}$$

Writing $F(\theta, z)$ and $\bar{F}(\theta, z)$ for $F_\theta(z)$ and $\bar{F}_\theta(z)$ respectively, for $\theta = 1, 2$, it follows that conditions (1.3) and (1.4) are equivalent to requiring F and $\bar{F}$ to be TP_2 functions. Later we will assume that θ takes values in an open interval contained in R.

Observe that condition (1.3) implies that for every z and $\delta > 0$,

$$\frac{\bar{F}_1(z)}{\bar{F}_1(z+\delta)} \geqslant \frac{\bar{F}_2(z)}{\bar{F}_2(z+\delta)},$$

which in turn implies that for every z,

$$\frac{f_1(z)}{\bar{F}_1(z)} \geqslant \frac{f_2(z)}{\bar{F}_2(z)}. \tag{1.5}$$

Conversely if (1.5) holds then

$$-\frac{\mathrm{d}}{\mathrm{d}z} \ln \bar{F}_1(z) \geqslant -\frac{\mathrm{d}}{\mathrm{d}z} \ln \bar{F}_2(z),$$

so that (1.3) holds. Thus the ordering defined by (1.5) can be termed as *failure rate* (fr) ordering. Condition (1.4) can be given a similar interpretation. It is easy to see that (1.4) is equivalent to

$$\frac{f_1(z)}{F_1(z)} \leqslant \frac{f_2(z)}{F_2(z)}. \tag{1.6}$$

To interpret the ratios in (1.6), imagine that F_1 is the life distribution of a component. Given that the component has failed by time z, then the probability that it survives up to time $z - \delta$ is approximately δ times the ratio appearing on the left side of the inequality (1.6). The corresponding ordering can be termed as *survival rate* (sr) ordering. Keilson and Sumita (1982) show that the TP_2 ordering implies fr as well as sr ordering and it is easy to see that each of these two imply stochastic ordering. In the following we will write $X_2 >_{\text{fr}} X_1$, or $F_2 >_{\text{fr}} F_1$, to denote fr ordering. Similarly, $X_2 >_{\text{st}} X_1$ will denote stochastic ordering, $X_2 >_{\text{sr}} X_1$ will denote sr ordering and $X_2 >_{\text{l}} X_1$ will denote likelihood ratio ordering.

It is easy to verify that partial orderings discussed above are preserved under a common nondecreasing transformation. For example, if g is nondecreasing and $X_2 >_{\mathrm{fr}} X_1$ then $g(X_2) >_{\mathrm{fr}} g(X_1)$.

Two recent articles derive results related to the above partial orderings. Capéraà (1988) derived several characterizations of fr and sr orderings and gave applications to computing asymptotic efficiency of rank tests. One of the features of the characterizations given by Capéraà is that they involve functions which are allowed to assume negative values. This feature widens the scope of applications of these orderings.

Lynch et al. (1987) (LMP) extend the composition result of TP_2 density functions to corresponding results for distribution functions and survival functions. From these generalizations it follows that the fr ordering is preserved under the operation of convolution with a common distribution having the *increasing failure rate* (IFR) property. As a result, if F_i dominates G_i in fr ordering for $i = 1, 2$ and if F_2 and G_1 are IFR, then $F_1 * F_2$ dominates $G_1 * G_2$ in fr ordering. A similar preservation holds for sr ordering.

In this paper we achieve two main goals:

(1) We extend the results of LMP concerning the composition of TP_2 functions. LMP assume both components of the composition are distribution or survival functions. We assume a more general first component; it may be unbounded or even assume negative values.

(2) We obtain the characterizations of Capéraà (1988) from our composition theorem as simple transparent corollaries, much shorter in proof than the lengthy Capéraà proof.

2. A preservation theorem

As noted above we plan to use the TP_2 techniques. However, due to the fact that we will be dealing with functions taking on negative values, many of the familiar results have to be extended to accommodate such functions. Fortunately, the tools needed for extensions to negative valued functions do hold.

It is important to take note of the difference between the statements one generally obtains when one is considering a pair of densities possessing the monotone likelihood ratio property where the pair is taken from a family of densities and the pair of functions we will be considering. In the case we deal with, the ratio is formed by functions, the numerator of which may take negative values while the denominator is assumed to be positive valued. Clearly for this case, the pair does not come from a family of density or distribution functions. What we use mainly is the TP_2 like property given in relation (1.2).

Definition. A pair of measurable real functions, (g_1, g_2), is said to satisfy the DP_2 condition if

(i) g_1 is nonnegative while g_2 may take negative values.

(ii) for every $x_1 \leqslant x_2$,

$$g_1(x_1)g_2(x_2) \geqslant g_1(x_2)g_2(x_1).$$

Here DP_2 denotes the positivity of the second order determinant.

Lemma 2.1. *Suppose that a pair (g_1, g_2) satisfies the DP_2 condition. Let (a, b) and (c, d) be a pair of intervals such that $a \leqslant c$ and $b \leqslant d$. Then*

$$\int_a^b g_1(x)\,\mathrm{d}x \int_c^d g_2(x)\,\mathrm{d}x \geqslant \int_a^b g_2(x)\,\mathrm{d}x \int_c^d g_1(x)\,\mathrm{d}x. \tag{2.1}$$

Proof. First assume that the intervals are disjoint. If for some x, both g_1 and g_2 take the value 0, then the contribution to both sides of (2.1) is 0. As a result we may assume that the set where both functions are simultaneously 0 has been removed. Now the DP_2 condition implies that the ratio $g_2(x)/g_1(x)$ is nondecreasing in x. This monotonicity together with the nonnegativity of g_1 implies that either $g_2(x)$ has the same sign for all x or there exists an $x_- \leqslant x_+$ such that g_2 is negative for $x<x_-$ and positive for $x>x_+$. It also follows that the set $\{x: g_1(x)>0\}$ is an interval containing x_- and x_+.

From these observations the inequality (2.1) is seen to be trivially true if either $g_1(b)$ or $g_1(c)$ is 0. Hence it will be assumed in the following that both of these numbers are positive.

Due to the assumption of DP_2, it follows that

$$g_1(c)g_2(x) \geqslant g_1(x)g_2(c) \quad \text{for every } x \in (c, d),$$

$$g_1(b)g_2(c) \geqslant g_1(c)g_2(b)$$

and

$$g_1(x)g_2(b) \geqslant g_1(b)g_2(x) \quad \text{for every } x \in (a, b).$$

Hence using these inequalities in succession, it follows that

$$\begin{aligned} g_1(b)g_1(c)\int_a^b g_1(x)\,\mathrm{d}x \int_c^d g_2(x)\,\mathrm{d}x &\geqslant g_1(b)g_2(c)\int_a^b g_1(x)\,\mathrm{d}x \int_c^d g_1(x)\,\mathrm{d}x \\ &\geqslant g_1(c)g_2(b)\int_a^b g_1(x)\,\mathrm{d}x \int_c^d g_1(x)\,\mathrm{d}x \\ &\geqslant g_1(c)g_1(b)\int_a^b g_2(x)\,\mathrm{d}x \int_c^d g_1(x)\,\mathrm{d}x. \end{aligned}$$

Since $g_1(b)g_1(c)>0$, the desired inequality is established for the case of disjoint intervals.

To see that it extends to the case of overlapping intervals, suppose $c<b$ so that the inequality to be proved becomes

$$\begin{aligned} &\left(\int_a^c g_1(x)\,\mathrm{d}x + \int_c^b g_1(x)\,\mathrm{d}x\right)\left(\int_c^b g_2(x)\,\mathrm{d}x + \int_b^d g_2(x)\,\mathrm{d}x\right) \\ &\quad \geqslant \left(\int_a^c g_2(x)\,\mathrm{d}x + \int_c^b g_2(x)\,\mathrm{d}x\right)\left(\int_c^b g_1(x)\,\mathrm{d}x + \int_b^d g_1(x)\,\mathrm{d}x\right). \end{aligned}$$

Expanding the two sides and using the result for the disjoint intervals (a, c), (c,b), (b, d) the desired inequality follows. □

Remark. Lemma 2.1 points out that the DP_2 notion leads to what is known as a *local property*. If the pair of functions were positive then the *local property* would be equivalent to having the TP_2 property for the measures generated by the pair of functions.

The main use of Lemma 2.1 will be made in the following Corollary.

Corollary 2.1. *Suppose a pair of functions g_1, g_2 satisfies the DP_2 property, as in Lemma* 2.1. *Let h_i be the function obtained by convolving g_i with a uniform distribution; that is,*

$$h_i(x) = \frac{1}{2\theta}\int_{-\theta}^{\theta} g_i(x-u)\,\mathrm{d}u.$$

Then

(i) *the pair h_1, h_2 preserves the DP_2 property.*
(ii) *if g_i is increasing (decreasing) so is h_i.*

Proof. Suppose $x_2 > x_1$. Then the functions h_i evaluated at these points are integrals of the functions g_i over the intervals which satisfy the conditions in Lemma 2.1. The assertion (i) now follows from Lemma 2.1. The assertion (ii) follows from the fact that the intervals over which integrals are taken are of the same length and that one interval lies to the right of the other. □

The motivation for considering the above result is that we may encounter functions which are discontinuous. Corollary 2.1 says that the *smoothed* version of the given functions will persist in having the DP_2 property so that we can establish the results for the smoothed versions and then extend those to the general functions by letting θ go to 0. For this purpose the following result is useful.

Lemma 2.2. *Suppose g is a nonnegative increasing function, integrable with respect to measure μ, and let*

$$h(x;\theta)=\frac{1}{\theta}\int_0^{\theta} g(x-u)\,\mathrm{d}u.$$

Then the sequence

$$\int h(x;1/n)\,\mathrm{d}\mu(x)\uparrow\int g(x)\,\mathrm{d}\mu(x),$$

as $n\to\infty$ on the positive integers.

In the above if g is decreasing, then by redefining $h(x;\theta)$, where convolution with the uniform distribution on $(-\theta, 0)$ is used, it follows that monotone convergence of the sequence of integrals holds.

Proof. It is clear that $h(x;\theta)$ is bounded above by the integrable function $g(x)$ for every θ. Further, for $n>m$, $h(x;1/n)\geqslant h(x;1/m)$ for every x so that by the monotone convergence theorem the first part of Lemma 2.2 follows. The same argument holds for the case where g is decreasing. □

Theorem 2.1. *Let (g_1, g_2) be a pair of functions satisfying the DP_2 property and the survival functions $\bar{F}(\theta, t)$ be TP_2 in (θ, t). Suppose that for $i=1, 2$,*

$$\int g_i(t)\,\mathrm{d}F_\theta(t) \quad \textit{exists and is finite.}$$

Further suppose that $g_1(t)$ is increasing in t. Then for $i=1, 2$,

$$h_i(\theta)=\int g_i(t)\,\mathrm{d}F_\theta(t)$$

is DP_2, or equivalently,

$$\int g_1(t)\,\mathrm{d}F_1(t)\int g_2(t)\,\mathrm{d}F_2(t)\geqslant\int g_1(t)\,\mathrm{d}F_2(t)\int g_2(t)\,\mathrm{d}F_1(t). \tag{2.2}$$

Proof. Without loss of generality, θ may be assumed to take values 1 and 2. Suppose F_i has density f_i, and g_i possesses derivative g_i' for $i=1, 2$. Due to the assumption of monotonicity of g_1, the derivative g_i' will be nonnegative. Note that the assumed TP_2 property for $\bar{F}(\theta, t)$ implies that the family of distribution functions $F(\theta,\cdot)$ is stochastically ordered with respect to θ.

To prove the theorem we have to verify that

$$D=\begin{vmatrix}\int g_1(t)f_1(t)\,\mathrm{d}t & \int g_2(t)f_1(t)\,\mathrm{d}t\\ \int g_1(t)f_2(t)\,\mathrm{d}t & \int g_2(t)f_2(t)\,\mathrm{d}t\end{vmatrix}\geqslant 0.$$

Using the basic composition formula the above relation can be rewritten as

$$D=\iint_{s<t}\begin{vmatrix} g_1(s) & g_1(t)\\ g_2(s) & g_2(t)\end{vmatrix}\begin{vmatrix} f_1(s) & f_2(s)\\ f_1(t) & f_2(t)\end{vmatrix}\mathrm{d}s\,\mathrm{d}t.$$

When the above determinants are expanded one gets four terms. When each term is integrated by parts with respect to t, where $-\bar{F}_i$ is used as the integral of f_i, two terms arise. For example,

$$g_1(s)f_1(s)\int_s^\infty g_2(t)f_2(t)\,\mathrm{d}t=g_1(s)f_1(s)[-g_2(t)\bar{F}_2(t)|_s^\infty+\textstyle\int_s^\infty g_2'(t)\bar{F}_2(t)\,\mathrm{d}t]$$

$$=g_1(s)f_1(s)[g_2(s)\bar{F}_2(s)+\textstyle\int_s^\infty g_2'(t)\bar{F}_2(t)\,\mathrm{d}t].$$

The determinant formed by the first terms is easily seen to be 0 while the one formed by the second terms yields

$$D=\iint_{s<t}\begin{vmatrix} g_1(s) & g_1'(t)\\ g_2(s) & g_2'(t)\end{vmatrix}\begin{vmatrix} f_1(s) & f_2(s)\\ \bar{F}_1(t) & \bar{F}_2(t)\end{vmatrix}\mathrm{d}s\,\mathrm{d}t.$$

To show that the first determinant is nonnegative, note that the DP_2 property for the g functions implies that for every $\delta>0$

$$g_1(x)[g_2(x)-g_2(x-\delta)]\geqslant g_2(x)[g_1(x)-g_1(x-\delta)],$$

and hence,

$$g_1(x)g_2'(x)\geqslant g_2(x)g_1'(x).$$

Since g_1 is nonnegative, the above inequality implies

$$g_1(t)g_2'(t)g_1(s)\geqslant g_2(t)g_1'(t)g_1(s)\geqslant g_2(s)g_1'(t)g_1(t),$$

where the last inequality follows from the DP_2 property and the nonnegativity of g_1'.

Since the $\bar{F}$ functions are TP_2, a property stronger than DP_2, the nonnegativity of the second determinant follows easily by a similar argument. This completes the proof of the result when the g functions and $\bar{F}$ are assumed to be differentiable.

Suppose now that the g_i are not continuous. By Lemma 2.1, it follows that the function g_1 can be replaced by a smoothed version which satisfies the conditions of the theorem and then using Lemma 2.2 the required inequalities can be established by taking limits.

In order to replace g_2 by a smooth version, first notice that the function g_1 is nondecreasing, so that the set where g_1 is 0 of the form $(-\infty, l)$. Further the assumption regarding the monotonicity of the ratio $g_2(x)/g_1(x)$ implies that g_2 has to be nonpositive on this set. Consequently, on the set where g_1 is 0, the contribution to the left side of (2.2) is 0 while the right side is nonpositive. Thus the right side of (2.2) is increased by making g_2 equal to 0 whenever g_1 is 0. Since the set where both are 0 can be ignored, we may assume that the ratio $g_2(x)/g_1(x)$ is well defined everywhere.

Let $g_3(x)=g_2(x)/g_1(x)$. According to the assumption of the theorem, g_3 is monotone increasing. As seen earlier, there exists an x_0, such that g_3 is nonpositive for $x\leqslant x_0$ and nonnegative for $x\geqslant x_0$. Considering the positive and negative parts of g_3 it follows that one can find smoothed versions which are bounded above by these parts. Since g_1 is nonnegative multiplying these by g_1 we get a smoothed version of g_2. By an argument similar to the one used in Lemma 2.2, the convergence of the integrals will lead to the required result for discontinuous functions g_2.

The proof of the preservation theorem is now complete. □

Remark. In the above theorem if F is assumed to have the TP_2 property rather than $\bar{F}$, analogous results may be obtained for nonincreasing functions.

3. Characterizations of FR and SR orderings

Recently Capéraà (1988) derived four characterizations of fr and sr orderings. It will be shown below that these characterizations follow easily from Theorem 3.1. From the paper by Capéraà (1988) it seems that condition (*d*) below is very useful in applications. Since the treatment of sr ordering is very similar we present the results only for fr. As before it is assumed that the random variables X_i have distribution functions F_i for $i=1, 2$.

Theorem 3.1 (Capéraà). *The following conditions are equivalent.*

(a) $F_2 >_{\mathrm{fr}} F_1$.

(b) *For a pair of nondecreasing functions a and b, where b is nonnegative, if* $E[a(X_1)b(X_1)]=0$, *then* $E[a(X_2)b(X_2)]\geqslant 0$.

(c) *For every nonnegative nondecreasing function b, with finite expectations* $E[b(X_1)]$ *and* $E[b(X_2)]$, *the distribution function* $H_2(\cdot, b)$ *is stochastically larger than* $H_1(\cdot, b)$ *where*

$$H_i(x, b)=\frac{\int_{-\infty}^{x} b(u)\,\mathrm{d}F_i(u)}{E[b(X_i]}$$

is a weighted distribution corresponding to F_i *with weight function b for* $i=1, 2$.

(d) *Suppose* α *and* β *are functions having finite expected values under* F_1 *and* F_2. *Further suppose that* β *is nonnegative and* α/β *and* β *are nondecreasing. Then*

$$\frac{\int_{-\infty}^{\infty} \alpha(x)\,\mathrm{d}F_2(x)}{\int_{-\infty}^{\infty} \beta(x)\,\mathrm{d}F_2(x)} \geqslant \frac{\int_{-\infty}^{\infty} \alpha(x)\,\mathrm{d}F_1(x)}{\int_{-\infty}^{\infty} \beta(x)\,\mathrm{d}F_1(x)}.$$

Proof. To show that (a) implies (d), suppose that the functions α and β satisfy the monotonicity conditions stated in (d). Define $g_1(t)=\beta(t)$ and $g_2(t)=\alpha(t)$. Then from the assumptions, the pair g_1, g_2 is DP_2. Let $F(2, t)=\bar{F}_2(t)$ and $F(1, t)=\bar{F}_1(t)$. Then the conditions of Theorem 2.1 are met and the resulting pair h_1, h_2 is DP_2. This however is equivalent to the desired inequality.

To show the converse, suppose (d) holds and define

$$\beta(x)=\begin{cases}1, & x>t_1\\ 0, & x\leqslant t_1\end{cases} \qquad \alpha(x)=\begin{cases}1, & x>t_2\\ 0, & x\leqslant t_2,\end{cases}$$

where $t_1<t_2$. Then

$$\frac{\bar{F}_2(t_2)}{\bar{F}_1(t_2)} \geqslant \frac{\bar{F}_2(t_1)}{\bar{F}_1(t_1)},$$

and hence $F_2 >_{\mathrm{fr}} F_1$.

To show that (d) implies (c), we choose $\beta(x)=b(x)$ and $\alpha(x)=b(x)I_{(t,\infty)}(x)$, where I is the indicator function. When the ratio in (d) is formed with this choice of integrands and the integrations are carried out with respect to the measures defined by F_1 and F_2, the inequality is equivalent to the one leading to the stochastic ordering.

The assertion that (c) implies (a) is proved easily (as pointed out by Capéraà (1988)) by taking b to be an indicator function of $x>p$, and then observing that the stochastic ordering of the resulting distributions is equivalent to the monotonicity of the ratio of the survival functions.

The proof that (b) implies (d) is quite simple and is given by Capéraà (1988). We give it here for completeness. Define

$$e=\frac{\int_{-\infty}^{\infty}\alpha(x)\mathrm{d}F_2(x)}{\int_{-\infty}^{\infty}\beta(x)\mathrm{d}F_2(x)}.$$

Then the inequality in (d) is equivalent to

$$\int_{-\infty}^{\infty}(\alpha(x)-e\beta(x))\mathrm{d}F_1(x)\geqslant 0. \tag{3.1}$$

Choose functions a and b in (b) as follows. Let

$$a(x)=\frac{\alpha(x)}{\beta(x)}-e$$

and let $b(x)$ be $\beta(x)$. Then it is clear that $E[a(X_1)b(X_1)]=0$. Hence (3.1) follows from (b) and hence (d) holds.

Finally, to show that (d) implies (b), suppose $E[a(X_1)b(X_1)]=0$, where a is a nondecreasing function and b is a nonnegative nondecreasing function. Choose $\beta(x)=b(x)$ and $\alpha(x)=a(x)b(x)$. Then α and β satisfy the conditions in (d). However, in this case the right side of the inequality in (d) is 0 due to the assumed condition on X_1. Thus the inequality in (d) implies that

$$\int_{-\infty}^{\infty}\alpha(x)\,\mathrm{d}F_2(x)\geqslant 0, \tag{3.2}$$

and hence $E[a(X_2)b(X_2)]\geqslant 0$, where X_2 has distribution function F_2. This completes the proof of Theorem 3.1. □

Remark. The inequality in (d) above was proved by Bickel and Lehmann (1975) under the more stringent condition of monotone likelihood ratio.

4. Application to MRL ordering

Let

$$\mu_{F_i}(x)=\frac{\int_x^{\infty}\bar{F}_i(u)\,\mathrm{d}u}{\bar{F}_i(x)}$$

denote the mean residual life (mrl) function of F_i, the distribution of a non-negative random variable, $i=1,2$. Let μ_i be the expected value corresponding to F_i. We say that F_2 is larger than F_1 in mrl ordering ($F_2>_{mrl}F_1$), if

$$\mu_{F_1}(x)\leqslant\mu_{F_2}(x)\quad\text{for every } x. \tag{4.1}$$

Let

$$\bar{H}_i(x)=\frac{\int_x^{\infty}\bar{F}_i(u)\,\mathrm{d}u}{\mu_i}. \tag{4.2}$$

Then inequality (4.1) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}\log\bar{H}_2(x)\geqslant\frac{\mathrm{d}}{\mathrm{d}x}\log\bar{H}_1(x)\quad\text{for every } x.$$

Consequently, the mrl ordering is equivalent to

$$\frac{\bar{H}_2(x)}{\bar{H}_1(x)} \quad \text{is increasing in } x.$$

These H functions are called the *equilibrium survival functions*. Apart from a normalizing constant μ_i, relation (4.2) reveals that $\bar{F}_i$ plays the role of density function. From this consideration and from the discussion in Section 1 it follows that

$$F_2 >_{\mathrm{fr}} F_1 \ \Rightarrow\ F_2 >_{\mathrm{mrl}} F_1.$$

Since we are dealing with nonnegative random variables, the integrals in (d) of Theorem 3.1 are taken over 0 to ∞. Using condition (d) with this restriction on the H functions we see that the mrl ordering can be expressed as

$$\frac{\int_0^\infty \alpha(x)\bar{F}_2(x)\,\mathrm{d}x}{\int_0^\infty \beta(x)\bar{F}_2(x)\,\mathrm{d}x} \geqslant \frac{\int_0^\infty \alpha(x)\bar{F}_1(x)\,\mathrm{d}x}{\int_0^\infty \beta(x)\bar{F}_1(x)\,\mathrm{d}x}, \tag{4.3}$$

where the functions α and β satisfy the same conditions as in (d) of Theorem 3.1. If we restrict α to be nonnegative, then the condition α/β nondecreasing would force α to be nondecreasing. Define

$$\alpha^*(x) = \int_0^x \alpha(u)\,\mathrm{d}u$$

and

$$\beta^*(x) = \int_0^x \beta(u)\,\mathrm{d}u,$$

with $\alpha^*(0) = \beta^*(0) = 0$. Using integration by parts in (4.3), the inequality can be rewritten as

$$\frac{E\alpha^*(X_2)}{E\beta^*(X_2)} \geqslant \frac{E\alpha^*(X_1)}{E\beta^*(X_1)}, \tag{4.4}$$

where the random variables X_i are assumed to be distributed according to the distribution functions F_i. The inequality (4.4) may be taken as a criterion for mrl ordering where α^* and β^* are nonnegative nondecreasing convex functions tending to 0 as $x \to 0$ and the ratio of their derivatives is increasing.

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