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WITNESSING DIFFERENCES WITHOUT REDUNDANCIES

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ABSTRACT. We show that $n-1$ elements suffice to witness the differences of n pairwise distinct sets, and provide sufficient conditions for an infinite family of pairwise distinct sets to have a minimal collection of elements witnessing the differences between any two of its members.

By the Extensionality Axiom, the difference between two distinct sets a and b is witnessed by at least one element d such that $d \in a \setminus b$ or $d \in b \setminus a$; in fact any element in the symmetric difference $a\Delta b = (a \setminus b) \cup (b \setminus a)$ witnesses such a difference. For that reason we say that $a\Delta b$ is a differentiating set for $\{a, b\}$. Since all the elements in $a\Delta b$ but one are redundant for that purpose, unless $a\Delta b$ is a singleton, we say that $a\Delta b$ is a redundant or non-minimal differentiating set for $\{a, b\}$, while for any $d \in a\Delta b$, $\{d\}$ is an irredundant or minimal differentiating set for $\{a, b\}$. Suppose now that n pairwise distinct sets $a_1, \dots, a_n$ are given; how many elements do we need to witness their being different from each other? Equivalently, given a differentiating set D for $\{a_1, \dots, a_n\}$, how many redundant elements are to be found in D? Two extreme cases immediately come under attention. If $a_1, \dots, a_n$ can be arranged into an increasing chain with respect to inclusion, or else if $a_1, \dots, a_n$ are pairwise disjoint, then obviously we need exactly $n-1$ elements to witness their differences and any differentiating set for $\{a_1, \dots, a_n\}$ of cardinality m has at least $m-n+1$ redundant elements. In general it is obvious that we need at most $\binom{n}{2}$ elements to witness the differences of n pairwise distinct sets $a_1, \dots, a_n$. However $\binom{n}{2}$ is by far an excessively large bound; in this note we offer an extremely simple proof that $n-1$ elements always suffice to witness the differences among n distinct sets (see Proposition 1). For an earlier proof of this result in the special case in which the n sets are subsets of an n-elements domain see [Bon72, Bol86].

Even from the first rough estimate, it is clear that in the case of finitely many pairwise distinct sets $a_1, \dots, a_n$, an irredundant differentiating set can be obtained from any finite differentiating set by suppressing one after the other the elements which are redundant and remain so as the procedure goes on. It is quite natural to enquire whether that holds also for infinite families of pairwise distinct sets. Any sequence of sets densely ordered with respect to inclusion readily provides an example of a family of pairwise distinct sets for which no minimal differentiating set can exist (see Proposition 2 below). However, by making an essential use of the

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Axiom of Choice (AC), we single out two significant cases in which differences can be witnessed without redundancies:

- with possibly finitely many exceptions, any two sets in the family have finite symmetric difference;
- only finitely many sets in the family have a non-empty intersection with any given set in the family.

Definition 1. 1. d witnesses the difference between two sets a and b if $d \in a$ and $d \notin b$, or else $d \notin a$ and $d \in b$.

2. D is a differentiating set of a family of sets $\{a_i\}_{i \in I}$ if for every $i, j \in I$, if $i \neq j$, then there is $d \in D$ such that d witnesses the difference between a_i and a_j.
3. If D is a differentiating set of a family $\{a_i\}_{i \in I}$, then d is redundant if $D \setminus \{d\}$ is also a differentiating set of $\{a_i\}_{i \in I}$.
4. D is redundant or non-minimal if it has a redundant element, irredundant or minimal otherwise.

Proposition 1. *If D is a differentiating set of a finite family $\{a_1, \dots, a_n\}$, then there is a differentiating set $D_0 \subseteq D$ of $\{a_1, \dots, a_n\}$, such that $|D_0| \leq n - 1$.*

Proof. If $n = 1$, obviously $D_0 = \emptyset$ has the desired property.

Assume the stated proposition holds for n. Given $a_1, \dots, a_n, a_{n+1}$, by inductive hypothesis there is a subset D_0' of D such that $|D_0'| \leq n - 1$, which is a minimal differentiating set for $a_1, \dots, a_n$.

Since D_0' is a differentiating set for $a_1, \dots, a_n$, there can be at most one k, $1 \leq k \leq n$ such that:

$$D_0' \cap a_{n+1} = D_0' \cap a_k.$$

If there is no such k, then it suffices to let $D_0 = D_0'$. Otherwise letting k_0 be the unique such k we pick any $d \in D$ such that

$$d \in (a_{n+1} \cap D) \Delta (a_k \cap D),$$

and let $D_0 = D_0' \cup \{d\}$. D_0 is a differentiating set for $a_1, \dots, a_{n+1}$ and $|D_0| \leq n$. □

Remark. Note that Proposition 1 is implied by the following weaker form: every finite family of n pairwise distinct sets has a differentiating set of cardinality less than n. In fact given a differentiating set D for $\{a_1, \dots, a_n\}$ it suffices to apply this weaker form to $\{a_1 \cap D, \dots, a_n \cap D\}$ to establish the conclusion of Proposition 1. The same remark will apply also to our further results.

Clearly the previous result entails that if D is a differentiating set for $\{a_1, \dots, a_n\}$, then D contains minimal differentiating sets for $\{a_1, \dots, a_n\}$ of cardinality less than or equal to $n - 1$.

Note, however, that it is possible to have minimal differentiating sets of different cardinalities. For example $D = \{0, 1, \dots, n - 1\}$ is a minimal differentiating set for the family $\{a_1, \dots, a_{2^n}\}$ of all subsets of $\{0, 1, \dots, n - 1\}$; D is also a minimal differentiating set for $\{a_1 \cup \{n\}, a_2 \cup \{n + 1\}, \dots, a_{2^n} \cup \{n + 2^n - 1\}\}$, which however admits also $\{n + 1, \dots, n + 2^n - 1\}$ as another minimal differentiating set of cardinality $2^n - 1$. Of course by Proposition 1 there cannot be any bigger minimal differentiating set for the same family.

As mentioned, Proposition 1, for the special case in which $a_1, \dots, a_n$ are subsets of $\{1, \dots, n\}$ has been proved in [Bon72] via a graph-theoretic argument. That result is reported in [Bol86], which, besides Bondy's proof, provides also a different proof which directly applies to yield the general result stated here.

Turning now our attention to infinite families, we note that a minimal differentiating set does not necessarily exist.

Proposition 2. *If $\mathcal{F}$ is a family densely ordered with respect to $\subset$, and D is a differentiating set for $\mathcal{F}$, then every element of D is redundant.*

Proof. Given $a, b \in \mathcal{F}$ such that $a \subset b$, there exists a countable sequence $a_1, a_2, \ldots \in \mathcal{F}$ such that $a \subset a_1 \subset a_2 \subset \ldots \subset b$. Since D is a differentiating set, for every $n > 1$, $D \cap (a_{n+1} \setminus a_n) \neq \emptyset$. Thus $D \cap (b \setminus a)$ is infinite. The conclusion immediately follows. □

Infinite sets have to appear in any family which has no minimal differentiating set: that is among the consequences of the next proposition.

Proposition 3. *If D is a differentiating set for a family $\{a_i\}_{i \in I}$ such that for i and j in I, $i \neq j$, the symmetric difference $(a_i \cap D) \Delta (a_j \cap D)$ is finite, then D includes a minimal differentiating set for $\{a_i\}_{i \in I}$.*

Proof. Let $\mathcal{D}$ be the set of differentiating sets for $\{a_i\}_{i \in I}$ which are contained in D. Let $\mathcal{C}$ be a descending chain, with respect to inclusion, of elements in $\mathcal{D}$. Every $C \in \mathcal{C}$ has a non-empty intersection with the symmetric difference of any pair of distinct elements in $\{a_i\}_{i \in I}$. Moreover, since such symmetric differences are all finite, the same holds also for $\bigcap_{C \in \mathcal{C}} C$, which therefore belongs to $\mathcal{D}$. An application of Zorn Lemma guarantees the existence of a minimal element in $\mathcal{D}$. □

Corollary 1. *Every family of pairwise distinct finite sets has an irredundant differentiating set.*

Infinitely many infinite sets are necessarily present in any family of pairwise distinct sets lacking an irredundant differentiating set; that is among the consequences of the following strengthening of Proposition 3.

Proposition 4. *If D is a differentiating set for a family $\mathcal{F} = \mathcal{F}_0 \cup \{a_1, \ldots, a_n\}$ such that for all a and b in $\mathcal{F}_0$, $(a \cap D) \Delta (b \cap D)$ is finite, then D includes an irredundant differentiating set for $\mathcal{F}$.*

Proof. By induction on n. We distinguish two cases:

Case 1): There exists $D' \subseteq D$, differentiating set for $\mathcal{F}_0$, such that for some $1 \leq i \leq n$, $a_i \cap D'$ has a finite symmetric difference with $b \cap D'$ for some—and hence for all—$b \in \mathcal{F}_0$. Pick any such a_i and let $\mathcal{F}'_0 = \mathcal{F}_0 \cup \{a_i\}$. Obviously,

$$\mathcal{F} = \mathcal{F}'_0 \cup (\{a_1, \ldots, a_n\} \setminus \{a_i\}).$$

Using the same argument of the proof of Proposition 1, it follows that by adding to D' at most n elements in D, we obtain a differentiating set $D'' \subseteq D$ for $\mathcal{F}$. Hence we can apply the inductive hypothesis to conclude that there exists an irredundant differentiating set $D''' \subseteq D'' \subset D$ for $\mathcal{F}$.

Case 2): Suppose the assumption of case 1 does not hold. Let X be a differentiating set for $\{a_1, \ldots, a_n\}$. By applying Zorn's Lemma, as in the proof of Proposition 3, to the family of subsets Y of D such that $Y \cup X$ is a differentiating set for $\mathcal{F}_0$, we obtain a minimal subset D' of D such that $D' \cup X$ is a differentiating set for $\mathcal{F}_0$.

We claim that $D' \cup X$ is also a differentiating set for $\mathcal{F}$. The only non-trivial point to verify is that two elements $a_i \in \{a_1, \dots, a_n\}$, and $b \in \mathcal{F}$ are differentiated by $D' \cup X$. Indeed, by the case hypothesis, for every $a_i \in \{a_1, \dots, a_n\}$, and every $b \in \mathcal{F}$ the symmetric difference $a_i \cap (D' \cup X) \Delta b \cap (D' \cup X)$ is infinite.

While the elements in D' are certainly not redundant, some of the elements in X could be so. However, since X is finite, it suffices to remove the redundant elements of X to obtain a minimal differentiating set for $\mathcal{F}$. □

Remark. Given a family of finite sets $\mathcal{F} = \{a_i\}_{i \in I}$, in ZF, $\mathcal{F}$ can be transformed into $\mathcal{F}' = \{b_i\}_{i \in I}$ where $b_i = \{\langle x, a_i \rangle \mid x \in a_i\}$. The elements of $\mathcal{F}'$ are finite and pairwise distinct, hence by Corollary 1, $\mathcal{F}'$ has an irredundant differentiating set. Using such a set, since the elements of $\mathcal{F}'$ are in fact pairwise disjoint, it is quite straightforward to obtain, in ZF, a choice function for the original family $\mathcal{F}$. Therefore Proposition 3 entails, in ZF, the axiom of choice for families of finite sets.

We do not know whether this principle, which is weaker than AC, suffices to establish in ZF Proposition 3.

The proofs given for Proposition 3 make use of Zorn's Lemma on a family of subsets of D. We can provide different proofs for Propositions 3 and 4 of a more constructive character, which only assume that the given differentiating set D can be well ordered. As a consequence no form of the axiom of choice is required when D is a countable set.

Proposition 5. *If D is a well ordered differentiating set for a family $\mathcal{F}$ such that for all a and b in $\mathcal{F}_0$, $(a \cap D) \Delta (b \cap D)$ is finite, then D includes an irredundant differentiating set for $\mathcal{F}$.*

Proof. Let $\{d_0, d_1, \dots, d_\gamma, \dots\}$ be a well ordering of D.

Let

- $D^0 = D$;
- $D^{\alpha+1} = D^\alpha \setminus \{d_\delta\}$ where δ is the least ordinal s.t. d_δ is redundant (for $\mathcal{F}$) in D^α; if there is no redundant element in D^α, then $D^{\alpha+1} = D^\alpha$;
- $D^\lambda = \bigcap_{\alpha<\lambda} D^\alpha$ for λ a limit ordinal.

Since the D^α's are decreasing with respect to inclusion, there is a (least) ordinal α_0 s.t.

$$D^{\alpha_0} = D^{\alpha_0+1}.$$

Clearly D^{α_0} has no redundant element. Furthermore for every α, D^α is a differentiating set; in particular D^{α_0} is a minimal differentiating set for $\mathcal{F}$. In fact D^0 is a differentiating set by hypothesis and if D^α is a differentiating set for $\mathcal{F}$, then $D^{\alpha+1}$ is a differentiating set for $\mathcal{F}$ as well. Furthermore, due to the finiteness of $(a \Delta b) \cap D$ for all $a, b \in \mathcal{F}$, if for all $\alpha < \lambda$, $(a \Delta b) \cap D^\alpha \neq \emptyset$, then $(a \Delta b) \cap D^\lambda \neq \emptyset$. □

Proposition 6. *If D is a well ordered differentiating set for a family $\mathcal{F} = \mathcal{F}_0 \cup \{a_1, \dots, a_n\}$ such that for all a and b in $\mathcal{F}_0$, $(a \cap D) \Delta (b \cap D)$ is finite, then D includes an irredundant differentiating set for $\mathcal{F}$.*

Proof. Let $\{d_0, d_1, \dots, d_\gamma, \dots\}$ be a well ordering of D. Since D is, in particular, a differentiating set for $\mathcal{F}_0$, as in Proposition 5 we can determine a minimal differentiating set $D_0 \subseteq D$ for $\mathcal{F}_0$. If D_0 is a differentiating set for $\mathcal{F}$, we are done. Otherwise by the argument used in the proof of Proposition 1 there is a subset C_0

of $D \setminus D_0$ having at most n elements, such that $D_0 \cup C_0$ is a differentiating set for $\mathcal{F}$. However, $D_0 \cup C_0$ need not be minimal since the presence of elements in C_0 can make redundant some of the elements in D_0.

Let

- $D_0^0 = D_0$;
- $D_0^{\alpha+1} = D_0^\alpha \setminus \{d_\delta\}$ where δ is the least ordinal such that $d_\delta \in D_0^\alpha$ and is redundant (for $\mathcal{F}$) in $D_0^\alpha \cup C_0$; if no such ordinal exists, $D_0^{\alpha+1} = D_0^\alpha$;
- $D^\lambda = \bigcap_{\alpha<\lambda} D^\alpha$ for λ a limit ordinal;

and let α_0 be the least ordinal such that $D_0^{\alpha_0} = D_0^{\alpha_0+1}$.

$D_0^{\alpha_0}$ has no redundant element in $D_0^{\alpha_0} \cup C_0$. Furthermore no element in C_0 can be redundant in $D_0^{\alpha_0} \cup C_0$, since the elements of C_0 were not redundant in $D_0 \cup C_0$ to start with. However $D_0^{\alpha_0} \cup C_0$ need not be a differentiating set for $\mathcal{F}$.

If $a, c \in \mathcal{F}$ and $a \cap (D_0^{\alpha_0} \cup C_0) = c \cap (D_0^{\alpha_0} \cup C_0)$ then $(a \Delta c) \cap (D_0 \setminus D_0^{\alpha_0})$ must be infinite. For, otherwise, for some $\alpha < \alpha_0$

$$(a\Delta c) \cap (D_0 \cup C_0) \subseteq D_0 \setminus D_0^{\alpha+1},$$

and

$$(a\Delta c) \cap (D_0 \cup C_0) \not\subseteq D_0 \setminus D_0^{\alpha}.$$

This means that $D_0^{\alpha+1}$ is obtained from D_0^α by taking away from D_0^α an element which is not redundant, since it is the only element which witnesses the difference between a and c in D_0^α, contrary to the definition of $D_0^{\alpha+1}$. Obviously, from the fact that $(a\Delta c) \cap (D_0 \setminus D_0^{\alpha_0})$ is infinite it follows that $(a\Delta c) \cap D_0$ is infinite, so that a and c cannot be both in $\mathcal{F}_0$. Hence $D_0^{\alpha_0} \cup C_0$ is a (minimal) differentiating set for $\mathcal{F}_0$. By adding a set C_1 of at most n elements to $D_0^{\alpha_0} \cup C_0$ we can obtain a differentiating set for $\mathcal{F}$. If $D_0^{\alpha_0} \cup C_0 \cup C_1$ is minimal we are done, otherwise we repeat the procedure leading from D_0 to $D_0^{\alpha_0}$, starting with $D_1 = D_0^{\alpha_0} \cup C_0$. We claim that after finitely many steps we obtain a minimal differentiating set for $\mathcal{F}$. This follows from the fact that, if $a \in \{a_1, \dots, a_n\}$ and $a \cap D_i = c \cap D_i$ for some i, then either

i) $c \in \{a_1, \dots, a_n\}$ and $a \cap D_j \neq c \cap D_j$ for any $j > i$, or
ii) $c \in \mathcal{F}_0$ and $a \cap D_j \neq b \cap D_j$ for $j > i$ and $b \in \mathcal{F}_0$.

As for i) notice that if $a \cap D_i = c \cap D_i$, then $(a\Delta c) \cap D_k$ is finite for any $k \geq i$, in particular this holds for $k = j-1$, from which it follows that $a \cap D_j \neq c \cap D_j$.

As for ii), if for $j \geq i$ there were $b \in \mathcal{F}_0 \setminus \{c\}$ such that $a \cap D_j = b \cap D_j$, then $(a\Delta b) \cap D_{j-1}$ would be infinite. Since $a \cap D_i = c \cap D_i$ and $D_{j-1} \setminus D_i$ is finite, it would follow that $(b\Delta c) \cap D_i$ is finite, contradicting $b, c \in \mathcal{F}_0$. □

The full fledged Axiom of Choice AC is certainly needed to prove the following result, which provides another sufficient condition for a family of pairwise distinct objects to have an irredundant differentiating set.

Proposition 7. *If D is a differentiating set of a family $\mathcal{F}$ such that for all $a \in \mathcal{F}$ there are only finitely many b's in $\mathcal{F}$ such that $a \cap b \cap D \neq \emptyset$, then D includes an irredundant differentiating set of $\mathcal{F}$.*

Proof. Given $x \in D$ let

$$A_x = \{a \in \mathcal{F} \mid x \in a\},$$

and let

$$B_x = \{b \in \mathcal{F} \mid x \notin b \wedge b \cap \bigcap A_x \neq \emptyset\}.$$

From the assumption on $\mathcal{F}$ it follows that both A_x and B_x are finite. Moreover, let

$$a_x = \left(\bigcap A_x\right) \setminus \left(\bigcup B_x\right).$$

Clearly if $a_x \neq a_y$, then $a_x \cap a_y = \emptyset$, for every $x \in D$,

$$x \in a \in \mathcal{F} \quad \text{iff} \quad a_x \subseteq a,$$

and only finitely many pairwise distinct a_x's are included in any given element of $\mathcal{F}$.

For $a \in \mathcal{F}$ let $\bar{a} = \{a_x : x \in a\}$ and let $\bar{\mathcal{F}} = \{\bar{a} \mid a \in \mathcal{F}\}$. Since $\bar{\mathcal{F}}$ is a family of finite sets, by Corollary 1 it has a minimal differentiating set $\bar{D}$. The image of any choice function for $\bar{D}$ is a minimal differentiating set for $\mathcal{F}$. □

The previous proof shows how the problem of determining a minimal differentiating set for a given family $\mathcal{F}$ can be reduced, by using the Axiom of Choice, to the problem of determining a minimal differentiating set for the family $\bar{\mathcal{F}}$ whose elements are the quotients of the sets in $\mathcal{F}$ with respect to the equivalence relation $\sim_{\mathcal{F}}$ defined as follows:

$$x \sim_{\mathcal{F}} y \quad \text{iff} \quad \forall a \in \mathcal{F}(x \in a \leftrightarrow y \in b).$$

As for Proposition 3 and Proposition 4 we provide a more constructive proof also for Proposition 7. We first sketch a proof, using the countable axiom of choice, under the assumption that the given family of sets is countable and then point out how AC permits the reduction of the general case to this special one (see Corollary 2 below).

Proposition 8. *If D is a differentiating set of a countable family $\mathcal{F}$ such that $\forall a \in \mathcal{F}$ there are only finitely many b's in $\mathcal{F}$ such that $a \cap b \cap D \neq \emptyset$, then D includes an irredundant differentiating set of $\mathcal{F}$.*

Proof. (Sketch) Let $\mathcal{F} = \{a_i\}_{i\in\omega}$ be a countable family of pairwise distinct sets with $a_0 = \emptyset$.

Let $D_0 = \emptyset$. Assuming D_n has been defined and is a minimal differentiating set for $\{a_0, \dots, a_n\}$, there is at most one k such that $0 \leq k \leq n$ and $a_{n+1} \cap D_n = a_k \cap D_n$. If there is no such k, then we let $D_{n+1} = D_n$. Otherwise D_{n+1} is obtained by first adding to D_n an element of D in $a_{n+1} \setminus a_k$, if that is possible, or else an element of D in $a_k \setminus a_{n+1}$, and then removing the redundant elements until a minimal differentiating set for $\{a_0, \dots, a_{n+1}\}$ is obtained.

For every $k \in \omega$ let

- $f_k = \min\{j \mid \forall i \geq j \ (a_i \cap a_k = \emptyset)\}$,
- $F_k = \min\{j \mid \forall i \geq j \ \forall h < f_k \ (a_i \cap a_h = \emptyset)\}$.

Then it follows that

1. $\forall i > j > F_k \ (a_k \cap D_i \subseteq a_k \cap D_j)$,
2. $\forall i > k \ (a_k \cap D_i \neq \emptyset)$,
3. $D_\omega = \{d \in D : \exists k \forall i > F_k \ d \in a_k \cap D_i\}$ is a minimal differentiating set for $\mathcal{F}$.

Finally, the assumption that $a_0 = \emptyset$ can be discharged passing to the family $\mathcal{F}' = \mathcal{F} \cup \{\emptyset\}$. □

Corollary 2. *If D is a differentiating set of a family $\mathcal{F}$ such that for all $a \in \mathcal{F}$ there are only finitely many b's in $\mathcal{F}$ such that $a \cap b \cap D \neq \emptyset$, then D includes an irredundant differentiating set of $\mathcal{F}$.*

Proof. For $a, b \in F$, let $a \sim_0 b$ if there is a finite sequence of sets $a_0, \dots, a_n$ such that $a_0 = a$, $a_n = b$ and for $0 \leq i < n$, $a_i \cap a_{i+1} \neq \emptyset$. Clearly $\sim_0$ is an equivalence relation and, because of the assumption on $\mathcal{F}$, only countably many members of $\mathcal{F}$ belong to the same equivalence class. Proposition 8 ensures the existence of a minimal differentiating set for every such class, and using AC we can pick one of them. The union of the minimal differentiating sets chosen is a minimal differentiating set for $\mathcal{F}$. □

Remark. Since any family of sets of pairwise disjoint sets trivially fulfills the condition in Corollary 2, the same argument given in the remark following Proposition 4 shows that full AC is a consequence of Corollary 2. Therefore Corollary 2 is equivalent to AC, over ZF.

The above results are by no means limited to the case in which one is dealing with sets and the membership relation; they apply to all those ways in which the difference between distinct objects is witnessed through a binary relation which may or may not hold between elements of a possibly different kind and the given ones. For example, if we look at the (supposedly) distinct columns of an $m \times n$, $(0,1)$-entries matrix, since the difference between two columns is witnessed by one row at least, we have a lower bound on the number of rows that can be suppressed still leaving a matrix with distinct columns, namely $m - n + 1$. Similarly an $\omega \times \omega$ $(0,1)$ matrix with distinct columns such that every column has only finitely many 1's, admits a minimal submatrix, obtained from it by suppressing rows (if necessary), still having different columns.

We can also state some relations with minimal covers: given a family $\mathcal{F} = \{a_i\}_{i \in I}$ and a set D, if for $d \in D$ we let $C(d) = \{(i,j) \mid d \in a_i \Delta a_j\}$, then it is easy to see that D is a differentiating set for $\mathcal{F}$ if and only if $\{C(d) \mid d \in D\}$ covers $I \times I \setminus \Delta(I)$, where $\Delta(I) = \{(i,i) \mid i \in I\}$. Furthermore D is an irredundant differentiating set for $\mathcal{F}$ if and only if such a cover is in fact a *minimal* cover. If $\mathcal{F}$ satisfies the condition of Proposition 3, then $\{C(d) \mid d \in D\}$ is a cover of $I \times I \setminus \Delta(I)$ with the property that every infinite subfamily has an empty intersection. Every cover having such a property has a minimal subcover, and Proposition 3 can be derived from this principle. Incidentally, such a principle can be established by using essentially the same argument used in proving Proposition 6. Despite such connections, we note however that the existence of a minimal subcover of a given family of sets and the existence of an irredundant differentiating set for it, are in general unrelated. For example, since for every natural number n we have that $n = \{0, \dots, n-1\}$, the family $\mathbf{N}$ of the natural numbers is a cover of $\mathbf{N}$ itself, which has no minimal subcover, while $\mathbf{N}$ is an irredundant differentiating set for $\mathbf{N}$. On the other hand the family $\{a_q \mid q \in \mathbf{Q}\} \cup \{\mathbf{Q}\}$, where $a_q = \{p \in \mathbf{Q} \mid p \leq q\}$, has $\{\mathbf{Q}\}$ as a minimal subcover, but it has no irredundant differentiating set.

We should mention that the original motivation which led to the results in this note came from investigations into the decision problem for the satisfiability of formulae in the language with the equality and the membership relation (see [PP92]).

As a matter of fact the possibility of bounding to $n-1$ the number of sets that is necessary to add to given n distinct sets $a_1, \ldots, a_n$ to make the resulting structure extensional over $a_1, \ldots, a_n$, greatly improves the efficiency of the decision procedure for (an extension of) the class $MLSS$ (see [CFO89]).

Concerning the naturally arising question of how many successive addition of differentiating sets are needed to eventually obtain an extensional structure including the originally given sets $a_1, \ldots, a_n$, we point out that [PP88] provides an example of two sets ω' and ω'' for which there is no way of completing that task in finitely many steps.

It is on the ground of such an example that a way of stating the existence of infinite sets, which is remarkably simple from the point of view of logical complexity, becomes available, as shown in [PP88] and [PP90].

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