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On pathwise stochastic integration

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Abstract

In this article, we construct a mapping

$$\mathscr{I} : D[0,\infty)\times D[0,\infty)\to D[0,\infty)$$

such that if (X_t) is a semimartingale on a probability space $(\Omega, \mathscr{F}, P)$ with respect to a filtration $(\mathscr{F}_t)$ and if (f_t) is an r.c.l.l. $(\mathscr{F}_t)$ adapted process, then

$$\mathscr{I}(f.(\omega), X.(\omega)) = \int_0^{\cdot} f_- \, \mathrm{d}X(\omega) \quad \text{a.s.}$$

This is of significance when using stochastic integrals in statistical inference problems. Similar results on solutions to SDEs are also given.

Keywords: Brownian motion; Semimartingale; Stochastic integral

1. Introduction

The aim of this article is to draw the attention of researchers dealing with applications of stochastic integration, particularly those working in the area of inference for continuous time stochastic processes, to an aspect of stochastic integration that has not received sufficient attention even in recent texts on the subject – the fact that for a large class of integrands, the stochastic integral can be defined *pathwise*. Invariably, in problems of inference for continuous time stochastic processes, the solution involves stochastic integrals with respect to the observation path. Either the statistic under consideration involves the integral or the likelihood function involves it. For example, for the estimation problem for the parameter θ in the model

$$Y_t = \int_0^t h(\theta, Y_s)\,\mathrm{d}s + W_t, \quad 0 \leqslant t \leqslant T,$$

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where (W_t) is the standard Brownian motion and h is a known smooth function, the likelihood function (w.r.t. the Wiener measure) $L(\theta, Y.)$ is given by

$$L(\theta, Y.) = \exp\left\{\int_0^T h(\theta, Y_s)\,\mathrm{d}Y_s - \frac{1}{2}\int_0^T h^2(\theta, Y_s)\,\mathrm{d}s\right\}.$$

Thus for inference via the maximum likelihood method to be meaningful, it should be possible to *evaluate* the stochastic integral $\int_0^T h(\theta, Y_s)\,\mathrm{d}Y_s$ for a given observation path Y. Traditionally, it had been emphasised that one cannot talk of a stochastic integral for a given path. However, this can be done for a large class of integrands, as we will see below. This result is due to Bichteler (1981) and a simple proof for the case of continuous semimartingale integrands was given in Karandikar (1981). A simple proof for the general result along with new results on pathwise construction of solution to SDE were given in Karandikar (1989, 1991). The result on pathwise integration can be stated in the following form. We can construct a mapping

$$\mathscr{I} : D[0, \infty) \times D[0, \infty) \to D[0, \infty)$$

such that if (X_t) is a semimartingale on a probability space $(\Omega, \mathscr{F}, P)$ with respect to a filtration $(\mathscr{F}_t)$ and if (Z_t) is an r.c.l.l. $(\mathscr{F}_t)$ adapted process, then

$$\mathscr{I}(Z.(\omega), X.(\omega)) = \int_0^{\cdot} Z_-\,\mathrm{d}X(\omega) \quad \text{a.s.}$$

Here and in the sequel, $X.(\omega)$ denotes the ω-path $t \to X_t(\omega)$ of the process X. This would show also that the stochastic integral neither depends on the underlying filtration nor on the underlying probability measure in any essential way. This can be extremely useful when dealing, for example, with Markov processes which involve a family of probability measures indexed by the initial condition. Furthermore, in problems related to the Girsanov theorem mappings such as $\mathscr{I}$ play an important role (see Karandikar, 1983c).

Here we first present a simple proof of the pathwise integration formula for the case of Brownian motion integrals. A proof in the general case using minimum technical background follows. We then recast this result by constructing the mapping $\mathscr{I}$ mentioned earlier. Then we state similar results on the quadratic variation process of a semimartingale and on solution to SDEs.

2. Main result

Throughout the article, we fix a complete probability space $(\Omega, \mathscr{F}, P)$ and a filtration $(\mathscr{F}_t)$ satisfying the usual conditions.

Our first result is the following theorem.

Theorem 1. *Let (W_t) be a Brownian motion adapted to the filtration $(\mathscr{F}_t)$ such that $W_t - W_s$ is independent of $\mathscr{F}_s$ for all $0 \leqslant s \leqslant t < \infty$. Let f be an r.c.l.l. adapted process* *and for* $n \geqslant 1$, *let* $\{\tau_i^n : i \geqslant 0\}$ *be defined by* $\tau_0^n = 0$ *and for* $i \geqslant 0$

$$\tau_{i+1}^n = \inf\{t \geqslant \tau_i^n : |f_t(\cdot) - f_{\tau_i^n}(\cdot)| \geqslant 2^{-n}\}.$$

Let (Y_t^n) *be defined as follows. For* $\tau_k^n \leqslant t < \tau_{k+1}^n$, $k \geqslant 0$,

$$Y_t^n = \sum_{i=0}^{k-1} f_{\tau_i^n}(W_{\tau_{i+1}^n} - W_{\tau_i^n}) + f_{\tau_k^n}(W_t - W_{\tau_k^n}).$$

Then, for all $T < \infty$,

$$\sup_{0 \leqslant t \leqslant T} \left| Y_t^n - \int_0^t f \, \mathrm{d}W \right| \to 0 \quad \text{a.s.} \tag{1}$$

Proof. Note that $Y_t^n = \int_0^t f^n \, \mathrm{d}W$, where

$$f_t^n = f_{\tau_k^n}, \quad \text{for } \tau_k^n \leqslant t < \tau_{k+1}^n$$

and hence by the choice of $\{\tau_i^n\}$ we have

$$|f_t^n - f_t| \leqslant 2^{-n}.$$

Using the standard estimate

$$\mathbb{E} \sup_{0 \leqslant t \leqslant T} \left| \int_0^t g \, \mathrm{d}W \right|^2 \leqslant 4\mathbb{E} \left| \int_0^T g^2 \, \mathrm{d}t \right| \tag{2}$$

one gets

$$\mathbb{E} \sup_{0 \leqslant t \leqslant T} \left| Y_t^n - \int_0^t f \, \mathrm{d}W \right|^2 \leqslant 4T.2^{-2n}. \tag{3}$$

Let

$$U_n = \sup_{0 \leqslant t \leqslant T} \left| Y_t^n - \int_0^t f \, \mathrm{d}W \right|.$$

Then (3) implies that $EU_n \leqslant 2\sqrt{T}2^{-n}$ and hence it follows that

$$\begin{aligned} \mathbb{E} \sum_n U_n &= \sum_n \mathbb{E} U_n \\ &\leqslant \sum_n 2\sqrt{T}2^{-n} \\ &< \infty. \end{aligned}$$

As a consequence, one has

$$\sum_n U_n < \infty \quad \text{a.s.}$$

which gives the required conclusion (1). □

We will now prove a similar result for integrals w.r.t. semimartingales. We will need some elementary facts on martingales, semimartingales and predictable processes, which we recall here. These can be found in Jacod (1979), Metivier (1982) and Protter (1990).

For a locally square integrable martingale M, $\langle M, M\rangle$ denotes the predictable quadratic variation process. If σ_k are stopping times increasing to ∞ such that $(M_{t\wedge\sigma_k})$ is a square integrable martingale then $\mathbb{E}\langle M, M\rangle_{\sigma_k} < \infty$. For a predictable process f such that

$$\int_0^t f^2\, \mathrm{d}\langle M, M\rangle < \infty \quad \forall t < \infty,$$

$N = \int f \mathrm{d}M$ is defined, is a local martingale and

$$\langle N, N\rangle_t = \int_0^t f^2\, \mathrm{d}\langle M, M\rangle.$$

As a consequence, for any stopping time σ one has

$$\mathbb{E} \sup_{0\leqslant t\leqslant\sigma} \left|\int_0^t f \mathrm{d}M\right|^2 \leqslant 4\mathbb{E}\left|\int_0^\sigma f^2 \mathrm{d}\langle M, M\rangle\right|. \tag{4}$$

Here the only fact we need about predictable processes is that for any r.c.l.l. adapted process Z, the process Z_- which is defined as the left limit of the process Z at each time point is predictable.

A semimartingale is an (r.c.l.l.) process X which can be written as $X = M + A$, where M is a local martingale and A is a process whose paths are of bounded variation on bounded intervals. A deep result on the structure of jumps of a martingale implies that in the above decomposition we can take M to be locally square integrable martingale.

Thus if X is a semimartingale with the above decomposition and Z is an r.c.l.l. adapted process, the integral $\int Z_-\, \mathrm{d}X$ is defined as

$$\int Z_-\, \mathrm{d}X = \int Z_-\, \mathrm{d}M + \int Z_-\, \mathrm{d}A.$$

We are now in a position to prove the following theorem.

Theorem 2. *Let X be a semimartingale and let Z be an r.c.l.l. adapted process. For $n \geqslant 1$, let $\{\tau_i^n\colon i \geqslant 0\}$ be defined by $\tau_0^n = 0$ and for $i \geqslant 0$,*

$$\tau_{i+1}^n = \inf\{t \geqslant \tau_i^n\colon |Z_t(\cdot) - Z_{\tau_i^n}(\cdot)| \geqslant 2^{-n}\}.$$

Let (Y_t^n) be defined as follows. For $\tau_k^n < t \leqslant \tau_{k+1}^n$, $k \geqslant 0$,

$$Y_t^n = Z_0 X_0 + \sum_{i=0}^{k-1} Z_{\tau_i^n}(X_{\tau_{i+1}^n} - X_{\tau_i^n}) + Z_{\tau_k^n}(X_t - X_{\tau_k^n}).$$

Then

$$\sup_{0\leqslant t\leqslant T} \left|Y_t^n - \int_0^t Z_-\, \mathrm{d}X\right| \to 0 \quad \text{a.s.} \tag{5}$$

Proof. Note that $Y_t^n = \int_0^t Z_-^n \, \mathrm{d}X$, where

$$Z_t^n = Z_{\tau_k^n}, \quad \tau_k^n < t \leqslant \tau_{k+1}^n$$

for $k \geqslant 1$ and $Z_0^n = Z_0$. Hence by the choice of $\{\tau_i^n\}$ we have

$$\sup_t |Z_t^n - Z_{t-}| \leqslant 2^{-n}.$$

Let $X = M + A$ be a decomposition of the semimartingale X where M is a locally square integrable martingale and A is a process whose paths are of bounded variation on bounded intervals. Define stopping times σ_k increasing to ∞ such that $C_k = \mathbb{E}\langle M, M\rangle_{\sigma_k} < \infty$. Then using (4) one has

$$\mathbb{E} \sup_{0 \leqslant t \leqslant \sigma_k} \left| \int_0^t Z^n \, \mathrm{d}M - \int_0^t Z_- \, \mathrm{d}M \right|^2 \leqslant 2^{-2n} C_k \tag{6}$$

and, as in the earlier result, we can conclude that

$$\sup_{0 \leqslant t \leqslant \sigma_k} \left| \int_0^t Z^n \, \mathrm{d}M - \int_0^t Z_- \, \mathrm{d}M \right| \to 0 \quad \text{a.s.} \tag{7}$$

for all $k \geqslant 1$. Since σ_k increases to ∞, we get

$$\sup_{0 \leqslant t \leqslant T} \left| \int_0^t Z^n \, \mathrm{d}M - \int_0^t Z_- \, \mathrm{d}M \right| \to 0 \quad \text{a.s.} \tag{8}$$

for all $T < \infty$. As for the $\mathrm{d}A$ integral, uniform convergence of Z_n to Z_- yields

$$\sup_{0 \leqslant t \leqslant T} \left| \int_0^t Z^n \, \mathrm{d}A - \int_0^t Z_- \, \mathrm{d}A \right| \to 0. \tag{9}$$

Together, (8) and (9) yield the result. □

The conclusion in Theorem 2 above can be stated equivalently as follows. Write $\mathbb{D} = D[0, \infty)$. A generic point of $\mathbb{D}$ will be denoted by ρ. Define mappings $\mathscr{I}_n$ and $\mathscr{I}$ from $\mathbb{D} \times \mathbb{D}$ into $\mathbb{D}$ as follows:

Fix $\rho, \eta \in \mathbb{D}$. For $n \geqslant 1$, let $\{a_i^n \colon i \geqslant 1\}$ be defined by $a_0^n = 0$ and for $i \geqslant 0$

$$a_{i+1}^n = \inf\{t \geqslant a_i^n \colon |\rho(t) - \rho(a_i^n)| \geqslant 2^{-n}\},$$

$\mathscr{I}_n(\rho, \eta) \in \mathbb{D}$ is then defined as follows: For $a_k^n \leqslant t < a_{k+1}^n$, $k \geqslant 0$,

$$\mathscr{I}_n(\rho, \eta)(t) = \rho(0)\eta(0) + \sum_{i=0}^{k-1} \rho(a_i^n)(\eta(a_{i+1}^n) - \eta(a_i^n)) + \rho(a_k^n)(\eta(t) - \eta(a_k^n)).$$

Then define $\mathscr{I}$ as

$$\mathscr{I}(\rho, \eta) = \lim \mathscr{I}_n(\rho, \eta)$$

if the limit exists in the topology of uniform convergence on compact subsets of $[0, \infty)$, otherwise define $\mathscr{I}(\rho, \eta) \equiv 0$. With this notation, the result in Theorem 2 can be recast as the following theorem.

Theorem 3. *Let (X_t) be a semimartingale and let Z be an r.c.l.l. adapted process (on the same space). Then*

$$\mathscr{I}(Z.(\omega), X.(\omega)) = \left(\int_0^{\cdot} Z_- \,\mathrm{d}X\right)(\omega) \quad \text{a.s.}$$

We now give a similar formula for the quadratic variation $[X, X]$ of a semimartingale X. For $\rho \in \mathbb{D}$ let (a_i^n) be as defined above. Let $\mathscr{T}_n \in \mathbb{D}$ be defined by

$$\mathscr{T}_n(\rho)(t) = \sum_{i=0}^{\infty} (\rho(a_{i+1}^n \wedge t) - \rho(a_i^n \wedge t))^2$$

and let $\mathscr{T}(\rho) = \lim \mathscr{T}_n(\rho)$ if the limit exists in the topology of uniform convergence on compact subsets of $[0, \infty)$, otherwise define $\mathscr{T}(\rho) \equiv 0$. Then for any semimartingale (X_t), one has

$$\mathscr{I}(X.(\omega)) = [X, X](\omega) \quad \text{a.s.}$$

This can easily be deduced from the previous result. It is of interest to note that for a continuous martingale X, this construction of quadratic variation was given in Karandikar (1983a) and the proof did not use stochastic integration. It used only Doob's maximal inequality. Thus, it can be used as a starting point for developing the theory of stochastic integration w.r.t. continuous martingales and continuous semimartingales (see Karandikar, 1983b).

We conclude this article by giving a formula for the solution of a SDE driven by a semimartingale. This shows that a suitable modification of the Euler–Peano approximation converges a.s.

The SDE (or more appropriately, stochastic integral equation) considered as

$$Z_t = H_t + \int_0^t a(Z)_{s-} \,\mathrm{d}X_s, \tag{10}$$

where X is an $\mathbb{R}^d$ valued semimartingale, H is a given adapted r.c.l.l. $\mathbb{R}^d$ valued process and

$$a : \mathbb{D}([0, \infty), \mathbb{R}^d) \to \mathbb{D}([0, \infty), L(d, d)),$$

where $L(d, d)$ is the space of $d \times d$ matrices. When $a(\rho)(s) = f(s, \rho(s))$ (and $H_t \equiv z_0$) Eq. (10) can be written in the more familiar form

$$\mathrm{d}Z_t = f(t-, Z_{t-}) \,\mathrm{d}X_t.$$

We assume that the functional a satisfies the following Lipschitz condition: For each $T < \infty$ there exists a finite constant C_T such that

$$\| a(\rho_1)(t) - a(\rho_2)(t) \| \leqslant C_T \sup_{0 \leqslant s \leqslant t} \| \rho_1(s) - \rho_2(s) \| \tag{11}$$

for all $\rho_1, \rho_2 \in \mathbb{D}([0, \infty), \mathbb{R}^d)$ and for all $0 \leqslant t \leqslant T$. Here $\| \cdot \|$ denotes the Euclidian norm on $\mathbb{R}^d$ and on $L(d, d)$.

We now define mappings $S_n : \mathbb{D}([0, \infty), \mathbb{R}^d) \to \mathbb{D}([0, \infty), \mathbb{R}^d)$. Fix $n \geqslant 1, \rho, \eta \in \mathbb{D}([0, \infty), \mathbb{R}^d)$. Let $\{u_i \colon i \geqslant 1\}$ and $\xi^i \in \mathbb{D}([0, \infty), \mathbb{R}^d)$ be defined inductively by

$$u_0 = 0 \quad \text{and} \quad \xi^0_t \equiv \eta_0 \tag{12}$$

and having defined u_j, ξ^j for $j \leqslant i$, let

$$u_{i+1} = \inf\{t > u_i \colon \| \eta(t) - \eta(u_i) + a(\xi^i)(u_i)(\rho(t) - \rho(u_i)) \| \geqslant 2^{-n}. \\ \text{or } \| a(\xi^i)(t) - a(\xi^i)(u_i) \| \geqslant 2^{-n}\}$$

and

$$\xi^{i+1}(t) = \begin{cases} \xi^i(t) & \text{for } t < u_{i+1}, \\ \xi^i(u_i) + \eta(u_{i+1}) - \eta(u_i) + a(\xi^i)(u_i)(\rho(u_{i+1}) - \rho(u_i)) & \text{for } t \geqslant u_{i+1}. \end{cases}$$

Thus, ξ^{i+1} is a function that has jumps at $u_1, \ldots, u_{i+1}$ and is constant on the intervals

$$[0, u_1), \ldots, [u_j, u_{j+1}), \ldots, [u_i, u_{i+1}), [u_{i+1}, \infty).$$

Piece together these paths ξ^i, $i = 1, 2, \ldots$ to define a function $\mathscr{S}_n(\eta, \rho) \in \mathbb{D}$ as follows. Let $\mathscr{S}_n(\eta, \rho)(0) = \eta(0)$ and, for $u_i < t \leqslant u_{i+1}$, let

$$\mathscr{S}_n(\eta, \rho)(t) = \xi^i(u_i) + \eta_t - \eta_{u_i} + a(\xi^i)(\rho(t) - \rho(u_i)).$$

We now define

$$\mathscr{S}(\eta, \rho) = \lim \mathscr{S}_n(\eta, \rho),$$

whenever the limit exists in the topology of uniform convergence on compact subsets of $[0, \infty)$. Then one has the following.

Let (X_t) be a semimartingale and let (H_t) be an r.c.l.l. adapted process (on the same probability space). Then

$$Z_t(\omega) = \mathscr{S}(H.(\omega), X.(\omega))(t)$$

is the (unique) solution to the equation

$$Z_t = H_t + \int_0^t a(Z)_{s-} \, \mathrm{d}X_s.$$

In fact one has a much stronger result. If we denote the approximating sequence by

$$Z^n_t(\omega) = \mathscr{S}_n(H.(\omega), X.(\omega))(t),$$

then Z^n converges in the Emery topology (on the space of semimartingales) to Z. For the proof of this result as well as details on Emery topology, we refer the reader to Karandikar (1991).

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