NORTH-HOLLAND

Generalized Inverses With Respect to General Norms. III

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ABSTRACT

We explain the relations between the existence of a minimum-norm inverse, the existence of an approximate-norm inverse, and the linear approximation property, for any norm on R^n.

1. INTRODUCTION

This is a sequel to [1] and [2].

First we shall quickly recall the definitions. Let us consider an $m \times n$ real matrix A, a norm $\|\cdot\|_1$ on R^n and a norm $\|\cdot\|_2$ on R^m. A minimum-$\|\cdot\|_1$ inverse (m$\|\cdot\|_1$i) of A is a g-inverse G of A (i.e. $AGA = A$) such that for each y which makes the system $Ax = y$ consistent, we have that $\|Gy\|_1 \leqslant \|x\|_1$ for all solutions x of $Ax = y$. An approximate-$\|\cdot\|_2$ inverse (a$\|\cdot\|_2$i) of A is a g-inverse G of A such that for all y, $\|AGy - y\|_2 = \min_x \|Ax - y\|_2$. References [1–5] give some results on the existence and evaluations of the above generalized inverses.

In the present note we shall explain the relations between the above types of g-inverses for general norms. Let us also recall a definition from [1]. If $(X, \|\cdot\|)$ is a finite-dimensional normed linear space and $\mathscr{S}$ is a subspace of X, we say that $\mathscr{S}$ has the linear approximation property if for every x in X

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there is an $M(x)$ in $\mathscr{S}$ such that $\|x - M(x)\| \leqslant \|x - z\|$ for all z in $\mathscr{S}$ and such that the map $x \to M(x)$ is linear. The existence of a$\|\cdot\|$i's and m$\|\cdot\|$i's with respect to a general norm is closely related to various subspaces having the linear approximation property, as the present paper shows. Recall that [2] a finite-dimensional normed linear space of dimension $\geqslant 3$ has the property that every subspace has the linear approximation property if and only if the norm is given by an inner product.

2. RESULTS

We shall deal with norms on finite-dimensional real vector spaces, not just with R^n. Depending on the suitability, we shall state some of our results for linear transformations and some for matrices.

THEOREM 1. *Let $T : Y \to X$ be a linear transformation and $\|\cdot\|$ be a norm on X. Let $\mathscr{S} = T(Y)$. Then the following are equivalent:*

(i) *T has an a$\|\cdot\|$i.*
(ii) *$\mathscr{S}$ has the linear approximation property.*

Proof. (i) $\Rightarrow$ (ii): Let S be an a$\|\cdot\|$i of T. Then for any $x \in X$, $\|x - TS(x)\| = \inf_{z \in \mathscr{S}} \|x - z\|$. So $TS : X \to \mathscr{S}$ is a linear projection, which gives the linear approximation property of $\mathscr{S}$.

(ii) $\Rightarrow$ (i): If $\mathscr{S}$ has the linear approximation property, let $\pi : X \to \mathscr{S}$ be such that

$$\|x - \pi(x)\| = \inf_{z \in \mathscr{S}} \|x - z\|.$$

Let U be any g-inverse of T. If we define

$$S(x) = U\pi(x) \qquad \text{for all} \quad x \in X,$$

then

$$\|x - TS(x)\| = \|x - TU\pi(x)\| = \|x - \pi(x)\| = \inf_{z \in \mathscr{S}} \|x - z\|,$$

because TU is the identity on $\mathscr{S}$. So S is an approximate-$\|\cdot\|$ inverse of T. ■

THEOREM 2. *Let $T : X \to Y$ be a linear transformation and $\|\cdot\|$ be a norm on X. Let $\mathscr{S} = T^{-1}(\{0\})$, the kernel of T. Then the following are equivalent*:

(i) *T has a m$\|\cdot\|$i.*
(ii) *$\mathscr{S}$ has the linear approximation property.*

Proof. (i) $\Rightarrow$ (ii): Let S be a m$\|\cdot\|$i of T. If $x_0 \in X$, then all x such that $T(x) = T(x_0)$ are given by $x + z$ for $z \in \mathscr{S}$. So, if $x \in X$,

$$\|ST(x)\| = \inf_{z \in \mathscr{S}} \|x + z\| = \inf_{z \in \mathscr{S}} \|x - z\|.$$

So, if we denote $I - ST$ by π, then

$$\|ST(x)\| = \|x - \pi(x)\| = \inf_{z \in \mathscr{S}} \|x - z\|.$$

But $\pi = I - ST$ is a linear transformation from X to $\mathscr{S}$, because $T((I - ST)(x)) = 0$. This shows that $\mathscr{S}$ has the linear approximation property.

(ii) $\Rightarrow$ (i): If $\mathscr{S}$ has the linear approximation property, let $\pi : X \to \mathscr{S}$ be such that

$$\|x - \pi(x)\| = \inf_{z \in \mathscr{S}} \|x - z\|.$$

Let U be any g-inverse of T. If we define

$$S(y) = (I - \pi)U(y) \qquad \text{for all} \quad y \in Y,$$

then

$$\|ST(x)\| = \|(I - \pi)UT(x)\| = \inf_{z \in \mathscr{S}} \|U(T(x)) - z\| = \inf_{z \in \mathscr{S}} \|x - z\|;$$

the last equality follows because the sets $\{U(T(x)) - z : z \in \mathscr{S}\}$ and $\{x - z : z \in \mathscr{S}\}$ are identical. This S is a m$\|\cdot\|$i of T. ■

REMARK 1. The above two theorems say that, for the existence of m$\|\cdot\|$i's or a$\|\cdot\|$i's of T, the actual structure of T is unimportant and only the linear approximation property of the kernel or the range has to be decided, as the case may be.

In [1], it was observed that the existence of a m$\|\cdot\|$i of a matrix can be reduced to the existence of a m$\|\cdot\|$i of a matrix of the type $[I\ A]$, if the norm is permutation invariant. The following theorem relates the existence of a m$\|\cdot\|$i to the existence of an a$\|\cdot\|$i for an appropriate matrix if $\|\cdot\|$ is permutation invariant.

THEOREM 3. *Let $\|\cdot\|$ be a norm on R_n which is permutation invariant. Then a $k \times n$ matrix $[I\ A]$ has a m$\|\cdot\|$i if and only if the $n \times (n-k)$ matrix $\begin{pmatrix} I \\ -A \end{pmatrix}$ has an a$\|\cdot\|$i.*

Proof. Observe that the null space of $[I\ A]$ is the same as the range space of $\begin{pmatrix} -A \\ I \end{pmatrix}$, and use Theorems 1 and 2. ■

A more general result for any norm is the following.

THEOREM 4.

(a) *Let A be an $m \times n$ matrix and $\|\cdot\|$ be a norm on R^n. Let G be a g-inverse of A. Then A has a m$\|\cdot\|$i if and only if the $n \times n$ matrix $I - GA$ has an a$\|\cdot\|$i.*

(b) *Let A be an $m \times n$ matrix and $\|\cdot\|$ be a norm on R^m. Let G be a g-inverse of A. Then A has an a$\|\cdot\|$i if and only if the $m \times m$ matrix $I - AG$ has a m$\|\cdot\|$i.*

Proof. For (a) observe that the null space of A is same as the range space of $I - GA$, and for (b) observe that the range space of A is same as the null space of $I - AG$. ■

Using both the parts (a) and (b) of the above theorem, we get

THEOREM 5. *Let G be a g-inverse of A. Then*

(i) *A has a m$\|\cdot\|$i if and only if GA has a m$\|\cdot\|$i.*
(ii) *A has an a$\|\cdot\|$i if and only if AG has an a$\|\cdot\|$i.*

REMARK 2. The above results say that any result about m$\|\cdot\|$i's can be translated to a result about a$\|\cdot\|$i's and vice-versa, for the same norm.

The present paper and two earlier papers in this sequence grew out of a conversation the author had with Professor C. R. Rao some years ago.

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