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Some results on normalized spacings from restricted families of distributions

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Abstract

It is well known that the normalized spacings of a random sample from a DFR (IFR) distribution are stochastically increasing (decreasing). In this note we strengthen this result to show that if the parent distribution is DFR, the successive normalized spacing increase in the failure rate ordering (which implies stochastic ordering) sense. We also study the joint distribution of the normalized spacings when the parent observations are not necessarily identical. It is shown that when the observations are independent with (possibly different) log-convex densities, the joint distribution of the normalized spacings is arrangement increasing.

Key words: Order statistics; Joint likelihood ratio ordering; Failure rate ordering; Stochastic ordering; Schur functions; Majorization; Arrangement increasing functions

1. Introduction

In reliability theory and survival analysis, the nonparametric classes of *increasing failure rate* (IFR) and *decreasing failure rate* (DFR) distributions play an important role. There is a vast literature on stochastic inequalies and order relations between the various statistics when the observations come from such distributions.

Let $X_1, \ldots, X_n$ be a random sample of size n from an absolutely continuous distribution with density function f, failure rate function r_F, distribution function F and survival function $\bar{F} = 1 - F$. As is the convention, we shall denote by $X_{i:n}$ the ith order statistic of a sample of size n. Let $D_{i:n} = (n-i+1)(X_{i:n} - X_{i-1:n})$ denote the ith normalized spacing, $i = 1, \ldots, n$, with $X_{0:n} \equiv 0$. It is well known that $D_{1:n}, \ldots, D_{n:n}$ are independent and identically distributed if and only if $\{X_1, \ldots, X_n\}$

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is a random sample from an exponential distribution. Barlow and Proschan (1966) proved the following result on stochastic ordering between the successive normalized spacings from DFR distributions.

Theorem 1.1 (Barlow and Proschan, 1966). *Let $X_1, \ldots, X_n$ be a random sample of size n from a DFR distribution. Then*

(a) $D_{i:n} \overset{\text{st}}{\leqslant} D_{i+1:n}, \quad i=1, \ldots, n-1,$

(b) $D_{i:n+1} \overset{\text{st}}{\leqslant} D_{i:n}, \quad n \geqslant i$ *for fixed i.*

Similar results hold for the IFR case with the inequalities reversed in (a) and (b) above. Later Pledger and Proschan (1971) partially extended this result to the case when the random variables are independent with *proportional decreasing failure rates.* Kim and David (1990) have also obtained some results on spacings from IFR (DFR) distributions.

In Section 2 we strengthen the above result of Theorem 1.1 from stochastic ordering to *failure (or hazard) rate ordering.* If $\bar{F}(\bar{G})$ denotes the survival function of a random variable $X(Y)$, we say that X is greater than Y according to failure rate ordering (written as $X \overset{\text{fr}}{\geqslant} Y$) if $\bar{F}(x)/\bar{G}(x)$ is nondecreasing in x. In the case of continuous distributions, this is equivalent to the failure rate of F being uniformly smaller than that of G. If $f(g)$ is the density function of $F(G)$ and $f(x)/g(x)$ is nondecreasing in x, then we say that X is greater than Y according to *likelihood ratio ordering* and write it as $X \overset{\ell r}{\geqslant} Y$. Likelihood ratio ordering implies failure rate ordering, which in turn implies stochastic ordering. For some other properties of these orderings, see Bickel and Lehmann (1975) and Joag-dev et al. (1995). We also show in this section that, under the above conditions, the normalized spacings are also ordered according to *dispersive ordering.* In Section 3, we look at the vector of $D_{i:n}$'s as a whole and establish that they are *jointly likelihood ratio ordered* (cf. Shanthikumar and Yao, 1991) when the parent densities are log-convex.

2. Failure rate and dispersive orderings between normalized spacings from DFR distributions

We shall need the following lemmas to prove our main results.

Lemma 2.1. *Let $\psi_1(x, y)$ and $\psi_2(x, y)$ be positive real-valued functions such that*

(i) *$\psi_2(x, y)$ is TP_2, that is, for $y_1 \leqslant y_2$,*

$$\frac{\psi_2(x, y_2)}{\psi_2(x, y_1)} \quad \text{is nondecreasing in } x,$$

(ii) *for* $y_1 \leqslant y_2$,

$$\frac{\psi_1(x, y_2)}{\psi_2(x, y_1)} \quad \textit{is nondecreasing in } x,$$

(iii) *for each fixed* x,

$$\frac{\psi_1(x, y)}{\psi_2(x, y)} \quad \textit{is nondecreasing in } y.$$

Then for $x_1 \leqslant x_2$, $y_1 \leqslant y_2$,

$$0 \leqslant \psi_1(x_2, y_2)\psi_2(x_1, y_1) - \psi_1(x_1, y_1)\psi_2(x_2, y_2) \tag{2.1}$$

$$\geqslant \psi_1(x_1, y_2)\psi_2(x_2, y_1) - \psi_1(x_2, y_1)\psi_2(x_1, y_2). \tag{2.2}$$

Proof. Since $\psi_1(x,y)/\psi_2(x,y)$ is nondecreasing in x and y, it follows that for $x_1 \leqslant x_2$, $y_1 \leqslant y_2$,

$$\frac{\psi_1(x_2,y_2)}{\psi_2(x_2,y_2)} - \frac{\psi_1(x_1,y_1)}{\psi_2(x_1,y_1)} \geqslant \frac{\psi_1(x_1,y_2)}{\psi_2(x_1,y_2)} - \frac{\psi_1(x_2,y_1)}{\psi_2(x_2,y_1)}. \tag{2.3}$$

Also it follows from the TP_2 property of ψ_2 that for $x_1 \leqslant x_2$, $y_1 \leqslant y_2$,

$$\psi_2(x_2,y_2)\psi_2(x_1,y_1) \geqslant \psi_2(x_1,y_2)\psi_2(x_2,y_1). \tag{2.4}$$

The required result follows by multiplying the inequalities (2.3) and (2.4) and noting that the left-hand side of (2.3) is nonnegative. □

Lemma 2.2. *Let*

$$\psi_1(x, y) = \bar{F}^{n-i}\left(\frac{x}{n-i} + y\right),$$

$$\psi_2(x, y) = \bar{F}^{n-i+1}\left(\frac{x}{n-i+1} + y\right),$$

where F is DFR. Then ψ_1 *and* ψ_2 *satisfy the conditions of Lemma* 2.1 *for* $1 \leqslant i \leqslant n-1$.

Proof.

$$\ln\left\{\frac{\psi_2(x, y_2)}{\psi_2(x, y_1)}\right\} = (n-i+1)\left[\ln \bar{F}\left(\frac{x}{n-i+1} + y_2\right) - \ln \bar{F}\left(\frac{x}{n-i+1} + y_1\right)\right].$$

On differentiating this with respect to x, we get

$$\frac{\partial}{\partial x} \ln\left\{\frac{\psi_2(x, y_2)}{\psi_2(x, y_1)}\right\} = -r_F\left(\frac{x}{n-i+1} + y_2\right) + r_F\left(\frac{x}{n-i+1} + y_1\right)$$

$$\geqslant 0 \quad \text{for } y_1 \leqslant y_2$$

since $r_F(x)$, the failure rate of F, is decreasing in x.

Hence $\psi_2(x, y_2)/\psi_2(x, y_1)$ is nondecreasing in x for $y_1 \leqslant y_2$, thereby proving the first part.

The rest of the proof follows on the same lines. □

Lemma 2.3. *Let $\psi_1(x,y)$ and $\psi_2(x,y)$ satisfy the conditions of Lemma 2.1 and let (Y_1, Y_2) be a bivariate random vector with joint density $f(y_1, y_2)$ satisfying*

$$f(y_1, y_2) \geqslant f(y_2, y_1) \quad \text{for } y_1 \leqslant y_2. \tag{2.5}$$

Then for $x_1 \leqslant x_2$,

$$\frac{E[\psi_1(x_1, Y_2)]}{E[\psi_2(x_1, Y_1)]} \leqslant \frac{E[\psi_1(x_2, Y_2)]}{E[\psi_2(x_2, Y_1)]}.$$

Proof. For $x_1 \leqslant x_2$,

$$\begin{aligned}
&E[\psi_1(x_2, Y_2)\psi_2(x_1, Y_1)] - E[\psi_1(x_1, Y_2)\psi_2(x_2, Y_1)] \\
&= \iint_{y_1 \leqslant y_2} [\psi_1(x_2,y_2)\psi_2(x_1,y_1) - \psi_1(x_1,y_2)\psi_2(x_2,y_1)]\, f(y_1, y_2)\, dy_1\, dy_2 \\
&\quad + \iint_{y_1 > y_2} [\psi_1(x_2,y_2)\psi_2(x_1,y_1) - \psi_1(x_1,y_2)\psi_2(x_2,y_1)]\, f(y_1, y_2)\, dy_1\, dy_2 \\
&= \iint_{y_1 \leqslant y_2} [\{\psi_1(x_2,y_2)\psi_2(x_1,y_1) - \psi_1(x_1,y_2)\psi_2(x_2,y_1)\}\, f(y_1, y_2) \\
&\quad + \{\psi_1(x_2,y_1)\psi_2(x_1,y_2) - \psi_1(x_1,y_1)\psi_2(x_2,y_2)\}\, f(y_2, y_1)]\, dy_1\, dy_2 \\
&\quad \text{(on making a change of variables in the second integral)} \\
&\geqslant \iint_{y_1 \leqslant y_2} [\psi_1(x_2,y_2)\psi_2(x_1,y_1) - \psi_1(x_1,y_2)\psi_2(x_2,y_1) \\
&\quad + \psi_1(x_2,y_1)\psi_2(x_1,y_2) - \psi_1(x_1,y_1)\psi_2(x_2,y_2)]\, f(y_1, y_2)]\, dy_1\, dy_2 \\
&\geqslant 0.
\end{aligned} \tag{2.6}$$

The inequality in (2.6) follows by noting that for $x_1 \leqslant x_2$, $y_1 \leqslant y_2$,

$$\begin{aligned}
&\psi_1(x_2,y_2)\psi_2(x_1,y_1) - \psi_1(x_1,y_2)\psi_2(x_2,y_1) \\
&\quad \geqslant \psi_1(x_1,y_1)\psi_2(x_2,y_2) - \psi_1(x_2,y_1)\psi_2(x_1,y_2)
\end{aligned} \tag{2.7}$$

(from Lemma 2.1) and by multiplying (2.5) with (2.7). Since by Lemma 2.1, the integrand in (2.6) is nonnegative, the required result follows. □

Remarks. (1) If Y_1 and Y_2 are independent random variables such that $Y_1 \overset{\ell r}{\leqslant} Y_2$, then (2.5) is obviously satisfied and consequently Lemma 2.3 will hold in this case.

(2) Lemma 2.3 generalizes the result contained in Lemma 2 of Bickel and Lehmann (1975).

Now we prove our main theorem.

Theorem 2.1. *Let $X_1, \ldots, X_n$ be a random sample of size n from a DFR distribution. Then*

(a) $D_{i:n} \overset{\mathrm{fr}}{\leqslant} D_{i+1:n}, \quad i=1, \ldots, n-1,$ (2.8)

(b) $D_{i:n+1} \overset{\mathrm{fr}}{\leqslant} D_{i:n}, \quad n \geqslant i$ *for fixed i.* (2.9)

Proof. (a) Let $f_{i:n}$ denote the probability density function of $X_{i:n}$, $i=1, \ldots, n$. The survival function of $D_{i:n}$ is

$$
\begin{aligned}
P[D_{i:n}>x] &= P[(n-i+1)\{X_{i:n} - X_{i-1:n}\}>x] \\
&= \int \left[\frac{\bar{F}(x/(n-i+1)+u)}{\bar{F}(u)}\right]^{n-i+1} f_{i-1:n}(u)\,\mathrm{d}u \\
&= \int \left[\bar{F}\left(\frac{x}{n-i+1}+u\right)\right]^{n-i+1} F^{i-2}(u)\, f(u)\,\mathrm{d}u \\
&= C(i, n)\, E\left[\bar{F}^{n-i+1}\left\{\frac{x}{n-i+1}+Y_{(i-i)}\right\}\right], \qquad (2.10)
\end{aligned}
$$

where $C(i, n)$ is a normalizing constant and $Y_{(i-1)} = \max\{X_1, \ldots, X_{i-1}\}$, X_i's being independent copies of X.

We have to prove that for $x_1 \leqslant x_2$ and $1 \leqslant i \leqslant n-1$,

$$
\frac{P[D_{i+1:n}>x_2]}{P[D_{i:n}>x_2]} \geqslant \frac{P[D_{i+1:n}>x_1]}{P[D_{i:n}>x_1]},
$$

that is, to prove that,

$$
\frac{E[\bar{F}^{n-i}(x_2/(n-i)+Y^{\star}_{(i)})]}{E[\bar{F}^{n-i+1}(x_2/(n-i+1)+Y_{(i-i)})]} \geqslant \frac{E[\bar{F}^{n-i}(x_1/(n-i)+Y^{\star}_{(i)})]}{E[\bar{F}^{n-i+1}(x_1/(n-i+1)+Y_{(i-i)})]}, \qquad (2.11)
$$

where $Y^{\star}_{(i)} = \max\{X_i, \ldots, X_{2i-1}\}$ and where $(X_1, \ldots, X_{2i-1})$ are $2i-1$ independent copies of X. Let

$$
\psi_1(x, y) = \bar{F}^{n-i}\left(\frac{x}{n-i}+y\right),
$$

$$
\psi_2(x, y) = \bar{F}^{n-i+1}\left(\frac{x}{n-i+1}+y\right).
$$

Then (2.11) is equivalent to

$$\frac{E[\psi_1(x_2, Y^\star_{(i)})]}{E[\psi_2(x_2, Y_{(i-i)})]} \geqslant \frac{E[\psi_1(x_1, Y^\star_{(i)})]}{E[\psi_2(x_1, Y_{(i-i)})]}. \tag{2.12}$$

The proof of (2.12) follows from Lemmas 2.2 and 2.3 since $Y_{(i-1)}$ and $Y^\star_{(i)}$ are independent and $Y_{(i-1)} \overset{\ell r}{\leqslant} Y^\star_{(i)}$.

(b) We have to prove that for $x_1 \leqslant x_2$ and $i = 0, \ldots, n-1$,

$$\frac{P[D_{i+1:n} > x_2]}{P[D_{i+1:n+1} > x_2]} \geqslant \frac{P[D_{i+1:n} > x_1]}{P[D_{i+1:n+1} > x_1]}.$$

From (2.10)

$$P[D_{i+1:n} > x] = C(i+1, n) E\left[\bar{F}^{n-i}\left(\frac{x}{n-i} + Y_{(i)}\right)\right]$$

$$= C(i+1, n) E[\psi_1(x, Y_{(i)})] \tag{2.13}$$

and

$$P[D_{i+1:n+1} > x] = C(i+1, n+1) E\left[\bar{F}^{n-i+1}\left(\frac{x}{n-i+1} + Y^\star_{(i)}\right)\right]$$

$$= C(i+1, n+1) E[\psi_2(x, Y^\star_{(i)})]. \tag{2.14}$$

Proving (b) is equivalent to showing that

$$\frac{E[\psi_1(x_2, Y_{(i)})]}{E[\psi_2(x_2, Y^\star_{(i)})]} \geqslant \frac{E[\psi_1(x_1, Y_{(i)})]}{E[\psi_1(x_1, Y^\star_{(i)})]}, \tag{2.15}$$

where again $Y_{(i)}$ and $Y^\star_{(i)}$ are independent as in the first part. The proof of (2.15) follows from Lemmas 2.2 and 2.3. □

Barlow and Proschan (1966) have shown that spacings of i.i.d. DFR random variables have also DFR distributions. Bagai and Kochar (1986) proved that if $G \overset{\mathrm{fr}}{\leqslant} F$ and if either F or G is DFR then G is less dispersive than F ($G \overset{\mathrm{disp}}{\leqslant} F$) in the sense that $G^{-1}(v) - G^{-1}(u) \leqslant F^{-1}(v) - F^{-1}(u)$ for $0 \leqslant u \leqslant v \leqslant 1$. The proof of the following theorem follows from the above results.

Theorem 2.2. *If $X_1, \ldots, X_n$ is a random sample from a DFR distribution, then for $i = 1, \ldots, n-1$.*

(a) $D_{i:n} \overset{\mathrm{disp}}{\leqslant} D_{i+1:n}$,

(b) $\mathrm{var}(D_{i:n}) \leqslant \mathrm{var}(D_{i+1:n})$,

(c) $D_{i:n+1} \overset{\mathrm{disp}}{\leqslant} D_{i:n}$,

(d) $\mathrm{var}(D_{i:n+1}) \leqslant \mathrm{var}(D_{i:n})$.

3. Joint likelihood ratio ordering between the normalized spacing from distributions with log-convex densities

In the previous sections we discussed some stochastic orders between the normalized spacings in terms of their *marginal* distributions. We know that the normalized spacings are independent only if the parent distribution is exponential. It is argued that in the case of dependent random variables, studying only the stochastic ordering between the marginal distributions may not be very useful in revealing monotone tendencies between dependent variables because the dependence information is being ignored. Realizing this, Shanthikumar and Yao (1991) introduced some new orderings of random variables for studying the stochastic relationships between the components of a random vector. We start our discussion with the extension of the idea of likelihood ratio ordering. For two *independent* random variables X_1 and X_2, it is known that $X_1 \overset{\ell r}{\leqslant} X_2$ if and only if

$$E[\phi(X_1, X_2)] \geqslant E[\phi(X_2, X_1)], \quad \forall \phi \in \mathscr{G}_{\ell r}, \tag{3.1}$$

where

$$\mathscr{G}_{\ell r}: \quad \{\phi\colon \phi(x_2, x_1) \leqslant \phi(x_1, x_2), \forall x_1 \leqslant x_2\}. \tag{3.2}$$

Motivated by the above characterization of likelihood ratio ordering, Shanthikumar and Yao (1991) extended this concept to the bivariate case as follows.

Definition 3.1. For a bivariate random variable (X_1, X_2), X_1 is said to be smaller than X_2 according to *joint likelihood ordering* ($X_1 \overset{\ell r:j}{\leqslant} X_2$) if and only if (3.1) holds.

It can be seen that

$$X_1 \overset{\ell r:j}{\leqslant} X_2 \Leftrightarrow f \in \mathscr{G}_{\ell r},$$

where $f(\cdot,\cdot)$ denotes the joint density of (X_1, X_2).

As pointed out by Shanthikumar and Yao (1991), joint likelihood ratio ordering between the components of a bivariate random vector may not imply likelihood ratio ordering between their marginal distributions, but it does imply stochastic ordering between them (that is, $X_1 \overset{\ell r:j}{\leqslant} X_2 \Rightarrow X_1 \overset{\text{st}}{\leqslant} X_2$).

A bivariate function $\phi \in \mathscr{G}_{\ell r}$ is called *arrangement increasing* (AI). Hollander et al. (1977) have studied many interesting properties of such functions, although, apparently, they did not relate it to the notion of likelihood ratio ordering.

The above idea can be extended to compare the components of an n-dimensional vector $X = (X_1, \ldots, X_n)$. We define $X_1 \overset{\ell r:j}{\leqslant} \cdots \overset{\ell r:j}{\leqslant} X_n$ if the joint density $f(x_1, \ldots, x_n)$ of X is an *arrangement increasing function*. (See Marshall and Olkin (1979) for the definition of an arrangement increasing function on $\mathbb{R}^n$.)

In a different context, Robertson and Wright (1982) studied a subclass of arrangement increasing functions on $\mathbb{R}^n$, which they call ISO* functions, as described below. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two vectors on $\mathbb{R}^n$ such that $\sum_{i=1}^{j} y_i \leqslant \sum_{i=1}^{j} x_i$, $j=1,\ldots,n-1$, and $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$. We shall denote this partial ordering between the vectors by $\boldsymbol{x} \overset{*}{\leqslant} \boldsymbol{y}$.

Definition 3.2. A real-valued function ϕ defined on a set $\mathscr{A} \subset \mathbb{R}^n$ is said to be ISO* and $\mathscr{A}$ if $\phi(\boldsymbol{x}) \leqslant \phi(\boldsymbol{y})$, $\forall \boldsymbol{x} \overset{*}{\leqslant} \boldsymbol{y}$.

As mentioned earlier, an ISO* function is arrangement increasing but the converse is not true. It is easy to see that the joint density $f(x_1, x_2)$ of a bivariate random vector (X_1, X_2) is ISO* if and only if the conditional density of X_2 given $X_1 + X_2 = t$ is monotonically increasing for each fixed t.

We shall prove in this section that the joint density of the normalized spacings is ISO* when the joint density of the parent observations is convex. This will hold, in particular, when the X_i's are independent (but not necessarily identical) with log-convex densities. Shaked and Tong (1984) have obtained a different kind of result on spacings from dependent observations.

The above concepts are closely related to majorization and Schur-convexity of functions on $\mathbb{R}^n$. As we shall need them in the sequel, we define them below.

Let $\{x_{(1)} \leqslant x_{(2)} \leqslant \cdots \leqslant x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. The vector $\boldsymbol{y}$ is said to *majorize* the vector $\boldsymbol{x}$ (written as $\boldsymbol{x} \overset{\mathrm{m}}{\leqslant} \boldsymbol{y}$) if $\sum_{i=1}^{j} y_{(i)} \leqslant \sum_{i=1}^{j} x_{(i)}$, $j=1,\ldots,n-1$, and $\sum_{i=1}^{n} y_{(i)} = \sum_{i=1}^{n} x_{(i)}$.

Definition 3.3. A real-valued function ϕ defined on a set $\mathscr{A} \subset \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on $\mathscr{A}$ if $\boldsymbol{x} \overset{\mathrm{m}}{\leqslant} \boldsymbol{y} \Rightarrow \phi(\boldsymbol{x}) \leqslant (\geqslant) \phi(\boldsymbol{y})$.

We shall need the following lemma to prove our next theorem.

Lemma 3.1. *Let $d_i \geqslant 0$, $d_i' \geqslant 0$, $i=1,\ldots,n$, be real numbers and let*

$$u_j = \sum_{i=1}^{j} \frac{d_i}{n-i+1}, \quad v_j = \sum_{i=1}^{j} \frac{d_i'}{n-i+1}, \qquad j=1,\ldots,n.$$

Then

$$\boldsymbol{d}' \overset{\star}{\lll} \boldsymbol{d} \;\Rightarrow\; \boldsymbol{v} \overset{\mathrm{m}}{\leqslant} \boldsymbol{u}. \tag{3.3}$$

Proof. Obviously, the components of $\boldsymbol{u}$ ($\boldsymbol{v}$) are increasing as the d_i's (d_i''s) are nonnegative. Also $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d_i'$.

Let $d' \overset{*}{\leqslant} d$. Then

$$
\begin{aligned}
\sum_{i=1}^{j} u_i &= \sum_{i=1}^{j} \sum_{l=1}^{i} \frac{d_l}{n-l+1} \\
&= \sum_{i=1}^{j} \frac{(n-j)}{(n-i)(n-i+1)} (d_1 + d_2 + \cdots + d_i) \\
&\leqslant \sum_{i=1}^{j} \frac{(n-j)}{(n-i)(n-i+1)} (d'_1 + d'_2 + \cdots + d'_i) \\
&= \sum_{i=1}^{j} v_i \quad \text{for } j = 1, 2, \ldots, n,
\end{aligned} \tag{3.4}
$$

since $\sum_{i=1}^{j} d_i \leqslant \sum_{i=1}^{j} d'_i$, $j = 1, \ldots, n-1$, and $\sum_{i=1}^{n} d_i = \sum_{i=1}^{n} d'_i$.

It follows that $v \overset{m}{\leqslant} v$. □

Theorem 3.1. *Let the joint density $f_X(X_1, \ldots, X_n)$ of $X = (X_1, \ldots, X_n)$ be convex. Then the joint density of $D = (D_{1:n}, \ldots, D_{n:n})$ is* ISO*.

Proof. Let $Y_i = X_{i:n}$ denote the ith order statistic, $i = 1, \ldots, n$. Then the joint density of $Y = (Y_1, \ldots, Y_n)$ is

$$
g_Y(y_1, y_2, \ldots, y_n) = \begin{cases} \sum_P f_X(y_{j1}, y_{j2}, \ldots, y_{jn}) & \text{if } y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n, \\ 0 & \text{otherwise,} \end{cases} \tag{3.5}
$$

where $\sum_P$ denotes summation over all permutations $(j1, j2, \ldots, jn)$ of n integers $\{1, 2, \ldots, n\}$.

From this we obtain the joint density of the normalized spacings $D = (D_{1:n}, \ldots, D_{n:n})$ as

$$
\phi_D(d_1, d_2, \ldots, d_n) = \sum_P f_X(u_{j1}, u_{j2}, \ldots, u_{jn}), \tag{3.6}
$$

where $u_i = \sum_{j=1}^{i} d_j/(n-j+1)$, $j = 1, 2, \ldots, n$.

Since $\sum_P f_X(u_{j1}, u_{j2}, \ldots, u_{jn})$ is a Schur-convex function (cf. Marshall and Olkin, 1979, pp. 82–83), the required result follows from Lemma 3.1 above. □

In particular if X_i's are independent with log-convex densities, the following result holds.

Theorem 3.2. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with log-convex densities. Then the joint density of D is* ISO*.

Proof. Let $g_i(\cdot)$ denote the density of X_i, $i = 1, \ldots, n$. Since $g_i(\cdot)$'s are assumed to be log-convex and the variables are nonnegative, it follows from Marshall and Olkin

(1979, p. 85) that the joint density,

$$g_Y(y_1, y_2, \dots, y_n) = \sum_P \prod_{k=1}^{n} g_k(y_{ji}),$$

of Y is Schur-convex. The required result follows on the lines of Theorem 3.1. □

As pointed out earlier, the joint density of $\boldsymbol{D}$ being ISO* implies that its component random variables are ordered according to joint likelihood ratio ordering. This fact is stated in the following corollary.

Corollary 3.1. *Under the conditions of Theorems* 3.1 *and* 3.2,

$$D_{1:n} \overset{\ell r:j}{\leqslant} D_{2:n} \overset{\ell r:j}{\leqslant} \cdots \overset{\ell r:j}{\leqslant} D_{n:n}.$$

Remarks. (1) If a density is log-convex, it is DFR, but the converse is not true. Theorem 3.2 establishes a stronger ordering between the normalized spacings than does Theorem 2.1 under a stronger condition on the parent distributions.

(2) Results parallel to Theorem 3.2 can be obtained if the parent distributions are log-concave, but otherwise arbitrary. However, we do not know whether results parallel to Theorem 2.1 hold for the IFR case.

(3) As pointed out earlier, in general, $X_1 \overset{\ell r:j}{\leqslant} X_2$ may not imply $X_1 \overset{\ell r}{\leqslant} X_2$. However, one can show that if $X_1, \dots, X_n$ is a random sample from a distribution with log-convex density, then

$$D_{1:n} \overset{\ell r}{\leqslant} \cdots \overset{\ell r}{\leqslant} D_{n:n}.$$

The proof of this result is similar to that of Theorem 2.1 and hence the details are omitted.

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